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TAU FUNCTIONS, PRYM-TYURIN CLASSES AND LOCI OF DEGENERATE DIFFERENTIALS

DMITRI KOROTKIN, ADRIEN SAUVAGET, AND PETER ZOGRAF

Abstract. We study the rational Picard group of the projectivized moduli space $P\overline{\mathfrak{M}}_g^{(n)}$ of holomorphic n-differentials on complex genus g stable curves. We define $n-1$ natural classes in this Picard group that we call *Prym-Tyurin* classes. We express these classes as linear combinations of boundary divisors and the divisor of n-differentials with a double zero. We give two different proofs of this result, using two alternative approaches: an analytic approach that involves the Bergman tau function and its vanishing divisor and an algebro-geometric approach that involves cohomological computations on the universal curve.

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1. Introduction

1.1. **Moduli space of n-differentials.** Let g and n be positive integers with $g \geq 2$. Let $\mathcal{M}_g$ (respectively $\overline{\mathcal{M}}_g$) be the moduli space of smooth (respectively stable nodal) complex curves. Denote by $D_0 \subset \overline{\mathcal{M}}_g$ the closure of the locus of stable curves with one nonseparating node. Further, denote by $D_i \subset \overline{\mathcal{M}}_g$, $1 \leq i \leq [g/2]$, the closure of the locus of curves with a separating node and two irreducible components of genera i and $g-i$. Finally, denote by $\pi : \mathcal{C}_g \to \mathcal{M}_g$ or $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ the universal curve and by $\omega = \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ the relative dualizing sheaf.

Let

$$\Omega_g^{(n)} = R^0\pi_*\omega^{\otimes n} \to \overline{\mathcal{M}}_g$$

be the direct image of the nth tensor power of ω. Using the Riemann-Roch formula and Serre's duality, one can easily check that $h^1(C, \omega_C^{\otimes n}) = 0$ if $n \geq 2$ and $h^1(C, \omega_C) = 1$ for any stable curve C. Thus $\Omega_g^{(n)}$ is a vector bundle for any $n \geq 1$. For $n = 1$ we write Ω_g instead of $\Omega_g^{(1)}$ and call it the *Hodge bundle*. The Riemann-Roch formula implies:

$$\operatorname{rk} \Omega_g^{(n)} = \begin{cases} g & \text{if } n = 1; \\ (2n-1)(g-1) & \text{otherwise.} \end{cases}$$

We define the following elements in the rational Picard group of $\overline{\mathcal{M}}_g$:

- the Chern class $\lambda_n \in \operatorname{Pic}(\overline{\mathcal{M}}_g)$ of the determinant line bundle of $\Omega_g^{(n)}$;
- the Poincaré dual classes $\delta_0, \ldots, \delta_{[g/2]} \in \operatorname{Pic}(\overline{\mathcal{M}}_g)$ of the boundary divisors $D_0, D_1, \ldots, D_{[g/2]} \subset \overline{\mathcal{M}}_g$.

For $k = 1$ we write λ instead of λ_1 and call this class the *Hodge class*.

Definition 1.1. The total space of the vector bundle $\Omega_g^{(n)}$ is denoted by $\overline{\mathfrak{M}}_g^{(n)}$ and is called the *space of n-differentials*.

The points of $\overline{\mathfrak{M}}_g^{(n)}$ correspond to equivalence classes of pairs (C, w), where C is a stable genus g algebraic curve, and w is an n-differential on C. We recall (see [2] for example) that an n-differential w on C is a meromorphic n-differential on each irreducible component of the normalization of C such that

- w can only have poles at the preimages of the nodes;
- these poles are of order at most n;
- at every node the n-residues of w at the poles satisfy

$$\operatorname{res}_{p_1}(w) = (-1)^n \operatorname{res}_{p_2}(w)$$

  where p_1 and p_2 are the two preimages of the node.

We denote by $\nu : \overline{\mathfrak{M}}_g^{(n)} \to \overline{\mathcal{M}}_g$ the forgetful map and we will use the same notation for its restriction $\nu : \mathfrak{M}_g^{(n)} \to \mathcal{M}_g$ to the locus of smooth curves. We also denote by $\tilde{\nu} : P\overline{\mathfrak{M}}_g^{(n)} \to \overline{\mathcal{M}}_g$ the projectivized space of n-differentials.

In this paper we study the Picard group of $P\overline{\mathfrak{M}}_g^{(n)}$. We will work over rational numbers, so that Pic will always denote the *rational* Picard group.

By abuse of notation we will denote by λ, λ_n and δ_i both the elements of $\operatorname{Pic}(\overline{\mathcal{M}}_g)$ and their pull-backs in $\operatorname{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$. In addition, we introduce the first Chern class $\psi \in \operatorname{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ of the tautological line bundle $L \to P\overline{\mathfrak{M}}_g^{(n)}$.

The following lemma is standard (cf., e.g, [13], Lemma 1).

Lemma 1.2. *The classes $\lambda, \psi, \delta_0, \ldots, \delta_{[g/2]}$ form a basis of* $\operatorname{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$.

The goal of this paper is to define the Prym-Tyurin classes in $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ and express them in the above basis.

1.2. **Stratification of $P\mathfrak{M}_g^{(n)}$.** The space of n-differentials is naturally stratified according to the multiplicities of the zeroes of the differential.

Let $\mathbf{k} = (k_1, \dots, k_m)$ be a partition of $n(2g-2)$. We denote by $\mathfrak{M}_g^{(n)}[\mathbf{k}] \subset \mathfrak{M}_g^{(n)}$ the locus of pairs (C, w) such that the n-differential w has m pairwise distinct zeros of orders exactly k_i. This locus is $\mathbb{C}^*$-invariant, thus we can also define its projectivization $P\mathfrak{M}_g^{(n)}[\mathbf{k}] \subset P\mathfrak{M}_g^{(n)}$. The space $P\mathfrak{M}_g^{(n)}$ is the disjoint union of the strata $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ for all partitions $\mathbf{k}$ of $n(2g-2)$. The following properties of the strata were proved in [15] and [18].

- Each stratum $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is smooth.
- If at least one k_i is not divisible by n then either $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is empty or it has pure dimension $2g-3+m$.
- If all k_i's are divisible by n then $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ has components of dimension $2g-2+m$. A differential (C, w) lies in a component of dimension $2g-2+n$ if and only if w is the nth power of a holomorphic differential. The stratum $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ may also have irreducible components of dimension $2g-3+m$ composed of n-differentials that are not nth powers.

We denote by $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{k}]$ the closure of $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ in $P\overline{\mathfrak{M}}_g^{(n)}$. In particular we have $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{1}] = P\overline{\mathfrak{M}}_g^{(n)}$, where $\mathbf{1}$ stands for the partition $(1, 1, \dots, 1)$.

Definition 1.3. Let $g, n \geq 2$. The *divisor of degenerate n-differentials* is defined as

$$D_{\deg} = \begin{cases} P\overline{\mathfrak{M}}_g^{(n)}[2, 1, \dots, 1], & \text{if} \quad (g, n) \neq (2, 2), \\ P\overline{\mathfrak{M}}_g^{(n)}[2, 1, 1] + 2 \cdot P\overline{\mathfrak{M}}_g^{(n)}[2, 2], & \text{if} \quad g = n = 2. \end{cases}$$

We denote by $\delta_{\deg}$ the cohomology class that is Poincaré dual of $D_{\deg}$.

Remark 1.4. Heuristically, $D_{\deg}$ is the divisor of n-differentials with a double zero, and for $(g, n) \neq (2, 2)$ it is just the closure of $P\mathfrak{M}_g^{(n)}[2, 1, \dots, 1]$ in $P\overline{\mathfrak{M}}_g^{(n)}$. In the case $g = n = 2$, however, $D_{\deg}$ has a special component consisting of squares of holomorphic differentials. This is because in genus 2 each quadratic differential is invariant with respect to the hyperelliptic involution. The four simple zeroes of w are pairwise equivalent under the hyperelliptic involution, and when two non-equivalent zeroes coalesce, the other two ones also coalesce, giving a differential with two double zeroes. Since every such differential has two square roots that differ by a sign, the divisor $P\overline{\mathfrak{M}}_g^{(n)}[2, 2]$ comes with a factor of 2. (Note that when two equivalent zeroes coalesce, the differential in the limit has one double zero at a Weierstrass point and two simple zeroes.)

1.3. **First definition of Prym-Tyurin classes.** Let (C, w) be a point in the projectivized moduli space $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$. One can define a canonical cyclic ramified cover $f : \widehat{C} \to C$ of degree n, where

$$\widehat{C} = \{(x, v) | x \in C,\ v \in T_x^* C,\ v^n = w\}.$$

This cover is completely ramified over the zeros of w. The curve $\widehat{C}$ is smooth of genus $\widehat{g} = n^2(g-1)+1$. It comes with a *canonical holomorphic differential* v given by $v(x, v) = v$. This differential v on $\widehat{C}$ satisfies $v^n = f^* w$.

The action of $\mathbb{Z}/n\mathbb{Z}$ on the cover is given by $\rho^k : (x, v) \mapsto (x, \rho^k v)$, where $\rho = e^{\frac{2\pi\sqrt{-1}}{n}}$. We denote by $\sigma : \widehat{C} \to \widehat{C}$ the automorphism of $\widehat{C}$ corresponding to

$k = 1$. Now consider the natural map

$$\hat{\nu} : P\mathfrak{M}_g^{(n)}[\mathbf{1}] \to \mathcal{M}_{\widehat{g}},$$
$$(C, w) \mapsto \widehat{C}$$

($\widehat{C}$ remains the same when we multiply w by a non-zero constant). We consider the pull-back of the Hodge bundle $\Omega_{\widehat{g}}$ by the map $\hat{\nu}$. The automorphism σ induces an endomorphism σ^* of the vector bundle $\hat{\nu}^*\Omega_{\widehat{g}}$ given by: $((C, w), u) \mapsto ((C, w), \sigma^* u)$, where u is an element of $H^0(\widehat{C}, \omega_{\widehat{C}})$. The endomorphism σ^* satisfies $(\sigma^*)^n = \mathrm{Id}$. Hence we have a decomposition

$$\hat{\nu}^*\Omega_{\widehat{g}} = \bigoplus_{k=0}^{n-1} \Lambda^{(k)}, \tag{1.1}$$

where $\Lambda^{(k)}$ is the eigenbundle of $\hat{\nu}^*\Omega_{\widehat{g}}$ corresponding to the eigenvalue $\rho^k = e^{\frac{2\pi\sqrt{-1}k}{n}}$.

Remark 1.5. The space $\Lambda^{(k)}$ is a vector bundle because the dimension of the fiber of $\Lambda^{(k)}$ is upper semi-continuous for all k and $\hat{\nu}^*\Omega_{\widehat{g}}$ is a vector bundle thus the rank of each $\Lambda^{(k)}$ is constant. We will see further that $\mathrm{rank}(\Lambda^{(k)}) = (2n - 2k + 1)(g - 1)$.

Definition 1.6. The vector bundles $\Lambda^{(k)}$ are called the *Prym-Tyurin vector bundles*. The *Prym-Tyurin class* $\lambda_{PT}^{(k)}$ is the first Chern class $c_1(\Lambda^{(k)}) \in \mathrm{Pic}(P\mathfrak{M}_g^{(n)}[\mathbf{1}])$.

For $n = 2$ the study of vector bundles of this type was initiated by Prym [17] and for $n > 2$ by A. N. Tyurin [19].

Remark 1.7. By abuse of notation we denote by $\lambda_{PT}^{(k)} = \det \Lambda^{(k)}$ both determinant line bundle and its class in the Picard group.

Let us mention the following fact: for any $(C, w) \in \mathfrak{M}_g^{(n)}$ we can construct the object $(f : \widehat{C} \to C, \sigma)$ and the eigenspaces of σ^*, however the genus of $\widehat{C}$ and the dimensions of the eigenspaces of σ^* will depend on the orders of zeros of w. We will even prove in Section 5 that the map $\hat{\nu} : \mathfrak{M}_g^{(n)} \to \mathcal{M}_{\hat{g}}$, used to define the Prym-Tyurin vector bundles, admits no natural extension to $P\overline{\mathfrak{M}}_g^{(n)}$.

Nonetheless, in the next section we extend the definition of the Prym-Tyurin class to $P\overline{\mathfrak{M}}_g^{(n)}$ by a construction involving the space of admissible covers. This will allow to compactify the $P\overline{\mathfrak{M}}_g^{(n)}$ in such a way that the cover constructed above is of constant arithmetic genus.

1.4. **Admissible covers and Prym-Tyurin bundles.** We denote by Hur_g^n the moduli space whose geometric points are isomorphism classes of pairs $(f : \widehat{C} \to C, \sigma)$ where:

- C and $\widehat{C}$ are smooth curves;
- f is a cyclic ramified cover of degree n which is totally ramified over $N := \deg(\omega^{\otimes n}) = 2n(g - 1)$ distinct points of C;
- σ is an automorphism of $\widehat{C}$ that commutes with f.

We denote by $\overline{\mathrm{Hur}}_g^n$ the compactification of this space by admissible covers (see [7]). The space of admissible covers has two forgetful maps (source and target of the cover):

$$\begin{tikzcd} & \overline{\mathrm{Hur}}_g^n \arrow[dl, "\text{target}"'] \arrow[dr, "\text{source}"] & \\ \overline{\mathcal{M}}_{g,N}/S_N & & \overline{\mathcal{M}}_{\widehat{g}}. \end{tikzcd}$$

We consider the pull-back $\mathrm{source}^*\Omega_{\widehat{g}}$ of the Hodge bundle under the source map. This vector bundle is endowed with the automorphism

$$\sigma^* : \Big((\widehat{C} \to C, \sigma), u\Big) \quad \mapsto \quad \Big((\widehat{C} \to C, \sigma), \sigma^* u\Big).$$

Thus, as in (1.1), we have the decomposition

$$\mathrm{source}^*\Omega_{\widehat{g}} = \bigoplus_{k=0}^{n-1} \Lambda^{(k)}, \tag{1.2}$$

where $\Lambda^{(k)}$ is the eigenbundle corresponding to the eigenvalue $\rho^k = e^{\frac{2\pi\sqrt{-1}k}{n}}$.

In the previous section we have constructed an embedding $i : P\mathfrak{M}_g^{(n)}[\mathbf{1}] \hookrightarrow \overline{\mathrm{Hur}}_g^n$ and, by construction, the pull-back $i^*\Lambda^{(k)}$ is, indeed, isomorphic to the Prym-Tyurin bundle $\Lambda^{(k)}$ over $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ as defined in the previous section.

We will see that the Prym-Tyurin vector bundles, and therefore the Prym-Tyurin classes, have a natural extension to the compactification of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ by admissible covers. We would like, however, to extend the Prym-Tyurin classes to a different compactification of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$, namely, to $P\overline{\mathfrak{M}}_g^{(n)}$. To do that we will construct a bigger space with a projection to both $\overline{\mathrm{Hur}}_g^n$ and $P\overline{\mathfrak{M}}_g^{(n)}$ and use the push-forward of a pull-back.

1.5. **Space of admissible differentials.**

Definition 1.8. Let

$$I : \mathfrak{M}_g^{(n)}[\mathbf{1}] \hookrightarrow \overline{\mathfrak{M}}_g^{(n)} \times_{\overline{\mathcal{M}}_g} \overline{\mathrm{Hur}}_g^n$$

be the product of the two natural embeddings. The *moduli space of admissible n differentials* $X(g,n)$, is the Zariski closure of the image of I in $\overline{\mathfrak{M}}_g^{(n)} \times \overline{\mathrm{Hur}}_g^n$.

Remark 1.9. The space of admissible n-differentials is a compactification of the stratum $\mathfrak{M}_g^{(n)}[\mathbf{1}]$. In [2], the authors introduced and described another compactification, the *incidence variety compactification*. The incidence variety is obtained by replacing the space $\overline{\mathrm{Hur}}_g^n$ in Definition 1.8 by the space $\overline{\mathcal{M}}_{g,N}/S_N$. We denote the incidence variety by $X_{g,n}^{\mathrm{inc}}$.

There is a birational and finite map from $X(g,n)$ to $X_{g,n}^{\mathrm{inc}}$, however this map is not an isomorphism (see Example 4.3 of [2]). We will make use of $X_{g,n}^{\mathrm{inc}}$ once in this text (proof of Lemma 2.4).

The space $X(g,n)$ has two natural morphisms: $\mathrm{adm} : X(g,n) \to \overline{\mathrm{Hur}}_g^n$ and $\mathrm{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$. We summarize the notation on the following diagram.

$$\begin{tikzcd}
 & \bigoplus \Lambda^{(k)} \arrow[r, "\mathrm{source}^*"] \arrow[d] & \Omega_{\widehat{g}} \arrow[d] \\
X(g,n) \arrow[r, "\mathrm{adm}"] \arrow[d, "\mathrm{diff}"'] & \overline{\mathrm{Hur}}_g^n \arrow[r, "\mathrm{source}"] & \overline{\mathcal{M}}_{\widehat{g}} \\
P\overline{\mathfrak{M}}_g^{(n)} & &
\end{tikzcd}$$

By construction the forgetful map $\mathrm{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is birational and its restriction to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an isomorphism onto its image. In general the space $X(g,n)$ is not normal; thus the push-forward of classes in the Picard group under diff is ill-defined. However we will prove the following proposition in Section 2.

Proposition 1.10. *There exist two smooth open-dense substacks $j : V \hookrightarrow X(g,n)$ and $j' : U \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}$ fitting into the commutative diagram*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d] & X(g,n) \arrow[d, "\mathrm{diff}"] \\ U \arrow[r, "j'"'] & P\overline{\mathfrak{M}}_g^{(n)} \end{tikzcd}$$

such that

- *the map $V \to U$ is an isomorphism on the underlying coarse spaces;*
- *the complement of U is of codimension at least 2 in $P\overline{\mathfrak{M}}_g^{(n)}$.*

The existence of U and V as above allows one to define the induced push-forward diff_* in the Picard groups. Indeed, the first property ensures that $\mathrm{Pic}(U)$ and $\mathrm{Pic}(V)$ are isomorphic. Similarly, the second propery ensures that j' induces an isomorphism between $\mathrm{Pic}(U)$ and $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$. Thus, given an element in $\mathrm{Pic}(X(g,n))$, one defines its push-forward to $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ by first taking its pull-back by j, then the push-forward from V to U, and finally the push-forward by j'.

We will see that the definition of diff_* is independent of the choice of U and V satisfying the above hypothesis (see Lemma 2.2).

Definition 1.11. The *Prym-Tyurin class* $\lambda_{PT}^{(k)}$ on $P\overline{\mathfrak{M}}_g^{(n)}$ is defined by the formula $\lambda_{PT}^{(k)} = \mathrm{diff}_* c_1(\Lambda^{(k)})$.

1.6. **Statement of the results.** The main result obtained in this paper is the expression for the Prym-Tyurin classes and the class δ_{deg} of Definition 1.3 in the basis $(\lambda, \psi, \delta_i)$.

Theorem 1.12. *In the rational Picard group of $P\overline{\mathfrak{M}}_g^{(n)}$ we have*

$$\delta_{\mathrm{deg}} = 12n(n+1)\lambda - 2(g-1)(2n+1)\psi - n(n+1)\sum_{i=0}^{[g/2]} \delta_i; \tag{1.3}$$

$$\lambda_{PT}^{(n-k)} = (6k^2+6k+1)\lambda - \frac{g-1}{n}k(2k+1)\psi - \frac{1}{2}k(k+1)\sum_{i=0}^{[g/2]} \delta_i + c_k\ \delta_{\mathrm{deg}}, \tag{1.4}$$

where

$$c_k = \begin{cases} \frac{2k-n}{2n}, & \text{if } (n-1)/2 < k < n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

1.7. **Strategy of the proof.** Formula (1.3) of Theorem 1.12 is proved in two distinct ways.

- In Section 3 we introduce the Bergman tau function on the moduli space $\mathfrak{M}_g^{(n)}$. We study its transformation property and its asymptotic behavior at the boundary divisors D_{deg} and D_i, $0 \le i \le [g/2]$. We explicitly compute the vanishing order of the Bergmann tau function along these divisors. We use these results to express the divisor δ_{deg} in the $(\lambda, \delta_i, \psi)$ basis of the Picard group. This first proof is a further development of the ideas introduced in [13] and [14].

- In Section 4 we give an alternative proof of Formula (1.3) based on algebro-geometric computations. We consider the moduli space of n-differentials on genus g curves with one marked point. This space carries a vector bundle of 2-jets of an n-differential at the marked point. The Euler class of this vector bundle has a natural expression involving the locus A_2 of n-differentials with a double zero at the marked point. The locus A_2 pushes forward to $D_{\deg}$ under the forgetful map that forgets the marked point. This allows one to compute the cohomology class $\delta_{\deg}$ that is Poincaré dual to the divisor class of $D_{\deg}$.

To prove Formulas (1.4) and (1.5) we combine (1.3) with the following two facts.

- First, the well-known Mumford formula [16] expressing the first Chern class of the vector bundle of k-differentials on $\overline{\mathcal{M}}_g$ via the Hodge class:

$$\lambda_k = (6k^2 - 6k + 1)\lambda - \frac{k(k-1)}{2} \sum_{i=0}^{[g/2]} \delta_i \ . \tag{1.6}$$

- Second, the fact that the morphism

$$\begin{array}{rcl} \Phi_k : \Lambda^{(k)} \otimes T^{\otimes n-k} & \to & H^0(C, \omega_C^{n-k+1}) \\ (q, v^{n-k}) & \mapsto & qv^{n-k} \end{array} \tag{1.7}$$

is actually an isomorphism of vector bundles outside $D_{\deg}$.

The second fact allows one to compute the rank of the Prym-Tyurin vector bundle $\Lambda^{(k)}$:

$$\operatorname{rk} \Lambda^{(k)} = \operatorname{rk} \Omega^{(n-k+1)} = (2n - 2k + 1)(g - 1) \ , \qquad k = 1, \dots, n-1 \ . \tag{1.8}$$

It also implies that

$$\lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi + \text{const} \cdot \delta_{\deg} \ . \tag{1.9}$$

In Section 5, we study the asymptotic of the determinant of Φ_k along the divisor $D_{\deg}$ to obtain Expressions (1.4) and (1.5).

Plan of the paper. In Section 2 we prove Proposition 1.10 and thus complete the definition of the Prym-Tyurin classes. In Sections 3 and 4 we prove Formula (1.3) of Theorem 1.12 using the two different approaches described above. In Section 5 we discuss the relationship between Prym-Tyurin vector bundles and vector bundles of n-differentials and derive a relationship between corresponding determinant line bundles. This allows us to express the Prym-Tyurin classes in the $(\lambda, \delta_i, \psi)$ basis of the Picard group and complete the proof of Theorem 1.12.

2. The space of admissible n-differentials

In this Section we justify the definition of the Prym-Tyurin classes by proving the following extended version of Proposition 1.10.

Proposition 2.1. *There exist two smooth open-dense substacks $j : V \hookrightarrow X(g,n)$ and $j' : U \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}$ fitting into the commutative diagram*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d] & X(g,n) \arrow[d, "\text{diff}"] \\ U \arrow[r, "j'"'] & P\overline{\mathfrak{M}}_g^{(n)} \end{tikzcd}$$

such that

- *V and U contain the image of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ under the embeddings into $X(g,n)$ and $P\overline{\mathfrak{M}}_g^{(n)}$;*
- *the complement of U is of codimension at least 2 in $P\overline{\mathfrak{M}}_g^{(n)}$;*
- *the map $V \to U$ is an isomorphism on the underlying coarse spaces;*
- *there exists a line bundle $T \to V$ such that T is a sub-vector bundle of $\Lambda^{(1)}$, $T^{\otimes n} \simeq L$ and the restriction of T to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ coincides with the sub-vector bundle of $\hat{\nu}^*\Omega_{\widehat{g}}$ spanned by v.*

Before proving this proposition, let us prove the following lemma.

Lemma 2.2. *Let $(U \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}, V \hookrightarrow X(g,n))$ and $(U' \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}, V' \hookrightarrow X(g,n))$ be two sub-stacks satisfying the list properties of Proposition 1.10. Let diff_* and diff'_* be the morphisms $\mathrm{Pic}(X(g,n)) \to \mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ constructed as in Section 1.5. Then we have $\mathrm{diff}_* = \mathrm{diff}'_*$.*

Proof. Let us denote by $U'' = U \times_{P\overline{\mathfrak{M}}_g^{(n)}} U'$ and $V'' = V \times_{X(g,n)} V'$. Then the pair $(U'' \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}, V'' \hookrightarrow X(g,n))$ satisfies the list of properties of Proposition 1.10. Moreover the complement of U'' in U and U' is of co-dimension at least two, thus we have $\mathrm{Pic}(U) \simeq \mathrm{Pic}(U') \simeq \mathrm{Pic}(U'')$ (the same stands for V''). Moreover we have $V'' = V \times_U U''$ thus the left square on the following diagram is cartesian

$$\begin{tikzcd}
V'' \arrow[r, "i"] \arrow[d] & V \arrow[r, "j"] \arrow[d] & X(g,n) \arrow[d, "\mathrm{diff}"] \\
U'' \arrow[r, "i'"'] & U \arrow[r, "j'"'] & P\overline{\mathfrak{M}}_g^{(n)}
\end{tikzcd}$$

and we have $i^* \circ \mathrm{diff}_{|V''*} \circ i'_* = \mathrm{diff}_{|V*}$. Thus the morphism diff_* and diff''_* (from $\mathrm{Pic}(X(g,n))$ to $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$) constructed in Section 1.5 are equal. The same stands for diff'_* and diff''_* thus finishing the proof. □

2.1. **Distinguished local coordinates on a cyclic cover.** Consider a curve C endowed with an n-differential w with simple zeros. Let $f : \widehat{C} \to C$ be the associated cover and $v = w^{1/n}$ be the canonical abelian differential on $\widehat{C}$. Denote by $x_i \in C$, $i = 1, \dots, N = 2n(g-1)$, the branch points of $\widehat{C} \to C$, which coincide with the zeros of w. Denote by $\hat{x}_i$ the unique preimage of x_i in $\widehat{C}$. Here we describe a specific parametrization of the covering curve $\widehat{C}$ near the branching points of $f : \widehat{C} \to C$. It is easy to see that all zeros of the holomorphic 1-form v are situated at the ramification points $\hat{x}_i$ and have multiplicity n. In other words,

$$(v) = n\hat{x}_1 + \dots + n\hat{x}_N. \tag{2.1}$$

Introduce a local parameter ζ_i in a neighborhood of $x_i \in C$ and a local parameter ξ_i in a neighborhood of $\hat{x}_i \in \widehat{C}$ such that

$$w = \zeta_i (d\zeta_i)^n, \qquad \xi_i(\hat{x})^{n+1} = \int_{\hat{x}_i}^{\hat{x}} v. \tag{2.2}$$

Both parameters are defined up to an $(n+1)$st root of unity and we make one choice in such a way that

$$\zeta_i = \left(\frac{n+1}{n}\right)^{n/(n+1)} \xi_i^n.$$

The local parameters ξ_i on $\widehat{C}$ and ζ_i on C given by (2.2) are called *distinguished*.

Since $f^*v = \rho\, v$, the local parameter $\xi_i(x)$ transforms under the action of f as $\xi_i(f(x)) = \rho\xi_i(x)$.

2.2. **Extension of the Prym-Tyurin bundles to codimension 1 loci.** By construction, $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an open dense substack of $X(g,n)$. Thus $\mathrm{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is birational and its restriction to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an isomorphism onto its image.

By abuse of notation we denote by D_i, $0 \le i \le [g/2]$, the preimage in $P\overline{\mathfrak{M}}_g^{(n)}$ of the boundary divisor $D_i \subset \overline{\mathcal{M}}_g$. Then the complement of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ in $P\overline{\mathfrak{M}}_g^{(n)}$ is the union of the divisors D_i and $D_{\deg}$.

For each of these divisors we define a dense open locus $\widetilde{D} \subset D$ as follows.

- $\widetilde{D}_0$ is the locus of (C,w) such that the curve C has exactly one non-separating node and the differential w has poles of order n at the node and N simple zeros;
- $\widetilde{D}_i$ for $i \ge 1$ is the locus of (C,w) such that the curve C has exactly one separating node and the differential w has poles of order n at the node and N simple zeros.
- $\widetilde{D}_{\deg}$ if $(g,n) \neq (2,2)$ is the locus of (C,w) such that the curve C is a smooth curve and the differential w has one zero of order exactly 2 and its other zeros are simple;
- $\widetilde{D}_{\deg}$ if $(g,n) = (2,2)$ is the disjoint union of the locus $\widetilde{D}_{\deg}(2,1,1)$ described above and of the locus $\widetilde{D}_{\deg}(2,2)$ of pairs (C,w) where C is smooth and w is a square of a holomorphic differential with simple zeros.

We define U as the union of $P\mathfrak{M}_g^{(n)}[\mathbf{1}] \cup \widetilde{D}_{\deg} \cup_i \widetilde{D}_i \subset P\overline{\mathfrak{M}}_g^{(n)}$. We define V as $\mathrm{diff}^{-1}(U) \subset X(g,n)$. We will prove that U and V satisfy the properties of Proposition 2.1.

Property 1 is satisfied by construction.

Lemma 2.3 (Property 2)**.** *The stack U is an open substack of $P\overline{\mathfrak{M}}_g^{(n)}$ and its complement is of codimension at least 2.*

Proof. The complement of U is a union of closed substacks: the strata of curves with at least two nodes and the strata $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{k}]$ for all $\mathbf{k}$ except $(1,\dots,1)$ and $(2,1\dots,1)$. Thus U is an open substack in $P\overline{\mathfrak{M}}_g^{(n)}$ and by dimension count its complement is of codimension at least 2. □

Lemma 2.4 (Property 3)**.** *The restriction of* $\mathrm{diff} : V \to U$ *induces an isomorphism on the underlying schemes. Moreover the map of stacks* $\mathrm{diff} : V \to U$ *is of degree one over $U \setminus D_{\deg}$ and of degree $1/2$ over $D_{\deg}$.*

Remark 2.5. The reader should keep in mind the model case of the quotient $\mathrm{Spec}(\mathbb{C}[x])/(\mathbb{Z}/2\mathbb{Z})$. This quotient can be realized as a Deligne-Mumford (DM) stack or as a scheme. In the later case, it is isomorphic to $\mathrm{Spec}(\mathbb{C}[u])$. We have a map from the DM stack $\mathbb{C}/(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{C}$, defined by $u \mapsto x^2$. Then the divisor $\{x = 0\}$ is mapped to the divisor $\{u = 0\}$ with degree $1/2$.

Proof. The underlying scheme of U is smooth and the map $\mathrm{diff} : V \to U$ is birational. Thus the map diff of schemes is an isomorphism if and only if it is finite. We consider the incidence variety $X_{g,n}^{\mathrm{inc}}$ compactification defined in Remark 1.9. We have two birational map: $\mathrm{diff}' X_{g,n}^{\mathrm{inc}} \to X(g,n)$ and $\epsilon : X_{g,n}^{\mathrm{inc}} \to \overline{\mathfrak{M}}_g^{(n)}$ such that $\mathrm{diff} = \mathrm{diff}' \circ \epsilon$. The map diff' is obtained by forgetting the admissible cover (but not the markings) and ϵ is obtained by forgetting the markings. As we have already

stated in Remark 1.9, the map diff' is finite because $\overline{\mathrm{Hur}}_{g,n} \to \overline{\mathcal{M}}_{g,N}/S_N$ is finite. Therefore we need to check that the map ϵ restricted to $\epsilon^{-1}(U)$ is finite.

The restriction $\epsilon : \epsilon^{-1}(U) \to U$ is a bijection. Indeed, if (C, w) be a n-differential in $U \setminus D_{\mathrm{deg}}$, then the preimage of (C, w) under ϵ is the n-differential w with the marked simple zeros. Now if (C, w) is a n-differential in $\tilde{D}_{\mathrm{deg}}$ then the preimage of (C, w) is the point (C', w', x_i) where C' is the curve with two components: one component isomorphic to C and one rational component attached to C at the double zero; the differential w' is then given by w on the main component and vanishes identically on the rational component; finally the marked points are the simple zeros on the main component and two marked points on the rational component.

Therefore $\epsilon : \epsilon^{-1}(U) \to U$ is finite and $\mathrm{diff} : V \to U$ is birational and finite thus an isomorphism of the underlying schemes. Moreover the restriction of the map ϵ to $U \setminus D_{\mathrm{deg}}$ is obviously an isomorphism of stacks. The degree of diff along D_{deg} will be computed in the next paragraphs. □

Lemma 2.6 (Property 4)**.** *There exists a line bundle $T \to V$ such that $T^{\otimes n} \simeq L$, T is a sub-vector bundle of $\Lambda^{(1)}$, and the restriction of T to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ coincides with the sub-vector bundle of $\hat{\nu}^*\Omega_{\widehat{g}}$ spanned by v.*

The proof requires a detailed analysis of the inverse morphism $\mathrm{diff}^{-1} : U \to V$ and is contained in the next two subsections.

2.2.1. *Nodal curves.* Let $0 \le i \le [g/2]$ and let (C, w) be a point in $\widetilde{D}_i$. The n-fold cover associated to (C, w) is given by $\widehat{C} = \{(x, v) \in T^*_C / v^n = w\}$ and the canonical differential v is still defined by $v(x, v) = v$. We can describe the topology of $\widehat{C}$ and the singularities (zeros and poles) of v:

- If $i = 0$, then the curve $\widehat{C}$ is an irreducible curve with n self-intersections. The differential v has zeros of order n at the marked points and poles of order 1 at the nodes.
- If $i \ge 1$ then $\widehat{C}$ has two irreducible components intersecting at n distinct nodes. The differential v has also zeros of order n at the marked points and poles of 1 at the nodes.

The canonical differential v is well-defined on $U \setminus D_{\mathrm{deg}}$, thus the line bundle T can be extended to $\mathrm{diff}^{-1}(U \setminus D_{\mathrm{deg}})$.

2.2.2. *Degenerate differentials.* Here we describe the local structure of the stacks $X(g, n)$ and $\overline{\mathfrak{M}}_g^{(n)}$ close to D_{deg}. This allows us to make explicit the isomorphism of Lemma 2.4 and to describe the fiber of the canonical line bundle along $\widetilde{D}_{\mathrm{deg}}$. Let (C_0, w_0) be a point in $\widetilde{D}_{\mathrm{deg}}$ and let W be a neighborhood of (C_0, w_0) in D_{deg}. We will give a local parametrization of $X(g, n)$ and $\overline{\mathfrak{M}}_g^{(n)}$ around the point (C_0, w_0).

- *Parameters of $\overline{\mathfrak{M}}_g^{(n)}$.* A neighborhood of (C_0, w_0) is given by $W \times \Delta$ where Δ is a disk in $\mathbb{C}$ centered at zero. A point (u, a) in $W \times \Delta$ parametrizes an n-differentials (C, w) such that
$$w = (\zeta^2 + a) d\zeta^n.$$
where the parameter ζ of the curve C is uniquely determined by the choice of a. The parameter a is a tranverse local parameter of D_{deg} in $\overline{\mathfrak{M}}_g^{(n)}$.
- *Parameters of $X(g, n)$.* A neighborhood of the point (C_0, w_0) in $X(g, n)$ is parametrized by $W \times \Delta'/(\mathbb{Z}/2\mathbb{Z})$ where Δ' is a disk in $\mathbb{C}$ centered at zero. Indeed, suppose first that the two colliding zeros of a differential (C, w) are labeled x_1 and x_2. Let ζ be a local parameter of C such that the positions

of x_1 and x_2 are given by ζ_1 and ζ_2, respectively. To fix ζ uniquely we can define it by:

$$w(x) = (\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)(d\zeta(x))^n \ , \qquad x \in C.$$

The parameter $(\zeta_1 - \zeta_2)/\{\pm 1\}$ is a local transverse parameter to $D_{\deg}$ in $X(g,n)$ (See Lemma 3.6 for a proof).

With these two local parametrizations, the map diff : $X(g,n) \to \overline{\mathfrak{M}}_g^{(n)}$ is given by

$$\begin{array}{rcl} W \times \Delta'/(\mathbb{Z}/2\mathbb{Z}) & \to & W \times \Delta \\ (u, \zeta_1 - \zeta_2) & \mapsto & (u, (\zeta_1 - \zeta_2)^2). \end{array}$$

This map is indeed an isomorphism of the underlying schemes. However it is of degree 1/2 along $D_{deg} \subset X(g,n)$ once we consider the stack structures of $X(g,n)$ and $\overline{\mathfrak{M}}_g^{(n)}$. This finishes the proof of Lemma 2.4.

Finally we describe the extension of the canonical line bundle T to $D_{\deg}$. Let (C,w) be a family of differentials with simple zeros which tends to $(C_0, w_0) \in D_{\deg} \subset X(g,n)$. Once again we label the two coalescing zeros x_1 and x_2 and we use the local parameter of the curve ζ with $\zeta(x_i) = \zeta_i$ and $w(x) = (\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)(d\zeta(x))^n$

As (C,w) tends to a pair (C_0, w_0) the zeros x_1 and x_2 tend to the double zero x_0 of w_0. The limit curve C_0 is a nodal curve with two components: a Riemann sphere C_1 which gets naturally equipped with the *meromorphic* n-differential $w_1(\zeta)$ given by the formula

$$w_1(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)(d\zeta)^n \ , \qquad \zeta \in C_1 \ , \tag{2.3}$$

which is holomorphic outside of $\zeta = \infty$ and has two simple zeros at ζ_1 and ζ_2. The Riemann surface C_2 is equipped with the holomorphic n-differential w_0. The nodal point on C_0 is formed by identifying the point $\zeta = \infty$ on C_1 with the point x_0 on C_2.

The limit n-differential (C_0, w_0) determines a canonical n-sheeted cover $\widehat{C}_0 \to C_0$. The curve $\widehat{C}_0$ consists of two components $\widehat{C}_1$ and $\widehat{C}_2$. The canonical cover $\widehat{C}_1$ of C_1 is given by equation

$$v_1^n = w_1(\zeta) \tag{2.4}$$

which, if we write $v_1 = y d\zeta$, is the curve

$$y^n = (\zeta - \zeta_1)(\zeta - \zeta_2) \tag{2.5}$$

of genus $\widehat{g}_1 = [(n-1)/2]$. The canonical cover $\widehat{C}_2$ of C_2 is defined by the equation

$$v_2^n = w_2 \ ; \tag{2.6}$$

its genus equals $\widehat{g}_2 = \widehat{g} - [n/2]$. Therefore, for odd n we have $\widehat{g} = \widehat{g}_1 + \widehat{g}_2$ while for even n we have $\widehat{g} = \widehat{g}_1 + \widehat{g}_2 + 1$.

The difference between the case of even n and the case of odd n is due to the fact that for odd n the covers $\widehat{C}_1$ and $\widehat{C}_2$ intersect at only one nodal point while for even n the nodal point on C_0 has two pre-images on $\widehat{C}_0$ i.e. for even n $\widehat{C}_1$ and $\widehat{C}_2$ intersect at two nodal points, $x_0^{(1)}$ and $x_0^{(2)}$ (see Figure 1 below).

With this description of the limit cover, we define the limit canonical differential v_0 as follows: it is given by $v_0 = v_2$ on the component $\widehat{C}_2$ and vanishes identicaly on the rational component $\widehat{C}_1$. It satisfies $v_0^n = f^* w_0$. Thus the canonical line bundle T can be extended to the open set V. This completes the proof of Lemma 2.6.

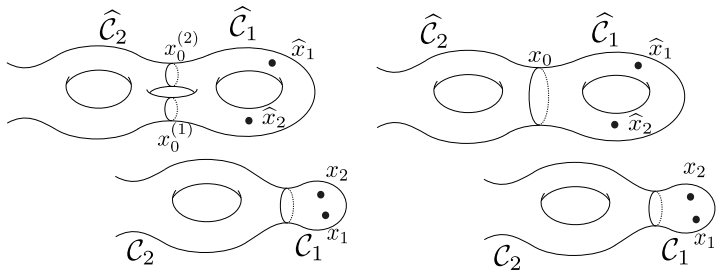

FIGURE 1. Examples of degeneracy of canonical n-covers (before colliding the zeros): on the left $n = 4$ and on the right $n = 3$.

3. Bergman tau function and Hodge class on $P\overline{\mathfrak{M}}_g^{(n)}$

Tau functions play an important role in the theory of integrable systems providing canonical generators for commuting flows on the phase space [1]. In some cases tau functions carry interesting algebro-geometric information, like the isomonodromic tau function of the Riemann-Hilbert problem that plays a noticeable role in the theory of Frobenius manifolds [4].

By means of the Bergman tau function on the Hurwitz space [11], the Hodge class on the space of admissible covers of the complex projective line was explicitly expressed as a linear combination of the boundary divisors [12]. This result was rederived in [8] by pure algebro-geometric methods (namely, using the Grothendieck-Riemann-Roch theorem) and was applied in [9] to answer a question of Harris-Mumford [7] about the classes of Hurwitz divisors in the moduli space $\overline{\mathcal{M}}_g$ of stable complex algebraic curves of even genus g.

A version of the Bergman tau function for the moduli space of holomorphic abelian differentials on algebraic curves [10] was used in [13] to obtain new relations in the rational Picard group. These results appeared to be relevant to the Kontsevich-Zorich theory of Teichmüller flow, cf. [13], [5]. In [14], Bergman tau functions helped to express the Hodge and Prym classes on the moduli space of holomorphic quadratic differentials in terms of other divisor classes. In this section we continue developing these ideas further for the moduli space of holomorphic n-differentials.

3.1. Bergman tau function on strata of n-differentials.

We begin with defining the Bergman tau function for each stratum $\mathfrak{M}_g^{(n)}[\mathbf{k}]$ where $\mathbf{k} = (k_1, \dots, k_m)$ is a partition of $N = 2n(g-1)$. We introduce the following notation (see [6] for precise definitions):

- $v_1, \dots, v_g$ – the normalized basis of holomorphic abelian differentials with respect to a given Torelli marking (or cut system) on C;
- Ω – the corresponding period matrix;
- $\Theta(z, \Omega)$ – the theta function associated with Ω;
- $W(x)$ – the Wronskian determinant of differentials $v_1, \dots, v_g$;
- $\tilde{C}$ – the fundamental polygon corresponding to the chosen cut system on C;
- $E(x, y)$ – the prime form on $C \times C$;
- $\mathcal{A}_x$ – the Abel map corresponding to the initial point x;
- K^x – the vector of Riemann constants corresponding to the initial point x.

The distinguished local parameters on C in a neighborhood of the points x_i (zeroes of the n-differential w) are given by

$$\zeta_i(x) = \left(\int_{x_i}^{x} v\right)^{n/(k_i+n)} \tag{3.1}$$

where k_i is the order of zero of w at the point x_i (in terms of these parameters $w \sim \zeta_i^{k_i}(d\zeta_i)^n$ near $x_i \in C$, and $v = w^{1/n} \sim \zeta_i^{k_i/n} d\zeta_i$ near $\hat{x}_i = f^{-1}(x_i) \in \widehat{C}$). The local parameters (3.1) are defined up to an nth root of unity, and different sheets of $\widehat{C}$ correspond to different choices of this root. Then for the prime form $E(x,y)$ on $C \times C$ we have

$$E(x,y) = \frac{E(\zeta(x),\zeta(y))}{\sqrt{d\zeta}(x)\sqrt{d\zeta}(y)}\,,$$

and we put

$$\begin{aligned} E(\zeta, x_k) &= \lim_{y\to x_k} E(\zeta(x),\zeta(y))\sqrt{\frac{d\zeta_k}{d\zeta}}(y), \\ E(x_k, x_l) &= \lim_{\substack{x\to x_k\\ y\to x_l}} E(\zeta(x),\zeta(y))\sqrt{\frac{d\zeta_k}{d\zeta}}(x)\sqrt{\frac{d\zeta_l}{d\zeta}}(y)\,. \end{aligned}$$

We define two vectors Z, $Z' \in \frac{1}{n}\mathbb{Z}^g$ by the condition

$$\frac{1}{n}\mathcal{A}_x((w)) + 2K^x = \Omega Z + Z'\,. \tag{3.2}$$

Definition 3.1. The Bergman tau function on the space $\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is given by

$$\tau(C,w) = \tag{3.3}$$

$$c(x)^{2/3} e^{-\frac{\pi}{6}\langle \Omega Z, Z\rangle - \frac{2\pi\sqrt{-1}}{3}\langle Z, K^x\rangle} \left(\frac{w(x)}{\prod_{i=1}^m E^{k_i}(x,x_i)}\right)^{(g-1)/3n} \prod_{i<j} E(x_i,x_j)^{\frac{k_ik_j}{6n^2}}\,,$$

where

$$c(x) = \frac{1}{W(x)}\left(\sum_{i=1}^g v_i(x)\frac{\partial}{\partial z_i}\right)^g \theta(z;\Omega)\Big|_{z=K^x}\,.$$

From Eq. (3.3) it is clear that the tau function is nowhere vanishing and non-singular on $\mathfrak{M}_g^{(n)}[\mathbf{k}]$. Other properties of τ follow below.

Proposition 3.2. *Under the change of Torelli marking on C*

$$\begin{pmatrix} \tilde{\boldsymbol{b}} \\ \tilde{\boldsymbol{a}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z})\,,$$

the tau function (3.3) transforms as follows:

$$\frac{\tau(C,w,\{\tilde{a}_i,\tilde{b}_i\})}{\tau(C,w,\{a_i,b_i\})} = \epsilon\det(C\Omega + D) \tag{3.4}$$

where ϵ is a root of unity of degree $48d$ with $d = \text{l.c.m.}(k_1+n,\dots,k_m+n)$.

The proof can be obtained by using standard transformation properties of all factors in (3.3) under the change of Torelli cut system on C (cf. [6]). The root of unity appears due to an ambiguity in the definition of the distinguished local parameters (3.1), which translates into an ambiguity in the definition of $E(x,x_i)$ and $E(x_i,x_j)$. The appearance of the term $\det(C\Omega + D)$ can also be seen from variational formulas for $\tau(C,w)$ discussed below, similarly to [10, 13, 14].

Proposition 3.3. *The tau function has the following quasi-homogeneity property:*

$$\tau(C,\delta w)=\delta^{\kappa}\tau(C,w) \tag{3.5}$$

with

$$\kappa=\frac{1}{12n^2}\sum_{i=1}^{m}\frac{k_i(k_i+2n)}{k_i+n}. \tag{3.6}$$

This proposition follows from the explicit formula (3.3), but can also be derived by applying the Riemann bilinear identity to variational formulas for τ as it was done in [12] in the context of Hurwitz spaces.

Combining Propositions 3.2 and 3.3, we arrive at the following

Theorem 3.4. *On the stratum $P\mathfrak{M}_g^{(n)}=P\mathfrak{M}_g^{(n)}[\mathbf{1}]\subset P\overline{\mathfrak{M}}_g^{(n)}$ of n-differentials with simple zeroes, the power $\tau^{48n(n+1)}$ of the tau function $\tau=\tau(C,w)$ is a nowhere vanishing holomorphic section of the line bundle $\lambda^{48n(n+1)}\otimes L^{-8(g-1)(2n+1)}\longrightarrow P\mathfrak{M}_g^{(n)}$.*

In order to find the divisor of the section $\tau^{48n(n+1)}$ on $P\overline{\mathfrak{M}}_g^{(n)}$, we will compute the asymptotics of τ at the boundary divisors $D_{\deg}$ and $D_j,\ j=0,1,\dots,[g/2]$. For that we need to study the tau function more carefully.

3.2. **Homological coordinates and variational formulas for the tau function.** The tau function $\tau(C,w)$ satisfies a system of linear differential equations on the space $\mathfrak{M}_g^{(n)}$ similar to the tau functions on Hurwitz spaces, or spaces of abelian or quadratic differentials [10, 12, 13, 14]. From now on we assume that all zeros of w are simple i.e. that all $k_i=1$ in (3.3).

The homology group $H_1(\widehat{C},\mathbb{C})$ can be decomposed into the eigenspaces of the automorphism σ_*:

$$H_1(\widehat{C},\mathbb{C})=\bigoplus_{i=0}^{n-1}\mathcal{H}_k\,,$$

where $\dim\mathcal{H}_0=2g$. Since all zeroes of w are assumed to be simple, the dimensions of $\mathcal{H}_k$ for $k=1,\dots,n-1$ are independent of k and we have

$$\dim\mathcal{H}_k=(2n+2)(g-1)\,,\quad k=1,\dots,n-1\,. \tag{3.7}$$

For any two classes $s_1\in\mathcal{H}_l$ and $s_2\in\mathcal{H}_k$ we have $s_1\cdot s_2=0$ unless $k+l=n$, where $\cdot$ denotes the intersection pairing. The spaces $\mathcal{H}_k$ and $\mathcal{H}_{n-k}$ are, therefore, dual to each other with respect to the standard intersection pairing (the space $\mathcal{H}_0$ can be identified with $H_1(C)$, and, therefore, it is self-dual). On the other hand, for any $q\in\Omega^{(k)}$ and $s\in\mathcal{H}_l$ we have $\int_s q=0$ unless $k=l$. In particular, since $v=w^{1/n}\in\Omega^{(1)}$, it can have non-trivial periods only over the cycles representing homology classes in $\mathcal{H}_1$. Actually, $\mathcal{H}_1$ can be naturally identified with the tangent space to the moduli space $\mathfrak{M}_g^{(n)}$. Choosing a basis $\{s_i\}_{i=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_1$ we introduce local coordinates $\mathcal{P}_i$ on $\mathfrak{M}_g^{(n)}$ given by

$$\mathcal{P}_i=\int_{s_i}v \tag{3.8}$$

that are called *homological coordinates* (cf. Corollary 2.3 of [2]).

Choose a Torelli marking on C and define the associated canonical bimeromorphic differential $B(x,y)=d_xd_y\log E(x,y)$ on C. The bidifferential B (also called the Bergman kernel) is symmetric with a second order pole with biresidue 1 on the diagonal $x=y$ and vanishing a-periods with respect to the both of its arguments.

Denote by $\widetilde{B}(x,y) = B(f(x), f(y))df(x)df(y)$ the pullback of $B(x,y)$ to $\widehat{C}\times\widehat{C}$. The bidifferential $B(x,y)$ has the following expansion near $x = y$:

$$\widetilde{B}(x,y) = \left(\frac{1}{(\zeta(x)-\zeta(y))^2} + \frac{1}{6}S_{\widetilde{B}}(\zeta(x)) + \dots\right) d\zeta(x)d\zeta(y)\,, \tag{3.9}$$

where ζ is a local parameter and $\widetilde{B}$ is the Bergman projective connection (lifted to $\widehat{C}$).

To study the tau function on $\mathfrak{M}_g^{(n)}$ we will need a variational formula for $\widetilde{B}(x,y)$.

Consider a fundamental polygon $\widetilde{C}$ of C (that is, dissect C along the cuts representing the Torelli marking). Choose a system Γ of non-intersecting cuts that lie within $\widetilde{C}$ and connect the first zero x_1 with every other zero of w. Pick a connected component of $f^{-1}(\widetilde{C}\setminus\Gamma)\subset\widehat{C}$ and identify it with $\widetilde{C}\setminus\Gamma$. On $\widetilde{C}\setminus\Gamma$ introduce the coordinate

$$z(x) = \int_{x_1}^{x} v\,, \tag{3.10}$$

where the path connecting x with x_1 entirely lies in $\widetilde{C}\setminus\Gamma$.

For a basis $\{s_i\}_{i=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_1$ take the dual basis $\{s_j^*\}_{j=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_{n-1}$, so that $s_j^*\cdot s_i = \delta_{ij}$.

Theorem 3.5. *The following variational formula holds for $i = 1,\dots,(2n+2)(g-1)$:*

$$\frac{\partial}{\partial\mathcal{P}_i}\widetilde{B}(z(x),z(y)) = \frac{1}{2\pi\sqrt{-1}n}\int_{t\in s_i^*}\frac{\widetilde{B}(z(x),t)\widetilde{B}(t,z(y))}{v(t)}dt\,. \tag{3.11}$$

Proof. Formula (3.11) follows from the variational formula for the Bergman kernel on the stratum $\mathcal{H}_{\hat{g}}(n,\dots,n)$ of the moduli space of holomorphic 1-differentials of genus $\widehat{g} = n^2(g-1)+1$ (Theorem 3 of [10]), in a way similar to Lemma 5 of [14] and Proposition 3.2 of [3].

Namely, consider a pair $(\widehat{C},v)$ representing a point in the space $\mathcal{H}_{\hat{g}}(n,\dots,n)$. Denote by $\widehat{B}(x,y)$ the canonical bidifferential (the Bergman kernel) on $\widehat{C}\times\widehat{C}$. Then we have the following variational formula for $\widehat{B}(x,y)$ [10]:

$$\frac{\partial\widehat{B}(z(x),z(y))}{\partial(\int_{s_i}v)} = \frac{1}{2\pi\sqrt{-1}}\int_{t\in s_i^*}\frac{\widehat{B}(z(x),t)\widehat{B}(t,z(y))}{v(t)}dt\,. \tag{3.12}$$

Here $\{s_i\}$ is a canonical basis in $H_1\left(\widehat{C},\{x_i\}_{i=1}^{n(2g-2)}\right)$, $\{s_j^*\}$ is the dual basis in $H_1\left(\widehat{C}\setminus\{x_i\}_{i=1}^{n(2g-2)}\right)$ (so that $s_i\cdot s_j^* = \delta_{ij}$), and $x_1,\dots x_{n(2g-2)}$ are the zeroes of v.

Let now the pair $(\widehat{C},v)$ be the canonical n-fold cover of (C,w). We require the Torelli markings of $\widehat{C}$ and C be compatible (i.e. each connected component of $f^{-1}(a_i)$, $i = 1,\dots,a_g$, belongs to the subgroup generated by $\hat{a}_j$, $j = 1,\dots,a_{\hat{g}}$, where a_i and $\hat{a}_j$ are the a-cycles on C and $\widehat{C}$ respectively). Then averaging $\widehat{B}(x,y)$ in both variables over the action of the group $\mathbb{Z}/n\mathbb{Z}$ on $\widehat{C}$ we get the formula

$$\widetilde{B}(x,y) = \frac{1}{n}\sum_{l,k=1}^{n-1}(\sigma_x^*)^k(\sigma_y^*)^l\widehat{B}(x,y) \tag{3.13}$$

that can be checked by comparing the singularities of the both sides. Restricting now formula (3.12) from $\mathcal{H}_{\hat{g}}(n,\dots,n)$ to $\mathfrak{M}_g^{(n)}$ and using (3.13) we get (3.11). □

Consider the Schwarzian derivative $S_v = \frac{v''}{v} - \frac{3}{2}\left(\frac{v'}{v}\right)^2$ of the abelian integral $\int_{x_0}^{x} v$ with respect to the local coordinate z on $\widehat{C}$ introduced earlier. For the holomorphic 1-differential v, S_v is a meromorphic projective connection on C, so that the difference $S_{\widetilde{B}} - S_v$ is a meromorphic quadratic differential.

The tau function $\tau = \tau(C, w)$ satisfies the following system of differential equations with respect to the homological coordinates $\mathcal{P}_i$ on $P\mathfrak{M}_g^{(n)}$:

$$\frac{\partial}{\partial \mathcal{P}_i} \log \tau = -\frac{1}{12\pi\sqrt{-1}n} \int_{s_i^*} \frac{S_{\widetilde{B}} - S_v}{v} \tag{3.14}$$

(equations (3.14) for the tau function $\tau(C, w)$ are restrictions of the equations for the Bergman tau function on the stratum $\mathcal{H}_{\widehat{g}}(n, \dots, n)$ of the moduli space of Abelian differentials [10] to the space $\mathfrak{M}_g^{(n)}$). The compatibility of the system (3.14), namely, the fact that $\frac{\partial}{\partial \mathcal{P}_j}\left(\frac{\partial}{\partial \mathcal{P}_i} \log \tau\right) = \frac{\partial}{\partial \mathcal{P}_i}\left(\frac{\partial}{\partial \mathcal{P}_j} \log \tau\right)$ follows from the invariance of the expression

$$\frac{\partial}{\partial \mathcal{P}_j} \int_{s_i^*} \frac{S_{\widetilde{B}} - S_v}{v} = \frac{1}{12\pi\sqrt{-1}n} \int_{s_j^*} \int_{s_i^*} \frac{\widetilde{B}(x,y)\widetilde{B}(y,x)}{v(x)v(y)} dxdy$$

under the transposition of i and j.

3.3. **Asymptotics of tau function near the boundary of $\overline{\mathfrak{M}}_g^{(n)}$.** Here we compute the asymptotics of τ near generic point of every boundary stratum $\overline{\mathfrak{M}}_g^{(n)} \setminus \mathfrak{M}_g^{(n)}[\mathbf{1}]$ of $\overline{\mathfrak{M}}_g^{(n)}$.

3.3.1. *Coalescing simple zeros of w: divisor $D_{\deg}$.*

Lemma 3.6. *Let x_1 and x_2 be two zeros of w coalescing at $D_{\deg}$. Then a transverse local coordinate on $\overline{\mathfrak{M}}_g^{(n)}$ in a tubular neighborhood of $D_{\deg}$ is given by*

$$t_{\deg} = \left(\int_{x_1}^{x_2} v\right)^{2n/(n+2)} . \tag{3.15}$$

Proof. This is analogous to the proof of Lemma 8 of [14]. Denote by ζ a local coordinate in a small disk U containing the coalescing zeros $x_{1,2}$ and no other zeros. Then we can write in U

$$w(\zeta) = (\zeta - \zeta(x_1))(\zeta - \zeta(x_2))(d\zeta)^n , \tag{3.16}$$

so that

$$\int_{x_1}^{x_2} v = \int_{\zeta(x_1)}^{\zeta(x_2)} ((\zeta - \zeta(x_1))(\zeta - \zeta(x_2)))^{1/n} \, d\zeta = \mathrm{const} \cdot (\zeta(x_1) - \zeta(x_2))^{(n+2)/n} .$$

Since $(\zeta(x_1) - \zeta(x_2))^2$ is independent of labeling of zeroes, $t_{\deg}$ is a coordinate transverse to $D_{\deg}$. Expressing $t_{\deg}$ in terms of $\int_{x_1}^{x_2} v$ we get (3.15). □

Lemma 3.7. *The tau function $\tau(C, w)$ has the following asymptotics near $D_{\deg}$:*

$$\tau(C, w) = t_{\deg}^{\frac{1}{12n(n+1)}} \tau_0(C_0, w_0)(1 + o(1)) , \tag{3.17}$$

where $(C_0, w_0) \in D_{\deg}$ and $\tau_0(C_0, w_0)$ is the tau function (3.3) on the moduli space $\mathfrak{M}_g^{(n)}[2, 1, \dots, 1]$.

Proof. The asymptotics (3.17) can be derived by computing the asymptotics of all factors in the explicit formula (3.3). Alternatively, using the system of equations (3.14) we see that in the limit $t_{\deg} \to 0$ the tau function $\tau(C, w)$ behaves like $t_{\deg}^p \tau(\mathbb{C}_0, w_0)$ for some power p, where w_0 is a differential with one zero of order two and every other zero being simple. To find p explicitly, we look at the transformation properties of τ, τ_0 and $t_{\deg}$ under the rescaling $w \mapsto \delta w$. The homogeneity degrees κ of τ and κ_0 of τ_0 are given by the formula (3.6), so that

$$\kappa - \kappa_0 = \frac{1}{6n(n+1)(n+2)}$$

On the other hand, the local parameter $t_{\deg}$ has the homogeneity degree $2/(n+2)$, which gives $p = \frac{1}{12n(n+1)}$. □

3.3.2. *Asymptotics of τ near D_0.* Take two loops a and b on C intersecting transversely at one point; their homology classes we will also denote by a, $b \in H_1(C, \mathbb{Z})$. Let us pinch a to a point, then C degenerates to a nodal curve C_0 that we represent by a smooth curve of genus $g-1$ with two points (say, x_0 and y_0) identified (we assume that all zeros of w remain far from the node).

The holomorphic n-differential w degenerates to a meromorphic n-differential w_0 on C_0 with poles of degree n at x_0 and y_0 on the normalization of C_0 such that the corresponding n-residues differ by $(-1)^n$. Since locally near both x_0 and y_0 the n-differential w behaves like $(d\zeta/\zeta)^n$, no branching over the node occurs.

Because of that the canonical cover $\widehat{C}$ of C degenerates to a nodal curve $\widehat{C}_0$ with exactly n nodes that can be thought of as n pairs of points $x_0^{(m)} = \sigma^m(x_0)$ and $y_0^{(m)} = \sigma^m(y_0)$ $(m = 0, \dots, n-1)$ on the normalization of $\widehat{C}_0$ that are pairwise identified (here σ is the covering automorphism of $\widehat{C}_0$). The differential v on $\widehat{C}$ degenerates to a meromorphic differential v_0 on $\widehat{C}_0$ with simple poles at the preimages of nodal points with residues at $x_0^{(m)}$ and $y_0^{(m)}$ that differ by a sign.

Choose one of the n simple closed loops in the preimage $f^{-1}(a) \subset \widehat{C}$. Let us assume that this loop pinches to the first node on $\widehat{C}_0$.

Consider the classes $\alpha, \beta \in \mathcal{H}_1$ given by

$$\alpha = \sum_{m=0}^{n-1} \rho^{-m} \sigma_*^{-m} a \,, \quad \beta = \sum_{m=0}^{n-1} \rho^{-m} \sigma_*^{-m} b \tag{3.18}$$

where σ is the covering automorphism of $\widehat{C}$. Introduce the homological coordinates $\mathcal{P}_\alpha = \int_\alpha v$ and $\mathcal{P}_\beta = \int_\beta v$ associated with α and β. We have $\int_\alpha v = n \int_a v$ and $\int_\beta v = n \int_b v$.

Since all $\mathcal{P}_j$ are independent local coordinates, we can choose such a path towards D_0 that all coordinates $\mathcal{P}_i$ remain constant under the degeneration of C to C_0 while $\mathcal{P}_\beta \to \infty$. Then a local coordinate on $\mathfrak{M}_g^{(n)}$ transverse to D_0 in a tubular neighborhood of D_0 can be chosen as

$$t_0 = e^{2\pi i \mathcal{P}_\beta / \mathcal{P}_\alpha} \tag{3.19}$$

(notice that $\mathrm{Im}(\mathcal{P}_\beta/\mathcal{P}_\alpha) > 0$ near D_0).

To show this one can use Fay's "plumbing construction", cf. p. 52 of [6]. Roughly speaking, the degenerating curve $\widehat{C}$ can be obtained from the nodal curve $\widehat{C}_0$ by cutting off a neighborhood of each node and connecting the boundaries with long thin tubes. More precisely, in terms of a local coordinate t near D_0 the differential v can be written as $\frac{\mathcal{P}_\alpha}{2\pi i} \frac{dx}{\sqrt{x^2 - t}} + \dots$. Then the coordinate t_0 (3.19) is given by $t(\mathrm{const} + o(1))$.

Let $\omega_{x,y}$ be the abelian differential of the 3rd kind on $\widehat{C}_0$ with simple poles at points x and y of residues $+1$ and -1 respectively, normalized with respect to some canonical basis $\{a_i, b_i\}$ in $H_1(\widehat{C}_0, \mathbb{Z})$. Since $v_0 \in \mathcal{H}^1$, it can be written as

$$v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}} \sum_{j=0}^{n-1} \rho^m \omega_{\sigma^m(x_0),\,\sigma^m(y_0)} + \text{holomorphic terms}\,.$$

In the limit $t_0 \to 0$ the bidifferential $B(x,y)$ on $C \times C$ tends to the meromorphic bidifferential $B_0(x,y)$ on $C_0 \times C_0$ with the same properties. We denote by $\widetilde{B}_0(x,y)$ the pullback of $B_0(x,y)$ to $\widehat{C}_0 \times \widehat{C}_0$.

To find the asymptotics of the tau function τ as $t_0 \to 0$ (i.e. $\mathcal{P}_\beta \to \infty$), consider the equation

$$\frac{\partial \log \tau}{\partial \mathcal{P}_\beta} = -\frac{1}{12\pi\sqrt{-1}n} \int_{\beta^*} \frac{S_{\widetilde{B}} - S_v}{v} \underset{t_0 \to 0}{\longrightarrow} -\frac{1}{6n} \mathrm{Res}_{x_0} \frac{S_{\widetilde{B}_0} - S_{v_0}}{v_0}\,, \tag{3.20}$$

where $\beta^* = \frac{1}{n}\sum_{m=0}^{n-1} \rho^m \sigma_*^m a \in \mathcal{H}_{n-1}$ is the class dual to $\beta \in \mathcal{H}_1$.

To compute the residue, choose a local coordinate ζ near x_0 on C_0 such that $S_{B_0} = 0$. Then we have $v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}n}\frac{d\zeta}{\zeta} + O(1)$ and $S_{v_0} = \left(\frac{v_0'}{v_0}\right)' - \frac{1}{2}\left(\frac{v_0'}{v_0}\right)^2 = \frac{1}{2\zeta^2} + O(1)$ as $\zeta \to 0$, so that

$$\frac{S_{v_0}}{v_0} = \frac{\pi\sqrt{-1}n}{\mathcal{P}_\alpha}\frac{d\zeta}{\zeta} + O(1).$$

Therefore, (3.20) implies

$$\frac{\partial \log \tau}{\partial \mathcal{P}_\beta}\Big|_{t_0=0} = \frac{\pi\sqrt{-1}}{6\mathcal{P}_\alpha} \tag{3.21}$$

and $\tau \sim e^{\pi\sqrt{-1}\mathcal{P}_\beta/6\mathcal{P}_\alpha}$, i.e.

$$\tau = t_0^{1/12}(\text{const} + o(1)) \tag{3.22}$$

as $t \to 0$.

3.3.3. *Asymptotics of τ near D_j.* Contracting a homologically trivial simple loop γ on C we get a reducible nodal curve C_0 that splits into two irreducible components C_1 and C_2 of genera $g_1 = j$ and $g_2 = g - j$ respectively, $j = 1, \dots, [g/2]$. Denote by $x_0 \in C_1$ and $y_0 \in C_2$ the intersection point of C_1 and C_2 (the node of C_0). The holomorphic n-differential w on C degenerates to a pair of meromorphic n-differentials w_1 and w_2 on C_1 and C_2 respectively, with poles of order n at $x_0 \in C_1$ and $y_0 \in C_2$ whose n-residues differ by $(-1)^n$ (we assume that under generation the zeroes of w stay away from the vanishing cycle γ).

Denote by $f_i : \widehat{C}_i \to C_i$ the canonical n-fold cover of the curve C_i $(i = 1, 2)$, and let $x_0^{(0)}, \dots, x_0^{(n-1)}$ (resp. $y_0^{(0)}, \dots, y_0^{(n-1)}$) denote the preimages of the node in $\widehat{C}_1$ (resp. $\widehat{C}_2$) that are cyclically ordered relative to the covering maps $\sigma_i : \widehat{C}_i \to \widehat{C}_i$. The canonical cover $\widehat{C}_0$ of the nodal curve C_0 is obtained from $\widehat{C}_1$ and $\widehat{C}_2$ by identifying $x_0^{(m)}$ with $y_0^{(m)}$ for each $m = 0, \dots, n-1$ (since the zeroes of both w_1 and w_2 stay away from the node of C_0, no branching over $x_0 \in C_1$ and $y_0 \in C_2$ can occur).

Define the differentials v_i $(i = 1, 2)$ on $\widehat{C}_i$ by putting $v_i^n = f_i^* w_i$ and choosing the roots of unity in such a way that $\mathrm{Res}|_{x_0^{(m)}} v_1 = -\mathrm{Res}|_{y_0^{(m)}} v_2$ (note that v_1 and v_2 have first order poles at the n preimages of the node). Moreover, applying m times the cover map σ_0, we see that $\mathrm{Res}|_{x_0^{(m)}} v_1 = \rho^{-m}\mathrm{Res}|_{x_0^{(0)}} v_1$ (resp. $\mathrm{Res}|_{y_0^{(m)}} v_2 = \rho^{-m}\mathrm{Res}|_{y_0^{(0)}} v_2$).

The preimage $f^{-1}(\gamma) \subset \widehat{C}$ of the loop γ on C is the disjoint union of n loops $\gamma_m,\ m = 0, \dots, n-1$ (we enumerate them in such a way that $\gamma_{m+1} = \sigma(\gamma_m)$, assuming that $\gamma_n = \gamma_0$). Note that the union of $\gamma_m,\ m = 0, \dots, n-1$, is homologically trivial on $\widehat{C}$. Consider also a simple loop η_0 on $\widehat{C}$ such that $\gamma_0 \cdot \eta_0 = 1$, $\gamma_1 \cdot \eta_0 = -1$, and $\gamma_k \cdot \eta_0 = 0$ for $k = 2, \dots, n-1$, where $\cdot$ stands for the intersection pairing of 1-cycles on $\widehat{C}$.

Introduce the loops $\eta_m = \sigma^m(\eta_0),\ m = 1, \dots, n-1$, and consider the cycles

$$\alpha = \sum_{m=1}^{n-1} (\rho^{-m} - 1)\gamma_m\ , \quad \beta = \frac{1}{\rho - 1} \sum_{m=1}^{n-1} (1 - \rho^{-m})\eta_m \tag{3.23}$$

(for homology classes in $H_1(\widehat{C}, \mathbb{Z})$ represented by γ_m and η_m we use the same notation). Clearly, $\alpha, \beta \in \mathcal{H}_1$, and the class $\beta^* \in \mathcal{H}_{n-1}$ dual to β is given by

$$\beta^* = \frac{1}{n} \sum_{m=1}^{n-1} (\rho^m - 1)\gamma_m\ . \tag{3.24}$$

Consider the homological coordinates

$$\mathcal{P}_\alpha = \int_\alpha v = n \int_{\gamma_0} v\ , \quad \mathcal{P}_\beta = \int_\beta v = \frac{n}{1-\rho} \int_{\eta_0} v\ . \tag{3.25}$$

By Fay's plumbing construction [6], the degenerating covering curve $\widehat{C}$ can be represented by $\widehat{C}_0$ with nodes replaced by n long thin cylinders in the flat metric on $\widehat{C}$. Therefore, while C degenerates to C_0 all homological coordinates except $\mathcal{P}_\beta$ remain finite. Similarly to the case of D_0, $(1/n)\mathcal{P}_\alpha$ equals to the integral of v along the vanishing cycle on $\mathcal{C}$. Therefore, in complete analogy to the case of D_0 the local parameter transverse to $D_j \subset \overline{\mathfrak{M}}_g^{(n)}$ can be chosen as

$$t_j = e^{2\pi\sqrt{-1}\mathcal{P}_\beta/\mathcal{P}_\alpha}\ . \tag{3.26}$$

To find the asymptotics of τ when $t_j \to 0$, we use Eq. (3.14) once again:

$$\begin{aligned} \frac{\partial \log \tau}{\partial \mathcal{P}_\beta} &= -\frac{1}{12\pi\sqrt{-1}n} \int_{\beta^*} \frac{S_{\widetilde{B}} - S_v}{v} \\ &= -\frac{1}{12\pi\sqrt{-1}n} \int_{\gamma_0} \frac{S_{\widetilde{B}} - S_v}{v} \longrightarrow -\frac{1}{6n} \mathrm{Res}\big|_{x_0^{(0)}} \frac{S_{\widetilde{B}_0} - S_{v_0}}{v_0}\ . \end{aligned} \tag{3.27}$$

Pick a coordinate ζ near $x_0^{(0)}$ such that $S_{\widetilde{B}_0} = 0$, then $v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}n}\frac{d\zeta}{\zeta} + O(1)$ and $\frac{S_{v_0}}{v_0} = \frac{\pi\sqrt{-1}n}{\mathcal{P}_\alpha}\frac{d\zeta}{\zeta} + O(1)$ as $t_j \to 0$. Therefore, (3.27) implies

$$\frac{\partial \log \tau}{\partial \mathcal{P}_\beta}\Big|_{t_j=0} = \frac{\pi\sqrt{-1}}{6\mathcal{P}_\alpha}\ .$$

Thus, $\tau \sim e^{\pi\sqrt{-1}\mathcal{P}_\beta/6\mathcal{P}_\alpha}$ as $t_j \to 0$, and

$$\tau = t_j^{1/12}(\mathrm{const} + o(1))\ . \tag{3.28}$$

3.4. **Hodge class on $P\overline{\mathfrak{M}}_g^{(n)}$.** A straightforward combination of Theorem 3.4 with asymptotic formulas (3.17), (3.22) and (3.28) yields

Theorem 3.8. (Formula (1.3) of Theorem 1.12) *The Hodge class λ on $\overline{\mathfrak{M}}_g^{(n)}$ is a linear combination of the tautological class ψ and the classes of boundary divisors D_{deg} and $D_j,\ j = 0, 1, \dots, [g/2]$, as follows:*

$$\lambda = \frac{(g-1)(2n+1)}{6n(n+1)}\psi + \frac{1}{12n(n+1)}\delta_{\mathrm{deg}} + \frac{1}{12}\sum_{j=0}^{[g/2]} \delta_j\ . \tag{3.29}$$

4. An alternative computation of $\delta_{\deg}$

Let $g, n \geq 2$. In order to compute the class $\delta_{\deg}$ in $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ we begin with marking one point on C, i.e. we study the space $P\overline{\mathfrak{M}}_{g,1}^{(n)}$. In $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ we define the loci

$$Z_i = \{(C, x_1, w) \mid C \text{ is smooth and } x_1 \text{ is a zero of } w \text{ of order at least } i\}.$$

We denote by $\overline{Z}_i$ the closure of Z_i. This is a closed substack of $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ of pure codimension i for $1 \leq i < N = (2g-2)n$, while for $i = N$ it has two components of codimensions $N-1$ and N respectively, cf Section 1.2.

Let $\pi : P\overline{\mathfrak{M}}_{g,1}^{(n)} \to P\overline{\mathfrak{M}}_g^{(n)}$ be the forgetful map of the marked point. Then it is easy to see that the image of $\overline{Z}_2$ under π is the divisor $D_{\deg}$. This statement takes into account the multiplicities of the components of $D_{\deg}$. Indeed, if $(g, n) \neq (2, 2)$ then the restriction of π to Z_2 is of degree 1 and if $(g, n) = (2, 2)$ then π is of degree one onto $P\overline{\mathfrak{M}}_g^{(n)}(2, 1, 1)$ and two onto $P\overline{\mathfrak{M}}_g^{(n)}(2, 2)$. Therefore $\pi_*[\overline{Z}_2] = \delta_{\deg}$. Thus to find an expression of $\delta_{\deg}$ it suffices to compute the class $[\overline{Z}_2] \in A^2(P\overline{\mathfrak{M}}_{g,1}^{(n)})$.

Computation of $[\overline{Z}_2]$. Let $\mathcal{L}_1 \to \overline{\mathcal{M}}_{g,1}$ be the line bundle whose fiber is the cotangent line to the curve C at x_1, and put $\psi_1 = c_1(\mathcal{L}_1)$. Consider the following line bundle over $P\overline{\mathfrak{M}}_{g,1}^{(n)}$:

$$\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n} \simeq \mathrm{Hom}\left(L, \mathcal{L}_1^{\otimes n}\right).$$

This line bundle has a natural section

$$s_1 : (C, w) \mapsto w(x_1).$$

In other words, s_1 is the evaluation of w at the marked point. The class of the vanishing locus of s_1 is given by the first Chern class of the line bundle:

$$\{s_1 = 0\} \;=\; c_1\left(\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n}\right) \;=\; -\psi + n\psi_1.$$

It is easy to see that $\overline{Z}_1 \subset \{s_1 = 0\}$. In the next section we will show that the vanishing locus has no other components and that the vanishing order of s_1 along $\overline{Z}_1$ is equal to 1. It will follow that $[\overline{Z}_1] = -\psi + n\psi_1$.

Now we restrict to $\overline{Z}_1$ and study the line bundle $\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n+1}$. This line bundle has a natural section

$$s_2 : (C, w) \mapsto w'(x_1).$$

In other words, assuming that w vanishes at x_1, the section s_2 assigns to w its derivative at x_1. It is easy to see that $\overline{Z}_2$ is a component of the vanishing locus $\{s_2 = 0\}$. In the next section we will show that the vanishing locus has no other components and that the vanishing order of s_2 along $\overline{Z}_2$ is equal to 1. Thus we have

$$[\overline{Z}_2] = (-\psi + (n+1)\psi_1)[\overline{Z}_1] = (-\psi + n\psi_1)(-\psi + (n+1)\psi_1).$$

(This computation seems standard, but we were not able to find a proper reference in the literature.)

Recall that $\delta_{\deg}$ is the push-forward by π of this expression. To compute this push-forward we use

- ψ is a pull-back under π;
- $\pi_*\psi_1 = 2g - 2$;
- $\pi_*\psi_1^2 = \kappa_1 = 12\lambda_1 - \sum_{i=1}^{[g/2]} \delta_i$.

The last equality is the well-known Mumford's formula.

Applying these equalities we get

$$\begin{aligned}\delta_{\deg} &= \pi_*\left((-\psi+n\psi_1)(-\psi+(n+1)\psi_1)\right)\\ &= n(n+1)\pi_*(\psi_1^2)-(2n+1)\pi_*(\psi_1)\psi\\ &= 12n(n+1)\lambda_1-n(n+1)\sum_{i=1}^{[g/2]}\delta_i-(2n+1)(2g-2)\psi.\end{aligned}$$

This coincides with the expression of Theorem 3.8.

In order to complete the proof of Theorem 3.8, it remains to prove that the vanishing locus of s_1 (respectively s_2) is exactly $\overline{Z}_1$ (respectively $\overline{Z}_2$) and that the vanishing order of s_1 and s_2 is 1.

Vanishing loci of s_1 and s_2. Let W be an irreducible divisor of $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ in the vanishing locus of s_1. Let k be the number of nodes of the curve represented by a generic point of W. The vanishing locus of s_1 is of codimension 1 in $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ thus $k=0$ or 1. We investigate both cases.

- Let $k=0$. Then a dense subset of W is contained in Z_1 and thus W is a component of $\overline{Z}_1$.
- Let $k=1$. Then the divisor W is contained in $\tilde{\nu}^{-1}(D)$ for some irreducible boundary divisor D of the moduli space of stable curves with one marked point (we recall that $\tilde{\nu}: P\overline{\mathfrak{M}}_{g,1}^{(n)}\to\overline{\mathcal{M}}_{g,1}$ is the forgetful map). Since the D is irreducible, and $\tilde{\nu}$ is the projectivization of a vector bundle, we necessarily have $W=\tilde{\nu}^{-1}(D)$. However there exists an n-differential in $\tilde{\nu}^{-1}(D)$ which is not identically zero on the component containing the marked point. Therefore, there exists a point in $\tilde{\nu}^{-1}(D)$ which is not in the vanishing locus of s_1. Thus the case $k=1$ does not occur.

To study the vanishing locus of s_2 we follow the same strategy. First we can check by dimension count that no irreducible component of Z_1 is in the zero locus of s_2. Now let W be an irreducible divisor in the vanishing locus of s_2 and let k be the number of nodes of the curve represented by a generic point of W. We have now 3 cases to study: $k=0,1$ and 2.

- Let $k=0$. Then a dense subset of W is contained in Z_2 and thus W is a component of $\overline{Z}_2$.
- Let $k=2$. Then $W=\tilde{\nu}^{-1}(D)$ where D is a codimension 2 boundary stratum of $\overline{\mathcal{M}}_{g,1}$. As above $\tilde{\nu}^{-1}(D)$ is not contained in the vanishing locus of s_2. Thus the case $k=2$ cannot occur.
- Let $k=1$. Then W is a co-dimension 1 locus inside $\tilde{\nu}^{-1}(D)$ for a boundary divisor D of $\overline{\mathcal{M}}_{g,1}$. The generic curve has two components of genera g' and $g-g'$ with $1\le g'\le g-1$. We assume that the marked point is carried by the component of genus g'. The rank of the bundle of n-differentials with a pole of order at most n at the node is $n(2g'-2+1)>1$. Thus the divisor D is not contained in the locus of differentials that vanish identically on this component. Thus D is not contained in the vanishing locus of s_2.

We conclude that $\{s_i=0\}=\overline{Z}_i$ for $i=1$ and 2.

Vanishing order of s_1. Let $y_0=(C_0,w_0,x_0)$ be a point in Z_1. We recall that $P\overline{\mathfrak{M}}_{g,1}^{(n)}\to P\overline{\mathfrak{M}}_g^{(n)}$ is isomorphic to the universal curve. Thus a neighborhood of y_0 in Z_1 is given by $U\times\Delta$ where U is a neighborhood of (C_0,w_0) in $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ and $\zeta\in\Delta$ is the distinguished parameter around x_0 in C_0 (cf Section 2.1). Let (C,w,x) be an n-differential in $U\times\Delta$. In coordinates $(u,\zeta)\in U\times\Delta$, the differential

w is given by $w = \zeta d\zeta^n$, the locus Z_1 is $\{\zeta = 0\}$ and the section s_1 is given by $s_1(u, \zeta) = \zeta$. Therefore the vanishing order of s_1 along Z_1 is 1.

Vanishing order of s_2. Let $y_0 = (C_0, w_0, x_0)$ be a point in Z_2. A neighborhood of y_0 in $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ is now given by $U \times \Delta \times \Delta'$ where U is a neighborhood of y_0 in Z_2 and Δ and Δ' are disks in the complex plane centered at 0 and parametrized by ζ and a such that:

$$w = (\zeta^2 + a)d\zeta^n.$$

With the parameters $(u, \zeta, a) \in U \times \Delta \times \Delta'$, the locus Z_1 is defined by $\zeta^2 + a = 0$. Moreover with these parameters, the section s_2 is given by $s_2(u, \zeta, a) = a$. Thus the vanishing order of s_2 along Z_2 is again 1.

5. Prym-Tyurin differentials on $\widehat{C}$ and holomorphic n-differentials on C

Here we relate Prym-Tyurin vector bundles to vector bundles of holomorphic k-differentials on the base Riemann surface C. We use this relation to finish the proof of Theorem 1.12.

We also prove that the Prym-Tyurin bundle is *not* a pullback from $P\overline{\mathfrak{M}}_g^{(n)}$ in general.

5.1. **Prym-Tyurin bundles and n-differentials.** Consider two vector bundles $\Lambda^{(k)}$ and $\tilde{\nu}^*\Omega_g^{(n-k+1)}$ over $X(g, n)$. The fiber of $\Lambda^{(k)}$ is the kth eigenspace in the space of abelian differentials on $\widehat{C}$. The fiber of Ω_{n-k+1} is the space of $(n-k+1)$-differentials on C. There are natural morphisms:

$$\begin{array}{llll} \Phi_0: & \Lambda^{(0)} & \to & \tilde{\nu}^*\Omega_g\,, \\ \Phi_k: & \Lambda^{(k)} \otimes T^{\otimes(n-k)} & \to & \tilde{\nu}^*\Omega_g^{(n-k+1)} \qquad \text{for } 1 \le k \le n. \end{array} \tag{5.1}$$

Indeed, let (C, w) be a point in $U \setminus D_{\deg}$ and let q be a differential in the fiber of $\Lambda^{(k)}$. The $n-k+1$ differential qv^{n-k} is invariant under the action of the automorphism group of the cover, thus qv^{n-k} is the pull-back of $n-k+1$ differential on C. For $k = 0$ the differential q is already invariant under the action of the automorphism group of the cover, so there is no need to multiply it by a power of v.

Lemma 5.1. *The map Φ_k is an isomorphism over $V \setminus D_{\deg}$.*

Proof. The maps Φ_k for $0 \le k \le n-1$ are injective because v does not vanish identically on any component of the nodal curve $\widehat{C}$. They are bijective because the sum of ranks of the Prym-Tyurin bundles $\Lambda^{(k)}$ for $0 \le k \le n-1$ is equal to the sum of ranks of the vector bundles Ω_{n-k+1}. Indeed,

$$\sum_{k=0}^{n-1} \operatorname{rk} \Lambda^{(k)} = \operatorname{rk} \Omega_{\widehat{g}} = n^2(g-1) + 1$$

and

$$\operatorname{rk} \Omega_g + \sum_{k=1}^{n-1} \operatorname{rk} \Omega_g^{(n-k+1)} = g + \sum_{k=1}^{n-1} (2n - 2k + 1)(g-1) = n^2(g-1) + 1.$$

□

Corollary 5.2. *The rank of the Prym-Tyurin bundle is g for $k = 0$ and $(2n-2k+1)(g-1)$ for $1 \le k \le n-1$.*

Corollary 5.3. *In* $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)} \setminus D_{\deg})$ *we have*

$$\lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi \tag{5.2}$$

for $1 \leq k \leq n-1$.

Proof. On the locus where Φ_k is an isomorphism we have

$$\begin{aligned}
\lambda_{PT}^{(k)} &= c_1(\Lambda^{(k)}) \\
&= c_1(\tilde{\nu}^*\Omega_k \otimes T^{\otimes -(n-k)}) \\
&= \lambda_{n-k+1} - (g-1)(n-k)(2n-2k+1)c_1(T) \\
&= \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi,
\end{aligned}$$

where the last equality is due to $T^{\otimes n} = L$.

The locus where Φ_k is an isomorphism contains $P\overline{\mathfrak{M}}_g^{(n)} \setminus D_{\deg}$ up to codimension 2 substacks that are immaterial for the first Chern class computations. □

To study the extension of the formula (5.2) to $P\overline{\mathfrak{M}}_g^{(n)}$ we need to study the behavior of Φ_k along the boundary divisor $D_{\deg}$. The determinant of Φ_k is a global section of

$$\det(\Lambda^{(k)} \otimes T^{n-k})^{-1} \otimes \det(\tilde{\nu}^*\Omega_g^{(n-k+1)}).$$

Thus the difference between $\lambda_{PT}^{(k)}$ and $\lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi$ is an effective divisor defined as the vanishing locus of $\det \Phi_k$.

In Section 2.2, we have described a parametrization of the cyclic covers along $D_{\deg}$. We use here this parametrization to prove the following Lemma.

Lemma 5.4. *If* $[(n-1)/2] + 1 \leq k \leq n-1$ *or* $k = 0$, *then the morphism* Φ_k *is an isomorphism of vector bundles over* $V \subset X(g,n)$. *Otherwise,* $\det(\Phi_k)$ *vanishes along* $D_{\deg}$ *with order* $(1 - \frac{2k}{n})$.

This lemma implies the following

Corollary 5.5. *The following relations between Prym-Tyurin class* $\lambda_{PT}^{(k)}$, *the class* $\psi = c_1(L)$ *and the pullback of class* λ_{n-k+1} *from* $\overline{\mathcal{M}}_g$ *to* $P\overline{\mathfrak{M}}_g^{(n)}$ *holds:*

$$\lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi + \left(\frac{1}{2} - \frac{k}{n}\right)\delta_{\deg}\,, \qquad 1 \leq k \leq [(n-1)/2]\,, \tag{5.3}$$

$$\lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi\,, \qquad [(n-1)/2] + 1 \leq k \leq n-1\,. \tag{5.4}$$

This corollary together with Theorem 3.8 and Formula (1.6) completes the proof of Theorem 1.12.

Remark 5.6. Note the difference of a factor 2 between the vanishing order of $\det \Phi_k$ and the contribution of $\delta_{\deg}$ in $\lambda_{PT}^{(k)}$. This is due to the fact that $V \to U$ is of degree 1/2 along $D_{\deg}$.

The proof of Lemma 5.4 will occupy the two following sections. We consider separately the cases of even n and odd n.

5.2. **Odd** $n = 2m+1$.

5.2.1. *Kernel and cokernel of* Φ_k. Let (C_0, w_0) be a point in $D_{\deg}$ and $f : \widehat{C}_0 \to C_0$ be the associated admissible cover. We recall that $\widehat{C}_0$ is a curve with two components intersecting in one point. The two components $\widehat{C}_1$ and $\widehat{C}_2$ are of genera $\widehat{g}_1 = m$ and $\widehat{g}_2 = \widehat{g} - m$ (see Section 2.1). We have denoted by w_1 the meromorphic n-differential on C_1 given by $(\zeta - \zeta_1)(\zeta - \zeta_2)d\zeta^n$. The curve $\widehat{C}_1$ is parametrized by $y^n = (\zeta - \zeta_1)(\zeta - \zeta_2)$ and the nth root of f^*w_1 is given by $v_1 = yd\zeta$. The cover $\widehat{C}_2 \to C_2$ is defined by $v_2^{\otimes n} = w_0$. Finally, let v be the canonical differential satisfying $v^{\otimes n} = f^*w_0$: it is determined by $v = v_2$ on the component $\widehat{C}_2$ and vanishes identically on the component $\widehat{C}_1$.

Denote the fiber of kth Prym-Tyurin vector bundle $\Lambda^{(k)}$ over the point $(C_0, w_0) \in D_{\deg}$ by $\Omega_0^{(k)}$. We can decompose

$$\Omega_0^{(k)} = \Omega_1^{(k)} \oplus \Omega_2^{(k)}$$

where $\Omega_1^{(k)}$ is the space of holomorphic differentials on $\widehat{C}_1$ which get multiplied by ρ^k under the action of the automorphism $(y, \zeta) \to (\rho^k y, \zeta)$; the space $\Omega_2^{(k)}$ is the analogous space of holomorphic differentials on $\widehat{C}_2$.

The canonical differential v vanishes identically on $\widehat{C}_1$. Thus the kernel of Φ_k contains the space $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. On the other hand, the restriction of the morphism Φ_k to the linear subspace $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ is injective. Indeed the differential v does not vanish identically on $\widehat{C}_2$. Thus

$$\ker \Phi_k \simeq \Lambda_1^{(k)} \otimes T^{(n-k)}.$$

over a generic point of $D_{\deg}$.

We have $\dim \Omega_1^{(k)} = 1$ for $k = 1, \dots, m$. These one-dimensional spaces are generated by the holomorphic differentials on $\widehat{C}_1$ given by $q_1^{(k)} = \frac{d\zeta}{y^{2m+1-k}}$. For $k = m+1 \dots 2m+1$ the eigenspaces $\Omega_1^{(k)}$ are trivial. Therefore,

$$\dim\Omega_2^{(k)} = \dim\Omega_0^{(k)} - 1 \ , \qquad k = 1, \dots, m \ ;$$

$$\dim\Omega_2^{(k)} = \dim\Omega_0^{(k)} \ , \qquad k = m+1, \dots, 2m.$$

We can also describe the images of Φ_k. For $k = m+1, \dots, 2m$, the morphism Φ_k is an isomorphism from $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ to $H^0(C_0, \omega_{C_0}^{n+k-1})$. However, for $k = 1, \dots, m$ the image of $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ is the space of holomorphic $n-k+1$ differentials vanishing at x_0.

We have proved the following

Lemma 5.7. *The kernel of* Φ_k *over a generic point of* D_{deg} *is the vector bundle* $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. *This kernel is trivial for* $k > m$ *and* $k = 0$. *If* $1 \le k \le m$, *the image of* Φ_k *is the vector bundle whose fibers are the* $H^0(C, \omega^{n+k-1}(-x_0))$ *where* x_0 *is the unique zero of order* 2.

Therefore the first part of Lemma 5.4 and Formula (5.4) are valid for odd n.

5.2.2. *Description of* Φ_k *for* $k = 1 \dots m$. We fix $1 \le k \le m$. We have seen that the kernel and cokernel of Φ_k are of dimension 1. Let (C_0, w_0) be a generic point in $D_{\deg}$. Let W be an open neighborhood of (C_0, w_0) in $X(g, n)$ with a non-vanishing section $q_0^{(k)}$ of $\Lambda^{(k)}|_W$ such that $q_0^{(k)}|_{D_{\deg}}$ spans $\ker \Phi_k|_{W \cap D_{\deg}}$. The section Φ_k is of co-rank 1 along $D_{\deg}$, thus the vanishing order of $\det \Phi_k$ is equal to the vanishing order of $\Phi_k(q_0^{(k)} \otimes v^{\otimes n-k})$ along $D_{\deg}$. Therefore, we will construct such a local section $q_0^{(k)}$ of $\Lambda^{(k)}$ and study the asymptotic behavior of $q_0^{(k)} \otimes v^{\otimes n-k}$ along $D_{\deg}$.

Let $\tilde{q}_0^{(k)}$ be a non-vanishing section of $\tilde{\nu}^*(\Omega^{(n-k+1)})$ over W such that: for all $(C,w) \in D_{\deg}$, $q_0^{(k)}(C,w)$ is a differential that does not vanish at the double zero of w. Up to a choice of a smaller W, such a section exists. We label the coalescing zeros by x_1 and x_2. We chose the parameter of the curve ζ such that position of x_1 and x_2 are ζ_1 and ζ_2 and $w = (\zeta-\zeta_1)(\zeta-\zeta_2)(d\zeta)^n$ (see Section 2.2). We define

$$q_0^{(k)} = (\zeta_1-\zeta_2)^{-1+2k/n} \cdot \frac{f^*(\tilde{q}_0^{(k)})}{v^k}.$$

We recall that root $(\zeta_1-\zeta_2)^{2/n}$ is well defined, it is the integral of v between $\hat{x}_1$ and $\hat{x}_2$ (see Lemma 3.6).

Over $W \setminus D_{\deg}$, the differential $q_0^{(k)}$ is obviously a non-vanishing section of $\hat{\nu}^*(\Lambda^{(k)})$. Besides, at a generic point of $D_{\deg}$ the differential $q_0^{(k)}$ vanishes identically on $\widehat{C}_2$ because of the factor $(\zeta_1-\zeta_2)^{-1+2k/n}$. To compute the limit of $q_0^{(k)}$ on $\widehat{C}_1$, we remark that $q_0^{(k)}$ is given by

$$(\zeta_1-\zeta_2)^{1-2k/n}[(\zeta(x)-\zeta_1)(\zeta(x)-\zeta_2)]^{k/n-1}d\zeta \tag{5.5}$$

in coordinate ζ. The differential (5.5) is invariant under simultaneous rescaling $\zeta \to \epsilon\zeta$, $\zeta_i \to \epsilon\zeta_i$, $i=1,2$. Therefore, as $x_{1,2} \to x_0$, the differential $q_0^{(k)}$ tends to the holomorphic differential

$$q_1^{(k)} = y^{-n+k}dx = (\zeta_1-\zeta_2)^{1-2k/n}[(\zeta(x)-\zeta_1)(\zeta(x)-\zeta_2)]^{k/n-1}d\zeta$$

on the curve $\widehat{C}_1$ (the generator of $\Omega_1^{(k)}$).

The image of $q_0^{(k)} \otimes v^{\otimes n-k}$ under Φ_k is equal to $(\zeta_1-\zeta_2)^{1-2k/n}\tilde{q}_0^{(k)}$ by construction. Thus the determinant of Φ_k is equivalent to a constant times $(\zeta_1-\zeta_2)^{1-2k/n}$. The parameter $(\zeta_1-\zeta_2)^2$ being a transverse parameter to $D_{\deg}$, we conclude that the vanishing order of Φ_k along $D_{\deg}$ is $1-\frac{2k}{n}$. This finishes the proof of Lemma 5.4 for odd n. □

5.3. **Even $n = 2m$.** The proof of Lemma 5.4 is essentially identical to the odd case. The difference between the odd and the even case is in the topolocial type of the degenerate cover and on the orders of singularities of the limit differentials. Thus we only need to transpose correctly the notation from the odd case taking into account this differences in orders of singularities.

Let (C_0,w_0) be a generic point in $D_{\deg}$. Now we have $\widehat{g}_1 = m-1$ and $\widehat{g}_2 = \widehat{g}-m$ and the two components intersect in two points. Let Ω_0^k be the PT bundle over (C_0,w_0) and let $\Omega_i^{(k)}$ be the subspace of Ω_0^k of holomorphic differentials supported on $\widehat{C}_i$. We still have the decomposition:

$$\Omega_0^{(k)} = \Omega_1^{(k)} \oplus \Omega_2^{(k)}$$

except for $k=m$. The kernel Φ_k is $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. This kernel is trivial for $k = m+1,\dots,2m-1$. For $k=1,\dots,m-1$, the map Φ_k has co-rank one and the generator of $\Omega_1^{(k)}$ is the differential $q_1^{(k)} = y^{k-2m}d\zeta$ on the canonical cover $\widehat{C}_1$ $y^{2m} = (\zeta-\zeta_1)(\zeta-\zeta_2)$ (see Section 2.2 for definition of the parameters).

For $k=m$, the space $\Omega_0^{(k)}$ contains a 1-dimension subspace of differentials of a third kind. These are differentials with simple poles at the nodal points $x_0^{(1)}$ and $x_0^{(2)}$ (both on $\widehat{C}_1$ and $\widehat{C}_2$) and opposite residues. The image of such differential under Φ_m is a holomorphic m-differential. Beside $\Omega_1^{(m)}$ is trivial. Therefore the morphism Φ_k is also bijective for $k=m$.

Similarly to the case of odd n, for $k=1,\dots,m-1$, let $\tilde{q}_0^{(k)}$ be a section of $\tilde{\nu}^*(\Omega^{(n-k+1)})$ over a neighborhood of (C_0,w_0) that does not vanish at the double

zero of w_0. We define $q_0^{(k)} = (\zeta_1 - \zeta_2)^{-1+2k/n} \cdot \frac{f^*(\tilde{q}_0^{(k)})}{v^k}$. and we study the asymptotic behavior of $\Phi_k(q_0^{(k)} \otimes v^{n-k})$ along $D_{\deg}$.

As in the odd case, the differential $q_0^{(k)}$ is a non vanishing section of $\Lambda^{(k)}$ and $\Phi_k(q_0^{(k)} \otimes v^{n-k}) = (\zeta_1 - \zeta_2)^{-1+2k/n} \cdot \tilde{q}_0^{(k)}$ by construction. Therefore the vanishing order of $\det(\Phi_k)$ is given by $1 - \frac{2k}{n}$ for $k = 1, \dots, m-1$. □

5.4. **Obstruction to the extension of the Prym bundles to $\overline{\mathfrak{M}}_g^{(n)}$.** In order to define the Prym-Tyurin classes we have constructed the space of admissible differentials $X(g,n)$ (see 1.5). Indeed, the Prym-Tyurin bundles are naturally defined over $X(g,n)$. The following theorem explains the necessity of the introduction of the space $X(g,n)$.

Theorem 5.8. *Let $g > 2$ and $k > 0$. There exists no vector bundle $\tilde{\Lambda} \to P\overline{\mathfrak{M}}_g^{(n)}$ such that $\Lambda^{(k)} = \mathrm{diff}^*\tilde{\Lambda}^{(k)}$, where $\mathrm{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is the forgetful map.*

Proof. Suppose that there exists $\tilde{\Lambda} \to P\overline{\mathfrak{M}}_g^{(n)}$ such that $\Lambda^{(k)} = \mathrm{diff}^*\tilde{\Lambda}^{(k)}$. Then in particular $\lambda_{PT}^k = c_1(\tilde{\Lambda}^{(k)})$ and thus $c_1(\Lambda^k) = \mathrm{diff}^*\lambda_{PT}^k$. We will prove that this equality does not hold for $g > 2$ and $k > 0$.

Let $m > 2$ be such that $\gcd(m,n) = 1$. Let μ be the partition of $n(2g-2)$ given by $(m, 1, \dots, 1)$. We denote by $\mathfrak{M}_g^{(n)}[m]$ the locus $\mathfrak{M}_g^{(n)}[\mu] \subset \overline{\mathfrak{M}}_g^{(n)}$. We denote by D_m the preimage of $\mathfrak{M}_g^{(n)}[m]$ in $X(g,n)$ under diff. The locus D_m is a divisor whose generic points are elements $(C, w, x_i, f : \widehat{C} \to C) \in X(g,n)$ such that:

- the curve C is a nodal curve with two components: a component C_2 isomorphic to C with $n(2g-2) - m$ marked points intersecting a rational component carrying m marked points in one point that is located at the zero of order m of w on C_2.
- the n-differential w is identically zero on C_1 and has profile μ on C_1;
- the cover curve $\widehat{C} \to C$ has two component $\widehat{C}_2$ and $\widehat{C}_1$. The component $\widehat{C}_2$ determined by the w as in Section 2.2 and $\widehat{C}_1 \to C_1$ is the unique n-sheeted ramified cover maximally ramified at the marked points and the node.

Moreover the canonical root v of f^*w vanishes identically on $\widehat{C}_1$ and has a zero of order $m+n-1$ at the preimage of the zero of order m. Thus the morphism $\Phi_k : \Lambda^{(k)} \otimes T^{\otimes(n-k)} \to \tilde{\nu}^*\Omega_g^{(n-k+1)}$ has a non-empty co-kernel along D_m for m large enough. Therefore the line bundle

$$\det(\Lambda^{(k)}) \otimes \mathrm{diff}^*\left(\det\tilde{\Lambda}^{(k)\vee}\right)$$

has a global section which vanishes along divisors contained in $X(g,n) \setminus V$. Thus $\Lambda^{(k)} \neq \mathrm{diff}^*\tilde{\Lambda}^{(k)}$. □

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