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► **To cite this version:**

Sara Mazzonetto. Rates of convergence to the local time of Oscillating and Skew Brownian Motions. 2021. hal-03367822v1

HAL Id: hal-03367822

<https://hal.science/hal-03367822v1>

Preprint submitted on 6 Oct 2021 (v1), last revised 3 Apr 2024 (v2)

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Rates of convergence to the local time of Oscillating and Skew Brownian Motions

Sara Mazzonetto*

Abstract

In this paper, a class of statistics based on high frequency observations of oscillating and skew Brownian motions is considered. Their convergence rate towards the local time of the underlying process is obtained in form of a Central Limit Theorem. Oscillating and skew Brownian motions are solutions to stochastic differential equations with *singular* coefficients: piecewise constant diffusion coefficient or drift involving the local time. The result is applied to provide estimators of the parameter of skew Brownian motion and study their asymptotic behavior. Moreover, in the case of the classical statistic given by the normalized number of crossings, the result is proved to hold for a larger class of Itô processes with singular coefficients.

Keywords: Skew Brownian motion, Oscillating Brownian motion, Local time, Functional limit theorems, Central Limit Theorem, Parameter estimation.

AMS 2010: 60F17, 62F12, 60J55, 60F05.

1 Introduction

It is well known that the normalized number of crossings of the level $r \in \mathbb{R}$ of the time discretization (high frequency) of a Brownian motion (BM) provides an estimator for its local time at r . Roughly speaking the local time at the point r measures the time the process spends around r (see (1.7) below for a precise definition), so a rescaled number of crossings for high frequency data is a natural approximation of the local time also for more general processes. The normalized number of crossings has been extensively studied as an approximation of the local time of Brownian diffusions whose drift and diffusion coefficient σ are sufficiently regular (in particular σ is continuous). The convergence was proven, for instance, in [5, 6]. In this document we allow the presence of some singularities and examine the asymptotic behavior in Theorem 2 below.

More general functionals of discrete observations can also be considered. Given a stochastic process $(X_t)_{t \in [0, \infty)}$, let us consider the following statistics for high frequency observations:

$$\varepsilon_{n,t}^{(r,f,X)} := \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} f(\sqrt{n}(X_{k/n} - r), \sqrt{n}(X_{(k-1)/n} - r)) \quad (1.1)$$

where $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a measurable function satisfying suitable integrability conditions. The normalized number of crossings of the level r corresponds to considering the function $f(x, y) := \mathbf{1}_{(-\infty, 0)}(xy)$ (the statistic is explicitly provided in (3.1) below).

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In the case X is a BM, for the kind of estimators in (1.1), convergence towards the local time and Central Limit Theorem (CLT) were obtained in [11, 12]. In the context of Brownian diffusions with regular coefficients mentioned above, convergence results are proven for specific functions f in [17] and for more general statistics of multivariate diffusions in [24]. In the latter article the associated CLT is proved. It is shown via semigroup estimates and martingale limit theorems, that $\varepsilon_{n,t}^{(r,f,X)}$ behaves asymptotically (in n) as a mixed Gaussian distribution (see (1.2) below). The techniques developed so far can be adapted to study more general processes, for instance fractional BM in [41] and [3, Corollary 14].

In this document we focus on one-dimensional Itô processes solutions of stochastic differential equations (SDEs) with singular coefficients: discontinuous coefficients or distributional drift in form of a weighted local time of the process at a given level. Solutions to such SDEs are often called *threshold* or *skew* diffusions. Two key cases belonging to this class of processes are oscillating Brownian motion (OBM) and skew Brownian motion (SBM). They are generalizations of BM and of reflected BM as well, with distributions which are possibly singular with respect to BM. They change behavior when they reach a point, called barrier or *threshold*, which then becomes a *discontinuity point* for the local time $x \mapsto L_t^x(X)$. More precisely, OBM behaves like a BM with a different volatility above and below the threshold (causing a *regime-switch*) while SBM behaves like a BM everywhere except when it reaches the threshold, which plays the role of semi-permeable and semi-reflecting barrier. Note that OBM and SBM are *null-recurrent* processes, hence results for ergodic processes cannot be applied.

Let us have a look at the following simplified statement of our key result, that is Theorem 1 below: let X be a SBM/OBM and r be the threshold, let $(L_t^r(X))_{t \geq 0}$ be its symmetric local time at the threshold. Then for appropriate constants $c, K \in \mathbb{R}$ it holds for all $t \in [0, \infty)$ that

$$n^{1/4} \left(\frac{\varepsilon_{n,t}^{(r,f,X)} - c L_t^r(X)}{K^2 \sqrt{L_t^r(X)}} \right) \stackrel{n \rightarrow \infty}{\sim} \mathcal{N}(0, 1). \quad (1.2)$$

Although, at the threshold r , OBM and SBM behave differently with respect to BM and in particular the local time $x \mapsto L_t^x(X)$ is discontinuous at r , the speed of convergence is the same as for BM: $n^{1/4}$. Heuristically the convergence is different from what one might expect (i.e. $n^{1/2}$), because the local time in r and its estimator change only when the process reaches r . Indeed, as $n \rightarrow \infty$, among n observations of the process on a fixed interval, the number of those which are sufficiently close to r to matter is of order $n^{1/2}$.

The result in Theorem 1 is actually more general. Firstly it is a *functional limit theorem*: the processes is seen as random variables with values in the Skorokhod space of càdlàg functions. Secondly it holds also for drifted OBM and SBM, under some suitable assumptions on the drift. Therefore, this result extends the existing results on BM to solutions to SDEs with singular coefficients such as OBM, SBM, and reflected BM (with suitable drift, by Girsanov). Extensions to more general statistics of solutions to SDEs with more general discontinuous diffusion coefficient is object of further research. Nevertheless, in Theorem 2, we prove the analogous of Theorem 1 for the well known estimator of the normalized number of crossings when the process is a more general threshold diffusion.

1.1 Motivation and applications

Since the seventies OBM and SBM together with their local time have been studied in the context of threshold diffusions. In probability and stochastic analysis [7, 30, 43, 20, 18, 47], . . . , recently in SPDEs [13, 4], in simulation [16, 45], . . . , and we refer the reader to the introduction of [32] for some more applications of threshold diffusions in astrophysics, brain imaging, ecology, geophysics, fluid/gas dynamics, meteorology, molecular dynamics, oceanography.

Some models in financial mathematics and econometrics are threshold diffusions, for instance continuous-time versions of SETAR (self-exciting threshold auto-regressive) models, see e.g. [14, 38]. The study of SBM (or OBM) and its local time has been recently investigated in the context of option pricing, as for instance in [19] and [15]. In [34] it is shown that a time series of threshold diffusion type captures leverage and mean-reverting effects. We refer to the introduction of the latter article for further references.

Statistical studies for threshold diffusions are partially motivated by calibration of such econometric models (see e.g. [46, 33]). Indeed, study of (quasi) maximum likelihood estimators (MLE) of drift coefficients from high frequency observations depends on the approximations of occupation times and local times of the process. This is quite naturally explained by the fact that the behavior of the process changes at the threshold. Less heuristically, and more quantitatively, since the process behaves differently on two semi-axes, say $(-\infty, 0)$ and $(0, +\infty)$, it is natural to look at the dynamics of the process in these semi-axes and this means considering $\max\{X_t, 0\}$ and $\min\{X_t, 0\}$. And if X_t satisfies a SDE then by Itô-Tanaka formula $\max\{X_t, 0\}$ and $\min\{X_t, 0\}$ satisfy a SDE involving the local time of the process X_t . In [39], Theorem 1 has been applied to exhibit the asymptotic behavior in high frequency of (quasi) MLE of the drift parameters of a threshold diffusion which is a continuous-time SETAR model. Similar applications are possible for other econometric models.

Despite the fact that the latter application was the original motivation of this document, Theorem 1 contributes to parameter estimation of SBM. Statistical analysis for SBM and OBM is quite recent: estimators based on high frequency observations of the skewness parameter of SBM are provided in [36, 37] and of the diffusion coefficients of OBM in [33]. The estimators proposed are based on approximations of the local time and occupation times from high frequency observations of the process itself (SBM or OBM). Hence the speed of convergence of such estimators can now be obtained thanks to Theorem 1. The result also allows to establish the speed of convergence and limit distribution of the MLE of the skewness parameter from high frequency observation of SBM conjectured in [37]. To give an example, in Theorem 3 (new) estimators of the skewness parameter of SBM are introduced and their asymptotic behavior is established. This should be the first result in the literature to prove the convergence speed for estimators of the skewness parameter of SBM.

In Section 3.1, we apply Theorem 1, to two classical approximations of the local time of BM such as the normalized number of crossings. Since standard BM (and reflected BM as well) is a special case of OBM and SBM, as a by-product, we recover the classical results on the convergence (rates) for BM. In the case of the number of crossings, we can consider threshold diffusions with diffusion coefficient which is piecewise differentiable and admits a finite jump and/or involving the local time of the process (see Theorem 2 below). In a series of works Gikhman, Portenko, and Goshko study the convergence in law towards the local time of the normalized number of crossings in a setting allowing for singularities (see [42] and references therein) which seems to include OBM, SBM, and also sticky BM. Theorem 2 below implies a stronger convergence (which was first proven in [37] for SBM) and exhibits also the rate of convergence. Up to our knowledge, Theorem 2 is the first result of its kind for such general one-dimensional threshold diffusion.

1.2 Outline of the paper

The paper is organized as follows. First we introduce the processes SBM and OBM as unique strong solutions to some SDE in Section 2.1, then we state the main result: Theorem 1 in Section 2.3. Section 3.1 is devoted to applications.

Proofs are provided in Section 4. An element of the proof of Theorem 1 is Proposition 5 below. Its proof is so technical that it is provided in Appendix B. Appendix A deals with useful

properties of OBM relevant in this article and in Appendix B.

1.3 Notation and notions of convergence

Throughout this document for every measurable functions $g: \mathbb{R} \rightarrow \mathbb{R}$ and measure μ on the Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we denote by $\langle \mu, g \rangle$ the integral of g with respect to the measure μ :

$$\langle \mu, g \rangle := \int_{-\infty}^{\infty} g(x) \mu(dx). \quad (1.3)$$

For every $\gamma \in [0, \infty)$ let $\lambda^{(\gamma)}$ be the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ absolutely continuous with respect to the Lebesgue measure satisfying $\lambda^{(\gamma)}(dx) = |x|^\gamma dx$ and let

$$(\mathbf{L}^1(\lambda^{(\gamma)}), \|\cdot\|_{1,\gamma}) \quad (1.4)$$

the set of Borel measurable $\lambda^{(\gamma)}$ -integrable functions and its norm. If $\lambda = 0$, we simply denote by $(\mathbf{L}^1, \|\cdot\|_1) := (\mathbf{L}^1(\lambda^{(0)}), \|\cdot\|_{1,0})$ the normed space of Lebesgue integrable functions.

Definition 1. Let $\gamma \in [0, \infty)$. We denote by $\mathbf{L}^{1,b}(\lambda^{(\gamma)})$ the following subspace of \mathbf{L}^1 :

$$\mathbf{L}^{1,b}(\lambda^{(\gamma)}) = \{f: \mathbb{R} \rightarrow \mathbb{R}, \text{ measurable and bounded s.t. } f \in \mathbf{L}^1(\lambda^{(\gamma)})\}. \quad (1.5)$$

We denote by \mathbf{L}_γ the following space of bi-variate functions

$$\mathbf{L}_\gamma = \{h: \mathbb{R}^2 \rightarrow \mathbb{R}, \exists \bar{h} \in \mathbf{L}^{1,b}(\lambda^{(\gamma)}), \exists a \in [0, \infty) \text{ s.t. } \forall x, y \in \mathbb{R} : |h(x, y)| \leq \bar{h}(x)e^{a|y-x|}\}. \quad (1.6)$$

Let us give a more rigorous definition of the local time process. Let $t \in [0, \infty)$ and let $(X_s)_{s \in [0, \infty)}$ be a one-dimensional semi-martingale. The *symmetric local time at the point x* accumulated on the time interval $[0, t]$ by the semi-martingale X satisfies a.s.

$$L_t^x(X) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{-\epsilon \leq X_s - x \leq \epsilon\}} d\langle X \rangle_s \quad (1.7)$$

and if $x = 0$ we denote $L_t^0(X)$ by $L_t(X)$.

As already mentioned, the main aim of this article is studying, as $n \rightarrow \infty$, the convergence towards the local time together with its rate of the statistics $\varepsilon_{n,\cdot}^{(r,f,X)}$, with X being an OBM or a SBM and f suitable function. Let us recall the notions of convergence used for the results of this paper. The statement of the CLT involves the notion of *stable convergence* which was introduced and studied first in [44] and [2]. We now specify it in the case used in this document.

Definition 2. Let (D, d) be a metric space, $(\Omega', \mathcal{F}', \mathbb{P}')$ be an extension of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X_n: \Omega \rightarrow D$, $n \in \mathbb{N}$, be a sequence of random variables, and let $X: \Omega' \rightarrow D$ be a random variable. Then we say that X_n converges stably in law to X if for all $f: D \rightarrow \mathbb{R}$ continuous and bounded and all bounded random variable $Y: \Omega \rightarrow \mathbb{R}$ it holds that

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(X_n)Y] = \mathbb{E}'[f(X)Y]. \quad (1.8)$$

Let $t \in [0, \infty)$, let \mathbb{D}_t , resp. \mathbb{D}_∞ , be the Skorokhod space of càdlàg functions from $[0, t]$, resp. $[0, \infty)$, to \mathbb{R} endowed with the Skorokhod topology. When $D = \mathbb{D}_t$, $t \in [0, \infty]$ the functional stable convergence in law is usually denoted by

$$X_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}-s} X. \quad (1.9)$$

Finally we recall the notion used in the convergence results, i.e. the *convergence in probability locally uniformly in time* or convergence *uniform on compacts in probability* (u.c.p.): let X, X_n , $n \in \mathbb{N}$, be random variables with values in \mathbb{D}_∞ , then

$$X_n \xrightarrow{\text{u.c.p.}} X, \quad (1.10)$$

if for all $t \in [0, \infty)$ it holds that $\sup_{s \in [0, t]} |X_n(s) - X(s)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

2 Rates of convergence to the local time

In the entire document let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ be a stochastic basis (i.e. a complete filtered probability space whose filtration satisfies the usual conditions) and W be an $(\mathcal{F}_t)_{t \in [0, \infty)}$ -adapted standard BM.

In this section we first introduce the processes Skew and Oscillating Brownian as solutions to SDEs respectively with discontinuous diffusion coefficient, see (2.4), and involving the local time of the process in (2.1). Note that the local time formally arises from a distributional drift: the Dirac δ at the threshold. Then we introduce a wider class of processes in Section 2.1.4. Finally we provide what can be considered the main result of this article, Theorem 1.

2.1 The framework

2.1.1 Skew Brownian motion

Roughly speaking a SBM can be described trajectoryally as a standard BM transformed by flipping its excursions from the origin with a certain probability. In this document we refer to the characterization as solution to a SDE involving the local time, which was first considered by [21]. We refer the reader to a somehow recent survey paper on SBM [31].

The SBM with skewness parameter $\beta \in [-1, 1]$ at the threshold $r \in \mathbb{R}$ is the diffusion which is strong solution to the following SDE

$$X_t = X_0 + W_t + \beta L_t^r(X) \quad (2.1)$$

where $L_t^r(X)$ is the symmetric local time of the process at r , $X_0 \in \mathbb{R}$, and $\beta X_0 \geq 0$ if $|\beta| = 1$. Some properties of the local time of SBM are object of the recent paper [10].

We call *standard SBM* a SBM with threshold $r = 0$ starting at 0. In this paper a SBM with skewness parameter $\beta \in (-1, 1)$ is also denoted by β -SBM. Note that a 0-SBM is a BM. Moreover the ± 1 -SBM is a positively/negatively reflected BM.

The following quantities are important for the next sections. Let us denote by μ_β the stationary measure associated to the β -SBM with threshold $r = 0$, that is

$$\mu_\beta(dx) := ((1 + \beta)\mathbf{1}_{(0, \infty)}(x) + (1 - \beta)\mathbf{1}_{(-\infty, 0)}(x)) dx = (1 + \operatorname{sgn}(x)\beta) dx, \quad (2.2)$$

and $p_\beta(t, x, y)$ denotes its transition density (first computed in [48]):

$$p_\beta(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) + \beta \operatorname{sgn}(y) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(|x| + |y|)^2}{2t}\right). \quad (2.3)$$

2.1.2 Oscillating Brownian motion

Let $X_0 \in \mathbb{R}$. The strong solution to the SDE

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s, \quad t \geq 0, \quad (2.4)$$

is called OBM with threshold $r \in \mathbb{R}$ when the diffusion coefficient σ is the positive two-valued function discontinuous at the threshold:

$$\sigma := \sigma_- \mathbf{1}_{(-\infty, r)} + \sigma_+ \mathbf{1}_{[r, +\infty)}. \quad (2.5)$$

In this document, we also denoted this process by σ_\pm -OBM. This process has been first defined and studied in [28]. Note that pathwise uniqueness for the SDE follows for instance from the results of [40].

We call *standard OBM* an OBM Y with threshold $r = 0$ and starting point $Y_0 = 0$. We can allow either σ_- or σ_+ to be infinity: If $\sigma_+ = 1$, $\sigma_- = +\infty$, $Y_0 \geq r$ (resp. $\sigma_- = 1$, $\sigma_+ = +\infty$, $Y_0 \leq r$) then if $r = 0$ it is a positively (resp. negatively) reflected BM.

The stationary measure for the OBM with threshold 0 is

$$\lambda_\sigma(dx) := \frac{1}{(\sigma(x-r))^2} dx = \left(\frac{1}{\sigma_-^2} \mathbf{1}_{(-\infty, 0)} + \frac{1}{\sigma_+^2} \mathbf{1}_{[0, \infty)} \right) dx \quad (2.6)$$

and its transition density, here denoted by $q_\sigma(t, x, y)$, satisfies

$$q_\sigma(t, x, y) = \frac{1}{\sigma(y-r)} p_{\beta_\sigma} \left(t, \frac{x}{\sigma(x-r)}, \frac{y}{\sigma(y-r)} \right) \quad \text{with} \quad \beta_\sigma := \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}, \quad (2.7)$$

where p_{β_σ} is the density of the SBM recalled in (2.3) (see, e.g. equation (3) in [33] for an explicit expression).

2.1.3 The interplay between SBM and OBM

For simplicity let $r = 0$. SBM and OBM are strongly related in the following sense: Let σ be the function in (2.5), let Y the solution to (2.4) and X be a SBM solution to (2.1) with skewness parameter $\beta_\sigma := \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}$ and suitable initial condition: Solution to the SDE

$$X_t = \frac{Y_0}{\sigma(Y_0)} + W_t + \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} L_t(X). \quad (2.8)$$

It holds that $Y_t = \sigma(X_t)X_t$, or equivalently $\frac{Y_t}{\sigma(Y_t)} = X_t$, and the local times satisfy

$$L(X) = \frac{\sigma_+ + \sigma_-}{2\sigma_+\sigma_-} L(Y). \quad (2.9)$$

This follows from applying Itô-Tanaka formula (see, e.g. [33, page 3573]).

2.1.4 Itô process with singular coefficients

We consider processes satisfying

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma(X_s) dW_s + \beta L_t^r(X) \quad t \geq 0 \quad \mathbb{P}\text{-a.s.} \quad (2.10)$$

where the diffusion coefficient $\sigma \in C^1(\mathbb{R} \setminus \{r\})$ is strictly positive and admits a finite jump at a fixed *threshold* $r \in \mathbb{R}$, $L_t^r(X)$ is the symmetric local time of the process at the fixed level $r \in \mathbb{R}$, $\beta \in [-1, 1]$, and b is a “suitable” drift (think of $b_s = b(X_s)$ or $b_s = b(s, X_s)$ with b a bounded measurable function).

SBM and OBM correspond to the driftless $b = 0$ SDE (2.10) whence σ is piecewise constant and $\beta(\sigma_- - \sigma_+) = 0$.

We are not concerned with (strong) existence and uniqueness results, for it we refer to the literature on SDEs and Itô processes with singular drift and diffusion coefficients (see e.g. [29, 8, 26] and references therein). This will provide conditions on the coefficients to get existence and uniqueness results.

We possibly need additional assumptions. Let Y_t be the strong solution to (2.10) with $\beta = 0$ and null drift. Then \mathbb{P} -a.s. for all $t \geq 0$ it holds $Y_t = Y_0 + \int_0^t \sigma(Y_s) dW_s$. Assume the drift b is such that the Doléans-Dade exponential $\mathcal{E}(\xi)$ with

$$\xi_t := \int_0^t \frac{b_s}{\sigma(r + (Y_s - r)(1 + \text{sgn}(Y_s - r)\beta)/2)} dW_s \quad (2.11)$$

is an exponential martingale (see e.g. Novikov’s condition in [27, Corollary 3.5.14]).

2.2 Convergence towards the local time

Let X be either the σ_{\pm} -OBM solution to (2.4) or the β -SBM solution to (2.1), let μ_X and p_X denote respectively the stationary measure λ_{σ} , resp. μ_{β} , and transition density q_{σ} , resp. p_{β} .

Given two measurable functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$, let

$$\mathbf{E}_{f,g}^X(x) = \int_{-\infty}^{\infty} f(x,y)g(y)p_X(1,x,y) dy \quad \text{and} \quad \mathbf{E}_f^X := \mathbf{E}_{f,1}^X \quad (\text{i.e. } g \equiv 1). \quad (2.12)$$

Note that, if X^x denotes the solution with $X_0 = x$ and $r = 0$, then

$$\mathbf{E}_{f,g}^X(x) = \mathbb{E}[f(x, X_1^x)g(X_1^x)]. \quad (2.13)$$

Hypothesis 1. The measurable bi-variate function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies that $\mathbf{E}_f^X, \mathbf{E}_{f^2}^X \in \mathbf{L}^{1,b}(\lambda^{(2)})$.

Proposition 1 (Convergence towards the local time). *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying Hypothesis 1 and let X be either the OBM solution to (2.4) or the SBM solution to (2.1). Then*

$$\varepsilon_{n,\cdot}^{(r,f,X)} \xrightarrow[n \rightarrow \infty]{\text{u.c.p.}} \langle \mu_X, \mathbf{E}_f^X \rangle L^r(X). \quad (2.14)$$

Note that the constant $\langle \mu_X, \mathbf{E}_f^X \rangle$ can be rewritten as

$$\langle \mu_X, \mathbf{E}_f^X \rangle = \mathbb{E}[f(X_0, X_1) | X_0 \sim \mu_X]. \quad (2.15)$$

Moreover observe that actually the convergence in Proposition 1 is uniform in the parameter. Let $\theta := (\sigma_-, \sigma_+) \in (0, \infty]^2 =: \Theta$ for OBM, $\theta := \beta \in [-1, 1] = \Theta$ for SBM, and let $X^{(\theta)}$ denote the solution associated to the parameter $\theta \in \Theta$, then for all $t \in (0, \infty)$ it holds for all $\varepsilon \in (0, \infty)$ that

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{s \in [0,t]} \left| \varepsilon_{n,s}^{(r,f,X^{(\theta)})} - \langle \mu_X, \mathbf{E}_f^X \rangle L_s^r(X^{(\theta)}) \right| \geq \varepsilon \right) = 0. \quad (2.16)$$

In the case of SBM the latter equation and Proposition 1 follows from [37, Proposition 2] (with $T = 1$) and the scaling property.

2.3 Rate of convergence to the local time

We refine the above convergence showing that the speed of convergence is of order 1/4.

Theorem 1. *Let $f \in \mathbf{I}_{\gamma}$, $\gamma > 3$, let X be either the σ_{\pm} -OBM solution to (2.4) (in this case take $\beta = 0$) or the β -SBM solution to (2.1) (in this case take $\sigma_{\pm} = 1$). Then there exists (possibly on an extension of the probability space) a BM B independent of X such that*

$$n^{1/4} \left(\varepsilon_{n,\cdot}^{(r,f,X)} - \langle \mu_X, \mathbf{E}_f^X \rangle L^r(X) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-s}} \sqrt{K_f} B_{L^r(X)}, \quad (2.17)$$

where

$$\begin{aligned} K_f &= \langle \mu_X, \mathbf{E}_{f^2}^X + 2\mathbf{E}_{f,p_f^X}^X \rangle + \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \frac{8}{3\sqrt{2\pi}} (\langle \mu_X, \mathbf{E}_f^X \rangle)^2 \\ &\quad - 2\sqrt{\frac{2}{\pi}} \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \langle \mu_X, \mathbf{E}_f^X \rangle \int_{-\infty}^{\infty} \left(e^{-\frac{y^2}{2}} - \sqrt{2\pi}|y|\Phi(-|y|) \right) \mathcal{P}_f^X(\sigma(y)y) \mu_X(dy) \\ &\quad - 2\langle \mu_X, \mathbf{E}_f^X \rangle \left(\frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \right)^2 \\ &\quad \cdot \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \frac{\sigma(x)\sigma(y)|x|e^{-\frac{x^2}{2}}\Phi(-|y|)}{\sqrt{2\pi}} \sqrt{\frac{1}{t} - 1} f(\sigma(x)x\sqrt{t}, \sigma(y)y\sqrt{1-t}) \mu_X(dy) dt \mu_X(dx), \end{aligned} \quad (2.18)$$

Φ is the cumulative distribution function of a standard Gaussian random variable,

$$\mathcal{P}_f^X(x) = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} p_X(j, x, y) \left(\mathbf{E}_f^X(y) - \langle \mu_X, \mathbf{E}_f^X \rangle \mathbf{E}_{g_\beta}^X(y) \right) dy, \quad (2.19)$$

and $g_\beta(x, y) = \frac{1}{1 + \operatorname{sgn}(y)\beta} \left(|y| - \frac{1 + \operatorname{sgn}(y)\beta}{1 + \operatorname{sgn}(x)\beta} |x| \right)$.

Remark 1 (Results for drifted OBM and SBM). Proposition 1 and Theorem 1 hold also in the case of drifted OBM and drifted SBM that are solutions to (2.10): respectively with σ piecewise constant and $(\sigma_- - \sigma_+)\beta = 0$.

Remark 2. If $\beta = 0$ and $\sigma \equiv 1$ we recover the known result for BM: e.g. [11, 12] and a special case of the already cited [24, Theorem 1.2]. The expression for the constant K_f we propose is slightly more explicit.

Remark 3. Note that the function f is allowed to depend only on the first variable. In particular we would have $\mathbf{E}_f^X = f$.

Remark 4. Theorem 1 implies a weaker version of Proposition 1. Proposition 1 requires Hypothesis 1 which is satisfied if for instance $f \in \mathbf{I}_2$. Theorem 1 instead assumes $f \in \mathbf{I}_\gamma$, $\gamma > 3$, which is a stronger condition.

Let us comment on how to derive the u.c.p. convergence from Theorem 1: The notions of convergence in law/stably in law/probability coincide when the limit is constant and so $\varepsilon_{n,\cdot}^{(r,f,X)} - \langle \mu_X, \mathbf{E}_f^X \rangle L^r(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ in the Skorokhod topology. Since $L^r(X)$ is (a.s.) continuous and increasing it can be proven (splitting into positive and negative part of f , and so of \mathbf{E}_f^X) that $\varepsilon_{n,\cdot}^{(r,f,X)} \xrightarrow[n \rightarrow \infty]{\text{u.c.p.}} \langle \mu_X, \mathbf{E}_f^X \rangle L^r(X)$ (see e.g. (2.2.16) in [25]).

3 Applications

Let $r \in \mathbb{R}$ be a fixed threshold, and let X be a stochastic process. Let $T \in (0, \infty)$, $N \in \mathbb{N}$, we observe the process on the discrete time grid $i \frac{T}{N}$. We denote by $X_i = X_{i \frac{T}{N}}$.

3.1 Estimating the local time via number of crossings of the threshold

In this section we consider some classical estimators of the local time of BM and show that they are still estimators, up to a multiplicative constant, of the local time $L_T^r(X)$ of SBM or OBM or a more general class of processes.

Let $\alpha \in [0, \infty)$ and note that the function h_α given by $h_\alpha(x, y) = |y|^\alpha \mathbf{1}_{(-\infty, 0)}(xy)$ is in \mathbf{I}_γ for all $\gamma \in [0, \infty)$. In fact $h_\alpha(x, y) \leq c_\alpha e^{-|x|} e^{|y-x|}$ for some constant c_α depending on α . We consider two estimators obtained considering the functions proportional to h_0 and h_1 :

$$\mathcal{L}_{T,N}^r(X) = \varepsilon_{\frac{N}{T}, T}^{(r, h_0, X)} = \sqrt{\frac{T}{N}} \sum_{i=0}^{N-1} \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} \quad \text{and} \quad (3.1)$$

$$L_{T,N}^r(X) = \varepsilon_{\frac{N}{T}, T}^{(r, 2h_1, X)} = 2 \sum_{i=0}^{N-1} \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} |X_{i+1} - r|. \quad (3.2)$$

The first is concerned with the number of crossings of the threshold and the second takes into account the distance from it. Moreover note that $L_{T,N}^r$ requires only the knowledge of the $N + 1$ observations X_i , $i = 0, \dots, N$, and not of T/N .

As mentioned in the introduction, in the case of BM, and more general Brownian diffusions, these are consistent estimators of the local time up to a constant.

We now show the consequence of Theorem 1 for these estimators and, in the case of the statistic (3.1), we extend the result to processes satisfying (2.10) in Theorem 2 below.

Throughout this section let the function $\Phi: \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function of a standard Gaussian random variable: for all $x \in \mathbb{R}$ it holds $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$.

3.1.1 Estimator counting the number of crossings of the threshold

Theorem 2. *Let X satisfy (2.10) and let $\sigma_{\pm} := \lim_{x \rightarrow r_{\pm}} \sigma(x)$. Then $\varepsilon^{(r, h_0, X)}$ satisfies (2.14) and (2.17) with $\langle \mu_X, \mathbf{E}_{h_0}^X \rangle =: c_{\sigma_{\pm}, \beta}$ and $K_{h_0} =: K_{\sigma_{\pm}, \beta}$ given by*

$$c_{\sigma_{\pm}, \beta} = \frac{2(1 - \beta^2)}{(1 + \beta)\sigma_- + (1 - \beta)\sigma_+} \sqrt{\frac{2}{\pi}} \quad (3.3)$$

and

$$\begin{aligned} \frac{K_{\sigma_{\pm}, \beta}}{c_{\sigma_{\pm}, \beta}} &= 1 + \sqrt{2\pi} \int_{-\infty}^{\infty} \Phi(-|x|) \mathcal{P}_{\sigma, \beta}(x) dx + \frac{4\sigma_- \sigma_+ (1 - \beta^2)}{(\sigma_- (1 + \beta) + \sigma_+ (1 - \beta))^2} \frac{8 - 3\pi}{3\pi} \\ &\quad - 4\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{(\sigma_+ \mathbf{1}_{(-\infty, r)}(x) + \sigma_- \mathbf{1}_{[r, +\infty)}(x))(1 + \operatorname{sgn}(x)\beta)}{(\sigma_- (1 + \beta) + \sigma_+ (1 - \beta))} (e^{-\frac{x^2}{2}} - \sqrt{2\pi}|x|\Phi(-|x|)) \mathcal{P}_{\sigma, \beta}(x) dx \end{aligned} \quad (3.4)$$

with,

$$\mathcal{P}_{\sigma, \beta}(x) = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} p_{\frac{\sigma_- (1 + \beta) - \sigma_+ (1 - \beta)}{\sigma_- (1 + \beta) + \sigma_+ (1 - \beta)}}(j, x, y) \mathfrak{G}_{\sigma_{\pm}, \beta}(y) dy, \quad (3.5)$$

and

$$\begin{aligned} \mathfrak{G}_{\sigma, \beta}(y) &= \frac{2(\sigma_- \mathbf{1}_{(-\infty, r)}(y) + \sigma_+ \mathbf{1}_{[r, +\infty)}(y))(1 - \operatorname{sgn}(y)\beta)}{\sigma_+ (1 - \beta) + \sigma_- (1 + \beta)} \\ &\quad \cdot \left(\Phi(-|y|) - \frac{2}{\pi} \frac{2(\sigma_+ \mathbf{1}_{(-\infty, r)}(y) + \sigma_- \mathbf{1}_{[r, +\infty)}(y))(1 + \operatorname{sgn}(y)\beta)}{\sigma_+ (1 - \beta) + \sigma_- (1 + \beta)} \left(e^{-\frac{y^2}{2}} - \sqrt{2\pi}|y|\Phi(-|y|) \right) \right). \end{aligned} \quad (3.6)$$

In particular $\mathcal{L}_{T, N}^r(X)$ in (3.1), counting the number of times the process X crosses its threshold r , satisfies

$$\mathcal{L}_{T, N}^r(X) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} c_{\sigma_{\pm}, \beta} L_T^r(Y) \quad (3.7)$$

and, as N goes to infinity,

$$(N/T)^{1/4} (\mathcal{L}_{T, N}^r(X) - c_{\sigma_{\pm}, \beta} L_T^r(X)) \text{ converges stably in law to } \sqrt{K_{\sigma_{\pm}, \beta}} B_{L_T^r(X)} \quad (3.8)$$

where B is a BM, possibly on an extension of the probability space, independent from X (independent from \mathcal{F}_T).

3.1.2 Another estimator

Let us consider the estimator $L_{T, N}^r$ in (3.2).

For an OBM, say Y , a proof that $L_{T, N}^r(Y) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} L_T^r(Y)$ can be found in [35, Lemma 1].

Applying Proposition 1 to $\varepsilon^{(r, 2h_1, X)}$ (with $\beta = 0$) we obtain a more general result. And applying Theorem 1 we obtain the convergence rate. We specify the asymptotic properties of $L_{T, N}^r(Y)$ in Proposition 2 below.

Proposition 2. *Let Y be the OBM solution to (2.4) and let $L_{T,N}^r(Y)$ be the estimator of the local time $L_T^r(Y)$ in (3.2). Then there exists (possibly on an extension of the probability space) a BM B independent of Y (independent from \mathcal{F}_T) such that, as N tends to infinity,*

$$(N/T)^{1/4}(L_{T,N}^r(Y) - L_T^r(Y)) \text{ converges stably in law to } \sqrt{\frac{16}{3\sqrt{2\pi}} \frac{\sigma_-^2 + \sigma_+^2}{\sigma_- + \sigma_+}} B_{L_T^r(Y)}. \quad (3.9)$$

The following proposition specifies, if the process is a SBM, the constants in Proposition 1 and Theorem 1 in case of the estimator (3.2) and shows its asymptotic properties.

Proposition 3. *Let X be the solution to (2.1) and $L_{T,N}^r(X)$ be the estimator in (3.2). There exists (possibly on an extension of the probability space) a BM B independent of X (of \mathcal{F}_T) such that $L_{T,N}^r(X) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} (1 - \beta^2)L_T^r(X)$ and $(N/T)^{1/4}(L_{T,N}^r(X) - (1 - \beta^2)L_T^r(X))$ converges stably in law to $\sqrt{K_\beta} B_{L_T^r(X)}$ where*

$$\frac{1}{1 - \beta^2} K_\beta = \frac{16}{3\sqrt{2\pi}} - 4\beta \int_{-\infty}^{\infty} x \Phi(-|x|) \mathcal{P}_\beta(x) dx + \frac{4}{\sqrt{2\pi}} (1 - \beta)\beta, \quad (3.10)$$

with $\mathcal{P}_\beta(x) := \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} p_\beta(j, x, y) \mathfrak{G}_\beta(y) dy$ and

$$\mathfrak{G}_\beta(y) := \text{sgn}(y)(1 - \text{sgn}(y)\beta)\beta \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} - \sqrt{2\pi}|y|\Phi(-|y|) \right). \quad (3.11)$$

3.2 Estimating the parameters of skew and oscillating Brownian motion

Let X be a β -SBM with threshold r . Recall that $L_{T,N}^r$ is given in (3.2) and its asymptotic properties are provided in Proposition 3. Let us introduce the following function $f_1(x, y) := \mathbf{1}_{(0,1)}(x)$. Then both $\mathbf{E}_{f_1}^X$ in (2.12) and f_1^2 coincide with f and satisfy Hypothesis 1. The same holds for the function $|f_1|(x) := f_1(|x|)$.

We consider the following estimators for the parameters of a the β -SBM solution to (2.1):

$$\hat{\beta}_N := 1 - \frac{L_{T,N}^r(X)}{\sqrt{\frac{T}{N}} \sum_{k=0}^{N-1} \mathbf{1}_{(0,1)}(\sqrt{N/T}(X_k - r))} \quad (3.12)$$

or

$$\hat{\beta}_N := \frac{\sum_{k=0}^{N-1} \text{sgn}(X_k - r) \mathbf{1}_{(-1,1)}(\sqrt{N/T}(X_k - r))}{\sum_{k=0}^{N-1} \mathbf{1}_{(-1,1)}(\sqrt{N/T}(X_k - r))}. \quad (3.13)$$

Note that both estimators above require the knowledge of the discretization step T/N .

The following result is a direct application of Theorem 1. Indeed, a sort of Slutsky lemma holds replacing converge in law by stable converge.

Theorem 3. *Let X be the β -SBM solution to (2.1). Then $\hat{\beta}_N$ in (3.12) (resp. (3.13)) is a consistent estimator of β and as $N \rightarrow \infty$*

$$N^{1/4} \left(\hat{\beta}_N - \beta \right) \text{ converges stably in law to } K T^{1/4} \frac{B_{L_T^r(X)}}{L_T^r(X)} \quad (3.14)$$

with B a BM independent of X and $K = \frac{\sqrt{K_{(1-\beta)f_1-2h_1}}}{1+\beta}$ where $K_{(1-\beta)f_1-2h_1}$ is provided by (2.18) (resp. $K = \frac{1}{2} \sqrt{K_{(\text{sgn}(\cdot)-\beta)\cdot|f_1|}}$).

The result above tells that $\hat{\beta}_N$ is a consistent estimator of the skewness parameter β , and $N^{1/4}(\hat{\beta}_N - \beta)$ for N large behaves like a “mixed” Gaussian law. The constant K can be computed in a more explicit expression depending on β , as in the previous applications. Note that, in the same way, we can provide other estimators satisfying Theorem 3 (with a suitable constant K). Moreover, as in [36], this paves the way to hypothesis testing to check whether some observations come from (drifted) BM or (drifted) SBM.

Remark 5. Similarly, we can provide an estimator for the parameters of OBM. Let Y be the solution to (2.4), $f_+ : x \rightarrow \mathbf{1}_{(0,1)}(x)$ and $f_- : x \rightarrow \mathbf{1}_{(-1,0)}(x)$. Then

$$\hat{\sigma}_{\pm}^2 := \frac{L_{T,N}^r(Y)}{\sqrt{\frac{T}{N} \sum_{k=0}^{N-1} f_{\pm}(\sqrt{N/T}(X_k - r))}} \quad (3.15)$$

is a consistent estimator of σ_{\pm}^2 and $(N/T)^{1/4}(\hat{\sigma}_{\pm}^2 - \sigma_{\pm}^2)$ converges stably in law to a (mixed) normal law. Since the estimator proposed relies mostly on the behavior of the process around the threshold, it is slower than the ones based on quadratic variations and occupation times of the positive and negative part of the process proposed in [33] for OBM.

4 Proofs of the main results

In this section we comment the results and their proof. We first deal with the convergence in probability to the local time in Proposition 1, which was already known for SBM. Then with the rate of (stable) convergence in the case of SBM and OBM in Theorem 1 whose proof is provided in Section 4.2 relying on an a well known CLT. Finally we prove Theorem 2. The relationship between SBM and OBM recalled in Section 2.1.3 is crucial in the proofs.

From now on we can take $r = 0$ for simplicity. Indeed if X is a β -SBM/ σ_{\pm} -OBM with threshold $r = 0$ then $X - r$ is a β -SBM/ σ_{\pm} -OBM with threshold $r = 0$.

4.1 Proof of Proposition 1

As already mentioned, in the case of SBM, Proposition 1 follows from [37, Proposition 2] (with $T = 1$) and the scaling property. By the transformation provided in Section 2.1.3, we are now ready to prove Proposition 1 for the OBM Y solution to (2.4).

Let us recall, given a measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a skew or oscillating BM X (with $r = 0$), we denote by p_X its transition density and the function \mathbf{E}_f^X in (2.12) is computed using it. Moreover μ_X denotes its stationary measure.

Proof of Proposition 1 for OBM. Let X be the SBM associated to the σ -OBM Y , i.e. X is a β_{σ} -SBM with skewness parameter $\beta_{\sigma} = \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}$ solution to (2.8). Let f_{σ} be the function satisfying $f_{\sigma}(x) := f(\sigma(x)x)$. Note that a function f satisfies Hypothesis 1 for Y if and only if f_{σ} satisfies Hypothesis 1 for X . Moreover it holds that $\mathbf{E}_{f_{\sigma}}^Y(\sigma(x)x) = \mathbf{E}_{(f_{\sigma})}^X(x)$ and so $\langle \mu_{\beta_{\sigma}}, \mathbf{E}_{(f_{\sigma})}^X \rangle = \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \langle \lambda_{\sigma}, \mathbf{E}_{f_{\sigma}}^Y \rangle$. Applying Proposition 1 for the SBM X and the function $f := f_{\sigma}$ and taking into account the latter equalities and the relationship between local times (2.9) complete the proof. \square

4.2 Proof of the Central Limit Theorem: Theorem 1

We introduce now Proposition 5 and show its key role in the proof of the main Theorem 1. Before to do it we reformulate, in Proposition 4, a special case of Theorem 3.2 in [23] which provides a suitable CLT to be combined with Proposition 5.

Proposition 4 (cf. Theorem 3.2 in [23]). *Let $(Y_t)_{t \in [0,1]}$ be an $(\mathcal{F}_t)_{t \in [0,1]}$ -local martingale on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$. Let $\mathcal{Z}^n = \sum_{k=1}^{\lfloor nt \rfloor} \chi_k^n$ where χ_k^n are square integrable $\mathcal{F}_{\frac{k}{n}}$ -measurable, $n \in \mathbb{N}$, and assume that there are E and F continuous processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ such that E has bounded variation and it holds*

- i) $\sup_{s \in [0,1]} \left| \sum_{k=1}^{\lfloor ns \rfloor} \mathbb{E}[\chi_k^n | \mathcal{F}_{(k-1)/n}] - E_s \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$,
- ii) for all $t \in [0, 1]$ that $\sum_{k=1}^{\lfloor nt \rfloor} \left(\mathbb{E}[(\chi_k^n)^2 | \mathcal{F}_{(k-1)/n}] - (\mathbb{E}[\chi_k^n | \mathcal{F}_{(k-1)/n}])^2 \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} F_t$,
- iii) for all $t \in [0, 1]$ that $\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\chi_k^n (Y_{k/n} - Y_{(k-1)/n}) | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$,
- iv) for all $\varepsilon \in (0, \infty)$ that $\sum_{k=1}^n \mathbb{E}[|\chi_k^n|^2 \mathbf{1}_{\{|\chi_k^n| \geq \varepsilon\}} | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$, and
- v) for all $t \in [0, 1]$ and M bounded $(\mathcal{F}_t)_{t \in [0,1]}$ -martingale such that for all $s \in [0, 1]$ the cross variation satisfies $\mathbb{P}(\langle M, Y \rangle_s = 0) = 1$ that

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\chi_k^n (M_{k/n} - M_{(k-1)/n}) | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (4.1)$$

Then there exists a BM B , possibly on an extension of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, such that B and Y are independent and

$$\mathcal{Z}^n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^{-s}} E + B_F. \quad (4.2)$$

Proposition 5. *Let $(Y_t)_{t \in [0,1]}$ be the OBM with threshold $r = 0$ strong solution to (2.4) on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, let $\gamma > 3$, and let $h \in \mathbf{I}_\gamma$.*

Then there exist sequences of stochastic process $(\mathcal{V}_t^n)_{t \in [0,1]}$, $n \in \mathbb{N}$, with

$$\sup_{s \in [0,1]} m^{1/4} |\mathcal{V}_s^m| \xrightarrow[m \rightarrow \infty]{\mathbb{P}} 0, \quad (4.3)$$

$(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$ -martingales $(\mathcal{M}_t^n)_{t \in [0,1]}$, $n \in \mathbb{N}$, and random variables $(\chi_k^n)_{k \in \{1, \dots, \lfloor nt \rfloor\}}$, $n \in \mathbb{N}$, such that for all $t \in [0, 1]$, $n \in \mathbb{N}$ it holds that $\mathcal{M}_t^n = \sum_{k=1}^{\lfloor nt \rfloor} \chi_k^n$ and

$$n^{1/4} \left(\varepsilon_{n,t}^{(0,h,Y)} - \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle L_t(Y) \right) = \mathcal{M}_t^n + n^{1/4} \mathcal{V}_t^n, \quad (4.4)$$

and it holds

- i) for all $k \in \{1, \dots, \lfloor nt \rfloor\}$ χ_k^n is square integrable $\mathcal{F}_{\frac{k}{n}}$ -measurable and $\mathbb{E}[\chi_k^n | \mathcal{F}_{(k-1)/n}] = 0$,
- ii) for all $t \in [0, 1]$ that $\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[(\chi_k^n)^2 | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} K_h L_t(Y)$ with K_h given by (2.18),
- iii) for all $t \in [0, 1]$ that $\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\chi_k^n (Y_{k/n} - Y_{(k-1)/n}) | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$,
- iv) for all $\varepsilon \in (0, \infty)$ that $\sum_{k=1}^n \mathbb{E}[|\chi_k^n|^2 \mathbf{1}_{\{|\chi_k^n| \geq \varepsilon\}} | \mathcal{F}_{(k-1)/n}] \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

The proof of this result consists in generalizing to the case of OBM the fundamental procedure used in [24] for BM. It is quite technical, therefore it is provided in Appendix B.

Let us now assume that Proposition 5 holds and let us prove Theorem 1 first for OBM and then for SBM.

Proof of Theorem 1 for OBM. Let Y be the OBM strong solution to (2.4). Recall we can take threshold $r = 0$ in all proofs.

Without loss of generality we can reduce ourselves to prove Theorem 1 on the interval $[0, 1]$ for $n \in \mathbb{N}$ tending to infinity.

The scaling property for the OBM and its local time (see (A.1)) yields the result for all non-negative times: as processes on $\mathbb{D}_{[0,t]}$. (The scaling property also ensures that in Theorem 1 (for SBM and OBM) n is not necessarily a natural number, but it can stay for a positive real number tending to infinity.) And $n^{1/4} \left(\varepsilon_{n,\cdot}^{(0,f,Y)} - \langle \lambda_\sigma, \mathbf{E}_f^Y \rangle L_\cdot(Y) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-s}} \sqrt{K_f} B_{L_\cdot(Y)}$ in \mathbb{D}_∞ if and only if for all $t \in [0, \infty)$ $n^{1/4} \left(\varepsilon_{n,\cdot}^{(0,f,Y)} - \langle \lambda_\sigma, \mathbf{E}_f^Y \rangle L_\cdot(Y) \right) |_{[0,t]} \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-s}} \sqrt{K_f} B_{L_\cdot(Y)} |_{[0,t]}$ in \mathbb{D}_t (indeed since $B_{L_\cdot(Y)}$ has (a.s.) continuous trajectories this follows, e.g. combining [9, Theorem 16.2] and [25, Proposition 2.2.4]).

Proposition 5 implies that there exists a decomposition as in (4.4) and its desired stable limit as $n \in \mathbb{N}$ goes to infinity coincides with the stable limit of the sequence \mathcal{M}^n of càdlàg $(\mathcal{F}_{\lfloor ns \rfloor})_{s \in [0,1]}$ -martingales, $n \in \mathbb{N}$. Indeed the fact that $\sup_{s \in [0,1]} n^{1/4} |\mathcal{V}_s^n| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$, implies that for every $h: \mathbb{D}_1 \rightarrow \mathbb{R}$ continuous and bounded it holds that $|h(\mathcal{M}^n + n^{1/4} \mathcal{V}^n) - h(\mathcal{M}^n)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ and so for every bounded continuous function $h: \mathbb{D}_1 \rightarrow \mathbb{R}$ and bounded measurable random variable $Y: \Omega \rightarrow \mathbb{R}$ it holds that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[|h(\mathcal{M}^n + n^{1/4} \mathcal{V}^n) - h(\mathcal{M}^n)| | Y \right] = 0.$$

Proposition 5 also ensures that \mathcal{M}^n , $n \in \mathbb{N}$, satisfies all assumptions, except Item (v), of Proposition 4 (with local martingale $M = Y$, $\mathcal{Z}^n = \mathcal{M}^n$, $E \equiv 0$ and $F = K_f L(Y)$ where K_f is the constant in equation (2.18)). Item (v) of Proposition 4 is trivial because such martingale M orthogonal to OBM is nothing but a constant.

Therefore, applying Proposition 4 as described above completes the proof of Theorem 1. \square

Since Theorem 1 for OBM holds, we now prove it for SBM.

Proof of Theorem 1 for SBM. Let $\beta \in [-1, 1]$, $\gamma > 3$, X the β -SBM solution to (2.1), and $f \in \mathbf{I}_\gamma$. Recall we take threshold $r = 0$ in all proofs.

Take $\sigma_+^{(\beta)}, \sigma_-^{(\beta)} \in (0, \infty]$ such that $\beta = \frac{\sigma_-^{(\beta)} - \sigma_+^{(\beta)}}{\sigma_-^{(\beta)} + \sigma_+^{(\beta)}} \left(\sigma_\pm^{(\beta)} = \frac{2}{1 \pm \beta} \right)$ and construct the diffusion coefficient $\sigma_\beta := \sigma_-^{(\beta)} \mathbf{1}_{(-\infty, 0)} + \sigma_+^{(\beta)} \mathbf{1}_{[0, +\infty)}$ in (2.5). Let Y be the $\sigma_\beta^{(\beta)}$ -OBM and initial condition $Y_0 = \sigma_\beta(X_0)X_0$ and let $h_\beta: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function satisfying $h_\beta(x) := f(x/\sigma_\beta(x))$. Note that $f \in \mathbf{I}_\gamma$ if and only if $h_\beta \in \mathbf{I}_\gamma$. Theorem 1 can be applied to the OBM Y and the function h_β to obtain that

$$n^{1/4} \left(\varepsilon_{n,\cdot}^{(0,h_\beta,Y)} - \langle \lambda_{\sigma_\beta}, \mathbf{E}_{h_\beta}^Y \rangle L_\cdot^0(Y) \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}\text{-s}} \sqrt{K_{h_\beta}} B_{L_\cdot^0(Y^\beta)} \quad (4.5)$$

with K_{h_β} from (2.18) (with $\sigma_\pm = \sigma_\pm^{(\beta)}$, $\beta = 0$). The relationship between X and Y , in particular between their local times (2.9) and their transition densities (2.7), and the fact that $\langle \lambda_{\sigma_\beta}, \mathbf{E}_{h_\beta}^Y \rangle = \frac{\sigma_-^{(\beta)} + \sigma_+^{(\beta)}}{2\sigma_+^{(\beta)}\sigma_-^{(\beta)}} \langle \mu_\beta, \mathbf{E}_f^X \rangle$ are used to rewrite the constant K_f (with $\sigma_\pm = 1$ and $\beta = \beta$ in the notation of (2.18)). \square

4.3 Proof of Theorem 2

With no surprise the proof relies on Girsanov's transform and Itô-Tanaka formula and reduces to prove the result for OBM, i.e. applying Proposition 1 and Theorem 1 for $f = h_0$.

Proof of Theorem 2. Let $\sigma_{\pm} := \lim_{x \rightarrow r^{\pm}} \sigma(x)$.

We first prove the statement for $\beta = 0$ in several steps.

Note that if X is a σ_{\pm} -OBM, then the statement follows from Proposition 1 and Theorem 1 taking $f = h_0$. The result holds with constants $c_{\sigma_{\pm},0}$ and $K_{\sigma_{\pm},0}$.

Next, let us consider the process Z satisfying that \mathbb{P} -a.s. for all $t \in [0, \infty)$

$$Z_t = Z_0 + \int_0^t \sigma(Z_s) dW_s + \frac{1}{2} \int_0^t \sigma(Z_s) \sigma'(Z_s) ds$$

where $0 < \inf_{x \in \mathbb{R}} \sigma(x) \leq \sup_{x \in \mathbb{R}} \sigma(x) < \infty$ and $\sup_{x \in \mathbb{R}} |\sigma'(x)| < \infty$. For simplicity consider threshold $r = 0$, indeed $Z - r$ is so. Let $S(x) := \int_0^x \frac{\sigma_{\pm}}{\sigma(y)} dy$. It follows from the Itô-Tanaka formula that $Y_t := S(Z_t)$ is an OBM with threshold $r = 0$ starting at $S(Z_0)$. Moreover Itô-Tanaka formula (applied to $|Y|$ and to $|S(Z)|$) yields that the local times at 0 of Y and Z coincide. The function S is invertible and $xS^{-1}(x) \geq 0$ for all $x \in \mathbb{R}$. Hence $\mathcal{L}_{T,N}^0(Y) = \mathcal{L}_{T,N}^0(Z)$. Therefore we proved that Theorem 2 holds for the process Z with the same constants $c_{\sigma_{\pm},0}$ and $K_{\sigma_{\pm},0}$.

The same result holds for Z without the boundedness conditions for σ and σ' , it can be seen using a localization argument. Moreover it holds for $r \neq 0$. And standard arguments combining stable converge and Girsanov's transform ensure that the same result holds for X solution to (2.10) if $\beta = 0$ with the same constants $c_{\sigma_{\pm},0}$ and $K_{\sigma_{\pm},0}$.

Now let us consider $\beta \neq 0$.

Let $\sigma_{\pm}^{(\beta)} \in (0, +\infty]$ constants such that $\beta = \frac{\sigma_-^{(\beta)} - \sigma_+^{(\beta)}}{\sigma_-^{(\beta)} + \sigma_+^{(\beta)}} (\sigma_{\pm}^{(\beta)} = \frac{2}{1 \pm \beta})$. Let $\sigma^{(\beta)} := \sigma_-^{(\beta)} \mathbf{1}_{(-\infty, r)} + \frac{\sigma_+^{(\beta)} + \sigma_-^{(\beta)}}{2} \delta_{\{r\}} + \sigma_+^{(\beta)} \mathbf{1}_{(r, +\infty)}$ and consider the process $\eta_s - r := \sigma^{(\beta)}(X_s)(X_s - r)$. Note that $\mathcal{L}_{T,N}^r(\eta) = \mathcal{L}_{T,N}^r(X)$. By Itô-Tanaka formula the latter process satisfies:

$$\eta_t = r + \sigma^{(\beta)}(X_0)(X_0 - r) + \int_0^t \sigma^{(\beta)}(\eta_s) \sigma(r + (\eta_s - r)/\sigma^{(\beta)}(\eta_s)) dW_s + \int_0^t \sigma^{(\beta)}(\eta_s) b_s ds \quad (4.6)$$

and

$$L_t^r(\eta) = \frac{2\sigma_+^{(\beta)}\sigma_-^{(\beta)}}{\sigma_+^{(\beta)} + \sigma_-^{(\beta)}} L_t^r(X) = \frac{1 - \beta^2}{2} (\sigma_+^{(\beta)} + \sigma_-^{(\beta)}). \quad (4.7)$$

We have proved above that the process η satisfies Theorem 2 with constants $c_{\sigma_{\pm}^{(\beta)}\sigma_{\pm},\beta}$ and $K_{\sigma_{\pm}^{(\beta)}\sigma_{\pm},\beta}$ hence X satisfies the theorem with constants

$$c_{\sigma_{\pm},\beta} := c_{\sigma_{\pm}^{(\beta)}\sigma_{\pm},0} \frac{1 - \beta^2}{2} (\sigma_+^{(\beta)} + \sigma_-^{(\beta)}) = \frac{\sigma_+^{(\beta)} + \sigma_-^{(\beta)}}{\sigma_-^{(\beta)}\sigma_- + \sigma_+^{(\beta)}\sigma_+} \sqrt{\frac{2}{\pi}} (1 - \beta^2) \quad (4.8)$$

and $K_{\sigma_{\pm},\beta} := \frac{K_{\sigma_{\pm}^{(\beta)}\sigma_{\pm},0}}{c_{\sigma_{\pm}^{(\beta)}\sigma_{\pm},0}} c_{\sigma_{\pm},\beta}$. The proof is thus completed. \square

A Properties of oscillating Brownian motion

In this section we consider Y to be an OBM with threshold $r = 0$.

A.1 Scaling property

In this section Y^{Y_0} denotes the OBM with threshold $r = 0$ starting from a deterministic point Y_0 , let $c \in (0, \infty)$. Let us mention the following well known diffusive scaling properties

for OBM: $\left(\frac{1}{\sqrt{c}}Y_{ct}^{Y_0}\right)_{t \geq 0} \stackrel{\text{law}}{=} (Y_t^{Y_0/\sqrt{c}})_{t \geq 0}$ (i.e. “the rescaled OBM is still a OBM with rescaled starting point”) and

$$\left(\frac{1}{\sqrt{c}}Y_{ct}^{\sqrt{c}Y_0}, \frac{1}{\sqrt{c}}L_{ct}(Y^{\sqrt{c}Y_0})\right)_{t \geq 0} \stackrel{\text{law}}{=} (Y_t^{Y_0}, L_t(Y^{Y_0}))_{t \geq 0}. \quad (\text{A.1})$$

A.2 The joint density of a standard OBM and its local time

The joint density of a standard OBM and its local time at time t , $\rho_t^\sigma(y, \ell)$ coincides with

$$\rho_t^\sigma(y, \ell) = \frac{1}{(\sigma(y))^2} \rho_t\left(\frac{y}{\sigma(y)}, \frac{\sigma_- + \sigma_+}{2\sigma_- \sigma_+} \ell\right) \quad (\text{A.2})$$

for $y \neq 0$, where ρ is the joint density of the BM and its local time at time t :

$$\rho_t(y, \ell) = \frac{|y| + \ell}{\sqrt{2\pi t^3}} \exp\left(-\frac{(|y| + \ell)^2}{2t}\right) \mathbf{1}_{(0, \infty)}(\ell). \quad (\text{A.3})$$

In particular $\rho_t^\sigma(y, \ell) dy d\ell = \rho_t\left(\frac{y}{\sigma(y)}, \frac{\sigma_- + \sigma_+}{2\sigma_- \sigma_+} \ell\right) \lambda_\sigma(dy) d\ell$.

A.3 Bounds for the semigroup

For a measurable, bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ set

$$Q_t^\sigma f(x) := \int_{-\infty}^{\infty} q_\sigma(t, x, y) f(y) dy \quad \text{and} \quad P_t^{\beta_\sigma} f(x) := \int_{-\infty}^{\infty} p_{\beta_\sigma}(t, x, y) f(y) dy \quad (\text{A.4})$$

for all $t \in [0, \infty)$. They are respectively the semigroup of the standard OBM and of the standard β_σ -SBM with skewness parameter $\beta_\sigma := \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}$ and they satisfy $Q_t^\sigma f(x) = P_t^{\beta_\sigma} f_\sigma(x/\sigma(x))$. Note that $P_t := P_t^0 = Q_t^1$ is the semigroup of the BM and

$$Q_t^\sigma f(x) = P_t f_\sigma(x/\sigma(x)) + \beta_\sigma P_t(f_\sigma \mathbf{1}_{[0, \infty)})(-|x|/\sigma(x)) - \beta_\sigma P_t(f_\sigma \mathbf{1}_{(-\infty, 0)})(|x|/\sigma(x)) \quad (\text{A.5})$$

where $f_\sigma(x) = f(\sigma(x)x)$. From this relationship between the semigroups of OBM and BM we derive the following properties.

Lemma 1. *Let $f \in \mathbf{L}^{1,b}(\lambda^{(2)})$, and let us denote by $p(t, \cdot)$ is the density of a Gaussian random variable with variance t . Then there exists a positive constant $K \in (0, \infty)$ such that for all $x, y \in \mathbb{R}$, $0 \leq s \leq t$ it holds that*

$$i) |Q_t^\sigma f(x)| \leq \frac{1}{\min\{\sigma_-, \sigma_+\}} \left(1 + \frac{|\sigma_- - \sigma_+|}{\sigma_- + \sigma_+}\right) \frac{1}{\sqrt{2\pi}} \frac{\|f\|_1}{\sqrt{t}},$$

$$ii) \left| Q_t^\sigma f(x) - \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \langle \lambda_\sigma, f \rangle p(t, x/\sigma(x)) \right| \leq \frac{K}{\sqrt{t^3}} (\|f\|_{1,2} + \|f\|_{1,1}|x|),$$

iii) *for all $\zeta \geq 0$ there exists a positive constant K_ζ such that*

$$\left| Q_t^\sigma f(x) - \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \langle \lambda_\sigma, f \rangle p(t, x/\sigma(x)) \right| \leq \frac{K_\zeta}{t} \left(\frac{\|f\|_{1,1}}{1+(|x|/(\sigma(x)\sqrt{t}))^\zeta} + \frac{\|f\|_{1,1+\zeta}}{1+(|x|/\sigma(x))^\zeta} \right),$$

$$iv) |Q_t^\sigma f(x) - Q_t^\sigma f(y)| \leq \frac{1}{\min\{\sigma_-, \sigma_+\}} \left(1 + \frac{|\sigma_- - \sigma_+|}{\sigma_- + \sigma_+}\right) K \frac{|x-y|}{t} \|f\|_1, \text{ and}$$

$$v) |Q_t^\sigma f(x) - Q_s^\sigma f(x)| \leq \frac{1}{\min\{\sigma_-, \sigma_+\}} \left(1 + \frac{|\sigma_- - \sigma_+|}{\sigma_- + \sigma_+}\right) K \frac{t-s}{\sqrt{s^3}} \|f\|_1.$$

Proof. Item (i) is a straightforward consequence of (2.7) and of the fact that

$$p_{\beta_\sigma}(t, x, y) \leq (1 + |\beta_\sigma|)p(t, x - y) \leq \frac{1 + |\beta_\sigma|}{\sqrt{2\pi t}}. \quad (\text{A.6})$$

To prove the other items we also use the fact for all $\alpha \geq 0$ $\|f(\sigma(\cdot)\cdot)\|_{1,\alpha} \leq \frac{\|f\|_{1,\alpha}}{(\min\{\sigma_-, \sigma_+\})^{1+\alpha}}$ and $\|f\mathbf{1}_{[0,\infty)}\|_{1,\alpha} + \|f\mathbf{1}_{(-\infty,0)}\|_{1,\alpha} = \|f\|_{1,\alpha}$. Item (ii) follows from (A.5) and [37, Lemma 1] for SBM. Item (iii) follows from (A.5) and the analogous result for BM: equation (3.2) in [24, Lemma 3.1]. Item (iv) and Item (v) follow from (A.5) and equations (3.4)-(3.5) in [24, Lemma 3.1]. \square

The proof of the following lemma follows from Lemma 1 and it is analogous to the one of [24, Lemma 3.3]. It is therefore omitted.

Lemma 2. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\Gamma_t(n, f) := \sum_{k=1}^{\lfloor nt \rfloor - 1} Q_k^\sigma f(\sqrt{n}Y_0)$. Then there exists a positive constant K (depending on σ_\pm) such that $|\Gamma_t(n, f)| \leq K\|f\|_1\sqrt{nt}$ and if $\langle \lambda_\sigma, f \rangle = 0$ then $|\Gamma_t(n, f)| \leq K(\|f\|_{1,2} + \|f\|_{1,1}|Y_0|\sqrt{n})$ $|\Gamma_t(n, f)| \leq K\|f\|_{1,1}(1 + \log(nt))$.*

A.4 Behavior of the local time

In this section we explore some properties of the local time of OBM and its moments.

Lemma 3. *For all $q \in (2, \infty)$, $\alpha \in (0, \frac{q-2}{2q})$ it holds that (the pathwise continuous version of) the local time $L(Y)$ is locally α -Hölder continuous. In particular for all $\delta \in (-\infty, \frac{1}{2})$, $T \in [0, \infty)$ it holds that*

$$\sup_{t \in [0, T]} n^\delta \left(L_{t+\frac{1}{n}}(Y) - L_t(Y) \right) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (\text{A.7})$$

This statement is not surprising since it is well known for the local time of BM. The proof is standard (based on Itô-Tanaka formula and Burkholder-Davis-Gundy inequality) and therefore omitted.

Lemma 4. *Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the real function satisfying $g(x, y) = |y| - |x|$. Then for all $t \in [0, \infty)$ it holds that $\langle \lambda_\sigma, \mathbf{E}_g^Y \rangle = 1$,*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{E}_g^Y(\sqrt{n}Y_{k/n}) = \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbb{E}[L_{(k+1)/n}(Y) - L_{k/n}(Y) | \mathcal{F}_{k/n}] \quad (\text{A.8})$$

and $\sup_{s \in [0, t]} \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor ns \rfloor - 1} \mathbf{E}_g^Y(\sqrt{n}Y_{k/n}) - L_s(Y) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

Proof. Some computations, that we decide to omit, show that $\langle \lambda_\sigma, \mathbf{E}_g^Y \rangle = 1$. The scaling property A.1, the Markov property, and Itô-Tanaka formula show $\frac{1}{\sqrt{n}} \mathbf{E}_g^Y(\sqrt{n}Y_{k/n}) = \mathbb{E}[L_{(k+1)/n}(Y) - L_{k/n}(Y) | \mathcal{F}_{k/n}]$. Lemma 2.14 in [22] ensures the desired convergence in probability. \square

In the remainder of this section Y^x , $x \in \mathbb{R}$, denotes the OBM with threshold $r = 0$ starting from $Y_0 = x$. For every $p \in [0, \infty)$, $x \in \mathbb{R}$, and function $f: \mathbb{R} \rightarrow \mathbb{R}$ either non-negative or such that $(L_1(Y^x))^p f(Y_1^x) \in \mathbf{L}^1(\mathbb{P})$ let

$$\mathbb{L}^{(p)}(f, x) := \mathbb{E}[(L_1(Y^x))^p f(Y_1^x)]. \quad (\text{A.9})$$

In this document we only consider functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying that there exist $K, \alpha \in [0, \infty)$ such that $|f(x)| \leq Ke^{\alpha|x|}$ for all $x \in \mathbb{R}$, so $\mathbb{L}^{(\cdot)}(f, \cdot)$ is well defined. The scaling property (A.1) in Appendix A.1 implies that

$$\mathbb{L}^{(p)}(f, \sqrt{n}Y_{\frac{k-1}{n}}^{Y_0}) = n^{\frac{p}{2}} \mathbb{E} \left[\left(L_{\frac{k}{n}}(Y^{Y_0}) - L_{\frac{k-1}{n}}(Y^{Y_0}) \right)^p f(\sqrt{n}Y_{\frac{k}{n}}^{Y_0}) | \mathcal{F}_{\frac{k-1}{n}} \right]. \quad (\text{A.10})$$

In particular note that $\mathbb{L}^{(1)}(1, \cdot) = \mathbf{E}_g^Y(\cdot)$ with $g(x, y) = |y| - |x|$.

Lemma 5. Let $x \in \mathbb{R}$, let W be a BM with $W_0 = 0$, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying that there exist $K, \alpha \in [0, \infty)$ such that $|f(y)| \leq Ke^{\alpha|y|}$ for all $y \in \mathbb{R}$. Then for all $p \in \mathbb{N}$ it hold that

$$\begin{aligned} \mathbb{L}^{(p)}(f, x) &= \int_0^1 \frac{|x|}{\sigma(x)} \frac{(1-t)^{\frac{p}{2}}}{\sqrt{2\pi t^{\frac{3}{2}}}} e^{-\frac{x^2}{2(\sigma(x))^2 t}} \mathbb{E}[(L_1(Y^0))^p f(Y_1^0 \sqrt{1-t})] dt \\ &= \left(\frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \right)^{(p+1)} \frac{|x|}{\sigma(x)} \int_0^1 \frac{(1-t)^{\frac{p}{2}}}{\sqrt{2\pi t^{\frac{3}{2}}}} e^{-\frac{x^2}{2(\sigma(x))^2 t}} \mathbb{E}[(L_1(W))^p (\sigma(W_1))^{-1} f(\sigma(W_1)W_1 \sqrt{1-t})] dt, \end{aligned} \quad (\text{A.11})$$

if $x \neq 0$, and $\mathbb{L}^{(p)}(f, 0) = \left(\frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \right)^{(p+1)} \mathbb{E}[(L_1(W))^p (\sigma(W_1))^{-1} f(\sigma(W_1)W_1)]$.

(If $\sigma_{\pm} = \infty$ then replace f in the right hand side of last two equalities with $f\mathbf{1}_{\mathbb{R}_{\mp}}$.)

Proof. We reduce to consider the case $x \neq 0$ because if $x = 0$ then the statement follows from simple computations using the joint density of the OBM Y and its local time (A.2).

Let X be the β_{σ} -SBM satisfying (2.8). In particular $X_0 = x/\sigma(x) \neq 0$. Let B be a BM starting at $x/\sigma(x)$ and let X_t^0 a standard β_{σ} -SBM independent of B . For a process ξ let us denote by $T_0(\xi) := \inf(\{\infty\} \cup \{t \geq 0: \xi_t = 0\})$ the first time it hits 0.

One well known property of SBM is that the process behaves as a BM until it reaches the barrier, which is 0. This means that $T_0(X) \stackrel{\text{law}}{=} T_0(B)$. Moreover, by the Markov property, it holds that $X_{t+T_0(X)}$ conditioned on $T_0(X)$ is distributed as X_t^0 .

This and the relationship between the local times of OBM and SBM (2.9) show that

$$\begin{aligned} \mathbb{L}^{(p)}(f, x) &= \left(\frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \right)^p \mathbb{E}[\mathbf{1}_{\{T_0(X) \leq 1\}} \mathbb{E}[(L_1(X))^p f(\sigma(X_1)X_1) | T_0(X)]] \\ &= \left(\frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \right)^p \mathbb{E}[\mathbf{1}_{\{T_0(B) \leq 1\}} (L_{1-T_0(B)}(X^0))^p f(\sigma(X_{1-T_0(B)}^0)X_{1-T_0(B)}^0)]. \end{aligned} \quad (\text{A.12})$$

Let us recall the well known fact that the random variable $T_0(B)$ has density w.r.t. the Lebesgue measure given by $(0, \infty) \ni t \mapsto \frac{|x|}{\sigma(x)} \frac{1}{\sqrt{2\pi t^{\frac{3}{2}}}} e^{-\frac{x^2}{2(\sigma(x))^2 t}}$. Then the relationship between the local times of OBM and associated SBM (2.9), the scaling property (A.1) and simple changes of variables imply that

$$\mathbb{L}^{(p)}(f, x) = \int_0^1 \frac{|x|}{\sigma(x)} \frac{(1-t)^{\frac{p}{2}}}{\sqrt{2\pi t^{\frac{3}{2}}}} e^{-\frac{x^2}{2(\sigma(x))^2 t}} \mathbb{E}[(L_1(Y^0))^p f(Y_1^0 \sqrt{1-t})] dt. \quad (\text{A.13})$$

The relationship between the joint density of the standard OBM and its local time (A.2) and the one for BM and its local time (A.3) yield the conclusion. \square

B Proof of the key Proposition 5

In this section we prove Proposition 5 which was stated in Section 4.2. The section is organized as follows: We first introduce, in Sections B.1-B.2, some auxiliary results and functions. Then we split the proof of Proposition 5 into two parts. The first part, in Section B.3, consists in proving the decomposition (4.4) into a sum of a vanishing term and a martingale part. In the second part, in Section B.4, we demonstrate that the martingale part satisfies Items (i)-(iv) of Proposition 5. Figure 1 show how the results intervenes in the proof.

In this section, $(Y_t)_{t \in [0,1]}$ is a standard OBM.

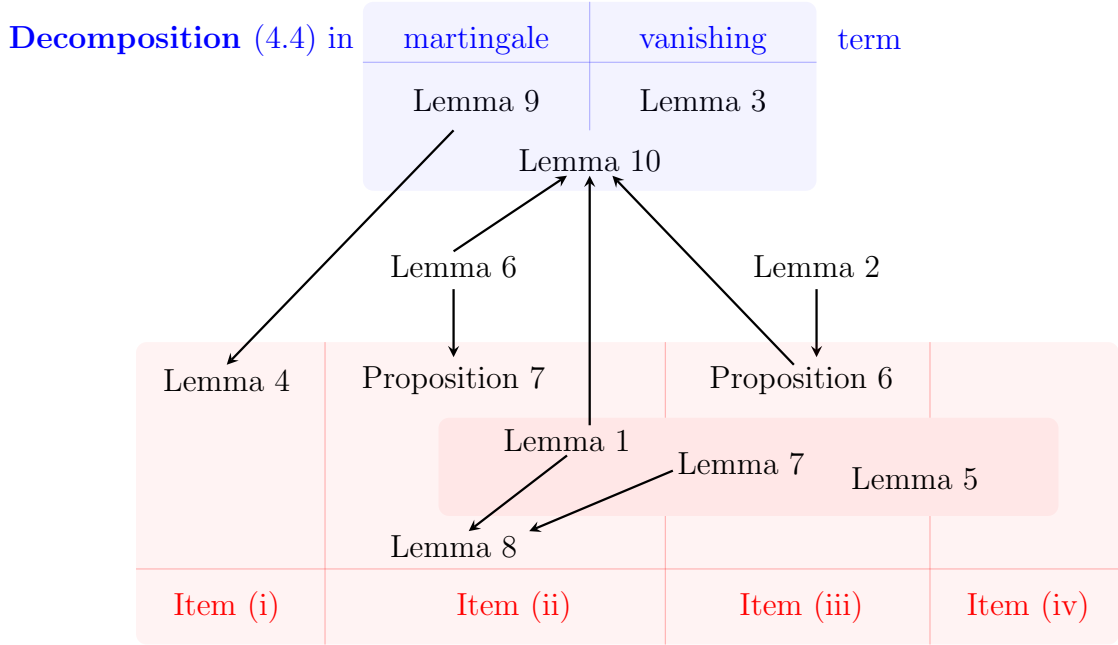


Figure 1: Map of the proof of Proposition 5.

In the appendix we introduce many auxiliary results. This map show how they intervene in the proof of Proposition 5 and of other results of the appendix.

B.1 Auxiliary convergence results

The following is the generalization to the case of OBM of Lemma 4.2 in [24]. The proof is analogous and therefore omitted.

Lemma 6. *Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of real functions satisfying that $\langle \lambda_\sigma, g_n \rangle = 0$ and for all $x \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \frac{g_n(x\sqrt{n})^2}{n} + \frac{\|g_n^2\|_1}{\sqrt{n}} + \frac{\|g_n\|_{1,1}|g_n(x\sqrt{n})|\log(n)}{n} + \frac{\|g_n\|_{1,1}\|g_n\|_1\log(n)}{\sqrt{n}} = 0. \quad (\text{B.1})$$

Then $\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} g_n(\sqrt{n}Y_{\frac{k}{n}}) \right)^2 \right] = 0$ for all $t \in [0, 1]$.

The following propositions correspond to Theorem 4.1 a) and b) in [24]. The proof of the first is step by step an adaptation to OBM of the proof of Theorem 4.1.a) for BM and it relies on Lemma 2. The proof of Proposition 6 is therefore omitted.

Proposition 6. *Let $g_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions satisfying $\lim_{n \rightarrow \infty} \|g_n\|_1 = 0$ and for all $x \in \mathbb{R}$ it holds that $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} g_n(\sqrt{n}x) = 0$. Then*

$$\lim_{n \rightarrow \infty} \sup_{s \in [0,1]} \mathbb{E} \left[\left| n^{-\frac{1}{2}} \sum_{k=0}^{\lfloor ns \rfloor - 1} g_n(\sqrt{n}Y_{\frac{k}{n}}) \right| \right] = 0. \quad (\text{B.2})$$

Proposition 7. *Let $g_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions satisfying (B.1) and there exists $\lambda \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, g_n \rangle = \lambda$. Then for all $t \in [0, 1]$ it holds that*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} g_n(\sqrt{n}Y_{\frac{k}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \lambda L_t(Y). \quad (\text{B.3})$$

If in addition $\sup_{n \in \mathbb{N}} \|g_n\|_1 < \infty$ then

$$\sup_{s \in [0,1]} \left| n^{-\frac{1}{2}} \sum_{k=0}^{\lfloor ns \rfloor - 1} g_n(\sqrt{n} Y_{\frac{k}{n}}) - \lambda L_s(Y) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (\text{B.4})$$

Proof. Let us set the sequence $f_n := g_n - \langle \lambda_\sigma, g_n \rangle \mathbf{E}_g^Y$ with $g(x, y) := |y| - |x|$. Note that Lemma 4 ensures that $\langle \lambda_\sigma, \mathbf{E}_g^Y \rangle = 1$ and that $\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{E}_g^Y(\sqrt{n} Y_{\frac{k}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} L_t(Y)$. Hence $\langle \lambda_\sigma, f_n \rangle = 0$ and one can easily show that f_n satisfies (B.1). Lemma 6 yields the result. The additional statement is the same as [24, Theorem 4.1]. \square

Remark 6 (Proposition 6 and 7 for a constant sequence of functions). Let $f \in \mathbf{L}^1$ such that for all $x \in \mathbb{R}$ it holds that $\lim_{n \rightarrow \infty} \frac{f(\sqrt{nx})}{\sqrt{n}} = 0$ (e.g. $f \in \mathbf{L}^{1,b}(\lambda^{(0)})$). Then Proposition 6 states that if $\|f\|_1 = 0$ then

$$\lim_{n \rightarrow \infty} \sup_{s \in [0,1]} \mathbb{E} \left[\left| n^{-\frac{1}{2}} \sum_{k=0}^{\lfloor ns \rfloor - 1} f(\sqrt{n} Y_{\frac{k}{n}}) \right| \right] = 0. \quad (\text{B.5})$$

And Proposition 7 states that if $f^2 \in \mathbf{L}^1$ and $f \in \mathbf{L}^1(\lambda^{(1)})$ (e.g. $f \in \mathbf{L}^{1,b}(\lambda^{(1)})$) then

$$\sup_{s \in [0,1]} \left| n^{-\frac{1}{2}} \sum_{k=0}^{\lfloor ns \rfloor - 1} f(\sqrt{n} Y_{\frac{k}{n}}) - \langle \lambda_\sigma, f \rangle L_s(Y) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (\text{B.6})$$

B.2 Auxiliary functions

Lemma 7. Let $\gamma \in (0, \infty)$, $h \in \mathbf{I}_\gamma$, and let \mathbf{G}_h be the function

$$\mathbf{G}_h := \mathbf{E}_h^Y - \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle \mathbf{E}_g^Y \quad (\text{B.7})$$

with $g(x, y) = |y| - |x|$. Then $\langle \lambda_\sigma, \mathbf{G}_h \rangle = 0$ and $\mathbf{G}_h \in \mathbf{I}_\gamma$.

Proof. Throughout this proof let $K_\sigma = \frac{1}{\min\{\sigma_-^2, \sigma_+^2\}} \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \in (0, \infty)$. First note that the fact that $g_\sigma(1, x, y) \leq K_\sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}}$ implies that $|\mathbf{E}_g^Y(x)| \leq K_\sigma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |y - x| e^{-\frac{(x-y)^2}{2}} dy \in \mathbf{I}_\alpha$ for all $\alpha \geq 0$. And it also implies, together with the fact that $h \in \mathbf{I}_\gamma$, that

$$|\mathbf{E}_h^Y(x)| \leq K_\sigma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |h(x, y)| e^{-\frac{(x-y)^2}{2}} dy \leq K_\sigma \bar{h}(x) e^{-\frac{a^2}{2}} \quad (\text{B.8})$$

with $\bar{h} \in \mathbf{L}^{1,b}(\lambda^{(\gamma)})$ positive function and a a non negative constant. Hence it holds that $\mathbf{E}_h^Y \in \mathbf{L}^{1,b}(\lambda^{(\gamma)})$. In particular it holds $\langle \lambda_\sigma, \mathbf{E}_h^Y \rangle \leq \|\mathbf{E}_h^Y\|_1 < \infty$. Therefore $|\mathbf{G}_h| \leq |\mathbf{E}_h^Y| + \|\mathbf{E}_h^Y\|_1 |\mathbf{E}_g^Y| \in \mathbf{I}_\gamma$ and so $\mathbf{G}_h \in \mathbf{I}_\gamma$.

It remains to prove that $\langle \lambda_\sigma, \mathbf{G}_h \rangle = 0$. This follows from the fact that $\langle \lambda_\sigma, \mathbf{G}_h \rangle = \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle (1 - \langle \lambda_\sigma, \mathbf{E}_g^Y \rangle)$ and $\langle \lambda_\sigma, \mathbf{E}_g^Y \rangle = 1$ by Lemma 4. \square

In the remainder of this section let $\gamma \in [1, \infty)$, $h \in \mathbf{I}_\gamma$, let \mathbf{G}_h be the function in (B.7), and for all $i, j \in \{0, 1\}$, $\eta \in [0, \infty)$, $n \in [1, \infty)$ let

$$\mathcal{Q}_{n,i,j}^{(\eta)} := \sum_{k=i}^{\lfloor n^\eta \rfloor + j} Q_k^\sigma \mathbf{G}_h \quad \text{and} \quad \mathcal{Q}_{n,i,j} := \mathcal{Q}_{n,i,j}^{(\frac{1}{4})} \quad (\text{B.9})$$

where Q^σ is the semigroup of the OBM given in (A.4). Let $n \in \mathbb{N}$ be fixed.

The following facts are consequences of Lemma 7, Lemma 1, and the fact that $\sum_{j=1}^n \frac{1}{j} \leq 2 \log(n)$ for $n \geq 2$.

For every $\zeta \in [0, \gamma - 1]$, for every $\eta \in (0, 1)$ the fact that $\lambda(\mathbf{G}_h) = 0$ and Item (iii) in Lemma 1 imply that for all $x \in \mathbb{R}$ it holds that

$$|\mathcal{Q}_{n^{\eta+1}}^\sigma \mathbf{G}_h(x)| \leq \left(1 + \frac{|\sigma_- - \sigma_+|}{\sigma_- + \sigma_+}\right) K_\zeta n^{-\eta} \left(\frac{1}{1 + |xn^{-\eta/2}/\sigma(x)|^\zeta} + \frac{1}{1 + |x/\sigma(x)|^\zeta} \right) \quad (\text{B.10})$$

and

$$|\mathcal{Q}_{n,1,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,1,1}^{(\eta)}(x)| \leq 2K_\zeta \log(n) \left(\frac{1}{1 + |xn^{-\frac{\eta}{2}}/\sigma(x)|^\zeta} + \frac{1}{1 + |x/\sigma(x)|^\zeta} \right) \quad (\text{B.11})$$

for some $K_\zeta \in (0, \infty)$ depending also on η . Hence (B.11) with $\zeta = 0$ and Item (ii) in Lemma 1 imply that for all $x \in \mathbb{R}$ it holds that

$$|\mathcal{Q}_{n,1,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,1,1}^{(\eta)}(x)| \leq 2K \min \{\log(n), (1 + |x|)\}, \quad (\text{B.12})$$

for some constant $K \in (0, \infty)$ depending on $\eta \in (0, 1)$. This and the fact that \mathbf{G}_h is bounded ensures that for some positive constant $K \in (0, \infty)$ it holds that $|\mathcal{Q}_{n,0,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,0,1}^{(\eta)}(x)| \leq 2|\mathbf{G}_h(x)| + |\mathcal{Q}_{n,1,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,1,1}^{(\eta)}(x)| \leq 2K(\log(n) + 1)$ and

$$|\mathcal{Q}_{n,0,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,0,1}^{(\eta)}(x)| + |\mathcal{Q}_{n,1,0}^{(\eta)}(x)| + |\mathcal{Q}_{n,1,1}^{(\eta)}(x)| \leq 2K(\log(n) + 1). \quad (\text{B.13})$$

The following lemma is a straightforward consequence of Item (ii) in Lemma 1 and of the fact that $\langle \lambda_\sigma, \mathbf{G}_h \rangle = 0$ (see Lemma 7).

Lemma 8. *Pointwise $\lim_{m \rightarrow \infty} \mathcal{Q}_{m,0,0} = \mathcal{P}_h^Y$. (Recall that \mathcal{P}_h^Y is given by (2.19).)*

B.3 The decomposition as sum martingale and vanishing terms

In this section let γ be an arbitrary non-negative number to be specified in each statement and let $h \in \mathbf{I}_\gamma$. We are now determining the terms of the decomposition in equation (4.4).

For every $n \in \mathbb{N}$ let $M^{n,1}$, $M^{n,2}$, and N^n be the processes satisfying for all $t \in [0, 1]$ that

$$M_t^{n,1} = \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left(h(\sqrt{n}Y_{\frac{k}{n}}^Y, \sqrt{n}Y_{\frac{k+1}{n}}^Y) - \mathbf{E}_h^Y(\sqrt{n}Y_{\frac{k}{n}}^Y) + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle \mathbf{E}_g^Y(\sqrt{n}Y_{\frac{k}{n}}^Y) \right) - L_{\frac{\lfloor nt \rfloor}{n}}(Y), \quad (\text{B.14})$$

$$M_t^{n,2} = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \left(\mathcal{Q}_{n,0,0}(\sqrt{n}Y_{\frac{k}{n}}^Y) - \mathcal{Q}_{n,1,1}(\sqrt{n}Y_{\frac{k-1}{n}}^Y) \right), \quad \text{and} \quad N_t^n = \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{G}_h(\sqrt{n}Y_{\frac{k}{n}}^Y) \quad (\text{B.15})$$

(recall the definition for \mathbf{G}_h in (B.7), \mathcal{Q} in (B.9), and $g(x, y) = |y| - |x|$). Trivially it holds for all $n \in \mathbb{N}$, $t \in [0, 1]$ that

$$\begin{aligned} \varepsilon_{n,t}^{(0,h,Y)} - \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle L_t(Y) &= M_t^{n,1} + N_t^n + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle (L_{\lfloor nt \rfloor/n}(Y) - L_t(Y)) \\ &= M_t^{n,1} + M_t^{n,2} - M_t^{n,2} + N_t^n + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle (L_{\lfloor nt \rfloor/n}(Y) - L_t(Y)). \end{aligned} \quad (\text{B.16})$$

The right-hand-side of the latter equality is a sum of an $(\mathcal{F}_{\lfloor nt \rfloor/n})_{t \geq 0}$ -martingale (denoted by $n^{-\frac{1}{4}}\mathcal{M}^n$) and of a vanishing term with rate of order at least $1/4$ (denoted by \mathcal{V}^n):

$$\mathcal{V}_t^n := N_t^n - M_t^{n,2} + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle (L_{\frac{\lfloor nt \rfloor}{n}}(Y) - L_t(Y)) \quad \text{and} \quad \mathcal{M}_t^n := n^{1/4} (M_t^{n,1} + M_t^{n,2}). \quad (\text{B.17})$$

Lemma 10 below, together with Lemma 3, ensures that $\sup_{t \in [0,1]} n^{1/4} |\mathcal{V}_t^n| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$. The two lemmas below prove the martingale property of \mathcal{M}^n , for $n \in \mathbb{N}$.

Lemma 9. $M^{n,1}$ is a martingale with respect to the filtration $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$.

Proof. Throughout this proof let A^n and B^n be the processes given by

$$A_t^n = \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \mathbf{E}_g^Y(\sqrt{n}Y_{\frac{k}{n}}) - L_{\frac{\lfloor nt \rfloor}{n}}(Y) \text{ and } B_t^n = M_t^{n,1} - \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle A_t^n. \quad (\text{B.18})$$

So $B_t^n = \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left(h(\sqrt{n}Y_{\frac{k}{n}}, \sqrt{n}Y_{\frac{k+1}{n}}) - \mathbf{E}_h^Y(\sqrt{n}Y_{\frac{k}{n}}) \right)$. It suffices to show the martingale property for A^n and B^n . Let $t \in [0, 1]$ be fixed. The martingale property for A^n is an immediate consequence of Lemma 4. Let us explicit the case of the process B^n . For all $j \in \{0, \dots, \lfloor nt \rfloor - 1\}$ it can be easily shown that

$$\mathbb{E} \left[n^{\frac{1}{2}} B_t^n | \mathcal{F}_{\frac{j}{n}} \right] = \sum_{k=j}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[h(\sqrt{n}Y_{\frac{k}{n}}, \sqrt{n}Y_{\frac{k+1}{n}}) - \mathbf{E}_h^Y(\sqrt{n}Y_{\frac{k}{n}}) | \mathcal{F}_{\frac{j}{n}} \right] + B_{\frac{j}{n}}^n = B_{\frac{j}{n}}^n. \quad (\text{B.19})$$

This completes the proof. \square

Lemma 10. Let $h \in \mathbf{I}_3$. Then $M^{n,2}$ is a martingale with respect to the filtration $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$ and it holds that $\sup_{s \in [0,1]} n^{1/4} |N_s^n - M_s^{n,2}| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

Proof. Using (B.9) we rewrite $M^{n,2}$ in (B.15) as

$$\begin{aligned} M_t^{n,2} &= \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j=0}^{\lfloor n^{\frac{1}{4}} \rfloor} \left(\mathbb{E} \left[\mathbf{G}_h(\sqrt{n}Y_{(j+k)/n}) | \mathcal{F}_{k/n} \right] - \mathbb{E} \left[\mathbf{G}_h(\sqrt{n}Y_{(j+k)/n}) | \mathcal{F}_{(k-1)/n} \right] \right) \\ &= N_t^n - \frac{1}{\sqrt{n}} \left(\mathcal{Q}_{n,0,0}(\sqrt{n}Y_0) - \mathcal{Q}_{n,0,0}(\sqrt{n}Y_{\lfloor nt \rfloor / n}) \right) - \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor - 1} \mathcal{Q}_{\lfloor n^{\frac{1}{4}} \rfloor + 1}^\sigma \mathbf{G}_h(\sqrt{n}Y_{j/n}). \end{aligned} \quad (\text{B.20})$$

The first equality of the latter equation makes clear that that $M^{n,2}$ is a martingale with respect to the filtration $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$. Let

$$m_t^n := N_t^n - \frac{1}{\sqrt{n}} \left(\mathcal{Q}_{n,0,0}^{(3/4)}(\sqrt{n}Y_0) - \mathcal{Q}_{n,0,0}^{(3/4)}(\sqrt{n}Y_{\lfloor nt \rfloor / n}) \right) - \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor - 1} \mathcal{Q}_{\lfloor n^{\frac{3}{4}} \rfloor + 1}^\sigma \mathbf{G}_h(\sqrt{n}Y_{j/n}). \quad (\text{B.21})$$

As for (B.20) it is clear that m^n is a martingale with respect to the filtration $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$. Therefore $n^{1/4}(M_t^{n,2} - m_t^n)$ as well. Let us denote by $D_t^n := n^{1/4}(N_t^n - M_t^{n,2})$ and $d_t^n := n^{1/4}(N_t^n - m_t^n)$. Then $D^n - d^n = n^{1/4}(w^n - M^{n,2})$ is a martingale with respect to the filtration $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$. In this notation, the goal is to prove $\sup_{s \in [0,1]} |D_s^n| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

First step: For every $t \in [0, 1]$, let us show that $\lim_{n \rightarrow \infty} \mathbb{E}[(D_t^n - d_t^n)^2] = 0$ by demonstrating the stronger fact that $\lim_{n \rightarrow \infty} \mathbb{E}[(D_t^n)^2 + (d_t^n)^2] = 0$.

Let $t \in [0, 1]$ and $\eta \in \{\frac{1}{4}, \frac{3}{4}\}$ be fixed. Inequality (B.13) implies for all $\omega \in \Omega$ that

$$\sup_{s \in [0,1]} n^{-\frac{1}{4}} \left(\mathcal{Q}_{n,0,0}^{(\eta)}(\sqrt{n}Y_0(\omega)) - \mathcal{Q}_{n,0,0}^{(\eta)}(\sqrt{n}Y_{\lfloor ns \rfloor / n}(\omega)) \right) \xrightarrow[n \rightarrow \infty]{} 0, \quad (\text{B.22})$$

hence it holds also that $\lim_{n \rightarrow \infty} \mathbb{E} \left[n^{-\frac{1}{2}} \left(\mathcal{Q}_{n,0,0}^{(\eta)}(\sqrt{n}Y_0) - \mathcal{Q}_{n,0,0}^{(\eta)}(\sqrt{n}Y_{\lfloor nt \rfloor / n}) \right)^2 \right] = 0$. Next observe that, since λ_σ is the stationary measure, the sequences of functions $g_n^{(\eta)} := n^{1/4} \mathcal{Q}_{\lfloor n^\eta \rfloor + 1}^\sigma \mathbf{G}_h$,

$\eta \in \{\frac{1}{4}, \frac{3}{4}\}$ satisfy that $\langle \lambda_\sigma, g_n^{(\eta)} \rangle = n^{1/4} \langle \lambda_\sigma, \mathbf{G}_h \rangle$ which is equal to 0 by Lemma 7. This and inequality (B.10) (with $\zeta = 2$ since $h \in \mathbf{I}_{\zeta+1}$) ensure that (B.1) holds. Hence, Lemma 6 shows that $\lim_{n \rightarrow \infty} \mathbb{E} \left[n^{-\frac{1}{2}} \left(\sum_{j=0}^{\lfloor nt \rfloor - 1} Q_{\lfloor n\eta \rfloor + 1}^\sigma \mathbf{G}_h(\sqrt{n} Y_{j/n}) \right)^2 \right] = 0$.

Second step: It holds that $\sup_{s \in [0,1]} |D_s^n - d_s^n| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$.

This follows from [1, Proposition 1.2] as a consequence of the previous step and the martingale property of $D^n - d^n$.

Third step: It holds that $\sup_{s \in [0,1]} |d_s^n| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$. This follows from (B.22) and from applying Proposition 6 to the sequence $g_n := n^{1/4} Q_{\lfloor n\frac{3}{4} \rfloor + 1}^\sigma \mathbf{G}_h$. The assumptions are indeed satisfied: $\lim_{n \rightarrow \infty} \|g_n\|_1 = 0$ by Item (i) in Lemma 1 and in the first step we have proven that g_n satisfies (B.1) and in particular $\lim_{n \rightarrow \infty} \frac{g_n(\sqrt{nx})}{\sqrt{n}} = 0$.

Combining the two last steps yields the conclusion. \square

B.4 Final steps of the proof

In this section we complete the proof of Proposition 5. The arguments are sometimes sketched because analogous to the ones in [24, Section 6].

Let $\gamma > 3$ and $h \in \mathbf{I}_\gamma$ be fixed. By Definition 1 of \mathbf{I}_γ there exist a non-negative function $\bar{h} \in \mathbf{L}^{1,b}(\lambda^{(\gamma)})$ and a constant $a \in [0, \infty)$ such that $|h(x, y)| \leq \bar{h}(x) e^{a|y-x|}$. In this section \bar{h} and a are fixed.

Let us also recall some notation: let \mathbf{E}^Y the functional in (2.12), Q^σ the semigroup in (A.4), \mathbf{G}_h in (B.7), \mathcal{Q} in (B.9) and its limit \mathcal{P}_h^Y in (2.19), and \mathbb{L} in (A.9).

For all $n \in \mathbb{N}$ the $(\mathcal{F}_{\lfloor nt \rfloor / n})_{t \in [0,1]}$ -martingale \mathcal{M}^n in (B.17) rewrites as $\mathcal{M}_t^n = \sum_{k=1}^{\lfloor nt \rfloor} \chi_k^n$ where

$$\begin{aligned} \chi_k^n &:= n^{-\frac{1}{4}} \left(h(\sqrt{n} Y_{(k-1)/n}, \sqrt{n} Y_{k/n}) - \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle \sqrt{n} (L_{k/n}(Y) - L_{(k-1)/n}(Y)) \right) \\ &\quad + n^{-\frac{1}{4}} \left(\mathcal{Q}_{n,0,0}(\sqrt{n} Y_{k/n}) - \mathcal{Q}_{n,0,1}(\sqrt{n} Y_{(k-1)/n}) \right). \end{aligned} \quad (\text{B.23})$$

Now it remains to prove Items (i)-(iv) in Proposition 5. Item (i) in Proposition 5 follows from scaling property (A.1) and Lemma 4.

B.4.1 Proof of Item (ii) in Proposition 5

Let $t \in [0, 1]$ be fixed.

First step: It can be easily shown that

$$\begin{aligned} &\sqrt{n} \mathbb{E}[(\chi_k^n)^2 | \mathcal{F}_{(k-1)/n}] \\ &= f_n(\sqrt{n} Y_{(k-1)/n}) + (\langle \lambda_\sigma, \mathbf{E}_h^Y \rangle)^2 \mathbb{L}^{(2)}(1, \sqrt{n} Y_{(k-1)/n}) - 2 \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle h_n(\sqrt{n} Y_{(k-1)/n}) \end{aligned} \quad (\text{B.24})$$

where f_n and h_n are given by $f_n(x) := \mathbf{E}_{h^2}^Y(x) + 2\mathbf{E}_{h, \mathcal{Q}_{n,0,0}}^Y(x) + g_n(x)$ with $g_n(x) := Q_1^\sigma(\mathcal{Q}_{n,0,0})^2(x) - (\mathcal{Q}_{n,0,1}(x))^2$ and $h_n(x) := \mathbb{L}^{(1)}(h(x, \cdot), x) + \mathbb{L}^{(1)}(\mathcal{Q}_{n,0,0}, x)$.

Second step: It follows from applying Proposition 7 to the constant sequence of functions $\mathbb{L}^{(2)}(1, \cdot)$ that $\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{L}^{(2)}(1, \sqrt{n} Y_{(k-1)/n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{2\sigma - \sigma_+}{\sigma_- + \sigma_+} \frac{8}{3\sqrt{2\pi}} L_t(Y)$. The fact that the assumptions of the proposition (i.e. $\mathbb{L}^{(2)}(1, \cdot) \in \mathbf{L}^{1,b}(\lambda^{(2)})$) are satisfied and $\langle \lambda_\sigma, \mathbb{L}^{(2)}(1, \cdot) \rangle = \frac{2\sigma - \sigma_+}{\sigma_- + \sigma_+} \frac{8}{3\sqrt{2\pi}}$ follows from the fact that $\mathbb{E}[(L_1(W))^2] = 1$ and Lemma 5.

In the two final steps, we want to apply Proposition 7 to the sequences f_n and h_n .

Third step: We show that $\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} f_n(\sqrt{n} Y_{(k-1)/n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle \lambda_\sigma, \mathbf{E}_{h^2}^Y + 2\mathbf{E}_{h, \mathcal{P}_h^Y}^Y \rangle L_t(Y)$. applying Proposition 7. To do so we check that the sequence f_n satisfies (B.1) and that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, f_n \rangle = \langle \lambda_\sigma, \mathbf{E}_{h^2}^Y + 2\mathbf{E}_{h, \mathcal{P}_h^Y}^Y \rangle$.

The fact that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, g_n \rangle = 0$ follows from the fact that λ_σ is the stationary measure, inequality (B.10) with $\zeta = \gamma - 1 > 2$, and inequality (B.11). Note that $\mathbf{E}_{h, \mathcal{Q}_{n,0,0}}^Y = \mathbf{E}_{h, \mathbf{G}_h}^Y + \mathbf{E}_{h, \mathcal{Q}_{n,1,0}}^Y$. Lemma 7 (in particular the fact that \mathbf{G}_h is bounded) ensures that there exists a constant $K \in (0, \infty)$ such that $|\mathbf{E}_{h, \mathbf{G}_h}^Y(x)| \leq K\bar{h}(x)$. By (B.12) there exists constants $K_1, K_2 \in (0, \infty)$ (all depending on σ_\pm and K_2 depending also on the constant $a \in [0, \infty)$) such that

$$\begin{aligned} |\mathbf{E}_{h, \mathcal{Q}_{n,1,0}}^Y(x)| &\leq K_1 \bar{h}(x) \left(\int_{-\infty}^{\infty} (1 + |x| \mathbf{1}_{[-x, x]}(y) + |y| \mathbf{1}_{\mathbb{R} \setminus [-x, x]}(y)) e^{a|y-x|} q_\sigma(1, x, y) dy \right) \\ &\leq K_2 \bar{h}(x) (1 + |x|). \end{aligned} \quad (\text{B.25})$$

The fact that $h \in \mathbf{I}_\gamma$ ensures that $\bar{h} \in \mathbf{L}^{1,b}(\lambda^{(1)})$ and so $\mathbf{E}_{h, \mathcal{Q}_{n,0,0}}^Y \in \mathbf{L}^1$. Hence, dominated convergence and Lemma 8 show that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, f_n \rangle = \langle \lambda_\sigma, \mathbf{E}_{h^2}^Y + 2\mathbf{E}_{h, \mathcal{P}_h^Y}^Y \rangle$.

Let us now show that f_n satisfies equation (B.1). The fact that $\bar{h} \in \mathbf{L}^{1,b}(\lambda^{(2)})$ ensures that $\mathbf{E}_{h^2}^Y(x) \leq K\bar{h}(x)$ for some positive constant K and so $\mathbf{E}_{h^2}^Y \in \mathbf{L}^{1,b}(\lambda^{(2)})$.

Let us explore the contribution to (B.1) of the other parts of f_n . Let us first consider $\mathbf{E}_{h, \mathcal{Q}_{n,0,0}}^Y = \mathbf{E}_{h, \mathbf{G}_h}^Y + \mathbf{E}_{h, \mathcal{Q}_{n,1,0}}^Y$. Above we saw that $|\mathbf{E}_{h, \mathbf{G}_h}^Y| \leq K\bar{h}$ with K non negative constant. Hölder's inequality implies that

$$(\mathbf{E}_{h, \mathcal{Q}_{n,1,0}}^Y(x))^2 \leq \int_{-\infty}^{\infty} h(x, y)^2 q_\sigma(1, x, y) dy \int_{-\infty}^{\infty} (\mathcal{Q}_{n,1,0}(y))^2 q_\sigma(1, x, y) dy. \quad (\text{B.26})$$

It can be easily shown that the first factor is uniformly bounded by a finite constant. And by (B.11) (taking $\zeta = \gamma - 1 > 2$) there exist constants $K_1, K_2 \in (0, \infty)$ such that

$$\begin{aligned} \int_{-\infty}^{\infty} (\mathcal{Q}_{n,1,0}(y))^2 q_\sigma(1, x, y) dy &\leq K_1 \int_{-\infty}^{\infty} \frac{(\log n)^2 q_\sigma(1, x, y)}{(1 + |yn^{-\frac{1}{8}}/\sigma(y)|^{\gamma-1})^2} dy \\ &\leq \frac{K_2 (\log(n))^2}{1 + |xn^{-\frac{1}{8}}/\sigma(x)|^{2(\gamma-1)}}. \end{aligned} \quad (\text{B.27})$$

The last inequality are consequences of the upper bound for the transition density of OBM $q_\sigma(1, x, y) \leq \frac{(1 + |\frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}|)}{\sqrt{2\pi}\sigma(y)} e^{-\frac{1}{2}(\frac{y}{\sigma(y)} - \frac{x}{\sigma(x)})^2}$ and of [24, Lemma 3.2] (or some computations). Therefore $\mathbf{E}_{h, \mathcal{Q}_{n,0,0}}^Y(x) \leq K \left(\bar{h}(x) + \frac{\log(n)}{1 + |xn^{-\frac{1}{8}}/\sigma(x)|^{\gamma-1}} \right)$. Finally we consider the auxiliary function g_n : note that $|g_n| \leq Q_1^2(2\mathcal{Q}_{n,0,0}^2 + 2\mathbf{G}_h^2) + 2\mathcal{Q}_{n,1,1}^2 + 2\mathbf{G}_h^2$. Inequality (B.11) and inequality (B.27) imply that $Q_1^\sigma(\mathcal{Q}_{n,0,0}^2) + \mathcal{Q}_{n,1,1}^2 \leq \frac{K(\log(n))^2}{1 + |xn^{-\frac{1}{8}}/\sigma(x)|^{2(\gamma-1)}}$ for some non negative constant K . Lemma 7, the fact that $\gamma \geq 2$ and Item (iii) of Lemma 1 yields $\mathbf{G}_h^2 \leq \mathbf{G}_h$ and $Q_1^\sigma \mathbf{G}_h^2(x) \leq K(e^{-x^2/2} + \frac{1}{1 + (|x|/\sigma(x))^\gamma})$ for some non-negative constant K . Combining all terms yields that f_n satisfied (B.1).

Forth step: We show that $\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} h_n(\sqrt{n}Y_{(k-1)/n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} c_h L_t(Y)$ applying Proposition 7, where $2\langle \lambda_\sigma, \mathbf{E}_h^Y \rangle c_h := -K_h + \langle \lambda_\sigma, \mathbf{E}_{h^2}^Y + 2\mathbf{E}_{h, \mathcal{P}_h^Y}^Y \rangle + \frac{2\sigma_- \sigma_+}{\sigma_- + \sigma_+} \frac{8}{3\sqrt{2\pi}} (\langle \lambda_\sigma, \mathbf{E}_h^Y \rangle)^2$. To do so we check that the sequence h_n satisfies (B.1) and that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, h_n \rangle = c_h$.

Inequality (B.12), the fact that \mathbf{G}_h is bounded (see Lemma 7), the fact that

$$\mathbb{E}[L_1(W)(1 + |W_1| + e^{a\sigma(W_1)|W_1|} \mathbf{1}_{\{\sigma(W_1) \in \mathbb{R}\}})] < \infty \quad (\text{B.28})$$

(see the joint density of BM and its local time (A.3)), boundedness of \bar{h} , and the change of variable $s = \frac{x^2}{(\sigma(x))^2 t}$ show that there exists $K \in (0, \infty)$ such that

$$|\mathbb{L}^{(1)}(h(x, \cdot), x)| + |\mathbb{L}^{(1)}(\mathcal{Q}_{n,0,0}, x)| \leq 2K|x|e^{-\frac{x^2}{(\sigma(x))^2}} \in \mathbf{L}^1. \quad (\text{B.29})$$

Hence dominated convergence, and Lemma 8 demonstrates that $\lim_{n \rightarrow \infty} \langle \lambda_\sigma, \mathbb{L}^{(1)}(\mathcal{Q}_{n,0,0}, \cdot) \rangle = \langle \lambda_\sigma, \mathbb{L}^{(1)}(\mathcal{P}_h^Y, \cdot) \rangle$. Moreover the latter inequalities ensure also that h_n satisfy (B.1).

Lemma 5 allows us to rewrite $c_h := \langle \lambda_\sigma, \mathbb{L}^{(1)}(h(\cdot, \cdot), \cdot) \rangle + \langle \lambda_\sigma, \mathbb{L}^{(1)}(\mathcal{P}_h^Y, \cdot) \rangle$ as

$$\begin{aligned} c_h &= \sqrt{\frac{2}{\pi}} \frac{(\sigma_- + \sigma_+)}{2\sigma_- \sigma_+} \int_0^1 \sqrt{\frac{1}{t} - 1} \int_{-\infty}^{\infty} \int_0^{\infty} \rho_1^\sigma(y, \ell) \ell \mathcal{P}_h^Y(y\sqrt{1-t}) d\ell dy dt \\ &\quad + \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \int_0^{\infty} \frac{|x| e^{-\frac{x^2}{2(\sigma(x))^2}}}{\sqrt{2\pi}\sigma(x)} \sqrt{\frac{1}{t} - 1} \rho_1^\sigma(y, \ell) \ell h(x\sqrt{t}, y\sqrt{1-t}) d\ell dy dt \lambda_\sigma(dx) \end{aligned} \quad (\text{B.30})$$

where ρ_1^σ is the joint density of a standard OBM and its local time at time 1 (given in (A.2)). This expression can be easily checked to be the desired expression for c_h .

B.4.2 Proof of Item (iii) in Proposition 5

Let $t \in [0, 1]$ be fixed.

First step: Let us show that

$$\begin{aligned} &\mathbb{E}[\chi_k^n(Y_{k/n} - Y_{(k-1)/n}) | \mathcal{F}_{(k-1)/n}] \\ &= \frac{1}{\sqrt{n}} \left(n^{-\frac{1}{4}} f_1(\sqrt{n}Y_{(k-1)/n}) + g_n(\sqrt{n}Y_{(k-1)/n}) + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle n^{-\frac{1}{4}} f_2(\sqrt{n}Y_{(k-1)/n}) \right) \end{aligned} \quad (\text{B.31})$$

where f_1, f_2, g_n are given by

$$\begin{aligned} f_1(x) &:= \int_{-\infty}^{\infty} (h(x, y) + \mathbf{G}_h(y))(y - x) q_\sigma(1, x, y) dy, \quad f_2(x) := x \mathbb{L}^{(1)}(1, x), \quad \text{and} \\ g_n(x) &:= n^{-\frac{1}{4}} \int_{-\infty}^{\infty} \mathcal{Q}_{n,1,0}(y)(y - x) q_\sigma(1, x, y) dy. \end{aligned} \quad (\text{B.32})$$

Throughout the proof of this step let $\mathbb{I}: x \mapsto x$ denote the identity function. It follows from (B.23), the fact that Y is a martingale, (A.10), and the scaling property (A.1) that

$$\begin{aligned} &n^{\frac{3}{4}} \mathbb{E}[\chi_k^n(Y_{k/n} - Y_{(k-1)/n}) | \mathcal{F}_{(k-1)/n}] \\ &= f_1(\sqrt{n}Y_{(k-1)/n}) + \langle \lambda_\sigma, \mathbf{E}_h^Y \rangle f_2(\sqrt{n}Y_{(k-1)/n}) - \mathbb{L}^{(1)}(\mathbb{I}, \sqrt{n}Y_{(k-1)/n}) + n^{1/4} g_n(\sqrt{n}Y_{(k-1)/n}). \end{aligned} \quad (\text{B.33})$$

The proof of this step is completed if $\mathbb{L}^{(1)}(\mathbb{I}, \cdot) = 0$. This equality follows from the fact that for a standard BM, say W , it holds that $\mathbb{E}[L_1(W)W_1] = 0$ and from Lemma 5.

In the next steps we want to check that Proposition 6 can be applied to the sequences $n^{-\frac{1}{4}} f_1$, $n^{-\frac{1}{4}} f_2$ and g_n .

Second step: We show that f_1 is bounded and integrable: $f_1 \in \mathbf{L}^{1,b}(\lambda^{(0)})$.

The proof follows from Lemma 7, and Item (iii) in Lemma 1. In this proof the fact that $\gamma > 3$ is strongly used.

Third step: The fact that f_2 is bounded and integrable follows from Lemma 5.

Forth step: We prove that $\int_{-\infty}^{\infty} |g_n(x)| dx$ and $\frac{1}{\sqrt{n}} g_n(\sqrt{n}x)$ converge to 0.

Cauchy-Schwarz inequality and inequality (B.27) yield that there exists a constant $K \in (0, \infty)$ depending on γ and σ_\pm such that $|g_n(x)| \leq \frac{n^{-\frac{1}{4}} K_2 \log(n)}{1 + |xn^{-\frac{1}{8}}/\sigma(x)|^{\gamma-1}}$.

In the last steps we proved that Proposition 6 can be applied and this completes the proof.

B.4.3 Proof of Item (iv) in Proposition 5

Let $\varepsilon \in (0, \infty)$ be fixed. For every k , Hölder's inequality and Markov's inequality show

$$\mathbb{E}[|\chi_k^n|^2 \mathbf{1}_{\{|\chi_k^n| \geq \varepsilon\}} | \mathcal{F}_{(k-1)/n}] \leq \mathbb{E}[|\chi_k^n|^5 | \mathcal{F}_{(k-1)/n}] \varepsilon^{-3}. \quad (\text{B.34})$$

The fact that $h \in \mathbf{I}_\gamma \subseteq \mathbf{I}_0$ ensures that \bar{h} is bounded and integrable. This combined with Jensen's inequality and (B.13) ensures that $\sup_{x \in \mathbb{R}} \bar{h}(x) + |\langle \lambda_\sigma, \mathbf{E}_h^Y \rangle|$ is bounded by a constant $K \in (0, \infty)$ and for all $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$ it holds that

$$\begin{aligned} \mathbb{E}[|\chi_k^n|^5 | \mathcal{F}_{(k-1)/n}] &\leq 4^4 n^{-\frac{5}{4}} K^5 \left(\mathbb{E} \left[e^{5a\sqrt{n}|Y_{k/n} - Y_{(k-1)/n}|} | \mathcal{F}_{(k-1)/n} \right] \right. \\ &\quad \left. + \mathbb{E} \left[(\sqrt{n}|L_{k/n}(Y) - L_{(k-1)/n}(Y)|)^5 | \mathcal{F}_{(k-1)/n} \right] + (\log n)^5 \right). \end{aligned} \quad (\text{B.35})$$

The fact that density of the OBM has a Gaussian behavior together with the scaling property (A.1) ensure that $\mathbb{E} \left[e^{5a\sqrt{n}|Y_{k/n} - Y_{(k-1)/n}|} | \mathcal{F}_{(k-1)/n} \right] = \int_{-\infty}^{\infty} e^{5a|y-x|} q_\sigma(1, x, y) dy$ is bounded. By (A.9) and Lemma 5 we can show, similarly to the third step of the proof of Item (iii), that there exist constants $K_1, K_2 \in (0, \infty)$ such that

$$\begin{aligned} \mathbb{E} \left[(\sqrt{n}|L_{k/n}(Y) - L_{(k-1)/n}(Y)|)^5 | \mathcal{F}_{(k-1)/n} \right] &= \mathbb{L}^{(5)}(1, \sqrt{n}Y_{(k-1)/n}) \\ &\leq K_1 \int_0^1 \frac{(1-t)^{\frac{5}{2}}}{\sqrt{2\pi t^{\frac{3}{2}}}} \frac{\sqrt{n}|Y_{(k-1)/n}|}{\sigma(Y_{(k-1)/n})} e^{-\frac{n(Y_{(k-1)/n})^2}{2(\sigma(Y_{(k-1)/n}))^2 t}} dt \leq K_2. \end{aligned} \quad (\text{B.36})$$

We conclude that there exists a constant $K \in (0, \infty)$ such that

$$\sum_{k=1}^n \mathbb{E}[|\chi_k^n|^2 \mathbf{1}_{\{|\chi_k^n| \geq \varepsilon\}} | \mathcal{F}_{(k-1)/n}] \leq nKn^{-\frac{5}{4}} \log(n) \varepsilon^{-3} \xrightarrow[n \rightarrow \infty]{} 0. \quad (\text{B.37})$$

Acknowledgments: The author thanks Sylvie Roelly and Paolo Pigato for fruitful discussions on the topic of this paper.

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