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Abstract

We present an observer for linear time-invariant (LTI) systems with measurement delay. Our design ensures that the observer error converges to zero within a prescribed terminal time. To achieve this, we employ time-varying output gains that approach infinity at the terminal time, which can be arbitrarily short but no shorter than the sensor delay time. We model the sensor delay as a transport partial differential equation (PDE) and build upon the cascade ODE-PDE setting while accounting for the infinite dimensionality of the sensor. To construct our time-varying gains, the observer design needs to be conducted in a particular system representation. For this reason, we employ a sequence of state transformations (and their inverses) mapping the original observer error model into (1) the observer form, (2) a sensor delay-compensated observer error form via backstepping, and (3) a particular diagonal form that is amenable to the selection of time-varying gains for prescribed-time stabilization. Our construction of the time-varying observer gains uses (a) generalized Laguerre polynomials, (b) elementary symmetric polynomials, and (c) polynomial-based Vandermonde matrices. A simulation illustrates the results.

Key words: Infinite dimensional systems, prescribed-time convergence, delay-compensation, observer design.

1 Introduction

Most stabilization and estimation algorithms for practical engineering problems provide asymptotic convergence, yet in many cases (e.g., multi–agent rendezvous, missile guidance, weather forecasting) this transient process must occur within a given time. The need to meet time constraints and increase temporal performance has motivated finite– and fixed–time stabilization and estimation. These enhanced stabilization/estimation techniques have been extensively studied within the framework of linear and nonlinear ordinary differential equations (ODEs) (see, e.g., [5,17,20,21,29,35,37] and references therein). Finite–time convergence refers to a terminal time which depends on the system's initial conditions, whereas for fixed–time convergence, the terminal time depends on a *bounded* function of the initial conditions. More recently, the *prescribed–time* convergence concept has arisen to cope with a more demanding type of convergence, which allows the terminal time to be prescribed independently of initial conditions and parameters [18,19,26,41,44,47].

For partial differential equations (PDEs), these time–enhanced concepts have become an attractive research area since PDEs describe many complex systems (thermal and fluid dynamics, chemical reactions, batteries, etc). Meeting time constraints while realizing the well–known separation principle are central issues that can be achieved with finite–, fixed– and prescribed–time convergence methods.

For hyperbolic PDE systems, finite–time stabilization and estimation have been studied in, e.g., [3,8,34], and [10,11]. The latter two contributions deal with the problem of finite-time output regulation for hyperbolic systems by using the backstepping approach and by invoking the finite-time convergent observer design introduced in [13] (which is the first continuous, prescribed-time observer design for LTI systems, and exploits the infinite-dimensionality of an auxiliary, delayed state estimate to generate a determined algebraic system from which the exact state is reconstructed).

For linear parabolic PDEs, null controllability/finite–time stabilization [9,15,16,42] have been achieved by using the backstepping approach with time–varying kernels. For abstract infinite–dimensional systems, [36] proposes homogeneity arguments to achieve enhanced stabilization.

Time–delay systems are ubiquitous in engineering, where delays appear on the righthand side of the model dynamics [33], or in the inputs, outputs and network graph communication topologies as in [46]. For the latter type, exponential stabilization of LTI systems with input delay is performed based on predictor feedback. In [27,28], under an ODE-PDE cascade setting, the classical predictor is related to the backstepping approach. The backstepping PDE framework for time–delay systems has been extended to deal with

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delay–adaptive control, delay compensation and estimation problems, nonlinear systems with input delay, time–varying delays, and distributed input delays [4,6,7]. In [38], state estimation with time–varying measurement delays is studied. Other contributions deal with observer–based input/output delays for linear systems, e.g., [48] and references therein.

Nevertheless, results for finite–, fixed– and prescribed–time concepts for time–delay systems remain sparse. Some pioneering contributions for time–enhanced stability of time–delay systems are [23], [31]; more recent results are [12], [1], [30] and [14], the latter dealing with prescribed–time predictor *control* for input-delayed LTI systems. Yet numerous applications call for enhanced *estimator* convergence results by using delayed measurements: in meteorological and social systems, state estimation over a finite horizon is highly valuable for hurricane and election forecasting, where measurements are delayed. In such problems, control actions may only take place at the end of the finite horizon, so exponentially convergent estimation is insufficient.

In this work, we combine the ideas of [18], [27] and [14] to handle the problem of estimation of LTI systems in the presence of delayed measurements. The proposed observer is made up of time–varying output gains whose design relies on suitable polynomial–based functions blowing–up in a prescribed time. For the prescribed–time convergence analysis and observer design, we use compact formulations using generalized Laguerre polynomials, elementary symmetric polynomials, and polynomial–based Vandermonde matrices. We build upon the cascade ODE–PDE setting while taking into account the infinite dimensionality of the sensor. We perform a series of changes of variables and design a suitable target *observer error* system which exhibits the desired convergence property. Due to the bounded invertibility of the related transformations, we guarantee the convergence of the estimated states to the actual states, within the prescribed terminal time, irrespective of initial conditions. A unique feature of our design is that the convergence we achieve is highly smooth: successive derivatives of the estimate errors also converge to zero, which can be useful in applications (cf. Remark 3 for details). In contrast to [14], the observer gains we derive necessarily differ in structure from the ones for the dual problem (control with input delay). Considering that we specialize to LTI systems, this is a surprising feature caused by the time-varying nature of our approach. Moreover, the approach herein omits the use of a Volterra integral transformation (as was used in [14]); as a consequence, we need not impose the restriction that the terminal time exceed the delay time, as was required in [14].

This paper is organized as follows. In Section 2, we introduce some preliminaries and the problem statement. In Section 3, we recall some relevant formulas that are instrumental in our design, and we present the main result: the prescribed-time observer. In Section 4, we discuss the transformations employed in our design/analysis and derive the desired prescribed-time convergence property.

Notation We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. For nonzero integers m and n, let $0_{m\times n}$ be the $(m,n)-$matrix with zero entries and let I_m be the identity matrix of dimension m. For $1 \leq p \leq +\infty$, the induced norm of an $(n,n)-$matrix M is defined as $\|M\|_p = \sup\{\|Mx\|_p : x \in \mathbb{R}^n \text{ with } \|x\|_p = 1\}$, where $\|x\|_p$ denotes the $p-$vector norm of x. Thus, when $p = 1$ (respectively $p = +\infty$), $\|M\|_p$ corresponds to the maximum absolute row (respectively column) sum norm. We recall the relevant case that $\|M\|_2 \leq \sqrt{\|M\|_1\|M\|_\infty}$. We denote by $L_m^{(\alpha)}(\cdot)$ the generalized Laguerre polynomial, by $\binom{n}{k} := \frac{n!}{k!(n-k)!}$, $k = 1,\dots,n$ the binomial coefficients, and by $\sigma_n(\cdot)$ the elementary symmetric polynomials. The set of all functions $g : [0,h] \to \mathbb{R}^n$ satisfying $\int_0^h g(x)^2 dx < +\infty$ is denoted by $L^2((0,h),\mathbb{R}^n)$.

2 Problem statement

We consider the following general LTI plant:

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t), \\ Y(t) &= CX(t-h), \end{aligned} \tag{1}$$

where $t \geq t_0 \geq 0$, $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, with initial condition $X(\theta) = X_0(\theta)$ for all $\theta \in [t_0 - h, t_0]$, where t_0 is the initialization time and $U(t) \in \mathbb{R}^m$ is the input signal. We specialize to the single output case, i.e., $p = 1$.

We recall that the observability matrix is given by

$$\mathscr{O} = \begin{pmatrix} C^\top & A^\top C^\top & \dots & \left(A^{n-1}\right)^\top C^\top \end{pmatrix}^\top, \tag{2}$$

where we assume that (A,C) is an observable pair, i.e., that rank $\mathscr{O} = n$. We perform the coordinate transformation

$$X_{ob}(t) = Q^{-1}X(t), \tag{3}$$

where $Q = (\tilde{q}, A\tilde{q}, \dots, A^{n-1}\tilde{q})$, with $\tilde{q}$ being the n-th column of $\mathscr{O}^{-1}$. An application of (3) to (1) transforms it into the following system in the observer form [2]:

$$\begin{aligned} \dot{X}_{ob}(t) &= A_{ob}X_{ob}(t) + B_{ob}U(t), \\ Y_{ob}(t) &= C_{ob}X_{ob}(t-h), \end{aligned} \tag{4}$$

where $A_{ob} = Q^{-1}AQ$, $C_{ob} = CQ$, and $B_{ob} = Q^{-1}B$. More precisely,

$$A_{ob} = \begin{pmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ \vdots & \ddots & & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}, \quad C_{ob} = \begin{pmatrix} 0 & \dots & 1 \end{pmatrix}. \tag{5}$$

Our goal is to design an observer for the system (1) which converges within the prescribed terminal time. To this end, the methodology developed in this paper relies on representing the sensor delay as a linear transport PDE, and builds upon the cascade ODE-PDE setting of [27, Chapter 3]. We henceforth represent system (1) as

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t), \\ u_t(t,x) &= u_x(t,x), \\ u(t,h) &= CX(t), \\ Y(t) &= u(t,0), \end{aligned} \tag{6}$$

where $t \geq t_0 \geq 0$, $x \in [0,h]$, and $u(t,\cdot)$ is the transport PDE state at time t whose solution is $u(t,x) = CX(t-h+x)$ and, in particular, $u(t,0) = Y(t)$.

Under the change of coordinates (3), system (4) rewrites as

$$\begin{aligned} \dot{X}_{ob}(t) &= A_{ob}X_{ob}(t) + B_{ob}U(t), \\ u_t(t,x) &= u_x(t,x), \\ u(t,h) &= C_{ob}X_{ob}(t), \\ Y_{ob}(t) &= u(t,0). \end{aligned} \tag{7}$$

The above observer form reformulation will be instrumental in performing several other transformations to achieve a full delay-compensated observer design, which we discuss next.

3 Delay-compensated prescribed-time observer

We design an observer that converges in the prescribed terminal time $t_0 + h + T$, where $T \in \mathbb{R}_+$ is a priori fixed, h is the known sensor delay, and t_0 is the initialization time.

To achieve prescribed-time convergence, we use time-varying observer gains constructed with suitable blow-up functions. The approach builds upon the recent ones for prescribed-time stabilization and estimation of ODEs [18,41], and [14], which proposes predictor feedback for prescribed-time input delay compensation of LTI systems. In the latter work and in [15,42], we use time-varying damping.

Consider the following blow-up function (as in [15]):

$$c(t) = \frac{\bar{c}_0^2 T^2}{(T+t_0-t)^2}, \quad c(t_0) = \bar{c}_0^2. \tag{8}$$

with $\bar{c}_0 > 0$. As discussed in [14] and [15], the chosen power degree of the blow-up function (8) is crucial for the convergence analysis; moreover, it allows us to obtain explicit and compact closed-form gain formulas throughout the design/-analysis.

3.1 Generalized polynomial-based Vandermonde matrix and time-varying gains

We consider the following time-varying matrices $P(\cdot)$ and $M(\cdot)$ in companion canonical/observer form, respectively:

$$P(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-1}(t) \end{pmatrix}, \tag{9}$$

$$M(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & -\mu_0(t) \\ 1 & 0 & 0 & \cdots & -\mu_1(t) \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -\mu_{n-1}(t) \end{pmatrix}, \tag{10}$$

where $p_{i-1}(\cdot)$, $\mu_{i-1}(\cdot)$, $i = 1 \ldots n$, are time-varying functions that will be characterized below. There exists a time-varying matrix $R(\cdot)$ such that

$$R(t)M(t) = P(t)R(t) - \dot{R}(t). \tag{11}$$

Indeed, R is given as follows [40]:

$$R(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \pi_{n-1,n-1}(t) \\ \vdots & \iddots & & \vdots & \vdots \\ 1 & \pi_{1,1}(t) & \cdots & \pi_{n-2,1}(t) & \pi_{n-1,1}(t) \end{pmatrix}, \tag{12}$$

where the time-varying coefficients $\pi_{i,k}(\cdot)$ are characterized by the following recurrence relation

$$\begin{aligned} &\pi_{i,k} \\ &= \begin{cases} -p_{n-1}, & 1 \leq k = i \leq n, \\ \pi_{i-1,k-1} - \dot{\pi}_{i-1,k}, & 1 < k < i \leq n, \\ -p_{n-i} - \sum_{j=0}^{i-2} p_{n-1-j}\pi_{i-1,j+1} - \dot{\pi}_{i-1,1}, & k = 1 < i \leq n. \end{cases} \end{aligned} \tag{13}$$

Hence, from (11) along with (9), (10) and (12), the time-varying coefficients of M can be explicitly characterized as follows [45, Section 2], [32]:

$$\mu_k(t) = \sum_{i=k}^{n-1} (-1)^{i-k} \binom{i}{k} p_i^{(i-k)}(t), \quad k = 0, \ldots, n-1. \tag{14}$$

Using (14), the inverse of (12) is given by

$$R^{-1}(t) = \begin{pmatrix} \tilde{\pi}_{1,1}(t) & \tilde{\pi}_{1,2}(t) & \cdots & \tilde{\pi}_{1,n-1}(t) & 1 \\ \vdots & \vdots & & \iddots & \vdots \\ \tilde{\pi}_{n-1,1}(t) & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{15}$$

with

$$\tilde{\pi}_{i,k} = \begin{cases} \mu_{n-1}, & i+k = n, \\ \tilde{\pi}_{i+1,k-1} + \dot{\tilde{\pi}}_{i+1,k}, & 1 < k < i < n, \\ \mu_i + \dot{\tilde{\pi}}_{i+1,k}, & k = 1 \leq i < n. \end{cases} \tag{16}$$

Next, consider the *Generalized Vandermonde* matrix given by

$$V(t) = \begin{pmatrix} 1 & \cdots & 1 \\ (\delta^0(-r_1 c))(t) & \cdots & (\delta^0(-r_n c))(t) \\ \vdots & \ddots & \vdots \\ (\delta^{n-2}(-r_1 c))(t) & \cdots & (\delta^{n-2}(-r_n c))(t) \end{pmatrix}, \tag{17}$$

where the operator δ is defined recursively by [22]

$$\begin{aligned}(\delta^0(-r_ic))(t) &= -r_ic(t),\\ (\delta^n(-r_ic))(t) &= (\delta(\delta^{n-1}(-r_ic)))(t)\\ &= (-r_ic(t))(\delta^{n-1}(-r_ic))(t)\\ &\quad + \frac{d}{dt}(\delta^{n-1}(-r_ic))(t), \end{aligned} \tag{18}$$

where the function $c(t)$ is defined according to (8), and where the coefficients r_i, $i=1,\dots,n$, are distinct positive real numbers (i.e., $r_i>0, r_i\neq r_j$ for $i\neq j$, for $i=1,\dots n$).

We utilize the following relation from [49]:

$$V(t)D(t) = P(t)V(t) - \dot{V}(t), \tag{19}$$

where $D(t) = \texttt{diag}\Big(-r_1c(t),\cdots,-r_nc(t)\Big)$.

Some characterizations of V have been obtained in [14] and are given in terms of the Bell and generalized Laguerre polynomials [1] which, in turn, lead to characterizations of $P(\cdot)$ in terms of the elementary symmetric polynomials and the Lah numbers.

We recall the following important result.

Proposition 1 ([14, Section 2])**.** *Let $V(t)$ be given by* (17) *with δ defined in* (18) *and let $D(t)$ and $P(t)$ satisfy* (19)*. Then, we have the relation*

$$(\delta^n(-r_ic))(t) = \frac{-r_ic(t)(\sqrt{c(t)})^n}{(\bar{c}_0T)^n} n! L_n^{(1)}\left(r_i\bar{c}_0T\sqrt{c(t)}\right), \tag{20}$$

where $L_n^{(1)}(\cdot)$ is a generalized Laguerre polynomial, which is given by

$$L_n^{(1)}\left(r_i\bar{c}_0T\sqrt{c(t)}\right) = \sum_{l=0}^{n}\binom{n+1}{n-l}\frac{(-1)^l}{l!}\left(r_i\bar{c}_0T\sqrt{c(t)}\right)^l \tag{21}$$

Notice that V in (17) is made up of terms involving the generalized Laguerre polynomials, yielding a special polynomial-based Vandermonde matrix from which we have the following result.

Proposition 2 ([14, Section 2])**.** *Under the assumptions of Proposition 1, the entries $p_0(t)$, $p_j(t)$ ($j=1,\dots n-1$) in* (9) *are explicitly characterized by*

$$p_0(t) = \sigma_n(r_1,..,r_n)c^n(t), \tag{22}$$

and for $j=1,\dots,n-1$,

$$\begin{aligned}p_j(t) =& \frac{(\sqrt{c(t)})^{n-j}}{(\bar{c}_0T)^{n-j}}\sum_{k=j}^{n}(-1)^{k-j}\sigma_{n-k}(r_1,...,r_n)\\ &\times\binom{k-1}{j-1}\frac{k!}{j!}\left(\bar{c}_0T\sqrt{c(t)}\right)^{n-k},\end{aligned} \tag{23}$$

[1] see [16,15] for related studies but in the context of reaction-diffusion PDEs.

where $c(t)$ is given in (8)*, $\sigma_{n-k}(\cdot)$ are the elementary symmetric polynomials given by*

$$\sigma_0(r_1,...,r_n) = 1, \tag{24}$$

$$\sigma_k(r_1,...,r_n) = \sum_{1\le i_1\le i_2\le\dots i_k\le n} r_{i_1}r_{i_2}\dots r_{i_k}, \tag{25}$$

$$\sigma_n(r_1,...,r_n) = \prod_{i=1}^{n} r_i, \tag{26}$$

and $\sigma_k(r_1,...,r_n)=0$, for $k>n$.

3.2 Main result: Sensor delay-compensated prescribed-time observer

The proposed prescribed-time observer is given in the following theorem. It is an infinite-dimensional, full order observer as it estimates both the plant state X and the sensor state Y (i.e. $Y(t)$ is introduced in the estimation error).

Theorem 1. *Let Q be as in* (3)*, a_{i-1} be as in* (5)*, $c(t-h)$ be defined according to* (8)*, $\mu_{i-1}(t-h)$ be given by* (14)*,* (22)-(23) *(for $i=1,\dots,n$), and let $T>0$ be fixed. For the system* (6)*, the observer*

$$\begin{aligned}\dot{\hat{X}}(t) &= A\hat{X}(t) + BU(t) + Qe^{Q^{-1}AQh}\gamma(t-h)\left(Y(t)-\hat{u}(t,0)\right),\\ \hat{u}_t(t,x) &= \hat{u}_x(t,x) + CQe^{Q^{-1}AQx}\gamma(t-h)\left(Y(t)-\hat{u}(t,0)\right),\\ \hat{u}(t,h) &= C\hat{X}(t),\end{aligned} \tag{27}$$

with time-varying output gain $\gamma(t-h) = (\gamma_1(t-h),\dots,\gamma_n(t-h))^\top$ given by

$$\gamma_i(t-h) = -a_{i-1} + \mu_{i-1}(t-h), \quad for \quad i=1,\dots,n, \tag{28}$$

guarantees that $(\hat{X},\hat{u})$ converges to (X,u) within the prescribed terminal time t_0+T+h. More precisely, there exist a positive constant $\bar{\zeta}_X$ and a positive polynomial $\mathscr{P}(\sqrt{c(t-h)})$ such that, for any initial conditions $X(t_0),\hat{X}(t_0)\in\mathbb{R}^n$ and for any initial measurements $u(t_0,x) = CX(t_0-h+x), \hat{u}(t_0,x) = C\hat{X}(t_0-h+x)$, where $x\in[0,h]$, we have that the quantity $\|X(t)-\hat{X}(t)\|^2 + \|u(t,\cdot)-\hat{u}(t,\cdot)\|^2_{L^2}$ remains bounded for $t\in[t_0,t_0+h]$, whereas it satisfies the bound

$$\begin{aligned}&\|X(t)-\hat{X}(t)\|^2 + \|u(t,\cdot)-\hat{u}(t,\cdot)\|^2_{L^2}\\ &\quad\le \bar{\zeta}_X\mathscr{P}(\sqrt{c(t-h)})\exp\left(-r_{min}\bar{c}_0T\sqrt{c(t-h)}\right)\\ &\qquad\times\|X(t_0)-\hat{X}(t_0)\|^2\end{aligned} \tag{29}$$

for $t\in[t_0+h,t_0+T+h)$, with $r_{min} = \min_{i=1,\dots n}\{r_i\}$, and in particular,

$$\|X(t)-\hat{X}(t)\|^2 + \|u(t,\cdot)-\hat{u}(t,\cdot)\|^2_{L^2}\to 0 \tag{30}$$

as $t\to t_0+T+h$.

The rest of this work aims to establish Theorem 1.

Remark 1. The observer we present has several distinctive features: the observer gains (28) are explicit; the terminal time $t=t_0+T+h$ can be a priori prescribed regardless of

size of initial conditions; the convergence of the estimate errors is highly smooth (cf. Remark 3 for details); and the observer estimates both the plant states *and* the sensor state.

4 Intermediate transformations and prescribed-time convergence analysis

The proof of Theorem 1 requires various intermediate steps, consisting of transformations (suitable change of coordinates) along with the study of the prescribed-time convergence property of an *observer error system*, to be introduced in the sequel. The key idea is to design a target system (which we refer to as the *Target observer error system*) which exhibits the desired prescribed-time convergence property. To this end, we use the various changes of coordinates that are summarized in Figure 1.

We start by considering the following observer for (7) with time-varying output gain,

$$\begin{aligned}\dot{\hat{X}}_{ob}(t) &= A_{ob}\hat{X}_{ob}(t) + B_{ob}U(t)\\ &\quad + e^{A_{ob}h}\gamma(t-h)\left(Y_{ob}(t) - \hat{u}(t,0)\right),\\ \hat{u}_t(t,x) &= \hat{u}_x(t,x) + C_{ob}e^{A_{ob}x}\gamma(t-h)\left(Y_{ob}(t) - \hat{u}(t,0)\right),\\ \hat{u}(t,h) &= C_{ob}\hat{X}(t).\end{aligned} \tag{31}$$

Remark 2. Classical predictor-based techniques, which rely on the explicit solution of the linear ODE plant and a virtual non-delayed measurement, can be used to design the following *reduced*-order observer for (7):

$$\begin{aligned}\dot{\hat{X}}_{ob}(t) &= A_{ob}\hat{X}_{ob}(t) + B_{ob}U(t) + e^{A_{ob}h}\gamma(t-h)\Big(Y_{ob}(t)\\ &\quad + C_{ob}\int_{t-h}^{t} e^{A_{ob}(t-h-\theta)h}B_{ob}U(\theta)d\theta - C_{ob}e^{-A_{ob}h}\hat{X}_{ob}(t)\Big),\end{aligned} \tag{32}$$

where the time-varying observer gain γ, given by (28) and designed below, ensures prescribed-time convergence.

The reduced-order observer (32) does not estimate the sensor state, whereas (31) does—the convergence (30) ensures finite-time estimation performance for the sensor state. Our approach offers a few advantages over classical predictor-based techniques: backstepping-based predictor designs extend to plants with more complex sensor dynamics that are governed by other PDEs (as in [27, Chapter 17]); they can also treat nonlinear plants, and those with sensor/input state-dependent delays. As such, we opt to build upon the cascade ODE-PDE setting by presenting a detailed design method that accounts for the hyperbolic PDE dynamics.

We aim to show that (31) guarantees convergence of ($\hat{X}_{ob}$, $\hat{u}$) to (X_{ob}, u) in (7) within the prescribed terminal time $t_0 + T + h$. To this end, we use the error variables

$$\begin{aligned}\tilde{X}_{ob} &= X_{ob} - \hat{X}_{ob},\\ \tilde{u}(t,x) &= u(t,x) - \hat{u}(t,x),\end{aligned} \tag{33}$$

from which we obtain the following *Observer error system*:

$$\begin{aligned}\dot{\tilde{X}}_{ob}(t) &= A_{ob}\tilde{X}_{ob}(t) - e^{A_{ob}h}\gamma(t-h)\tilde{u}(t,0),\\ \tilde{u}_t(t,x) &= \tilde{u}_x(t,x) - C_{ob}e^{A_{ob}x}\gamma(t-h)\tilde{u}(t,0),\\ \tilde{u}(t,h) &= C_{ob}\tilde{X}_{ob}(t).\end{aligned} \tag{34}$$

Consider the change of coordinates

$$\tilde{Z}(t) = e^{-A_{ob}h}\tilde{X}_{ob}(t), \tag{35}$$

which transforms (34) into

$$\begin{aligned}\dot{\tilde{Z}}(t) &= A_{ob}\tilde{Z}(t) - \gamma(t-h)\tilde{u}(t,0),\\ \tilde{u}_t(t,x) &= \tilde{u}_x(t,x) - C_{ob}e^{A_{ob}x}\gamma(t-h)\tilde{u}(t,0),\\ \tilde{u}(t,h) &= C_{ob}e^{A_{ob}h}\tilde{Z}(t),\end{aligned} \tag{36}$$

where we've used the fact that $e^{-A_{ob}h}A_{ob}e^{A_{ob}h} = A_{ob}$. Next, consider the transformation

$$\tilde{w}(t,x) = \tilde{u}(t,x) - C_{ob}e^{A_{ob}x}\tilde{Z}(t) \tag{37}$$

from [27, Equ. 3.14], which maps (36) into the following system:

Target observer error system in "Observer" form

$$\begin{aligned}\dot{\tilde{Z}}(t) &= M(t-h)\tilde{Z}(t) - \gamma(t-h)\tilde{w}(t,0),\\ \tilde{w}_t(t,x) &= \tilde{w}_x(t,x),\\ \tilde{w}(t,h) &= 0,\end{aligned} \tag{38}$$

where we choose $M(t-h)$ as in (10). Comparing the ODE dynamics of (36) with (38) and using (37) at $x = 0$, we have

$$\gamma(t-h)C_{ob} = A_{ob} - M(t-h). \tag{39}$$

Combining (39) with (5) and (10) yields

$$\gamma_i(t-h) = -a_{i-1} + \mu_{i-1}(t-h), \quad \text{for} \quad i = 1,\dots,n. \tag{40}$$

On the other hand, we compute the time derivative of (37) to obtain

$$\tilde{w}_t(t,x) = \tilde{u}_x(t,x) - C_{ob}e^{A_{ob}x}A_{ob}\tilde{Z}(t); \tag{41}$$

since a matrix commutes with its corresponding matrix exponential, it follows from (37) that $\tilde{w}_t - \tilde{w}_x = 0$. Since $\tilde{w}(t,h) = 0$, it also follows from (37) that $\tilde{u}(t,h) = C_{ob}e^{A_{ob}h}\tilde{Z}(t)$, realizing the transformation between $\tilde{u}$ and $\tilde{w}$.

4.1 Prescribed-time convergence analysis of the target observer error system

The target observer error system in "Observer" form (38) has been carefully selected in order to meet the required convergence property: indeed, the next lemma states that the solutions of (38) converge to zero within the prescribed terminal time $t_0 + h + T$.

Let us introduce the "𝔈xponential"-like function

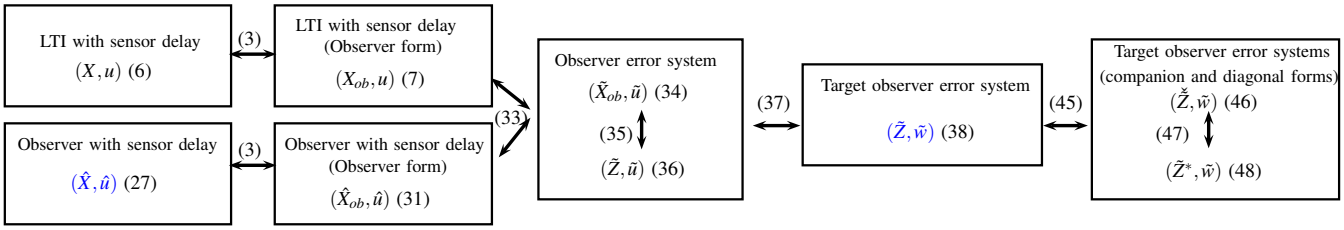

Figure 1. Overview of the four changes of coordinates and the resulting cascade ODE-PDE systems. In blue, we highlight the proposed prescribed-time observer (27), and we indicate the chosen target observer error system in the "Observer" form (38), whose suitable choice is the key in our approach.

$$\mathfrak{C}_r(t-h)=\exp\left(r\bar{c}_0T\sqrt{c(t-h)}\right), \tag{42}$$

for some $r\in\mathbb{R}$, and the "$\mathfrak{D}$iagonal" matrix

$$\mathfrak{D}(t-h)=\mathtt{diag}\Big(\mathfrak{C}_{-r_1}(t-h),\dots,\mathfrak{C}_{-r_n}(t-h)\Big). \tag{43}$$

Lemma 1. *Let a_{i-1} be as in (5), $\mu_{i-1}(t-h)$ as in (14), (22)-(23) (for $i=1,\dots,n$), $c(t-h)$ as in (8), and $M(t-h)$ as in (10). Then for any $\tilde{Z}(t_0)\in\mathbb{R}^n$, there exist $\zeta_{\tilde{Z}}>0$ and a positive polynomial $\mathscr{P}(\sqrt{c(t-h)})$ such that the solution $\tilde{Z}$ of (38) remains bounded for all $[t_0,t_0+h]$, whereas it satisfies the bound*

$$\|\tilde{Z}(t)\|^2\le 2\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|\tilde{Z}(t_0)\|^2 \tag{44}$$

for all $t\in[t_0+h,t_0+h+T)$, with $r_{\min}=\min_{i=1,\dots,n}\{r_i\}$. In particular, $\|\tilde{Z}(t)\|^2\to 0,\quad as\quad t\to t_0+h+T$.

Proof. Key to achieving prescribed-time stability of the target system is mapping to a system with the time-varying diagonal operator (43), generating "dynamic eigenvalues" (42). Consider the following change of variables:

$$\breve{\tilde{Z}}(t)=R(t-h)\tilde{Z}(t), \tag{45}$$

where $R(t-h)$ is given in (12) along with (16). Then, from (11), (14), the target observer error system (38) is transformed into the following *Target observer in "Companion" canonical form*

$$\begin{aligned}&\dot{\breve{\tilde{Z}}}(t)=P(t-h)\breve{\tilde{Z}}(t)-R(t-h)\gamma(t-h)\tilde{w}(t,0),\\&\tilde{w}_t(t,x)=\tilde{w}_x(t,x),\\&\tilde{w}(t,h)=0,\end{aligned} \tag{46}$$

where $P(t-h)$ is given in (9) and characterized by means of (22)-(23).

Consider next the following change of variables:

$$\tilde{Z}^*(t)=V^{-1}(t-h)\breve{\tilde{Z}}(t), \tag{47}$$

with $V(t-h)$ given in (17), (20). Its inverse V^{-1} exists and can be explicitly characterized (see [14]). Applying (47) to (46) yields the *Target observer error system in "Diagonal" form*

$$\begin{aligned}&\dot{\tilde{Z}}^*(t)=D(t-h)\tilde{Z}^*(t)-V^{-1}(t-h)R(t-h)\gamma(t-h)\tilde{w}(t,0),\\&\tilde{w}_t(t,x)=\tilde{w}_x(t,x),\\&\tilde{w}(t,h)=0,\end{aligned} \tag{48}$$

where $D(t-h)$, $M(t-h)$ and $V(t-h)$ satisfy (19). Specifically, $D(t-h)=\mathtt{diag}\Big(-r_1c(t-h),\dots,-r_nc(t-h)\Big)$. The solution of the $\tilde{Z}^*$- dynamics of (48) is given as follows:

$$\begin{aligned}\tilde{Z}^*(t)=&\ \tilde{\Phi}_D(t-h,t_0-h)\tilde{Z}^*(t_0)\\&-\int_{t_0}^t\tilde{\Phi}_D(t-h,\tau-h)V^{-1}(\tau-h)R(\tau-h)\gamma(\tau-h)\tilde{w}(\tau,0)d\tau,\end{aligned} \tag{49}$$

where $\tilde{\Phi}_D(t-h,t_0-h)=\exp\left(\int_{t_0}^t D(s-h)ds\right)=\exp\left(\int_{t_0-h}^{t-h}D(s)ds\right)$ is the state transition matrix of the underlying linear (diagonal-form) time-varying system ($\tilde{Z}^*$-dynamics). Using $\tilde{\Phi}_D(t-h,t_0-h)=\mathfrak{D}(t-h)\mathfrak{D}^{-1}(t_0-h)$ (by virtue of (42)-(43)), we get

$$\begin{aligned}\tilde{Z}^*(t)=&\ \mathfrak{D}(t-h)\mathfrak{D}^{-1}(t_0-h)\tilde{Z}^*(t_0)\\&-\int_{t_0}^t\mathfrak{D}(t-h)\mathfrak{D}^{-1}(\tau-h)V^{-1}(\tau-h)R(\tau-h)\gamma(\tau-h)\tilde{w}(\tau,0)d\tau.\end{aligned} \tag{50}$$

Then, by inverting (47), the solution of the $\breve{\tilde{Z}}$- dynamics of (38) is given as follows, for all $t\in[t_0,t_0+h+T)$:

$$\begin{aligned}\breve{\tilde{Z}}(t)=&\tilde{\Phi}_P(t-h,t_0-h)\breve{\tilde{Z}}(t_0)\\&-\int_{t_0}^t\tilde{\Phi}_P(t-h,\tau-h)R(\tau-h)\gamma(\tau-h)\tilde{w}(\tau,0)d\tau,\end{aligned} \tag{51}$$

where

$$\tilde{\Phi}_P(\xi,\eta)=V(\xi)\mathfrak{D}(\xi)\mathfrak{D}^{-1}(\eta)V^{-1}(\eta). \tag{52}$$

In addition, by inverting (45), the solution of the $\tilde{Z}$- dynamics of (46) is given as follows, for all $t\in[t_0,t_0+h+T)$:

$$\begin{aligned}\tilde{Z}(t)=&\tilde{\Phi}_M(t-h,t_0-h)\tilde{Z}(t_0)\\&-\int_{t_0}^t\tilde{\Phi}_M(t-h,\tau-h)\gamma(\tau-h)\tilde{w}(\tau,0)d\tau,\end{aligned} \tag{53}$$

where

$$\tilde{\Phi}_M(t-h,t_0-h)=\mathfrak{V}(t-h)\mathfrak{V}^{-1}(t_0-h), \tag{54}$$

$$\mathfrak{V}(t-h)=R^{-1}(t-h)V(t-h)\mathfrak{D}(t-h). \tag{55}$$

Furthermore, we denote by $\gamma^{\tilde{w}}(\tau-h)=\gamma(\tau-h)\tilde{w}(\tau,0)$, and by virtue of (28), we obtain

$$\gamma_i^{\tilde{w}}(\tau-h)=-a_{i-1}\tilde{w}(\tau,0)+\mu_{i-1}(\tau-h)\tilde{w}(\tau,0), \tag{56}$$

for $i=1,\dots,n$. Therefore, we have that the last term on the right-hand side of (53) is given compactly as

$$\phi_M^{\gamma}(t,t_0)=\begin{pmatrix}\int_{t_0}^{t}\sum_{j=1}^{n}\tilde{\Phi}_{M1j}(t-h,\tau-h)\gamma_j^{\tilde{w}}(\tau-h)d\tau\\ \vdots\\ \int_{t_0}^{t}\sum_{j=1}^{n}\tilde{\Phi}_{Mnj}(t-h,\tau-h)\gamma_j^{\tilde{w}}(\tau-h)d\tau\end{pmatrix}. \tag{57}$$

Now, using (16), (20), (42) and (43), the entries of $\mathfrak{V}(t-h)$ are, for $i=1,\dots,n-1$,

$$\mathfrak{V}_{ij}(t-h)=\Bigg(\sum_{k=1}^{n-i-1}\tilde{\pi}_{i,k+1}(t-h)(\delta^{k-1}(-r_jc))(t-h)+\tilde{\pi}_{i,1}(t-h)+(\delta^{n-i-1}(-r_jc))(t-h)\Bigg)\mathfrak{C}_{-r_j}(t-h), \tag{58}$$

and $\mathfrak{V}_{nj}(t-h)=\mathfrak{C}_{-r_j}(t-h)$. On one hand, using $\|\mathfrak{V}(t-h)\|_2\leq\sqrt{\|\mathfrak{V}(t-h)\|_1\|\mathfrak{V}(t-h)\|_\infty}$, we obtain

$$\begin{aligned}\|\mathfrak{V}(t-h)\|_2^2&\leq\|\mathfrak{V}(t-h)\|_1\|\mathfrak{V}(t-h)\|_\infty\\&\leq\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h),\end{aligned} \tag{59}$$

for $t\in[t_0,t_0+h+T)$, where $\mathscr{P}(\sqrt{c(t-h)})$ is a positive polynomial function of $\sqrt{c(t-h)}$ which can be obtained by computing $\|\mathfrak{V}(t-h)\|_1$ and $\|\mathfrak{V}(t-h)\|_\infty$ from (58) (from which it is clear that both $\tilde{\pi}_{i,j}(\cdot)$ and $(\delta^{k-1}(-r_jc))(\cdot)$ are polynomial functions of $\sqrt{c(t-h)}$). On the other hand, $\|\mathfrak{V}^{-1}(t_0-h)\|_2^2\leq\|\mathfrak{V}^{-1}(t_0-h)\|_1\|\mathfrak{V}^{-1}(t_0-h)\|_\infty,\leq\zeta_{\tilde{Z}}$, where $\zeta_{\tilde{Z}}$ is some positive constant. Therefore, we obtain

$$\|\tilde{\Phi}_M(t-h,t_0-h)\|^2\leq\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h), \tag{60}$$

for all $t\in[t_0,t_0+h+T)$. We exploit the cascade nature of the system (38) along with the fact that the $\tilde{w}$–transport PDE is fixed-time stable. Indeed, by virtue of the method of characteristics, $\tilde{w}(t,x)=\tilde{w}_0(x+t)$ for $t\leq t_0+h-x$ and $\tilde{w}(t,x)=0$ for $t\geq t_0+h-x$. Thus, we have an analytical expression for $\tilde{w}(t,0)$; that is $\tilde{w}(t,0)=\tilde{w}_0(t)$ for $t\leq t_0+h$. In addition, after $t=t_0+h$, one has $\tilde{w}(t,0)\equiv 0$.

Combining (53), (57) and (60) yields

$$\|\tilde{Z}(t)\|^2\leq 2\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|\tilde{Z}(t_0)\|^2+\tilde{b}(t,t_0), \tag{61}$$

for all $t\in[t_0,t_0+h+T)$, where we have used $\|\phi_M^{\gamma}(t,t_0)\|^2\leq n\|\phi_M^{\gamma}(t,t_0)\|_\infty^2\leq\frac{n}{2}\tilde{b}(t,t_0)$, with $\tilde{b}(t,t_0)$ given by

$$\tilde{b}(t,t_0)=2n\max_{1\leq k\leq n}\Big\{\int_{t_0}^{t}\Big|\sum_{j=1}^{n}\tilde{\Phi}_{Mkj}(t-h,\tau-h)\gamma_j^{\tilde{w}}(\tau-h)\Big|d\tau\Big\}^2 \tag{62}$$

for all $t\in[t_0,t_0+h)$, and $\tilde{b}(t,t_0)=0$ for all $t\in[t_0+h,t_0+h+T)$; hence, $\tilde{b}(t,t_0)<+\infty$ for all $t\in[t_0,t_0+h)$. This holds true irrespective of whether $T<h$ or $T\geq h$, since the finite escape time of $\tilde{\Phi}_M(t-h,\tau-h)\gamma(\tau-h)$ (see (53)) only occurs after $t>t_0+h$, namely as $t\to t_0+h+T$. Nevertheless, since $\tilde{w}\equiv 0$ for $t\geq t_0+h$, we have that $\tilde{b}\equiv 0$, ensuring boundedness of $\tilde{b}$. We conclude that $\|\tilde{Z}(t)\|^2<+\infty$, implying $\|\tilde{Z}(t_0+h)\|<+\infty$, for all $t\in[t_0,t_0+h]$. Next, from (61) and the definition of $\tilde{b}(t,t_0)$, it follows that for $t\in[t_0+h,t_0+h+T)$,

$$\|\tilde{Z}(t)\|^2\leq 2\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|\tilde{Z}(t_0)\|^2. \tag{63}$$

Notice that $\mathscr{P}(\sqrt{c(t-h)})$ diverges as $t\to t_0+h+T$; nevertheless, the decay of $\mathfrak{C}_{-r_{\min}}(t-h)$ dominates the growth of $\mathscr{P}(\sqrt{c(t-h)})$ uniformly. Thus, $\|\tilde{Z}(t)\|^2\to 0$ as $t\to t_0+h+T$. This concludes the proof. ■

Remark 3. The diagonal structure of (48) allows us to successively differentiate (49) $k\in\mathbb{N}$ times while retaining the prescribed-time stability result (63), since $\mathfrak{C}_{-r_{\min}}(t-h)$ is a monotonically decreasing smooth "bump-like" function and uniformly dominates the polynomial time-varying growth in (59) and (60). This is a desirable feature for, e.g., Euler-Lagrange systems, where generating estimates of higher order derivatives of the state is beneficial. E.g., consider (46) for $n=2$—an observer for the double integrator with sensor delay. To ensure the integrity of docking vessels and their payloads, convergence of estimate errors describing acceleration and jerk of the vessel—and not just position and velocity—are advantageous for control/monitoring tasks targeting "jerk-free" (or highly smooth) docking.

We can now proceed to establish Theorem 1.

5 Proof of Theorem 1

We establish the convergence property (30) as a consequence of Lemma 1 and the bounded invertibility of all underlying transformations on the interval $[t_0,t_0+h+T)$ (see Figure 1).

Proof. The inverse transformation of (37) is given by

$$\tilde{u}(t,x)=\tilde{w}(t,x)+C_{ob}e^{A_{ob}x}\tilde{Z}(t), \tag{64}$$

from which, after applying the triangle and Young's inequalities, the following estimate holds for all $t\in[t_0,t_0+h+T)$:

$$\|\tilde{u}(t,\cdot)\|_{L^2}^2\leq 2\|\tilde{w}(t,\cdot)\|_{L^2}^2+2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2\|\tilde{Z}(t)\|^2. \tag{65}$$

We recall that $C_{ob}=CQ$ and $A_{ob}=Q^{-1}AQ$. From Lemma 1,

$$\begin{aligned}\|\tilde{u}(t,\cdot)\|_{L^2}^2&\leq 2\|\tilde{w}(t,\cdot)\|_{L^2}^2+2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2\\&\times\Big(2\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|\tilde{Z}(t_0)\|^2+\tilde{b}(t,t_0)\Big),\end{aligned} \tag{66}$$

for all $t \in [t_0, t_0+h+T)$. It follows from (35) that

$$\|\tilde{Z}(t_0)\|^2 \leq \|e^{-A_{ob}h}\|^2 \|\tilde{X}_{ob}(t_0)\|^2. \tag{67}$$

In addition, using (3),(33), and (67),

$$\|\tilde{Z}(t_0)\|^2 \leq \|e^{-A_{ob}h}\|^2 \|Q^{-1}\|^2 \|X(t_0)-\hat{X}(t_0)\|^2. \tag{68}$$

By using (33), (69), and (68), we obtain the estimate

$$\begin{aligned} &\|u(t,\cdot)-\hat{u}(t,\cdot)\|_{L^2}^2 + \|X(t)-\hat{X}(t)\|^2 \leq 2\|\tilde{w}(t,\cdot)\|_{L^2}^2 \\ &+ \zeta_X \mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|X(t_0)-\hat{X}(t_0)\|^2 \\ &\quad + 2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2 \tilde{b}(t,t_0) + \|X(t)-\hat{X}(t)\|^2, \end{aligned} \tag{69}$$

where $\zeta_X = 4\zeta_{\tilde{Z}}\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2 \|e^{-A_{ob}h}\|^2\|Q^{-1}\|^2$.

From Lemma 1 and the various changes of variables (see Figure 1), we have

$$\begin{aligned} &\|X(t)-\hat{X}(t)\|^2 \leq 2\zeta_{\tilde{Z}}\mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h) \\ &\times \|Q\|^2\|e^{A_{ob}h}\|^2\|e^{-A_{ob}h}\|^2\|Q^{-1}\|^2\|X(t_0)-\hat{X}(t_0)\|^2. \end{aligned} \tag{70}$$

By combining (69) and (70), we finally obtain

$$\begin{aligned} &\|u(t,\cdot)-\hat{u}(t,\cdot)\|_{L^2}^2 + \|X(t)-\hat{X}(t)\|^2 \\ &\leq \left(1+\frac{\|Q\|^2\|e^{A_{ob}h}\|^2}{2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2}\right)\zeta_X \mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h) \\ &\times \|X(t_0)-\hat{X}(t_0)\|^2 + 2\|\tilde{w}(t,\cdot)\|_{L^2}^2 + 2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2\tilde{b}(t,t_0). \end{aligned} \tag{71}$$

Since $\|w(t,\cdot)\|_{L^2}^2 \equiv 0$ for all $t \geq t_0+h$ and by the definition of $\tilde{b}(t,t_0)$ (in (62)), it follows that

$$\begin{aligned} &\|u(t,\cdot)-\hat{u}(t,\cdot)\|_{L^2}^2 + \|X(t)-\hat{X}(t)\|^2 \\ &\leq \bar{\zeta}_X \mathscr{P}(\sqrt{c(t-h)})\mathfrak{C}_{-r_{\min}}(t-h)\|X(t_0)-\hat{X}(t_0)\|^2 \end{aligned} \tag{72}$$

for all $t \in [t_0+h, t_0+h+T)$, with $\bar{\zeta}_X = \left(1+\frac{\|Q\|^2\|e^{A_{ob}h}\|^2}{2\|C_{ob}e^{A_{ob}\cdot}\|_{L^2}^2}\right)\zeta_X$. As in the proof of Lemma 1, we use the fact that the decaying exponential $\mathfrak{C}_{-r_{\min}}(t-h)$ uniformly dominates the growth-in-time of $\mathscr{P}(\sqrt{c(t-h)})$. Thus, we deduce that (30) holds, concluding the proof. ■

6 Simulations

We consider a double integrator with single output as in [18] but now treat the case of sensor delay. More precisely, we simulate the system

$$\begin{aligned} &\dot{X}_1(t) = X_2(t), \\ &\dot{X}_2(t) = U(t), \quad \text{with} \quad Y(t) = X_1(t-h), \end{aligned} \tag{73}$$

with $h = 0.4$s. The initial conditions are $X(\theta) = 0$ for $\theta \in [t_0-h, t_0)$ and $X(t_0) = (5,10)^\top$ and the initialization time is set at $t_0 = 0$. Numerical simulations are carried out by discretizing the cascade ODE-PDE systems (6), (27) and (38). We utilize a two-step variant of the Lax-Friedrichs numerical method, presented in [39], and use its respective solver in Matlab. The parameters of the numerical scheme are selected so that the *Courant-Friedrich-Levy* condition for numerical stability holds. The time-varying output gains are given by (28) (after performing (3)), where $\mu_0(t-h)$ and $\mu_1(t-h)$ are computed using (14) along with (22)-(23). Hence,

$$\begin{aligned} &\gamma_1(t-h) = r_1 r_2 c^2(t-h) - \tfrac{2(r_1+r_2)}{\bar{c}_0 T}c^{3/2}(t-h) + \tfrac{2}{(\bar{c}_0 T)^2}c(t-h), \\ &\gamma_2(t-h) = (r_1+r_2)c(t-h) - \tfrac{2}{\bar{c}_0 T}c^{1/2}(t-h), \end{aligned}$$

for $r_1 = 1$, $r_2 = 2$, and $c(t-h)$ given by (8) with $\bar{c}_0 = 2.5$. We fix $T = 1$s.

Figure 2 shows the numerical solution of the system. On the left we can observe the evolution of the states of the double integrator and the estimated states subject to the input $U(t) = 10\sin(10t) + 10\cos(t)$ (depicted in blue line). On the right, the output measurement $Y(t)$ is compared to observer output $\hat{u}(t,0)$. Moreover, Figure 3 shows the numerical solution of the observer error system from which we can observe convergence of solutions to the origin. Figure 4 (blue line) shows the evolution of $\|X(t)-\hat{X}(t)\|^2 + \|u(t,x)-\hat{u}(t,x)\|_{L^2}^2$. The plot is in logarithmic scale to better illustrate that the error system converges in a prescribed time given by $T+h = 1.4$s.

Numerical implementation issues and sensitivity to measurement noise

Selecting the observer gain (28) for all $t \in [t_0, t_0+T+h)$ renders our approach quite sensitive to measurement noise: the output injection terms in (27) can grow unbounded. Additionally, it is infeasible to precisely implement the observer's output injection terms (which involve products of very small and very large quantities) due to machine precision. To resolve these practical issues, we saturate the observer gains (28) as they reach a certain threshold, which concedes prescribed-time convergence to the origin and instead achieves prescribed-time convergence to a *neighborhood of the origin*. To see this, we consider an additive measurement noise of zero mean, with 0.01 standard deviation and $20Hz$ bandwidth, which is generated by passing white noise through a low-pass Butterworth filter of bandwidth $20Hz$ (as in [26]). Figure 4 (red line) shows the result of simulating the evolution of $\|X(t)-\hat{X}(t)\|^2 + \|u(t,x)-\hat{u}(t,x)\|_{L^2}^2$ with this measurement noise. While estimate error convergence to a neighborhood of the origin can be achieved for LTI systems by classical observers with exponential convergence, doing so in a finite time causes undesirable "peaking" [25,43]: the large time-invariant gains cause the transient behavior of some states grow very large before converging. Our design does not exhibit "peaking"—observer gains *gradually* increase with time as the observer error decreases. However, sensitivity to measurement noisy may still be considerable after saturation. This is a common issue with high-gain observers (e.g., [24, Ch. 8]) and is a focal point of current research efforts. Alleviating the impact of noise remains challenging and requires novel tools specific to our framework.

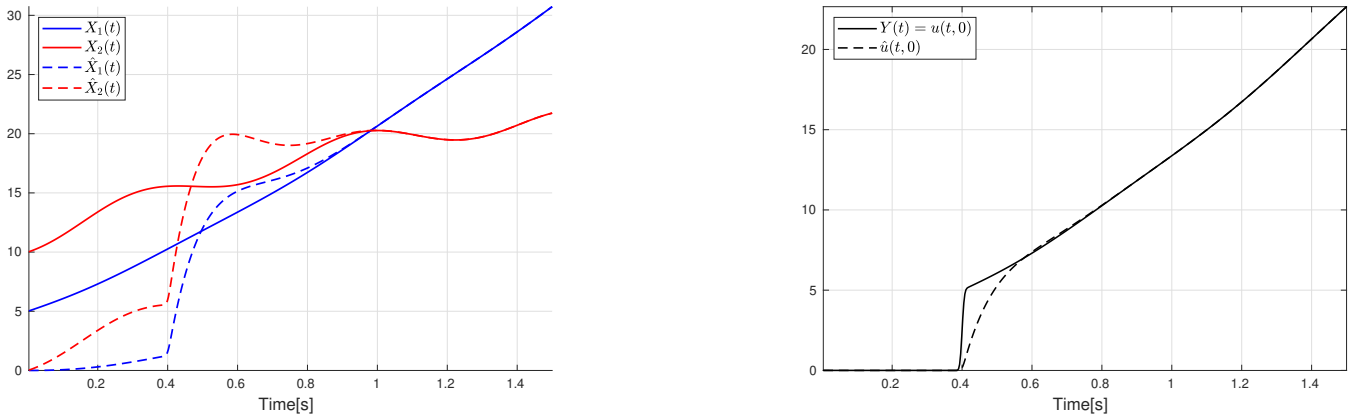

Figure 2. Numerical solution of the system (6) and its observer (27). On the left: evolution of the states of the double integrator and the observer states with sensor delay $h = 0.4$s, initial conditions $X(0) = (5,10)^\top$, $\hat{X}(0) = (0,0)^\top$ and input $U(t) = 10\sin(10t) + 10\cos(t)$. On the right: numerical solution of the output measurement $Y(t) = CX(t-h) = u(t,0)$ and the observer output $\hat{u}(t,0)$.

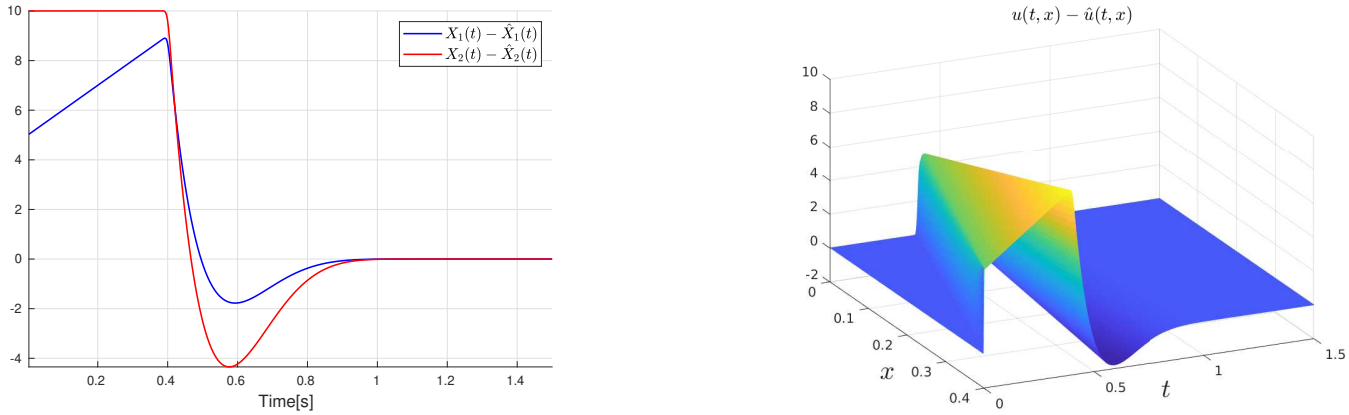

Figure 3. Numerical solution of the observer error system -ODE dynamics (on the left). Numerical solution of the observer error system sensor - PDE dynamics (on the right).

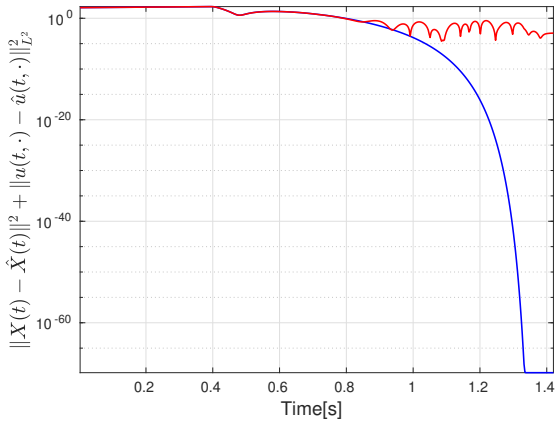

Figure 4. Evolution of $\|X(t) - \hat{X}(t)\|^2 + \|u(t,x) - \hat{u}(t,x)\|^2_{L^2}$ (on a logarithmic scale), where $h = 0.4$s and $T = 1$s. The noise-free and measurement noise (mean 0, std. dev. 0.01, bandwidth $20Hz$) simulations are plotted and blue and red, respectively.

7 Conclusion

We have addressed the problem of prescribed-time estimation of LTI systems with sensor delay. We advance the cascade ODE-PDE setup, accounting for the infinite dimensionality of the sensor. The key idea of our approach is the choice of a target system which is endowed with prescribed-time convergence.

Our time-varying observer gains are derived by using several transformations and the properties of the generalized Laguerre and elementary symmetric polynomials. Due to the bounded invertibility of the various transformations employed, we guarantee that the state estimates converge to the actual ones within the prescribed terminal time, irrespective of initial conditions.

Our design is sensitive to measurement noise, requiring further developments that are specific to our time-varying framework.

Future research directions include studying model and/or time-delay uncertainties via delay-adaptive and prescribed-time control design, and studying robustness with respect to

external disturbances via adaptive disturbance rejection.

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