Jump-drift and jump-diffusion processes: large deviations for the density, the current and the jump-flow and for the excursions between jumps

Cecile Monthus

Jump-Drift and Jump-Diffusion Processes : Large Deviations for the density, the current and the jump-flow and for the excursions between jumps

Cécile Monthus
Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA, 91191 Gif-sur-Yvette, France

For one-dimensional Jump-Drift and Jump-Diffusion processes converging towards some steady state, the large deviations of a long dynamical trajectory are described from two perspectives. Firstly, the joint probability of the empirical time-averaged density, of the empirical time-averaged current and of the empirical time-averaged jump-flow are studied via the large deviations at Level 2.5. Secondly, the joint probability of the empirical jumps and of the empirical excursions between consecutive jumps are analyzed via the large deviations at Level 2.5 for the alternate Markov chain that governs the series of all the jump events of a long trajectory. These two general frameworks are then applied to three examples of positive jump-drift processes without diffusion, and to two examples of jump-diffusion processes, in order to illustrate various simplifications that may occur in rate functions and in contraction procedures.

I. INTRODUCTION

Jump-drift and jump-diffusion processes play a major role in many applications, in particular in mathematical finance [1, 2]), in biology for integrate-and-fire neuronal models [3, 4] and in ecology to describe fires [5] or soil moisture [6–9]. They have also attracted a lot of interest in the field of intermittent search strategies (see the review [10] and references therein) and in the context of stochastic resetting (see the review [11] and references therein).

In the present paper, the goal is to analyze the large deviations (see the reviews [12–14] and references therein) of the one-dimensional jump-drift or jump-diffusion process, whenever the space-dependent parameters of the model, namely the drift $v(x)$, the diffusion coefficient $D(x)$, the jump rate $\lambda(x)$ and the jump probability $\Pi(x'|x)$ (see section II) produce a localized steady state.

On one hand, the fluctuations of the empirical time-averaged density, of the empirical time-averaged current and of the empirical time-averaged jump-flow of a long dynamical trajectory can be analyzed using the explicit large deviations at Level 2.5 for both continuous-time Markov Jump processes [15–31] and for Diffusion processes [18, 19, 22, 30–34]. This Level 2.5 can be then contracted to obtain the large deviations properties of any time-additive observable of the dynamical trajectory. The link with the studies of time-additive observables via deformed Markov operators [22, 34, 38–73] can be understood via the corresponding 'conditioned' process obtained from the generalization of Doob's h-transform.

On the other hand, the series of all the jump events of a long trajectory can be analyzed via the alternate Markov chain that governs the jumps and the excursions between consecutive jumps. As a consequence, the fluctuations of the empirical jump events and of the empirical excursions between consecutive jumps can be derived from the large deviations at Level 2.5 for discrete-time Markov chains [14, 29–31, 36, 37]. The large deviations for excusions between jumps have been studied previously for the special case of stochastic resetting to the origin [30, 74, 75] and for run-and-tumble processes [76].

The paper is organized as follows. In section II, the general one-dimensional jump-diffusion model is defined in terms of four space-dependent parameters, namely the drift $v(x)$, the diffusion coefficient $D(x)$, the jump rate $\lambda(x)$ and the jump probability $\Pi(x'|x)$. In section III, the large deviations at level 2.5 are analyzed for the joint distribution of the empirical density $\rho(x)$, of the empirical current $j(x)$ and of the empirical jump-flow $Q(x,y)$. In section IV, the contraction over the jump-flow $Q(x,y)$ for given in/out-flows $Q^{\pm}(.)$ is analyzed and explicit solutions are given for two simple cases. In section V, the jump-diffusion dynamics is analyzed from the point of view of the jump events and of the excursions between the consecutive jumps. In section VI, the large deviations for the empirical density of excursions between consecutive jumps is described. These various large deviations properties are then illustrated with three examples of positive jump-drift models without diffusion $D(x)=0$ in sections VII, VIII, IX, and with two examples of jump-diffusion models in sections X and XI. Our conclusions are summarized in XII.

II. JUMP-DIFFUSION PROCESS CONVERGING TOWARDS SOME LOCALIZED STEADY-STATE

A. Diffusion with drift $v(x)$ and diffusion coefficient $D(x)$; jumps with rate $\lambda(x)$ and probability $\Pi(x'|x)$

We consider the following one-dimensional jump-diffusion dynamics : when at position x, the particle experiences the drift $v(x)$ and diffuses with the diffusion coefficient $D(x)$, but with the jump rate $\lambda(x)$ per unit time, it can also

make a non-local jump toward some new position x' chosen with the probability distribution $\Pi(x'|x)$ normalized to unity for any x

$$\int dx' \Pi(x'|x) = 1 \tag{1}$$

As a consequence, the probability $\rho_t(x)$ to be at position x at time t satisfies the continuity equation

$$\frac{\partial \rho_t(x)}{\partial t} = -\frac{\partial j_t(x)}{\partial x} - Q_t^-(x) + Q_t^+(x) \tag{2}$$

with the following notations. The diffusive current $j_t(x)$ involves the drift $v(x)$ and the diffusion coefficient $D(x)$

$$j_t(x) \equiv v(x)\rho_t(x) - D(x)\frac{\partial \rho_t(x)}{\partial x} \tag{3}$$

The out-flow $Q_t^-(x)$ out of the position x involves the jump rate $\lambda(x)$

$$Q_t^-(x) \equiv \lambda(x)\rho_t(x) \tag{4}$$

The in-flow $Q_t^+(x)$ into the position x takes into account the arrival after a jump from any other position y, so that it involves both the jump rate $\lambda(y)$ and the jump probability $\Pi(x|y)$

$$Q_t^+(x) \equiv \int dy\ \Pi(x|y)\lambda(y)\rho_t(y) \tag{5}$$

The total probability n_t of a jump at time t can be computed either via integration over the out-flow $Q_t^-(x)$ or via integration over the in-flow $Q_t^+(x)$

$$n_t \equiv \int dx Q_t^-(x) = \int dx \lambda(x)\rho_t(x) = \int dx Q_t^+(x) \tag{6}$$

B. Existence of a normalizable steady state

In the whole paper, we will assume that the steady-state solution $\rho_*(x)$ of Eq. 2

$$0 = -\frac{d}{dx}\left[v(x)\rho_*(x) - D(x)\frac{d\rho_*(x)}{dx}\right] - \lambda(x)\rho_*(x) + \int dy\ \Pi(x|y)\lambda(y)\rho_*(y) \tag{7}$$

is normalizable

$$\int dx \rho_*(x) = 1 \tag{8}$$

The corresponding steady current reads

$$j_*(x) \quad = \rho_*(x)v(x) - D(x)\rho_*'(x) \tag{9}$$

while the steady out-flow and in-flow are given by

$$\begin{aligned} Q_*^-(x) \quad &= \lambda(x)\rho_*(x) \\ Q_*^+(x) \quad &= \int dy\ \Pi(x|y)\lambda(y)\rho_*(y) \end{aligned} \tag{10}$$

with the corresponding steady density of jumps

$$n_* \equiv \int dx Q_*^-(x) = \int dx \lambda(x)\rho_*(x) = \int dx Q_*^+(x) \tag{11}$$

The goal of the present paper is to analyze the possible fluctuations around these steady state properties.

III. LARGE DEVIATIONS AT LEVEL 2.5 FOR THE EMPIRICAL DENSITY AND FLOWS

A. Empirical density $\rho(x)$, empirical current $j(x)$ and empirical jump-flow $Q(x,y)$ with their constraints

For a very long dynamical trajectory $x(0 \leq t \leq T)$ of the jump-diffusion process of Eq. 2, the relevant empirical time-averaged observables are :

(i) the empirical time-averaged density

$$\rho(x) \quad \equiv \frac{1}{T}\int_0^T dt\ \delta(x(t)-x) \tag{12}$$

normalized to unity

$$\int dx\ \rho(x) \quad = 1 \tag{13}$$

(ii) the empirical time-averaged current $j(x)$ characterizing the drift-diffusive part of the dynamics

$$j(x) \equiv \frac{1}{T}\int_0^T dt\ \frac{dx(t)}{dt}\delta(x(t)-x) \tag{14}$$

(iii) the empirical time-averaged jump-flow $Q(x,y)$ measuring the density of jumps from y towards x

$$Q(x,y) \equiv \frac{1}{T}\sum_{t:x(t^+)\neq x(t^-)} \delta(x(t^+)-x)\delta(x(t^-)-y) \tag{15}$$

The corresponding empirical time-averaged out-flow $Q^-(.)$ and in-flow $Q^+(.)$ can be obtained via integration over one position

$$\begin{aligned} Q^-(y) &\equiv \frac{1}{T}\sum_{t:x(t^+)\neq x(t^-)} \delta(x(t^-)-y) = \int dx Q(x,y) \\ Q^+(x) &\equiv \frac{1}{T}\sum_{t:x(t^+)\neq x(t^-)} \delta(x(t^+)-x) = \int dy Q(x,y) \end{aligned} \tag{16}$$

while the total density n of jumps during $[0,T]$ corresponds to the integration over the two positions

$$n \equiv \frac{1}{T}\sum_{t:x(t^+)\neq x(t^-)} 1 = \int dx \int dy Q(x,y) = \int dx Q^+(x) = \int dy Q^-(y) \tag{17}$$

For any position x, the stationarity constraint involves the divergence of the current $j(x)$ and the difference between the in-flow $Q^+(x)$ and the out-flow $Q^-(x)$

$$0 = -\frac{dj(x)}{dx} - Q^-(x) + Q^+(x) \tag{18}$$

The integral version of this stationary constraint reads

$$j(x) = \int_x^{+\infty} dy \left[Q^-(y) - Q^+(y)\right] = \int_{-\infty}^{x} dy \left[Q^+(y) - Q^-(y)\right] \tag{19}$$

where the vanishing of the full integral is a consequence of Eq. 17

$$\int_{-\infty}^{+\infty} dy \left[Q^+(y) - Q^-(y)\right] = n - n = 0 \tag{20}$$

Eq. 19 can also be rewritten in terms of the jump-flow $Q(.,.)$ as

$$j(z) = \int_z^{+\infty} dy \int_{-\infty}^{z} dx\ Q(x,y) - \int_{-\infty}^{z} dy \int_z^{+\infty} dx\ Q(x,y) \tag{21}$$

with the following physical meaning : the backward jump-flow $Q(x,y)$ with $x<y$ have to be compensated by a positive current contribution at any point of the interval $z \in]x,y[$ (first term), while the forward jump-flow $Q(x,y)$ with $x>y$ have to be compensated by a negative current contribution at any point of the interval $z \in]y,x[$ (second term). In particular, the global integral over z of the current $j(z)$ of Eq. 21

$$\int dz\ j(z) = -\int dy \int dx\ (x-y) Q(x,y) \tag{22}$$

has to compensate the average amplitude $(x-y)$ of the jump-flow $Q(x,y)$.

B. Large deviations at Level 2.5 for the empirical time-averaged observables

The joint distribution of the empirical density $\rho(.)$, of the empirical current $j(.)$ and of the empirical jump-flow $Q(.,.)$, with the corresponding in-flow $Q^+(.)$, out-flow $Q^-(.)$ and density n satisfy the large deviation form

$$P_T^{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] \underset{T \to +\infty}{\simeq} C_{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] e^{-T I_{2.5}[\rho(.), j(.), Q(.,.)]} \tag{23}$$

The prefactor $C_{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)]$ contains the constitutive constraints of normalization (Eq. 13) and stationarity (Eq. 18), as well as the definitions of the in-flow $Q^+(.)$, of the out-flow $Q^-(.)$ and of the jump density n in terms of the jump flow $Q(.,.)$ (Eqs 16 and 17)

$$C_{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] = \delta\left(\int dx \rho(x) - 1\right) \left[\prod_x \delta\left(j'(x) + Q^-(x) - Q^+(x)\right)\right] \tag{24}$$

$$\delta\left(\int dy\ Q^-(y) - n\right) \delta\left(\int dx\ Q^+(x) - n\right) \left[\prod_x \delta\left(\int dy\ Q(x,y) - Q^+(x)\right)\right] \left[\prod_y \delta\left(\int dx\ Q(x,y) - Q^-(y)\right)\right]$$

The rate function $I_{2.5}[\rho(.), j(.), Q(.,.)]$ contains two contributions

$$I_{2.5}[\rho(.), j(.), Q(.,.)] = I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda,\Pi]}[\rho(.), Q(.,.)] \tag{25}$$

(i) The first contribution involving the diffusion coefficient $D(x)$ and the drift $v(x)$ corresponds to the usual rate function for diffusion processes [18, 19, 22, 30–34]

$$I_{2.5}^{[D,v]}[\rho(.), j(.)] = \int \frac{dx}{4D(x)\rho(x)} \left[j(x) - \rho(x)v(x) + D(x)\rho'(x)\right]^2 \tag{26}$$

(ii)The second contribution involving the jump rate $\lambda(y)$ and the jump probability $\Pi(x|y)$ corresponds to the usual rate function for Markov jump processes [15–31]

$$I_{2.5}^{[\lambda,\Pi]}[\rho(.), Q(.,.)] = \int dx \int dy \left[Q(x,y) \ln\left(\frac{Q(x,y)}{\Pi(x|y)\lambda(y)\rho(y)}\right) - Q(x,y) + \Pi(x|y)\lambda(y)\rho(y)\right] \tag{27}$$

Using the normalization of Eq. 1 and the constraints of Eq. 24, it is useful to introduce the out-flow $Q^-(y)$ in order to rewrite this jump rate function as a sum of two terms

$$I_{2.5}^{[\lambda,\Pi]}[\rho(.), Q^-(.), Q(.,.)] = I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] \tag{28}$$

The first term associated to the jump rate $\lambda(.)$

$$I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] = \int dy \left[Q^-(y) \ln\left(\frac{Q^-(y)}{\lambda(y)\rho(y)}\right) - Q^-(y) + \lambda(y)\rho(y)\right] \tag{29}$$

describes the possible fluctuations of the out-flow $Q^-(y)$ with respect to its typical value $\lambda(y)\rho(y)$. The second term associated to the jump probability $\Pi(.|.)$

$$I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] = \int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{\Pi(x|y)Q^-(y)}\right) \tag{30}$$

characterizes the possible fluctuations of the empirical jump probability $\frac{Q(x,y)}{Q^-(y)}$ with respect to its typical value $\Pi(x|y)$.

In summary, the rate function at Level 2.5 of Eq. 25 has been decomposed as the sum of the three terms associated to the diffusion parameters $[D(.), v(.)]$, to the jump rate $\lambda(.)$ and to the jump probability $\Pi(.|.)$ respectively

$$I_{2.5}[\rho(.), j(.), Q^-(.), Q(.,.)] = I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] \tag{31}$$

C. Alternative formulation for the inferred parameters that would make the empirical observables typical

Another point of view on the large deviations at Level 2.5 is based on the inverse problem of inference [31] : from the data of a long dynamical trajectory, one computes the empirical time-averaged observables described above, and one infers the best steady state $\hat{\rho}_*(x)$ and the best corresponding parameters $[\hat{v}(x), \hat{\lambda}(x), \hat{\Pi}(x|y)]$ of the model as follows (Note that the diffusion coefficient cannot fluctuate $\hat{D}(x) \equiv D(x)$ as discussed in detail in [31]).

(i) the best inferred steady state $\hat{\rho}_*(x)$ is simply the measured empirical density

$$\hat{\rho}_*(x) \equiv \rho(x) \tag{32}$$

(ii) the best inferred drift $\hat{v}(x)$ is the drift that would make vanish the diffusive rate function Eq. 26

$$\hat{v}(x) \equiv \frac{j(x) + D(x)\rho'(x)}{\rho(x)} \tag{33}$$

(iii) the best inferred jump rate $\hat{\lambda}(y)$ is the rate that would make vanish the rate function of Eq. 29

$$\hat{\lambda}(y) \equiv \frac{Q^-(y)}{\rho(y)} \tag{34}$$

(iv) the best inferred jump probability $\hat{\Pi}(x|y)$ is the jump probability that would make vanish the rate function of Eq. 30

$$\hat{\Pi}(x|y) \equiv \frac{Q(x,y)}{Q^-(y)} \tag{35}$$

Via this change of variables, the large deviations at Level 2.5 of Eq. 23 translates into the joint probability to infer the model parameters $[\hat{v}(x), \hat{\lambda}(x), \hat{\Pi}(x|y)]$ and the corresponding steady state $\hat{\rho}_*(x)$ that they produce together

$$P_T^{Infer}[\hat{\rho}_*(.), \hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] \underset{T\to+\infty}{\simeq} C_{Infer}[\hat{\rho}_*(.), \hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] e^{-T I_{Infer}[\hat{\rho}_*(.), \hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)]} \tag{36}$$

The constraints derived from Eq. 24

$$C_{Infer}[\hat{\rho}_*(.), \hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] = \delta\left(\int dx \hat{\rho}_*(x) - 1\right)\left[\prod_y \delta\left(\int dx\ \hat{\Pi}(x|y) - 1\right)\right]$$
$$\left[\prod_x \delta\left(\frac{d}{dx}\left[\hat{\rho}_*(x)\hat{v}(x) - D(x)\frac{d\hat{\rho}_*(x)}{dx}\right] + \hat{\lambda}(x)\hat{\rho}_*(x) - \int dy\ \hat{\Pi}(x|y)\hat{\lambda}(y)\hat{\rho}_*(y)\right)\right] \tag{37}$$

contains the normalization of the inferred steady state $\hat{\rho}_*(.)$ and the normalization of the inferred jump probability $\hat{\Pi}(.|.)$ on the first line, while the second line means that $\hat{\rho}_*(.)$ should be the steady state produced by the inferred parameters $\hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)$. The rate function translated from Eq. 31 contains three contributions

$$I_{Infer}[\hat{\rho}_*(.), \hat{v}(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] = I_{Infer}^{[D,v]}[\hat{\rho}_*(.), \hat{v}(.)] + I_{Infer}^{[\lambda]}[\hat{\rho}_*(.), \hat{\lambda}(.)] + I_{Infer}^{[\Pi]}[\hat{\rho}_*(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] \tag{38}$$

The first contribution governs the fluctuations of the inferred drift $\hat{v}(.)$ around the true drift $v(.)$

$$I_{Infer}^{[D,v]}[\hat{\rho}_*(.), \hat{v}(.)] = \int dx \frac{\hat{\rho}_*(x)}{4D(x)} [\hat{v}(x) - v(x)]^2 \tag{39}$$

The second contribution governs the fluctuations of the inferred jump rate $\hat{\lambda}(.)$ around the true jump rate $\lambda(.)$

$$I_{Infer}^{[\lambda]}[\hat{\rho}_*(.), \hat{\lambda}(.)] = \int dy \hat{\rho}_*(y) \left[\hat{\lambda}(y) \ln\left(\frac{\hat{\lambda}(y)}{\lambda(y)}\right) - \hat{\lambda}(y) + \lambda(y)\right] \tag{40}$$

The third contribution governs the fluctuations of the inferred jump probability $\hat{\Pi}(.|.)$ with respect to the true jump probability $\Pi(.|.)$

$$I_{Infer}^{[\Pi]}[\hat{\rho}_*(.), \hat{\lambda}(.), \hat{\Pi}(.|.)] = \int dy \hat{\rho}_*(y)\hat{\lambda}(y) \int dx \hat{\Pi}(x|y) \ln\left(\frac{\hat{\Pi}(x|y)}{\Pi(x|y)}\right) \tag{41}$$

D. Simplification of the Level 2.5 for jump-drift models without diffusion $D(x) = 0$

For jump-drift models without diffusion $D(x) = 0$, the diffusive rate function of Eq. 26 does not appear anymore but yields that the empirical current is directly related to the empirical density

$$j(x) = \rho(x)v(x) \quad \text{when} \quad D(x) = 0 \tag{42}$$

as a consequence of the deterministic motion at velocity $v(x)$ between jumps.

So the large deviations at Level 2.5 of Eq. 23 can be rewritten after the elimination of the current $j(.)$ via Eq. 42

$$P_T^{2.5[D(.)=0]}[\rho(.), n, Q^{\pm}(.), Q(.,.)] \underset{T\to+\infty}{\simeq} C_{2.5}^{[D(.)=0]}[\rho(.), n, Q^{\pm}(.), Q(.,.)] e^{-T I_{2.5}^{[D(.)=0]}[\rho(.), Q^-(.), Q(.,.)]} \tag{43}$$

with the constraints

$$C_{2.5}^{[D(.)=0]}[\rho(.), n, Q^{\pm}(.), Q(.,.)] = \delta\left(\int dx \rho(x) - 1\right)\left[\prod_x \delta\left(\frac{d}{dx}\left[\rho(x)v(x)\right] + Q^-(x) - Q^+(x)\right)\right] \tag{44}$$
$$\delta\left(\int dy\ Q^-(y) - n\right)\delta\left(\int dx\ Q^+(x) - n\right)\left[\prod_x \delta\left(\int dy\ Q(x,y) - Q^+(x)\right)\right]\left[\prod_y \delta\left(\int dx\ Q(x,y) - Q^-(y)\right)\right]$$

and the rate function involving only the two jump contributions associated to $\lambda(.)$ and $\Pi(.|.)$

$$\begin{aligned} I_{2.5}^{[D(.)=0]}\left[\rho(.), Q^-(.), Q(.,.)\right] &= I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] \\ &= \int dy \left[Q^-(y)\ln\left(\frac{Q^-(y)}{\lambda(y)\rho(y)}\right) - Q^-(y) + \lambda(y)\rho(y)\right] + \int dx \int dy Q(x,y)\ln\left(\frac{Q(x,y)}{\Pi(x|y)Q^-(y)}\right) \end{aligned} \tag{45}$$

Three examples of jump-drift models without diffusion $D(x) = 0$ will be described in sections VII, VIII, IX.

E. Simplification of the Level 2.5 when the jump probability $\Pi^{deter}(x|y) = \delta(x - \Phi(y))$ is deterministic

When the jump probability describes a deterministic rule for the position after the jump $x = \Phi(y)$ in terms of the position y before the jump

$$\Pi^{deter}(x|y) = \delta\left(x - \Phi(y)\right) \tag{46}$$

the jump flow involves the same deterministic function $\delta\left(x - \Phi(y)\right)$

$$Q(x,y) = \delta\left(x - \Phi(y)\right) Q^-(y) \tag{47}$$

and the rate function of Eq. 30 vanishes

$$I_{2.5}^{[\Pi^{deter}]}[Q^-(.), Q(.,.)] = 0 \tag{48}$$

One can also use Eq. 47 to eliminate the in-flow $Q^+(.)$ in terms of the out-flow $Q^-(.)$

$$Q^+(x) = \int dy Q(x,y) = \int dy\ \delta\left(x - \Phi(y)\right) Q^-(y) \tag{49}$$

Putting everything together, one obtains that the large deviations at Level 2.5 of Eq. 23 can be rewritten for the joint distribution of the density $\rho(.)$, of the current $j(.)$, of the density n and of the out-flow $Q^-(.)$ only as

$$\begin{aligned} &P_T^{2.5[\Pi^{deter}(x|y)=\delta(x-\Phi(y))]}[\rho(.), j(.), n, Q^-(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int dx\rho(x) - 1\right)\delta\left(\int dy\ Q^-(y) - n\right) \\ &\left[\prod_x \delta\left(j'(x) + Q^-(x) - \int dy\ \delta\left(x - \Phi(y)\right) Q^-(y)\right)\right] e^{-T\left[I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)]\right]} \end{aligned} \tag{50}$$

Two examples of deterministic jump probability $\Pi^{deter}(x|y) = \delta(x - \Phi(y))$ will be described in sections VII and VIII.

IV. CONTRACTION OVER THE JUMP-FLOW $Q(x,y)$ FOR GIVEN IN/OUT-FLOWS $Q^{\pm}(.)$

In the large deviations at Level 2.5 described in the previous section, the jump-flow $Q(x,y)$ is the only empirical observable that involves two positions. The goal of this section is to analyze whether the contraction of the jump-flow $Q(x,y)$ can be carried out.

A. Large deviations for the one-position empirical observables $[\rho(.), j(.), Q^{\pm}(.)]$ via contraction over $Q(.,.)$

The joint distribution of $\rho(.)$, $j(.)$, n and $Q^{\pm}(.)$ can be derived from the Level 2.5 of Eq. 23

$$P_T[\rho(.), j(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int dx \rho(x) - 1\right) \left[\prod_x \delta\left(j'(x) + Q^-(x) - Q^+(x)\right)\right] \tag{51}$$

$$\delta\left(\int dy\ Q^-(y) - n\right) \delta\left(\int dx\ Q^+(x) - n\right) e^{-T\left(I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)]\right)} K_T \tag{52}$$

if one can evaluate the remaining integral over the jump flow $Q(x,y)$

$$\begin{aligned} K_T \quad &\equiv \int \mathcal{D}Q(.,.) \left[\prod_x \delta\left(\int dy\ Q(x,y) - Q^+(x)\right)\right] \left[\prod_y \delta\left(\int dx\ Q(x,y) - Q^-(y)\right)\right] \\ &\quad e^{-T\int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{\Pi(x|y)Q^-(y)}\right)} \end{aligned} \tag{53}$$

for large T. In order to optimize over the jump flow $Q(.,.)$ the functional appearing in factor of T in the exponential of the second line in the presence of the constraints of the first line, let us introduce the following Lagrangian

$$\begin{aligned} \mathcal{L}[Q(.,.)] \quad &\equiv -\int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{\Pi(x|y)Q^-(y)}\right) \\ &\quad + \int dx \phi(x) \left(\int dy\ Q(x,y) - Q^+(x)\right) + \int dy \psi(y) \left(\int dx\ Q(x,y) - Q^-(y)\right) \end{aligned} \tag{54}$$

where the Lagrange multipliers $\phi(.), \psi(.)$ are associated to the two constraints. The optimization with respect to the jump flow $Q(x,y)$

$$0 = \frac{\partial \mathcal{L}[Q(.,.)]}{\partial Q(x,y)} = -\ln\left(\frac{Q(x,y)}{\Pi(x|y)Q^-(y)}\right) - 1 + \phi(x) + \psi(y) \tag{55}$$

leads to the optimal solution

$$Q^{opt}(x,y) = e^{-1} e^{\phi(x)} \Pi(x|y) Q^-(y) e^{\psi(y)} \tag{56}$$

that should satisfy the two constraints

$$\begin{aligned} Q^+(x) \quad &= \int dy Q^{opt}(x,y) = e^{-1} e^{\phi(x)} \int dy \Pi(x|y) Q^-(y) e^{\psi(y)} \\ Q^-(y) \quad &= \int dx Q^{opt}(x,y) = e^{-1} Q^-(y) e^{\psi(y)} \int dx e^{\phi(x)} \Pi(x|y) \end{aligned} \tag{57}$$

The second constraint can be used to eliminate the Lagrange multiplier $\psi(y)$ in terms of the other one $\phi(x)$

$$e^{\psi(y)} = \frac{1}{e^{-1}\int dx' e^{\phi(x')} \Pi(x'|y)} \tag{58}$$

Plugging this value into Eq. 56 yields

$$Q^{opt}(x,y) = \frac{e^{\phi(x)} \Pi(x|y)}{\int dx' e^{\phi(x')} \Pi(x'|y)} Q^-(y) \tag{59}$$

while the first constraint of Eq. 57 becomes

$$Q^+(x) = e^{\phi(x)} \int dy \frac{\Pi(x|y)}{\int dx' e^{\phi(x')} \Pi(x'|y)} Q^-(y) \tag{60}$$

Let us now describe two examples where this optimization problem has a simple solution.

B. Explicit contraction for resetting models $\Pi^{reset}(x|y) = R(x)$

When the jump probability is independent of the starting point y

$$\Pi^{reset}(x|y) = R(x) \tag{61}$$

the jumps correspond to the following stochastic resetting procedure (see the review [11] and references therein) : $\lambda(y)$ is the reset rate when in position y, while the normalized probability distribution $R(x)$ governs the choice of the new position x after each jump. Then Eq. 60 reduces to

$$Q^+(x) = \frac{e^{\phi(x)}R(x)}{\int dx' e^{\phi(x')}R(x')} \int dy Q^-(y) = \frac{e^{\phi(x)}R(x)}{\int dx' e^{\phi(x')}R(x')} n \tag{62}$$

and the optimal solution $Q^{opt}(x,y)$ of Eq. 59 can be rewritten in terms of the in-flow $Q^+(x)$ at x, of the out-flow $Q^-(y)$ at the position y and of the jump density n

$$Q^{opt}(x,y) = \frac{e^{\phi(x)}R(x)}{\int dx' e^{\phi(x')}R(x')} Q^-(y) = \frac{Q^+(x)Q^-(y)}{n} \tag{63}$$

The value of the Lagrangian of Eq. 54 for this optimal solution satisfying the constraints reduces to

$$\begin{aligned} \mathcal{L}[Q^{opt}(.,.)] &\equiv -\int dx \int dy Q^{opt}(x,y) \ln\left(\frac{Q^{opt}(x,y)}{R(x)Q^-(y)}\right) = -\int dx \int dy \frac{Q^+(x)Q^-(y)}{n} \ln\left(\frac{Q^+(x)}{nR(x)}\right) \\ &= -\int dx Q^+(x) \ln\left(\frac{Q^+(x)}{nR(x)}\right) \equiv -I^{[R]}[n, Q^+(.)] \end{aligned} \tag{64}$$

where the rate function

$$I^{[R]}[n, Q^+(.)] \equiv \int dx Q^+(x) \ln\left(\frac{Q^+(x)}{nR(x)}\right) \tag{65}$$

governs the possible fluctuations of the out-flow $Q^+(x)$ with respect to its typical value $nR(x)$ involving the normalized reset probability $R(x)$. The optimal value of Eq. 64 governs the exponential behavior in T of the integral of Eq. 53

$$K_T \underset{T\to+\infty}{\simeq} e^{T\mathcal{L}[\pi^{opt}(.|.)]} = e^{-TI^{[R]}[n,Q^+(.)]} = e^{-T\int dx Q^+(x) \ln\left(\frac{Q^+(x)}{nR(x)}\right)} \tag{66}$$

Plugging this result into Eq. 52 yields the large deviation form for the one-position empirical observables $[\rho(.), j(.), n, Q^{\pm}(.)]$

$$\begin{aligned} &P_T^{\Pi^{reset}(x|y)=R(x)}[\rho(.), j(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int dx \rho(x) - 1\right) \left[\prod_x \delta\left(j'(x) + Q^-(x) - Q^+(x)\right)\right] \\ &\delta\left(\int dy\ Q^-(y) - n\right) \delta\left(\int dx\ Q^+(x) - n\right) e^{-T\left(I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I^{[R]}[n, Q^+(.)]\right)} \end{aligned} \tag{67}$$

An example of jump-diffusion process with the resetting jump probability $\Pi^{reset}(x|y) = R(x)$ towards an aritrary function $R(x)$ will be described in section X.

C. Explicit contraction for the positive exponential jump probability $\Pi_\alpha^{exp}(x|y) = \alpha e^{-\alpha(x-y)}$ for $x \geq y$

Let us now consider the case where the jump probability

$$\Pi_\alpha^{exp}(x|y) = \alpha e^{-\alpha(x-y)} \quad \text{for} \quad x \in [y, +\infty[\tag{68}$$

describes positive jumps whose amplitude $z = x - y \geq 0$ is exponentially distributed. Then Eq. 59 reads for $x \geq y$

$$Q^{opt}(x,y) = \frac{e^{\phi(x)}e^{-\alpha x}}{\int_y^{+\infty} dx'' e^{\phi(x'')}e^{-\alpha x''}} Q^-(y) \tag{69}$$

while Eq. 60 becomes

$$Q^+(x) = \int_{-\infty}^{x} dy Q^{opt}(x,y) = e^{\phi(x)} e^{-\alpha x} \int_{-\infty}^{x} dy \frac{Q^-(y)}{\int_y^{+\infty} dx'' e^{\phi(x'')} e^{-\alpha x''}} \tag{70}$$

It is useful to introduce the corresponding negative current via Eq. 21

$$j(x) \quad = -\int_x^{+\infty} dx' \int_{-\infty}^{x} dy' Q^{opt}(x',y') = -\left[\int_x^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'}\right] \left[\int_{-\infty}^{x} dy \frac{Q^-(y)}{\int_y^{+\infty} dx'' e^{\phi(x'')} e^{-\alpha x''}}\right] \tag{71}$$

and to consider the ratio

$$\frac{Q^+(x)}{[-j(x)]} = \frac{e^{\phi(x)} e^{-\alpha x}}{\int_x^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'}} = -\frac{d}{dx} \ln\left[\int_x^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'}\right] \tag{72}$$

The integration

$$\int_y^{x} dz \frac{Q^+(z)}{[-j(z)]} = \ln\left[\frac{\int_y^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'}}{\int_x^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'}}\right] \tag{73}$$

yields

$$\int_y^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'} = e^{\int_y^x dz \frac{Q^+(z)}{[-j(z)]}} \int_x^{+\infty} dx' e^{\phi(x')} e^{-\alpha x'} \tag{74}$$

Using this integral and Eq. 72, one can rewrite the optimal solution of Eq. 69 without the Lagrange multiplier $\phi(.)$ as

$$Q^{opt}(x,y) \quad = \frac{e^{\phi(x)} e^{-\alpha x}}{\int_x^{+\infty} dx'' e^{\phi(x'')} e^{-\alpha x''}} e^{-\int_y^x dz \frac{Q^+(z)}{[-j(z)]}} Q^-(y) = \frac{Q^+(x)}{[-j(x)]} e^{-\int_y^x dz \frac{Q^+(z)}{[-j(z)]}} Q^-(y) \tag{75}$$

in terms of the in-flow $Q^+(.)$, of the out-flow $Q^+(.)$, and the current $j(.)$. The value of the Lagrangian of Eq. 54 for this optimal solution satisfying the constraints reads

$$\begin{aligned}
\mathcal{L}[Q^{opt}(.,.)] \quad &\equiv -\int dx \int_{-\infty}^{x} dy \; Q^{opt}(x,y) \ln\left(\frac{Q^{opt}(x,y)}{\alpha e^{-\alpha(x-y)} Q^-(y)}\right) \\
&= -\int dx \int_{-\infty}^{x} dy \; Q^{opt}(x,y) \ln\left(\frac{\frac{Q^+(x)}{[-j(x)]} e^{-\int_y^x dz \frac{Q^+(z)}{[-j(z)]}}}{\alpha e^{-\alpha(x-y)}}\right) \\
&= -\int dx \int_{-\infty}^{x} dy \; Q^{opt}(x,y) \ln\left(\frac{Q^+(x)}{\alpha[-j(x)]}\right) - \alpha \int dx \int_{-\infty}^{x} dy \; Q^{opt}(x,y) \, (x-y) \\
&+ \int_{-\infty}^{+\infty} dz \frac{Q^+(z)}{[-j(z)]} \int_z^{+\infty} dx \int_{-\infty}^{z} dy \; Q^{opt}(x,y)
\end{aligned} \tag{76}$$

Using the constraints, Eq. 21 and Eq. 22, Eq. 76 reduces to

$$\begin{aligned}
\mathcal{L}[Q^{opt}(.,.)] \quad &= -\int dx \; Q^+(x) \ln\left(\frac{Q^+(x)}{\alpha[-j(x)]}\right) - \alpha \int dz[-j(z)] + \int_{-\infty}^{+\infty} dz Q^+(z) \\
&= -\int dx \left[Q^+(x) \ln\left(\frac{Q^+(x)}{\alpha[-j(x)]}\right) - Q_+(x) + \alpha[-j(x)]\right] \equiv -I_\alpha^{exp}[Q^+(.), j(.)]
\end{aligned} \tag{77}$$

where the rate function

$$I_\alpha^{exp}[Q^+(.), j(.)] \equiv \int dx \left[Q^+(x) \ln\left(\frac{Q^+(x)}{\alpha[-j(x)]}\right) - Q_+(x) + \alpha[-j(x)]\right] \tag{78}$$

governs the possible fluctuations of the out-flow $Q^+(x)$. The final result is thus that the large deviation form for the one-position empirical observables $[\rho(.), j(.), n, Q^{\pm}(.)]$ reads

$$P_T^{[\Pi_\alpha^{exp}(x|y)=\alpha e^{-\alpha(x-y)}]}[\rho(.), j(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int dx \rho(x) - 1\right)\left[\prod_x \delta\left(j'(x) + Q^-(x) - Q^+(x)\right)\right] \tag{79}$$
$$\delta\left(\int dy\ Q^-(y) - n\right)\delta\left(\int dx\ Q^+(x) - n\right) e^{-T\left(I_{2.5}^{[D,v]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_\alpha^{exp}[Q^+(.), j(.)]\right)}$$

An example of jump-drift process with the exponential jump probability $\Pi_\alpha^{exp}(x|y) = \alpha e^{-\alpha(x-y)}$ will be described in section IX.

V. ANALYSIS OF THE DYNAMICS FROM THE POINT OF VIEW OF JUMP EVENTS

In this section, the jump-diffusion dynamics of Eq. 2 is analyzed instead from the point of view of the jump events only.

A. Alternate Markov chain governing the sequence of jump events

A very long trajectory $x(0 \leq t \leq T)$ is characterized by a large number $N = nT$ of jumps, where the density n of jumps had been already discussed in Eq. 17. Let us introduce the times t_i with $i = 1, .., N$ where the N jumps occur, between the positions just before the jump at $t = t_i^-$

$$y_i \equiv x(t_i^-) \tag{80}$$

and the positions just after the jump at $t = t_i^+$

$$x_i \equiv x(t_i^+) \tag{81}$$

The dynamics of these jump events is governed by the following alternate Markov chain

$$\begin{aligned} P_{t_i^+}(x_i) &= \int dy_i\ \Pi(x_i|y_i) P_{t_i^-}(y_i) \\ P_{t_{i+1}^-}(y_{i+1}) &= \int dx_i\ W(t_{i+1} - t_i, y_{i+1}|x_i) P_{t_i^+}(x_i) \end{aligned} \tag{82}$$

where the jump kernel $\Pi(x|y)$ is given by the definition of the model with the normalization of Eq. 1, while the kernel $W(\tau, y|x)$ for the excursions between two consecutive jumps with the normalization

$$\int_0^{+\infty} d\tau \int dy\ W(\tau, y|x) = 1 \tag{83}$$

is discussed in detail below.

B. Kernel $W(\tau, y|x)$ for the excursions between two consecutive jumps

The probability that an excursion starting at position x ends after the time τ at the position y involves the jump rate $\lambda(y)$ at position y

$$W(\tau, y|x) = \lambda(y) P_\tau^{surv}(y|x) \tag{84}$$

while $P_\tau^{surv}(y|x)$ is the probability to have diffused from the position x to the position y in the time τ without any jump. As a consequence, this probability satisfies the initial dynamical Eq. 2 with absorption only, i.e. without the last term representing the re-injection after the jumps

$$\frac{\partial P_\tau^{surv}(y|x)}{\partial \tau} = -\frac{\partial}{\partial y}\left[v(y) P_\tau^{surv}(y|x) - D(y)\frac{\partial P_\tau^{surv}(y|x)}{\partial y}\right] - \lambda(y) P_\tau^{surv}(y|x) \tag{85}$$

with the initial condition at $\tau = 0$

$$P_0^{surv}(y|x) = \delta(y-x) \tag{86}$$

The path-integral representation of the solution reads

$$P_\tau^{surv}(y|x) = \int_{z(0)=x}^{z(\tau)=y} \mathcal{D}z(.)\; e^{-\int_0^\tau ds \left(\lambda(z(s)) + \frac{[\dot{z}(s)-v(z(s))]^2}{4D(z(s))} - \frac{[D'(z(s))]^2}{16D(z(s))} + \frac{D''(z(s))}{4} + \frac{v'(z(s))}{2} \right)} \tag{87}$$

So the excursion kernel of Eq. 84 will be explicit whenever one can solve the time-dependent absorbing dynamics of Eqs 85 86 or equivalently compute the path-integral of Eq. 87. Let us now describe some simple examples.

C. Simplifications of the excursion kernel $W(\tau, y|x)$ for jump-drift models without diffusion $D(y) = 0$

For jump-drift models without diffusion $D(y) = 0$, the path-integral of Eq. 87 reduces to the single deterministic trajectory $z_x(0 \leq s \leq \tau)$ satisfying the equation of motion with the drift $v(.)$

$$\frac{dz_x(s)}{ds} = v(z_x(s)) \tag{88}$$

and starting at the position x at the initial time $s = 0$

$$z_x(0) \quad = x \tag{89}$$

The solution

$$P_\tau^{surv}(y|x) = \delta\left(y - z_x(\tau)\right) e^{-\int_0^\tau ds \lambda(z_x(s))} \tag{90}$$

can be rewritten using the separation of variables $ds = \frac{dz}{v(z)}$ of Eq. 88 as

$$P_\tau^{surv}(y|x) = \frac{\delta\left(\tau - \int_x^y \frac{dz}{v(z)}\right)}{|v(y)|} \; e^{-\int_x^y dz \frac{\lambda(z)}{v(z)}} \tag{91}$$

So the excursion kernel of Eq. 84 reduces to

$$W(\tau, y|x) = W^{sp}(y|x) \delta\left(\tau - \int_x^y \frac{dz}{v(z)}\right) \tag{92}$$

where

$$W^{sp}(y|x) = \frac{\lambda(y)}{|v(y)|} \; e^{-\int_x^y dz \frac{\lambda(z)}{v(z)}} \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) \tag{93}$$

represents the spatial probability that an excursion ends at position y if it starts at position x, the last Heaviside function ensuring that the corresponding duration is positive $\tau \geq 0$. In most models of interest, the drift $v(z)$ is a continuous function of the position z, so that the sign of the velocity $v(x)$ at the initial point will determine the sign of the velocity during the whole excursion and whether the position of the end-point y is greater or smaller than x. Then Eq. 93 can be rewritten more explicitly as

$$\begin{aligned} W^{sp}(y|x) \quad &= \frac{\lambda(y)}{|v(y)|} \; e^{-\int_x^y dz \frac{\lambda(z)}{v(z)}} \theta\left((y-x)v(x) \geq 0\right) \\ &= \theta\left(v(x) > 0\right) \theta\left(y > x\right) W_+^{sp}(y|x) + \theta\left(v(x) < 0\right) \theta\left(y < x\right) W_-^{sp}(y|x) \end{aligned} \tag{94}$$

in terms of the two cases :

(i) when the drift is positive $v(x) > 0$ at the initial point x, Eq. 93 reads more explicitly

$$W_+^{sp}(y|x) = \frac{\lambda(y)}{v(y)} \; e^{-\int_x^y dz \frac{\lambda(z)}{v(z)}} = -\frac{d}{dy} \; e^{-\int_x^y dz \frac{\lambda(z)}{v(z)}} \qquad \text{for} \;\; y \geq x \tag{95}$$

with the normalization

$$\int_x^{+\infty} dy W_+^{sp}(y|x) = 1 \tag{96}$$

(ii) When the drift is negative $v(x) < 0$ at the initial point x, Eq. 93 reads more explicitly

$$W_-^{sp}(y|x) = \frac{\lambda(y)}{|v(y)|} \; e^{-\int_y^x dz \frac{\lambda(z)}{|v(z)|}} = \frac{d}{dy} \; e^{-\int_y^x dz \frac{\lambda(z)}{|v(z)|}} \qquad \text{for} \;\; y \leq x \tag{97}$$

with the normalization

$$\int_{-\infty}^x dy W_-^{sp}(y|x) = 1 \tag{98}$$

Three examples of jump-drift models without diffusion $D(x) = 0$ will be described in sections VII, VIII, IX.

D. Simplifications of the excursion kernel $W(\tau, y|x)$ when the jump rate is uniform $\lambda(x) = \lambda$

When the jump rate is uniform $\lambda(x) = \lambda$, the path-integral of Eq. 87 reduces to

$$P_\tau^{surv}(y|x) = e^{-\tau\lambda} P_\tau^{free}(y|x) \tag{99}$$

where

$$P_\tau^{free}(y|x) = \int_{z(0)=x}^{z(\tau)=y} \mathcal{D}z(.) \; e^{-\int_0^\tau ds \left(\frac{[\dot{z}(s) - v(z(s))]^2}{4D(z(s))} - \frac{[D'(z(s))]^2}{16D(z(s))} + \frac{D''(z(s))}{4} + \frac{v'(z(s))}{2} \right)} \tag{100}$$

is the free propagator of the diffusion process

$$\begin{aligned} \frac{\partial P_\tau^{free}(y|x)}{\partial \tau} &= -\frac{\partial}{\partial y} \left[v(y) P_\tau^{free}(y|x) - D(y) \frac{\partial P_\tau^{free}(y|x)}{\partial y} \right] \\ P_0^{free}(y|x) &= \delta(y-x) \end{aligned} \tag{101}$$

normalized to unity at any time τ

$$\int dy P_\tau^{free}(y|x) = 1 \tag{102}$$

So the kernel of Eq. 84 is factorized

$$W(\tau, y|x) = E^{exc}(\tau) P_\tau^{free}(y|x) \tag{103}$$

into the normalized exponential probability to see the duration $\tau \in]0, +\infty[$

$$E^{exc}(\tau) = \lambda e^{-\tau\lambda} \tag{104}$$

and into the free propagator $P_\tau^{free}(y|x)$ discussed above. In conclusion, when the jump is uniform $\lambda(x) = \lambda$, the excursion kernel $W(\tau, y|x)$ is explicit whenever the free propagator $P_\tau^{free}(y|x)$ is known. Two example of jump-diffusion models with uniform jump rate $\lambda(x) = \lambda$ will be described in sections X and XI.

VI. LARGE DEVIATIONS FOR THE EMPIRICAL EXCURSIONS BETWEEN CONSECUTIVE JUMPS

A. Density $q(\tau, y, x)$ of empirical excursions between two consecutive jumps

For the jump events, we have already introduced the empirical jump-flow $Q(x,y)$ in Eq. 15 that can be rewritten more explicitely with the notations of Eqs 80 81

$$Q(x,y) = \frac{1}{T}\sum_{i=1}^{N} \delta(x - x(t_i^+))\delta(y - x(t_i^-)) \tag{105}$$

with the corresponding in-flow and out-flow of Eq. 16

$$\begin{aligned} Q^+(x) &\equiv \int dy Q(x,y) = \frac{1}{T}\sum_{i=1}^{N} \delta(x - x(t_i^+)) \\ Q^-(y) &\equiv \int dx Q(x,y) = \frac{1}{T}\sum_{i=1}^{N} \delta(y - x(t_i^-)) \end{aligned} \tag{106}$$

In this section, we are interested into the empirical density of the excursions between two consecutive jumps

$$q(\tau, y, x) \equiv \frac{1}{T}\sum_{i=1}^{N} \delta(\tau - (t_{i+1} - t_i))\delta(y - x(t_{i+1}^-))\delta(x - x(t_i^+)) \tag{107}$$

that also contains the information on the in-flow and out-flow of Eq. 106 after integration over one position and over the duration τ

$$\begin{aligned} Q^+(x) &= \frac{1}{T}\sum_{i=1}^{N} \delta(x - x(t_i^+)) = \int_0^{+\infty} d\tau \int dy\ q(\tau, y, x) \\ Q^-(y) &= \frac{1}{T}\sum_{i=0}^{N-1} \delta(y - x(t_{i+1}^-)) = \int_0^{+\infty} d\tau \int dx\ q(\tau, y, x) \end{aligned} \tag{108}$$

The total density of excursions of duration τ can be obtained via the integration over the two positions

$$q(\tau) \equiv \int dx \int dy q(\tau, y; x) = \frac{1}{T}\sum_{i=1}^{N} \delta(\tau - (t_{i+1} - t_i)) \tag{109}$$

The sum of the durations $\tau_i = t_{i+1} - t_i$ of all the excursions determines the normalization

$$1 = \frac{1}{T}\sum_{i=1}^{N} (t_{i+1} - t_i) = \int_0^{+\infty} d\tau \tau q(\tau) = \int_0^{+\infty} d\tau \tau \int dx \int dy q(\tau, y, x) \tag{110}$$

while the total density $n = \frac{N}{T}$ of jumps of Eq. 17 corresponds to the total density of excursions

$$n = \frac{N}{T} = \int dx \int dy Q(x,y) = \int dx\ Q^+(x) = \int dy\ Q^-(y) = \int_0^{+\infty} d\tau \int dx \int dy\ q(\tau, y, x) \tag{111}$$

It is important to stress that even the empirical current $j(.)$ can actually be reconstructed via the formula

$$j(z) = \int_0^{+\infty} d\tau \left[\int_z^{+\infty} dy \int_{-\infty}^{z} dx\ q(\tau, y, x) - \int_{-\infty}^{z} dy \int_z^{+\infty} dx\ q(\tau, y, x) \right] \tag{112}$$

whose derivative coincides with the stationarity condition of Eq 18

$$\begin{aligned} j'(z) &= \int_0^{+\infty} d\tau \left[-\int_{-\infty}^{z} dx\ q(\tau, z, x) + \int_z^{+\infty} dy\ q(\tau, y, z) - \int_z^{+\infty} dx\ q(\tau, z, x) + \int_{-\infty}^{z} dy\ q(\tau, y, z) \right] \\ &= \int_0^{+\infty} d\tau \left[\int_{-\infty}^{+\infty} dy\ q(\tau, y, z) - \int_{-\infty}^{+\infty} dx\ q(\tau, z, x) \right] = Q^+(z) - Q^-(z) \end{aligned} \tag{113}$$

The physical meaning of Eq. 112 is that the forward excursions $q(\tau, y, x)$ with $y > x$ correspond to a positive current contribution at any point of the interval $z \in]x, y[$ (first term), while the backward excursions $q(\tau, y, x)$ with $y < x$ correspond to a negative current contribution at any point of the interval $z \in]y, x[$ (second term).

Finally, for the empirical density $\rho(x)$, one must distinguish two cases :

(i) for jump-drift models without diffusion $D(x) = 0$, the empirical density can be reconstructed via $\rho(x) = \frac{j(x)}{v(x)}$ (Eq. 42) from the empirical current $j(x)$ of Eq. 112 as will be discussed in more detail in the subsection VI D.

(ii) for jump-diffusion models with non-vanishing diffusion $D(x) \neq 0$, the empirical density $\rho(.)$ cannot be reconstructed from the empirical excursions $q(\tau, y, x)$ alone, i.e. some information on the position during the excursions has been lost.

B. Large deviations for the empirical jumps and for the empirical excursions between jumps

The joint distribution of the density n of jumps, of the in-flow $Q^+(.)$, of the out-flow $Q^-(.)$, of jump-flow $Q(.,.)$, and of the density $q(.,.,.)$ of excursions between jumps, with its partial density $q(\tau)$ of the duration τ follows the large deviation form

$$\mathcal{P}_T[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)] \underset{T \to +\infty}{\simeq} \mathcal{C}[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)] e^{-T\mathcal{I}[Q^{\pm}(.); Q(.,.); q(.,.,.)]} \tag{114}$$

The constraints

$$\begin{aligned}
&\mathcal{C}[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)] = \delta\left(\int dy\ Q^-(y) - n\right) \delta\left(\int dx\ Q^+(x) - n\right) \\
&\delta\left(\int_0^{+\infty} d\tau \tau q(\tau) - 1\right) \left[\prod_{\tau>0} \delta\left(\int dy \int dx\ q(\tau, y, x) - q(\tau)\right)\right] \\
&\left[\prod_x \delta\left(\int dy\ Q(x, y) - Q^+(x)\right) \delta\left(\int_0^{+\infty} d\tau \int dy\ q(\tau, y, x) - Q^+(x)\right)\right] \\
&\left[\prod_y \delta\left(\int dx\ Q(x, y) - Q^-(y)\right) \delta\left(\int_0^{+\infty} d\tau \int dx\ q(\tau, y, x) - Q^-(y)\right)\right]
\end{aligned} \tag{115}$$

can be understood as follows : the first line contains the definition of the total density n of Eq. 111, the second line contains the normalization of Eq. 110 with the definition of $q(\tau)$ of Eq. 109, while the two last lines contain the definitions of the in-flow $Q^+(.)$ and of the out-flow $Q^-(.)$ of Eqs 106 and 108. The rate function corresponds to the alternate Markov chain of Eq. 82 and contains the two corresponding contributions :

$$\mathcal{I}[Q^{\pm}(.); Q(.,.); q(.,.,.)] = \mathcal{I}^{[\Pi]}[Q^-(.); Q(.,.)] + \mathcal{I}^{[W]}[Q^+(.); q(.,.,.)] \tag{116}$$

(i) The first contribution involving the jump kernel $\Pi(x|y)$

$$\mathcal{I}^{[\Pi]}[Q^-(.); Q(.,.)] = \int dx \int dy Q(x, y) \ln\left(\frac{Q(x, y)}{\Pi(x|y) Q^-(y)}\right) \equiv I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] \tag{117}$$

coincides with the contribution $I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)]$ of Eq. 30 discussed previously.

(ii) The second contribution involving the excursion kernel $W(\tau, y|x)$ of Eq. 84

$$\mathcal{I}^{[W]}[Q^+(.); q(.,.,.)] = \int_0^{+\infty} d\tau \int dx \int dy\ q(\tau, y, x) \ln\left(\frac{q(\tau, y, x)}{W(\tau, y|x) Q^+(x)}\right) \tag{118}$$

takes into account the diffusion coefficient $D(.)$, the drift $v(.)$ and the jump rate $\lambda(.)$ that determine the excursion kernel $W(\tau, y|x)$.

C. Steady state properties from the point of view of the jump events only

From the point of view of the jump events only, the steady state properties make the rate functions of Eq. 117 and Eq. 118 vanish

$$\begin{aligned}
Q_*(x, y) &= \Pi(x|y) Q_*^-(y) \\
q_*(\tau, y, x) &= W(\tau, y|x) Q_*^+(x)
\end{aligned} \tag{119}$$

and satisfy all the constraints of Eq. 115

$$\begin{aligned}
1 &= \int_0^{+\infty} d\tau\tau \int dx \int dy q_*(\tau,y,x) = \int_0^{+\infty} d\tau\tau \int dx \int dy W(\tau,y|x) Q_*^+(x) \\
n_* &= \int dy\ Q_*^-(y) \\
n_* &= \int dx\ Q_*^+(x) \\
Q_*^+(x) &= \int dy\ Q_*(x,y) = \int dy \Pi(x|y) Q_*^-(y) \\
Q_*^-(y) &= \int_0^{+\infty} d\tau \int dx\ q_*(\tau,y,x) = \int_0^{+\infty} d\tau \int dx W(\tau,y|x) Q_*^+(x)
\end{aligned} \tag{120}$$

while the two remaining constraints are automatically satisfied as a consequence of the normalizations of Eqs 1 and 83

$$\begin{aligned}
Q_*^+(x) &= \int_0^{+\infty} d\tau \int dy\ q_\tau^*(y,x) = \left[\int_0^{+\infty} d\tau \int dy W(\tau,y|x) \right] Q_*^+(x) = Q_*^+(x) \\
Q_*^-(y) &= \int dx\ Q_*(x,y) = \left[\int dx \Pi(x|y) \right] Q_*^-(y) = Q_*^-(y)
\end{aligned} \tag{121}$$

In summary, the steady state properties of the jump events alone can be found as follows. The in-flow $Q_*^+(x)$ is the steady state of the global composite kernel

$$Q_*^+(x) = \int dx' \left[\int dy \Pi(x|y) \int_0^{+\infty} d\tau W(\tau,y|x') \right] Q_*^+(x') \tag{122}$$

with the normalization

$$1 = \int_0^{+\infty} d\tau\tau \int dx \int dy W(\tau,y|x) Q_*^+(x) \tag{123}$$

The jump density n_* and the out-flow $Q^- * (y)$ can be then computed via

$$\begin{aligned}
n_* &= \int dx\ Q_*^+(x) \\
Q_*^-(y) &= \int_0^{+\infty} d\tau \int dx W(\tau,y|x) Q_*^+(x)
\end{aligned} \tag{124}$$

D. Simplifications of empirical excursions for jump-drift models without diffusion $D(y) = 0$

For jump-drift models without diffusion $D(y) = 0$, the factorization of Eq. 92 for the excursion kernel

$$W(\tau,y|x) = W^{sp}(y|x) \delta \left(\tau - \int_x^y \frac{dz}{v(z)} \right) \tag{125}$$

yields that the empirical excursions display the same factorization between the spatial part $q^{sp}(y,x)$ and the delta function for the corresponding duration τ

$$q(\tau,y,x) = q^{sp}(y,x) \delta \left(\tau - \int_x^y \frac{dz}{v(z)} \right) \tag{126}$$

The spatial part $q^{sp}(y,x)$ contains the same Heaviside function $\theta \left(\int_x^y \frac{dz}{v(z)} > 0 \right)$ as Eq. 93

$$q^{sp}(y,x) = \int_0^{+\infty} d\tau q(\tau,y,x) = q^{sp}(y,x) \theta \left(\int_x^y \frac{dz}{v(z)} > 0 \right) \tag{127}$$

As a consequence, one can rewrite in terms of the spatial part $q^{sp}(y,x)$ both the rate function of Eq. 118

$$\mathcal{I}^{[W^{sp}]}_{[D(.)=0]}[Q^+(.);q^{sp}(.,.)] = \int dx \int dy\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) \ln\left(\frac{q^{sp}(y,x)}{W^{sp}(y|x)Q^+(x)}\right) \tag{128}$$

and the constraints of Eq. 115

$$\begin{aligned}
&\mathcal{C}_{[D(.)=0]}[n;Q^{\pm}(.);Q(.,.);q^{sp}(.,.)] = \delta\left(\int dy\ Q^-(y) - n\right)\delta\left(\int dx\ Q^+(x) - n\right) \\
&\delta\left(\int dx \int dy q^{sp}(y,x)\left[\int_x^y \frac{dz}{v(z)}\right]\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) - 1\right) \\
&\left[\prod_x \delta\left(\int dy\ Q(x,y) - Q^+(x)\right)\delta\left(\int dy\ q^{sp}(y,x)\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) - Q^+(x)\right)\right] \\
&\left[\prod_y \delta\left(\int dx\ Q(x,y) - Q^-(y)\right)\delta\left(\int dx\ q^{sp}(y,x)\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) - Q^-(y)\right)\right]
\end{aligned} \tag{129}$$

The reconstruction of the empirical current via Eq. 112 becomes

$$j(z) = \theta(v(z)>0)\int_z^{+\infty} dy \int_{-\infty}^z dx\ q^{sp}(y,x) - \theta(v(z)<0)\int_{-\infty}^z dy \int_z^{+\infty} dx\ q^{sp}(y,x) \tag{130}$$

while the empirical density can be then obtained from Eq. 42 in the absence of diffusion $D(x)=0$

$$\begin{aligned}
\rho(z) &= \frac{j(z)}{v(z)} = \frac{1}{v(z)}\left[\theta(v(z)>0)\int_z^{+\infty} dy \int_{-\infty}^z dx\ q^{sp}(y,x) - \theta(v(z)<0)\int_{-\infty}^z dy \int_z^{+\infty} dx\ q^{sp}(y,x)\right] \\
&= \frac{1}{|v(z)|}\left[\theta(v(z)>0)\int_z^{+\infty} dy \int_{-\infty}^z dx\ q^{sp}(y,x) + \theta(v(z)<0)\int_{-\infty}^z dy \int_z^{+\infty} dx\ q^{sp}(y,x)\right]
\end{aligned} \tag{131}$$

Note that the normalization of this expression for the empirical density coincides with the second line of Eq. 129 concerning the durations of excursions

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} dz\rho(z) = \int_{-\infty}^{+\infty} dy \int_{-\infty}^y dx\ q^{sp}(y,x)\int_x^y \frac{dz}{v(z)} - \int_{-\infty}^{+\infty} dy \int_y^{+\infty} dx\ q^{sp}(y,x)\int_y^x \frac{dz}{v(z)} \\
&= \int dx \int dy\ q^{sp}(y,x)\left[\int_x^y \frac{dz}{v(z)}\right]\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right)
\end{aligned} \tag{132}$$

Putting everything together, one obtains that for jump-drift models without diffusion $D(x)=0$, the joint distribution of the empirical density $\rho(.)$, the jump density n, the in-flow and the out-flow $Q^{\pm}(.)$, the jump flow $Q(.,.)$, and the spatial excursion density $q^{sp}(.,.)$ follow the large deviations at Level 2.75

$$P_T^{2.75[D(.)=0]}[\rho(.),n,Q^{\pm}(.),Q(.,.),q^{sp}(.,.)] \underset{T\to+\infty}{\simeq} \mathcal{C}_{2.75}[\rho(.),n,Q^{\pm}(.),Q(.,.),,q^{sp}(.,.)]e^{-T\mathcal{I}_{2.75}[Q^{\pm}(.),Q(.,.),q^{sp}(.,.)]} \tag{133}$$

The constraints $\mathcal{C}_{2.75}[\rho(.),n,Q^{\pm}(.),Q(.,.),,q^{sp}(.,.)]$ at Level 2.75 include the constraints $C_{2.5}^{[D(.)=0]}[\rho(.),n,Q^{\pm}(.),Q(.,.)]$ of the Level 2.5 of Eq. 44 and contains in addition the definitions of the in-flow Q^+ and of the out-flow Q^- in terms of the spatial excursions $q^{sp}(.,.)$

$$\mathcal{C}_{2.75}[\rho(.),n,Q^{\pm}(.),Q(.,.),,q^{sp}(.,.)] = C_{2.5}^{[D(.)=0]}[\rho(.),n,Q^{\pm}(.),Q(.,.)] \tag{134}$$

$$\left[\prod_x \delta\left(\int dy\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) - Q^+(x)\right)\right]\left[\prod_y \delta\left(\int dx\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) - Q^-(y)\right)\right] \tag{135}$$

The rate function at Level 2.75 contains the jump contribution of Eq. 117 and the excursion contribution of Eq. 128

$$\mathcal{I}_{2.75}[Q^{\pm}(.),Q(.,.),q^{sp}(.,.)] = \mathcal{I}^{[\Pi]}[Q^-(.);Q(.,.)] + \mathcal{I}^{[W]}_{[D(.)=0]}[Q^+(.);q^{sp}(.,.)] \tag{136}$$

The comparison with the Level 2.5 of Eq. 43 yields that the conditional probability to see spatial excursions $q^{sp}(.,.)$ once all the other empirical observables are given reduces to

$$\mathcal{P}_T^{conditional}[q^{sp}(.,.)|\rho(.),n,Q^{\pm}(.),Q(.,.)] \equiv \frac{\mathcal{P}_T^{2.75}[\rho(.),j(.),n,Q^{\pm}(.),Q(.,.),q^{sp}(.,.)]}{P_T^{2.5}[\rho(.),n,Q^{\pm}(.),Q(.,.)]}$$
$$\underset{T\to+\infty}{\simeq} \left[\prod_x \delta\left(\int dy\ q^{sp}(y,x)\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) - Q^+(x)\right)\right]\left[\prod_y \delta\left(\int dx\ q^{sp}(y,x)\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) - Q^-(y)\right)\right]$$
$$e^{-T\mathcal{I}^{conditional}[\rho(.),Q^{\pm}(.),q^{sp}(.,.)]} \tag{137}$$

where the conditional rate function reads

$$\mathcal{I}^{conditional}[\rho(.),Q^{\pm}(.),q^{sp}(.,.)] = \mathcal{I}_{2.75}[Q^{\pm}(.),Q(.,.),q^{sp}(.,.)] - I_{2.5}^{[D(.)=0]}[\rho(.),Q^{\pm}(.),Q(.,.) \tag{138}$$
$$= \mathcal{I}_{[D(.)=0]}^{[W^{sp}]}[Q^+(.);q^{sp}(.,.)] - I_{2.5}^{[\lambda]}[\rho(.),Q^-(.)]$$
$$= \int dx \int dy\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) \ln\left(\frac{q^{sp}(y,x)}{W^{sp}(y|x)Q^+(x)}\right) - \int dy \left[Q^-(y)\ln\left(\frac{Q^-(y)}{\lambda(y)\rho(y)}\right) - Q^-(y) + \lambda(y)\rho(y)\right]$$

The explicit form of the spatial kernel $W^{sp}(y|x)$ of Eq. 93, can be translated for the effective spatial kernel $\hat{W}^{sp}(y|x)$ associated to the effective jump rate $\hat{\lambda}(y) \equiv \frac{Q^-(y)}{\rho(y)}$ of Eq. 34

$$\hat{W}^{sp}(y|x) = \frac{\hat{\lambda}(y)}{|v(y)|}\ e^{-\int_x^y dz \frac{\hat{\lambda}(z)}{v(z)}}\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) = \frac{Q^-(y)}{\rho(y)|v(y)|}\ e^{-\int_x^y dz \frac{Q^-(z)}{\rho(z)v(z)}}\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) \tag{139}$$

This effective spatial kernel $\hat{W}^{sp}(y|x)$ is useful to rewrite the conditional rate function of Eq. 138 as

$$\mathcal{I}^{conditional}[\rho(.),Q^{\pm}(.),q^{sp}(.,.)] = \mathcal{I}_{[D(.)=0]}^{[\hat{W}]}[Q^+(.);q^{sp}(.,.)]$$
$$= \int dx \int dy\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) \ln\left(\frac{q^{sp}(y,x)}{\hat{W}^{sp}(y|x)Q^+(x)}\right)$$
$$= \int dx \int dy\ \theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) q^{sp}(y,x) \ln\left(\frac{q^{sp}(y,x)}{\frac{Q^-(y)}{\rho(y)|v(y)|}\ e^{-\int_x^y dz \frac{Q^-(z)}{\rho(z)v(z)}}Q^+(x)}\right) \tag{140}$$

This factorized form shows that this conditional rate function vanishes for the optimal value

$$q_{opt}^{sp}(y,x) = \hat{W}^{sp}(y|x)Q^+(x) = Q^+(x)\frac{Q^-(y)}{\rho(y)|v(y)|}\ e^{-\int_x^y dz \frac{Q^-(z)}{\rho(z)v(z)}}\theta\left(\int_x^y \frac{dz}{v(z)} \geq 0\right) \tag{141}$$

once all the other one-position empirical observables $[\rho(.),Q^{\pm}(.)]$ are given. Three examples of jump-drift models without diffusion $D(x) = 0$ will be described in sections VII, VIII, IX.

VII. EXAMPLE: JUMP-DRIFT PROCESS $[v(x) > 0, \lambda(x)]$ WITH ORIGIN RESETTING $\Pi(x|y) = \delta(x)$

In this section, we consider the positive jump-drift process $x(t) \geq 0$ without diffusion $D(x) = 0$, with the space-dependent positive velocity $v(x) > 0$, and the space-dependent jump rate $\lambda(x)$, while the jump probability

$$\Pi(x|y) = \delta(x) \tag{142}$$

describes the stochastic resetting towards the origin $x = 0$ (see the review [10] on stochastic resetting and references therein). The large deviations at Level 2.5 for resetting towards the origin have been already discussed in detail in [30] for discrete-time Markov chains, continuous-time Markov jump processes and diffusion processes. However, the present example corresponding to the continuous-time continuous-space version of the Sisyphus Random Walk [77] in an arbitrary space-dependent landscape parametrized by drift $v(x) > 0$, and the jump rate $\lambda(x)$, is useful here as the simplest possible application of the present general formalism, and as a comparison for the more complicated examples considered in the next sections.

A. Normalizability of the steady state $\rho_*(x)$

The steady-state $\rho_*(x)$ satisfying Eq. 7

$$\frac{d}{dx}\left[v(x)\rho_*(x)\right] + \lambda(x)\rho_*(x) = \delta(x)\int_0^{+\infty} dy\lambda(y)\rho_*(y) \tag{143}$$

reads

$$\rho_*(x) = \rho_*(0)\frac{v(0)}{v(x)}e^{-\int_0^x dy\frac{\lambda(y)}{v(y)}} \tag{144}$$

It is normalizable if the following integral involving the drift $v(.)$ and the jump rate $\lambda(.)$ converges

$$1 = \int_0^{+\infty} dx\rho_*(x) = \rho_*(0)v(0)\int_0^{+\infty}\frac{dx}{v(x)}e^{-\int_0^x dy\frac{\lambda(y)}{v(y)}} \tag{145}$$

B. Large deviations at Level 2.5

For the present model, the large deviations at Level 2.5 are greatly simplified because there is no diffusion $D(x) = 0$ (Eq. 43) and because the jump probability is deterministic $\Pi(x|y) = \delta(x)$ (Eq. 50). As a consequence, one obtains that the joint distribution of the empirical density $\rho(.)$ and of the empirical out-flow $Q^-(.)$ with the corresponding density n reads

$$\begin{aligned} P_T[\rho(.), n, Q^-(.)] \quad &\underset{T\to+\infty}{\simeq} \delta\left(\int_0^{+\infty} dx\rho(x) - 1\right)\delta\left(\int_0^{+\infty} dy\ Q^-(y) - n\right)\left[\prod_{x\geq 0}\delta\left(\frac{d}{dx}\left[\rho(x)v(x)\right] + Q^-(x) - n\delta(x)\right)\right] \\ &e^{-T\int_0^{+\infty} dy\left[Q^-(y)\ln\left(\frac{Q^-(y)}{\lambda(y)\rho(y)}\right) - Q^-(y) + \lambda(y)\rho(y)\right]} \end{aligned} \tag{146}$$

while the in-flow $Q^+(.)$ can be obtained from the density n alone

$$Q^+(x) = n\delta(x) \tag{147}$$

and the jump-flow $Q(.,.)$ can be computed from the out-flow $Q^-(.)$ alone

$$Q(x,y) = \delta(x)Q^-(y) \tag{148}$$

1. *Large deviations at Level 2 for the empirical density $\rho(.)$ alone*

One can use the stationarity constraint in Eq. 146 to eliminate the out-flow $Q^-(.)$ in terms of the empirical density $\rho(.)$ for $x > 0$

$$Q^-(x) = -\frac{d}{dx}\left[\rho(x)v(x)\right] \tag{149}$$

The jump density n is then related to the empirical density $\rho(x=0)$ at the origin

$$n = \int_0^{+\infty} dxQ^-(x) = \rho(0)v(0) \tag{150}$$

So one obtains that the large deviations at Level 2 for the empirical density $\rho(.)$ alone reads

$$P_T[\rho(.)] \quad \underset{T\to+\infty}{\simeq} \delta\left(\int_0^{+\infty} dx\rho(x) - 1\right)e^{-TI_2[\rho(.)]} \tag{151}$$

where the rate function at Level 2 reads

$$I_2[\rho(.)] \equiv \int_0^{+\infty} dx \left[\left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right) \ln \left(\frac{\left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right)}{\lambda(x) \rho(x)} \right) + \frac{d}{dx} \left[\rho(x) v(x) \right] + \lambda(x) \rho(x) \right]$$

$$= \int_0^{+\infty} dx \left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right) \ln \left(\frac{\left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right)}{\frac{\lambda(x)}{v(x)}} \right) + \int_0^{+\infty} dx \frac{d}{dx} \left(\left[\rho(x) v(x) \right] \ln \left[\rho(x) v(x) \right] \right) + \int_0^{+\infty} dx \lambda(x) \rho(x)$$

$$= \int_0^{+\infty} dx \left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right) \ln \left(\frac{\left(-\frac{d}{dx} \left[\rho(x) v(x) \right] \right)}{\frac{\lambda(x)}{v(x)}} \right) - \left[\rho(0) v(0) \right] \ln \left[\rho(0) v(0) \right] + \int_0^{+\infty} dx \lambda(x) \rho(x) \tag{152}$$

2. *Large deviations for the large deviations for the out-flow $Q^-(.)$ and the density n alone*

One can instead use the stationarity constraint to eliminate the empirical density $\rho(.)$ in terms of the out-flow $Q^-(.)$

$$\rho(x) = \frac{1}{v(x)} \int_x^{+\infty} dy Q^-(y) \tag{153}$$

The normalization of the empirical density becomes

$$\int_0^{+\infty} dx \rho(x) = \int_0^{+\infty} dy Q^-(y) \int_0^y \frac{dx}{v(x)} \tag{154}$$

So one obtains the large deviation form

$$P_T[n, Q^-(.)] \underset{T \to +\infty}{\simeq} \delta \left(\int_0^{+\infty} dy Q^-(y) \int_0^y \frac{dx}{v(x)} - 1 \right) \delta \left(\int_0^{+\infty} dy \; Q^-(y) - n \right) e^{-TI[n, Q^-(.)]} \tag{155}$$

with the rate function translated from Eq. 152

$$\begin{aligned} I[n, Q^-(.)] &= \int_0^{+\infty} dx Q^-(x) \ln \left(\frac{Q^-(x)}{\frac{\lambda(x)}{v(x)}} \right) - n \ln n + \int_0^{+\infty} dx \frac{\lambda(x)}{v(x)} \int_x^{+\infty} dy Q^-(y) \\ &= \int_0^{+\infty} dy Q^-(y) \ln \left(\frac{Q^-(y)}{\frac{\lambda(y)}{v(y)}} \right) - n \ln n + \int_0^{+\infty} dy Q^-(y) \int_0^y dx \frac{\lambda(x)}{v(x)} \\ &= \int_0^{+\infty} dy Q^-(y) \ln \left(\frac{Q^-(y)}{\frac{\lambda(y)}{v(y)} e^{-\int_0^y dx \frac{\lambda(x)}{v(x)}} n} \right) \end{aligned} \tag{156}$$

C. Excursions between jumps

The excursion kernel reduces to Eq. 92 for $x = 0$ and $y \geq 0$

$$W(\tau, y|0) = W_+^{sp}(y|0) \delta \left(\tau - \int_0^y \frac{dz}{v(z)} \right) \tag{157}$$

with Eq. 95

$$W_+^{sp}(y|0) = \frac{\lambda(y)}{v(y)} \; e^{-\int_0^y dz \frac{\lambda(z)}{v(z)}} = -\frac{d}{dy} \; e^{-\int_0^y dz \frac{\lambda(z)}{v(z)}} \tag{158}$$

For the present model, the empirical excursions can be rewritten in terms of the out-flow $Q^-(.)$

$$\begin{aligned} q^{sp}(y, x) &= Q^-(y) \delta(x) \\ q(\tau, y, x) &= Q^-(y) \delta(x) \delta \left(\tau - \int_0^y \frac{dz}{v(z)} \right) \end{aligned} \tag{159}$$

and thus do not contain additional information with respect to the previous subsections.

VIII. EXAMPLE: JUMP-DRIFT PROCESS $[v(x) = v, \lambda(x) = \lambda]$ WITH $\Pi^{deter}(x|y) = \delta(x - \gamma y)$

As an example where the new position x after the jump follows some non-trivial deterministic rule $x = \Phi(y)$ (Eq. 46), let us consider the positive jump-drift process $x(t) \geq 0$ without diffusion $D(x) = 0$, with the uniform positive velocity $v(x) = v > 0$ and with the uniform jump rate $\lambda(y) = \lambda$, while the jump probability of parameter $\gamma \in]0, 1[$ (for instance $\gamma = \frac{1}{2}$)

$$\Pi^{deter}(x|y) = \delta\left(x - \gamma y\right) \tag{160}$$

describes backward jumps from y to $x = \gamma y < y$ [78].

A. Steady state $\rho_*(x)$ via its moments

The dynamics of Eq. 2

$$\frac{\partial \rho_t(x)}{\partial t} = -\frac{d}{dx}\left[v\rho_t(x)\right] - \lambda\rho_t(x) + \lambda \int dy\ \delta\left(x - \gamma y\right)\rho_t(y) \tag{161}$$

yields that the integer moments

$$\langle x^k \rangle_t \equiv \int_0^{+\infty} dx x^k \rho_t(x) \tag{162}$$

satisfy the closed dynamical equations

$$\frac{\partial \langle x^k \rangle_t}{\partial t} = kv\langle x^{k-1} \rangle_t - \lambda(1 - \gamma^k)\langle x^k \rangle_t \tag{163}$$

where the first terms $k = 1$ and $k = 2$ read

$$\begin{aligned} \frac{\partial \langle x \rangle_t}{\partial t} &= v - \lambda(1 - \gamma)\langle x \rangle_t \\ \frac{\partial \langle x^2 \rangle_t}{\partial t} &= 2v\langle x \rangle_t - \lambda(1 - \gamma^2)\langle x^2 \rangle_t \end{aligned} \tag{164}$$

As a consequence, the moments of the steady state $\rho_*(x)$ can be computed recursively

$$\begin{aligned} \langle x \rangle_* &= \frac{v}{\lambda(1 - \gamma)} \\ \langle x^2 \rangle_* &= \frac{2v}{\lambda(1 - \gamma^2)}\langle x \rangle_* = \frac{2v^2}{\lambda^2(1 - \gamma)(1 - \gamma^2)} \\ &... \\ \langle x^k \rangle_* &= \frac{kv}{\lambda(1 - \gamma^k)}\langle x^{k-1} \rangle_* = \left(\frac{v}{\lambda}\right)^k \frac{k!}{\prod_{k'=1}^{k}(1 - \gamma^{k'})} \end{aligned} \tag{165}$$

and lead to the serie representation of the Laplace transform

$$\hat{\rho}_*(s) \equiv \int_0^{+\infty} dx e^{-sx}\rho_*(x) = \sum_{k=0}^{+\infty} \frac{(-s)^k}{k!}\langle x^k \rangle_* = \sum_{k=0}^{+\infty} \frac{\left(-s\frac{v}{\lambda}\right)^k}{\prod_{k'=1}^{k}(1 - \gamma^{k'})} \tag{166}$$

The conditions for the existence of the steady state are analyzed in [78] for the more general models where both the drift $v(x)$ and the jump rate $\lambda(x)$ are polynomial functions of x.

B. Large deviations at Level 2.5

Since there is no diffusion $D(x) = 0$ and since the jump probability is deterministic $\Pi(x|y) = \delta(x - \gamma y)$, the large deviations at Level 2.5 simplify (Eqs 43 and 50) into the joint distribution of the empirical density $\rho(.)$ and of the empirical out-flow $Q^-(.)$ with the corresponding density n

$$\begin{aligned} P_T[\rho(.), n, Q^-(.)] \quad & \underset{T\to+\infty}{\simeq} \delta\left(\int_0^{+\infty} dx\rho(x) - 1\right) \delta\left(\int_0^{+\infty} dy\ Q^-(y) - n\right) \left[\prod_{x\geq 0} \delta\left(v\rho'(x) + Q^-(x) - \frac{1}{\gamma} Q^-\left(\frac{x}{\gamma}\right)\right)\right] \\ & e^{-T\int_0^{+\infty} dy \left[Q^-(y) \ln\left(\frac{Q^-(y)}{\lambda\rho(y)}\right) - Q^-(y) + \lambda\rho(y)\right]} \end{aligned} \tag{167}$$

while the jump-flow $Q(.,.)$ and the in-flow $Q^+(.)$ can be computed from the out-flow $Q^-(.)$

$$\begin{aligned} Q(x,y) &= \delta(x - \gamma y) Q^-(y) \\ Q^+(x) &= \frac{1}{\gamma} Q^-\left(\frac{x}{\gamma}\right) \end{aligned} \tag{168}$$

One can use the stationarity constraint to eliminate the empirical density $\rho(.)$ in terms of the out-flow $Q^-(.)$

$$\rho(x) = \frac{1}{v} \int_x^{\frac{x}{\gamma}} dy Q^-(y) \tag{169}$$

The normalization of the empirical density becomes

$$\int_0^{+\infty} dx\rho(x) = \frac{1}{v} \int_0^{+\infty} dy Q^-(y) \int_{\gamma y}^{y} dx = \frac{1-\gamma}{v} \int_0^{+\infty} dy y Q^-(y) \tag{170}$$

So the large deviations for the out-flow $Q^-(.)$ and the density n reduce to

$$\begin{aligned} P_T[n, Q^-(.)] \quad & \underset{T\to+\infty}{\simeq} \delta\left(\frac{1-\gamma}{v} \int_0^{+\infty} dy y Q^-(y) - 1\right) \delta\left(\int dy\ Q^-(y) - n\right) \\ & e^{-T\left[-n + \lambda + \int_0^{+\infty} dx Q^-(x) \ln\left(\frac{Q^-(x)}{\frac{\lambda}{v}\int_x^{\frac{x}{\gamma}} dy Q^-(y)}\right)\right]} \end{aligned} \tag{171}$$

C. Large deviations for excursions between jumps

Here the excursion kernel reduces to Eq. 92

$$W(\tau, y|x) = W_+^{sp}(y|x) \delta\left(\tau - \int_x^y \frac{dz}{v}\right) = W_+^{sp}(y|x) \delta\left(\tau - \frac{y-x}{v}\right) \tag{172}$$

with Eq. 95 for $y \geq x$

$$W_+^{sp}(y|x) = \frac{\lambda}{v}\ e^{-\int_x^y dz \frac{\lambda}{v}} = \frac{\lambda}{v}\ e^{-\frac{\lambda}{v}(y-x)} \tag{173}$$

The empirical excursions contain the same delta function for the duration τ as in Eq 172

$$q(\tau, y, x) = q^{sp}(y, x) \delta\left(\tau - \frac{y-x}{v}\right) \tag{174}$$

where the spatial part $q^{sp}(y, x)$ defined for $y \geq x$ can fluctuate according to the conditional probability of Eq. 137

$$\begin{aligned} & \mathcal{P}_T^{conditional}[q^{sp}(.,.)|\rho(.), Q^-(.)] \underset{T\to+\infty}{\simeq} e^{-T\mathcal{I}^{conditional}[\rho(.), Q^-(.), q^{sp}(.,.)]} \\ & \left[\prod_{x>0} \delta\left(\int_x^{+\infty} dy\ q^{sp}(y,x) - \frac{1}{\gamma} Q^-\left(\frac{x}{\gamma}\right)\right)\right] \left[\prod_{y>0} \delta\left(\int_0^y dx\ q^{sp}(y,x) - Q^-(y)\right)\right] \end{aligned} \tag{175}$$

where the conditional rate function (Eq. 140)

$$\mathcal{I}^{conditional}[\rho(.),Q^-(.),q^{sp}(.,.)]=\int_0^{+\infty}dx\int_x^{+\infty}dy\ q^{sp}(y,x)\ln\left(\frac{q^{sp}(y,x)}{\frac{Q^-(y)}{\rho(y)v}\ \ e^{-\int_x^y dz\frac{Q^-(z)}{\rho(z)v}}\frac{1}{\gamma}Q^-\left(\frac{x}{\gamma}\right)}\right) \tag{176}$$

governs the fluctuations around the optimal value

$$q^{sp}_{opt}(y,x)=\frac{Q^-(y)}{\rho(y)v}\ \ e^{-\int_x^y dz\frac{Q^-(z)}{\rho(z)v}}\frac{1}{\gamma}Q^-\left(\frac{x}{\gamma}\right) \tag{177}$$

once $[\rho(.),Q^-(.)]$ are given.

IX. EXAMPLE OF JUMP-DRIFT PROCESS $[v(x)=-x,\lambda(x)=\lambda]$WITH $\Pi(x|y)=\theta(x\geq y)\alpha e^{-\alpha(x-y)}$

In this section, we consider the case of the positive jump-drift process with the linear negative drift $v(x)=-x$, with the uniform jump rate $\lambda(x)=\lambda$, and where the jump probability describes positive jumps whose amplitude $z=x-y\geq 0$ is exponentially distributed

$$\Pi^{exp}_\alpha(x|y)=\alpha e^{-\alpha(x-y)}\ \ \text{for}\ \ x\in[y,+\infty[\tag{178}$$

This exponential distribution is often considered in soil moisture models in order to represent rainfall events [6–9].

A. Steady state $\rho_*(x)$

The steady-state $\rho_*(x)$ satisfying Eq. 7

$$\frac{d}{dx}\left[-x\rho_*(x)\right]+\lambda\rho_*(x)=\int_0^x dy\alpha e^{-\alpha(x-y)}\lambda\rho_*(y) \tag{179}$$

is the Gamma-law of shape parameter λ and of scale parameter $\frac{1}{\alpha}$

$$\rho_*(x)\ \ =\frac{\alpha^\lambda}{\Gamma(\lambda)}\ x^{\lambda-1}e^{-\alpha x} \tag{180}$$

B. Large deviations at Level 2.5

Since there is no diffusion $D(x)=0$ (Eq. 43) and since the jumps have a positive amplitude $z=x-y\geq 0$, the large deviations at Level 2.5 read

$$P_T^{2.5[D(.)=0]}[\rho(.),n,Q^\pm(.),Q(.,.)]\underset{T\to+\infty}{\simeq}C_{2.5}^{[D(.)=0]}[\rho(.),n,Q^\pm(.),Q(.,.)]e^{-T\left[I_{2.5}^{[\lambda]}[\rho(.),Q^-(.)]+I_{2.5}^{[\Pi]}[Q^-(.),Q(.,.)]\right]} \tag{181}$$

with the constraints

$$C_{2.5}^{[D(.)=0]}[\rho(.),n,Q^\pm(.),Q(.,.)]=\delta\left(\int_0^{+\infty}dx\rho(x)-1\right)\delta\left(\int_0^{+\infty}dy\ Q^-(y)-n\right)\delta\left(\int_0^{+\infty}dx\ Q^+(x)-n\right) \tag{182}$$
$$\left[\prod_{x\geq 0}\delta\left(\frac{d}{dx}\left[-x\rho(x)\right]+Q^-(x)-Q^+(x)\right)\right]\left[\prod_{x\geq 0}\delta\left(\int_0^x dy\ Q(x,y)-Q^+(x)\right)\right]\left[\prod_{y\geq 0}\delta\left(\int_y^{+\infty}dx\ Q(x,y)-Q^-(y)\right)\right]$$

and with the two contributions to the rate function

$$\begin{aligned}I_{2.5}^{[\lambda]}[\rho(.),Q^-(.)]\ \ &=\int_0^{+\infty}dy\left[Q^-(y)\ln\left(\frac{Q^-(y)}{\lambda\rho(y)}\right)-Q^-(y)+\lambda\rho(y)\right]\\ I_{2.5}^{[\Pi(x|y)=\alpha e^{-\alpha(x-y)}]}[Q^-(.),Q(.,.)]\ \ &=\int_0^{+\infty}dy\int_x^{+\infty}dxQ(x,y)\ln\left(\frac{Q(x,y)}{\alpha e^{-\alpha(x-y)}Q^-(y)}\right)\end{aligned} \tag{183}$$

Since the jump probability is given by the exponential form of Eq. 178, the contraction over the jump flow $Q(.,.)$ can be explicitly computed : the joint distribution of $[\rho(.), n, Q^{\pm}(.)]$ reads (Eq. 79)

$$P_T^{[\Pi_\alpha^{exp}(x|y)=\alpha e^{-\alpha(x-y)}]}[\rho(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int_0^{+\infty} dx\rho(x) - 1\right)\left[\prod_{x>0}\delta\left(\frac{d}{dx}\left[-x\rho(x)\right] + Q^-(x) - Q^+(x)\right)\right]$$
$$\delta\left(\int_0^{+\infty} dy\ Q^-(y) - n\right)\delta\left(\int_0^{+\infty} dx\ Q^+(x) - n\right) e^{-T\left(I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_\alpha^{exp}[Q^+(.), \rho(.)]\right)} \tag{184}$$

with the rate function contribution of Eq. 78 after taking into account $j(x) = -x\rho(x)$

$$I_\alpha^{exp}[Q^+(.), \rho(.)] \equiv \int_0^{+\infty} dx\left[Q^+(x)\ln\left(\frac{Q^+(x)}{\alpha x\rho(x)}\right) - Q_+(x) + \alpha x\rho(x)\right] \tag{185}$$

C. Large deviations for excursions between jumps

Here the excursion kernel reduces to Eq. 92

$$W(\tau, y|x) = W_-^{sp}(y|x)\delta\left(\tau - \int_y^x \frac{dz}{z}\right) = W_-^{sp}(y|x)\delta\left(\tau - \ln\left(\frac{x}{y}\right)\right) \tag{186}$$

with Eq. 97 for $0 \leq y \leq x$

$$W_-^{sp}(y|x) = \frac{\lambda}{y}\ e^{-\int_y^x dz\frac{\lambda}{z}} = \frac{\lambda y^{\lambda-1}}{x^\lambda} \quad \text{for} \quad y \in [0, x] \tag{187}$$

The conditional probability to see spatial excursions $q^{sp}(.,.)$ once all the other empirical observables are given reads (Eq. 137)

$$\mathcal{P}_T^{conditional}[q^{sp}(.,.)|\rho(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} e^{-T\mathcal{I}^{conditional}[\rho(.), Q^{\pm}(.), q^{sp}(.,.)]}$$
$$\left[\prod_{x\geq 0}\delta\left(\int_0^x dy\ q^{sp}(y,x) - Q^+(x)\right)\right]\left[\prod_{y\geq 0}\delta\left(\int_y^{+\infty} dx\ q^{sp}(y,x) - Q^-(y)\right)\right] \tag{188}$$

where the conditional rate function (Eq. 140)

$$\mathcal{I}^{conditional}[\rho(.), Q^{\pm}(.), q^{sp}(.,.)] = \int_0^{+\infty} dx\int_0^x dy q^{sp}(y,x)\ln\left(\frac{q^{sp}(y,x)}{\frac{Q^-(y)}{y\rho(y)}\ e^{-\int_y^x dz\frac{Q^-(z)}{z\rho(z)}}Q^+(x)}\right) \tag{189}$$

governs the fluctuations around the optimal value

$$q_{opt}^{sp}(y,x) = \frac{Q^-(y)}{y\rho(y)}\ e^{-\int_y^x dz\frac{Q^-(z)}{z\rho(z)}}Q^+(x) \tag{190}$$

once $[\rho(.), Q^{\pm}(.)]$ are given.

X. EXAMPLE : JUMP-DIFFUSION $[D(x) = D, v(x) = 0, \lambda(x) = \lambda]$ WITH RESETTING $\Pi^{reset}(x|y) = R(x)$

As an example of stochastic resetting (see the review [11] and references therein) towards an arbitrary probability distribution $R(x)$ (Eq. 61)

$$\Pi^{reset}(x|y) = R(x) \tag{191}$$

instead of the resetting towards the origin discussed in section VII, let us consider the jump-diffusion process without drift $v(x) = 0$, with uniform diffusion coefficient $D(x) = D$ and uniform jump rate $\lambda(x) = \lambda$.

A. Steady state $\rho_*(x)$

The steady-state $\rho_*(x)$ satisfying Eq. 7

$$-D\frac{d^2\rho_*(x)}{d^2x} + \lambda\rho_*(x) = \lambda R(x) \tag{192}$$

can be written as

$$\rho_*(x) = \int_{-\infty}^{+\infty} dx_0 G(x,x_0) R(x_0) \tag{193}$$

where the Green function $G(x,x_0)$ satisfying

$$-D\frac{d^2G(x,x_0)}{dx^2} + \lambda G(x,x_0) = \lambda\delta(x-x_0) \tag{194}$$

corresponds to the elementary solution associated to the deterministic resetting towards x_0. The solution that is well-behaved at $x \to \pm\infty$

$$G(x,x_0) = \frac{1}{2}\sqrt{\frac{\lambda}{D}} e^{-|x-x_0|\sqrt{\frac{\lambda}{D}}} \tag{195}$$

yields the steady state (Eq. 193)

$$\rho_*(x) = \frac{1}{2}\sqrt{\frac{\lambda}{D}} \left[e^{-x\sqrt{\frac{\lambda}{D}}} \int_{-\infty}^{x} dx_0 R(x_0) e^{x_0\sqrt{\frac{\lambda}{D}}} + e^{x\sqrt{\frac{\lambda}{D}}} \int_{x}^{+\infty} dx_0 R(x_0) e^{-x_0\sqrt{\frac{\lambda}{D}}} \right] \tag{196}$$

B. Large deviations at Level 2.5

For the present model, the large deviations at Level 2.5 of Eq. 23 read with the constraints $C_{2.5}[\rho(.), j(.), Q(.,.)]$ given in Eq. 24

$$\begin{aligned} &P_T^{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] \underset{T\to+\infty}{\simeq} C_{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] \\ &e^{-T\left[I_{2.5}^{[D]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)]\right]} \end{aligned} \tag{197}$$

with the three contributions to the rate function

$$\begin{aligned} I_{2.5}^{[D]}[\rho(.), j(.)] &= \int_{-\infty}^{+\infty} \frac{dx}{4D\rho(x)} \left[j(x) + D\rho'(x)\right]^2 \\ I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] &= \int_{-\infty}^{+\infty} dy \left[Q^-(y) \ln\left(\frac{Q^-(y)}{\lambda\rho(y)}\right) - Q^-(y) + \lambda\rho(y) \right] \\ I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] &= \int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{R(x)Q^-(y)}\right) \end{aligned} \tag{198}$$

Since the jump probability corresponds to the stochastic resetting form of Eq. 191, the contraction over the jump flow $Q(.,.)$ can be explicitly computed : the joint distribution of $[\rho(.), n, Q^{\pm}(.)]$ reads (Eq 67) reads

$$\begin{aligned} &P_T^{\Pi^{reset}(x|y)=R(x)}[\rho(.), j(.), n, Q^{\pm}(.)] \underset{T\to+\infty}{\simeq} \delta\left(\int dx\rho(x) - 1\right) \left[\prod_x \delta\left(j'(x) + Q^-(x) - Q^+(x)\right)\right] \\ &\delta\left(\int dy\ Q^-(y) - n\right) \delta\left(\int dx\ Q^+(x) - n\right) e^{-T\left(I_{2.5}^{[D]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I^{[R]}[n, Q^+(.)]\right)} \end{aligned} \tag{199}$$

with the last contribution of the rate function

$$I^{[R]}[n, Q^+(.)] \equiv \int_{-\infty}^{+\infty} dx Q^+(x) \ln\left(\frac{Q^+(x)}{nR(x)}\right) \tag{200}$$

C. Large deviations for the jumps and for the excursions between jumps

Since the jump rate is uniform $\lambda(x) = \lambda$, the excursion kernel of Eq. 103 is factorized

$$W(\tau, y|x) = E^{exc}(\tau) P_\tau^{free}(y|x) \tag{201}$$

into the normalized exponential probability to see the duration $\tau \in]0, +\infty[$

$$E^{exc}(\tau) = \lambda e^{-\tau\lambda} \tag{202}$$

and into the gaussian free propagator (uniform diffusion $D(y) = D$ without drift $v(y) = 0$)

$$P_\tau^{free}(y|x) = \frac{1}{\sqrt{4\pi D\tau}} e^{-\frac{(y-x)^2}{4D\tau}} \tag{203}$$

The large deviations for the empirical jumps and for the empirical excursions between jumps of Eq. 114 read with the constraints $\mathcal{C}[n; Q^\pm(.); Q(.,.); q(.); q(.,.,.)]$ given in Eq. 115

$$\begin{aligned} \mathcal{P}_T[n; Q^\pm(.); Q(.,.); q(.); q(.,.,.)] \quad & \underset{T\to+\infty}{\simeq} \mathcal{C}[n; Q^\pm(.); Q(.,.); q(.); q(.,.,.)] \\ & e^{-T\left[\int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{R(x)Q^-(y)}\right) + \mathcal{I}^{[W]}[Q^+(.); q(.,.,.)]\right]} \end{aligned} \tag{204}$$

where the rate function contribution involving the excursion kernel $W(\tau, y|x)$ of Eq. 201 reads

$$\mathcal{I}^{[W]}[Q^+(.); q(.,.,.)] = \int_0^{+\infty} d\tau \int dx \int dy \; q(\tau, y, x) \ln\left(\frac{q(\tau, y, x)}{\frac{\lambda}{\sqrt{4\pi D\tau}} e^{-\tau\lambda - \frac{(y-x)^2}{4D\tau}} Q^+(x)}\right) \tag{205}$$

XI. EXAMPLE OF JUMP-DIFFUSION $[D(x) = D, v(x) = -x, \lambda(x) = \lambda]$ WITH $\Pi(x|y) = H(x-y)$

As last example, let us consider the case of uniform diffusion $D(x) = D$ with the linear drift towards the origin $v(x) = -x$, with uniform jump rate $\lambda(x) = \lambda$, while the jump probability involves an arbitrary function $H(z)$ of the amplitude $z = x - y$

$$\Pi(x|y) = H(x-y) \tag{206}$$

A. Steady state $\rho_*(x)$ and its Fourier transform $\hat{\rho}_*(k)$

Eq. 7 for the steady-state $\rho_*(x)$

$$-D\frac{d^2\rho_*(x)}{d^2x} - \frac{d}{dx}\left[x\rho_*(x)\right] + \lambda\rho_*(x) = \lambda \int dy H(x-y)\rho_*(y) \tag{207}$$

can be translated in terms of the Fourier transforms

$$\begin{aligned} \hat{\rho}_*(k) &\equiv \int_{-\infty}^{+\infty} dx e^{ikx} \rho_*(x) \\ \hat{H}(k) &\equiv \int_{-\infty}^{+\infty} dz e^{ikz} H(z) \end{aligned} \tag{208}$$

into the first-order differential equation in k

$$\frac{d\hat{\rho}_*(k)}{dk} = -Dk\hat{\rho}_*(k) + \lambda\left(\frac{\hat{H}(k) - 1}{k}\right)\hat{\rho}_*(k) \tag{209}$$

Using the normalization of the steady state

$$\hat{\rho}_*(k=0) = \int_{-\infty}^{+\infty} dx \rho_*(x) = 1 \tag{210}$$

the solution of Eq. 209 reads

$$\hat{\rho}_*(k) = e^{-\frac{Dk^2}{2} - \lambda \int_0^k dk' \left(\frac{1-\hat{H}(k')}{k'} \right)} \tag{211}$$

1. *Example of symmetric Lévy jumps $H(z) = L_\mu(z)$ of index $\mu \in]0,2[$*

An interesting example is when the size $z = x - y$ of the jump is drawn with the Lévy symmetric stable law $H(z) = L_\mu(z)$ of index $\mu \in]0,2[$ and of characteristic scale Δ

$$H(z) = L_\mu(z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikz - \Delta^\mu |k|^\mu} \tag{212}$$

displaying the power-law decay of exponent $(1+\mu)$

$$L_\mu(z) \underset{z \to \pm\infty}{\simeq} \frac{\Gamma(1+\mu) \sin\left(\frac{\pi\mu}{2}\right) \Delta^\mu}{\pi |z|^{1+\mu}} \tag{213}$$

For instance, the value $\mu = 1$ corresponds to the Cauchy distribution

$$L_{\mu=1}(z) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikz - \Delta|k|} = \frac{\Delta}{\pi(z^2 + \Delta^2)} \tag{214}$$

Then the steady state of Eq. 211

$$\hat{\rho}_*(k) = e^{-\frac{Dk^2}{2} - \lambda \int_0^k dk' \left(\frac{1 - e^{-\Delta^\mu |k'|^\mu}}{k'} \right)} = e^{-\frac{Dk^2}{2} - \lambda \int_0^{|k|} dk' \left(\frac{1 - e^{-\Delta^\mu (k')^\mu}}{k'} \right)} \tag{215}$$

inherits the Lévy singularity in $|k|^\mu$ near the origin $k \to 0$

$$\hat{\rho}_*(k) \underset{k \to 0}{\simeq} 1 - \frac{\lambda \Delta^\mu}{\mu} |k|^\mu \tag{216}$$

so that the steady state $\rho_*(x)$ decays only as the power-law $|x|^{-1-\mu}$ in real space $x \to \pm\infty$.

2. *Example of symmetric exponential jumps $H(z) = \frac{\alpha}{2} e^{-\alpha|z|}$*

When the size $z = x - y$ of the jump is drawn with the symmetric exponential distribution

$$H(z) = \frac{\alpha}{2} e^{-\alpha|z|} \tag{217}$$

its Fourier transform

$$\hat{H}(k) = \frac{1}{1 + \frac{k^2}{\alpha^2}} \tag{218}$$

yields that Eq. 211 becomes

$$\hat{\rho}_*(k) = e^{-\frac{Dk^2}{2} - \frac{\lambda}{2} \ln\left(1 + \frac{k^2}{\alpha^2}\right)} = \frac{e^{-\frac{Dk^2}{2}}}{\left(1 + \frac{k^2}{\alpha^2}\right)^{\frac{\lambda}{2}}} \tag{219}$$

B. Large deviations at Level 2.5

For the present model, the large deviations at Level 2.5 of Eq. 23 read with the constraints $C_{2.5}[\rho(.), j(.), Q(.,.)]$ given in Eq. 24

$$P_T^{2.5}[\rho(.), j(.), n, Q^{\pm}(.), Q(.,.)] \underset{T\to+\infty}{\simeq} C_{2.5}[\rho(.), j(.), Q(.,.)] e^{-T\left[I_{2.5}^{[D]}[\rho(.), j(.)] + I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] + I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)]\right]} \tag{220}$$

with the three contributions to the rate function

$$\begin{aligned} I_{2.5}^{[D,v]}[\rho(.), j(.)] &= \int \frac{dx}{4D\rho(x)} \left[j(x) + x\rho(x) + D\rho'(x)\right]^2 \\ I_{2.5}^{[\lambda]}[\rho(.), Q^-(.)] &= \int dy \left[Q^-(y) \ln\left(\frac{Q^-(y)}{\lambda\rho(y)}\right) - Q^-(y) + \lambda\rho(y)\right] \\ I_{2.5}^{[\Pi]}[Q^-(.), Q(.,.)] &= \int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{H(x-y)Q^-(y)}\right) \end{aligned} \tag{221}$$

C. Large deviations for the empirical jumps and for the empirical excursions between jumps

Since the jump rate is uniform $\lambda(x) = \lambda$, the excursion kernel of Eq. 103 is factorized

$$W(\tau, y|x) = E^{exc}(\tau) P_\tau^{free}(y|x) \tag{222}$$

into the normalized exponential probability to see the duration $\tau \in]0, +\infty[$

$$E^{exc}(\tau) = \lambda e^{-\tau\lambda} \tag{223}$$

and into the free Ornstein-Uhlenbeck propagator ($[D(x) = D, v(x) = -x]$)

$$P_\tau^{free}(y|x) = \frac{1}{\sqrt{2\pi D(1-e^{-2\tau})}} e^{-\frac{\left(y - xe^{-\tau}\right)^2}{2D(1-e^{-2\tau})}} \tag{224}$$

The large deviations for the empirical jumps and for the empirical excursions between jumps of Eq. 114 read with the constraints $\mathcal{C}[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)]$ given in Eq. 115

$$\begin{aligned} &\mathcal{P}_T[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)] \underset{T\to+\infty}{\simeq} \mathcal{C}[n; Q^{\pm}(.); Q(.,.); q(.); q(.,.,.)] \\ &e^{-T\left[\int dx \int dy Q(x,y) \ln\left(\frac{Q(x,y)}{H(x-y)Q^-(y)}\right) + \mathcal{I}^{[W]}[Q^+(.); q(.,.,.)]\right]} \end{aligned} \tag{225}$$

where the rate function contribution involving the excursion kernel $W(\tau, y|x)$ of Eq. 222 reads

$$\mathcal{I}^{[W]}[Q^+(.); q(.,.,.)] = \int_0^{+\infty} d\tau \int dx \int dy \; q(\tau, y, x) \ln\left(\frac{q(\tau, y, x)}{\frac{\lambda}{\sqrt{2\pi D(1-e^{-2\tau})}} e^{-\tau\lambda - \frac{\left(y - xe^{-\tau}\right)^2}{2D(1-e^{-2\tau})}} Q^+(x)}\right) \tag{226}$$

XII. CONCLUSION

In this paper, we have considered one-dimensional Jump-Drift and Jump-Diffusion processes, defined in terms of four space-dependent parameters, namely the drift $v(x)$, the diffusion coefficient $D(x)$, the jump rate $\lambda(x)$ and the jump probability $\Pi(x'|x)$. We have assumed that these parameters produce some normalizable steady state and we have analyzed the large deviations of a long dynamical trajectory from two points of view. We have first applied the Large deviations at Level 2.5 to the joint probability of the empirical time-averaged density $\rho(x)$, of the empirical time-averaged current $j(x)$ and of the empirical time-averaged jump-flow $Q(x, y)$. We have then focused on the alternate

Markov chain that governs the series of all the jump events of a long trajectory in order to obtain the large deviations at Level 2.5 for the joint probability of the empirical jumps and of the empirical excursions between consecutive jumps. Finally, we have applied these two general frameworks to three examples of positive jump-drift processes without diffusion, and to two examples of jump-diffusion processes, in order to illustrate various simplifications that may occur in rate functions and in contraction procedures.

---

[1] R. Cont and P. Tankov, Financial Modelling with Jump Processes, Chapman and Hall / CRC Press, 2004.
[2] P. Tankov and E. Voltchkova, Banque et Marchés, vol. 99, 2009.
[3] G. Dumont, J. Henry, C. O. Tarniceriu, Journal of Mathematical Biology 73, 1413 (2016).
[4] C. E. Miles, J. P. Keener, J. Phys. A: Math. Theor. 50 425003 (2017)
[5] E. Daly and A. Porporato, PRE 74, 041112 (2006).
[6] I. Rodriguez-Iturbe, A. Porporato, L. Ridolfi, V. Isham and D. R. Cox, Proc. R. Soc. Lond. A (1999) 455, 3789.
[7] E. Daly and A. Porporato, PRE 73, 026108 (2006).
[8] E. Daly and A. Porporato, PRE 75, 011119 (2007).
[9] M. S. Bartlett, E. Daly, J. J. McDonnell, A. J. Parolari, and A. Porporato, Proc. R. Soc. A 471: 20150389 (2015).
[10] O. Bénichou, C. Loverdo, M. Moreau, R. Voituriez Reviews of Modern Physics 83, 81 (2011).
[11] M. R. Evans, S. N. Majumdar, G. Schehr, J. Phys. A: Math. Theor. 53 193001 (2020).
[12] Y. Oono, Progress of Theoretical Physics Supplement 99, 165 (1989).
[13] R.S. Ellis, Physica D 133, 106 (1999).
[14] H. Touchette, Phys. Rep. 478, 1 (2009).
[15] A. de La Fortelle, Problems of Information Transmission 37 , 120 (2001).
[16] C. Maes and K. Netocny, Europhys. Lett. 82, 30003 (2008)
[17] C. Maes, K. Netocny and B. Wynants, Markov Proc. Rel. Fields. 14, 445 (2008).
[18] B. Wynants, arXiv:1011.4210, PhD Thesis (2010), "Structures of Nonequilibrium Fluctuations", Catholic University of Leuven.
[19] A. C. Barato and R. Chétrite, J. Stat. Phys. 160, 1154 (2015).
[20] L. Bertini, A. Faggionato and D. Gabrielli, Ann. Inst. Henri Poincare Prob. and Stat. 51, 867 (2015).
[21] L. Bertini, A. Faggionato and D. Gabrielli, Stoch. Process. Appli. 125, 2786 (2015).
[22] R. Chétrite, HDR Thesis (2018) "Pérégrinations sur les phénomènes aléatoires dans la nature", Laboratoire J.A. Dieudonné, Université de Nice.
[23] C. Monthus, J. Stat. Mech. (2019) 023206
[24] C. Monthus, J. Phys. A: Math. Theor. 52, 135003 (2019)
[25] C. Monthus, J. Phys. A: Math. Theor. 52, 025001 (2019)
[26] C. Monthus, J. Phys. A: Math. Theor. 52, 485001 (2019)
[27] A. C. Barato, R. Chétrite, J. Stat. Mech. (2018) 053207
[28] L. Chabane, R. Chétrite, G. Verley, J. Stat. Mech. (2020) 033208.
[29] A. de La Fortelle, PhD (2000) "Contributions to the theory of large deviations and applications" INRIA Rocquencourt.
[30] C. Monthus, J. Stat. Mech. (2021) 033201
[31] C. Monthus, arXiv:2101.09045
[32] C. Maes, K. Netocny and B. Wynants Physica A 387, 2675 (2008).
[33] J. Hoppenau, D. Nickelsen and A. Engel, New J. Phys. 18 083010 (2016).
[34] C. Monthus, J. Stat. Mech. (2021) 033303
[35] F. Coghi, R. Chetrite and H. Touchette, arXiv:2103.14150.
[36] G. Fayolle and A. de La Fortelle, Problems of Information Transmission 38, 354 (2002).
[37] C. Monthus, Eur. Phys. J. B 92, 149 (2019) in the topical issue " Recent Advances in the Theory of Disordered Systems" edited by F. Igloi and H. Rieger.
[38] B. Derrida, J. Stat. Mech. P07023 (2007).
[39] R. L. Jack, P. Sollich, The European Physical Journal Special Topics 224, 2351 (2015).
[40] A. Lazarescu, J. Phys. A: Math. Theor. 48 503001 (2015).
[41] A. Lazarescu, J. Phys. A: Math. Theor. 50 254004 (2017).
[42] R. L. Jack, Eur. Phy. J. B 93, 74 (2020)
[43] V. Lecomte, PhD Thesis (2007) "Thermodynamique des histoires et fluctuations hors d'équilibre" Université Paris 7.
[44] V. Lecomte, C. Appert-Rolland and F. van Wijland, Phys. Rev. Lett. 95 010601 (2005).
[45] V. Lecomte, C. Appert-Rolland and F. van Wijland, J. Stat. Phys. 127 51-106 (2007).
[46] V. Lecomte, C. Appert-Rolland and F. van Wijland, Comptes Rendus Physique 8, 609 (2007).
[47] J.P. Garrahan, R.L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk, F. van Wijland, Phys. Rev. Lett. 98, 195702 (2007).
[48] J.P. Garrahan, R.L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk and F. van Wijland, J. Phys. A 42, 075007 (2009).
[49] K. van Duijvendijk, R.L. Jack and F. van Wijland, Phys. Rev. E 81, 011110 (2010).
[50] R. L. Jack, P. Sollich, Prog. Theor. Phys. Supp. 184, 304 (2010)
[51] D. Simon, J. Stat. Mech. (2009) P07017

[52] V. Popkov, G. M. Schuetz, D. Simon, J. Stat. Mech. P10007 (2010).
[53] D. Simon, J. Stat. Phys. 142, 931 (2011)
[54] V. Popkov, G. M. Schuetz, J. Stat. Phys 142, 627 (2011)
[55] V. Belitsky, G. M. Schuetz, J. Stat. Phys. 152, 93 (2013)
[56] O. Hirschberg, D. Mukamel, G. M. Schuetz, J. Stat. Mech. P11023 (2015).
[57] G. M. Schuetz, From Particle Systems to Partial Differential Equations II, Springer Proceedings in Mathematics and Statistics Volume 129, pp 371-393, P. Gonçalves and A.J. Soares (Eds.), (Springer, Cham, 2015).
[58] R. Chétrite and H. Touchette, Phys. Rev. Lett. 111, 120601 (2013).
[59] R. Chétrite and H. Touchette Ann. Henri Poincare 16, 2005 (2015).
[60] R. Chétrite, H. Touchette, J. Stat. Mech. P12001 (2015).
[61] P. T. Nyawo, H. Touchette, Phys. Rev. E 94, 032101 (2016)
[62] H. Touchette, Physica A 504, 5 (2018).
[63] F. Angeletti, H. Touchette, Journal of Mathematical Physics 57, 023303 (2016).
[64] P. T. Nyawo, H. Touchette, Europhys. Lett. 116, 50009 (2016);
P. T. Nyawo, H. Touchette, Phys. Rev. E 98, 052103 (2018).
[65] B. Derrida and T. Sadhu, Journal of Statistical Physics 176, 773 (2019);
B. Derrida and T. Sadhu, Journal of Statistical Physics 177, 151 (2019).
[66] K. Proesmans, B. Derrida, J. Stat. Mech. (2019) 023201.
[67] N. Tizon-Escamilla, V. Lecomte and E. Bertin, J. Stat. Mech. (2019) 013201.
[68] J. du Buisson, H. Touchette, Phys. Rev. E 102, 012148 (2020)
[69] E. Mallmin, J. du Buisson and H. Touchette, arXiv:2102.04846
[70] F. Carollo, J. P. Garrahan, I. Lesanovsky, C. Perez-Espigares, Phys. Rev. A 98, 010103 (2018).
[71] F. Carollo, R. L. Jack, J. P. Garrahan, Phys. Rev. Lett. 122, 130605 (2019).
[72] F. Carollo, J. P. Garrahan, R. L. Jack, arXiv:2101.04138.
[73] C. Monthus, arXiv:2102.10834
[74] J.M. Meylahn, S. Sabhapandit, H. Touchette, Phys. Rev. E 92, 062148 (2015).
[75] F. den Hollander, S.N. Majumdar, J.M. Meylahn, H. Touchette, J. Phys. A: Math. Theor. 52, 175001 (2019).
[76] C. Monthus, arXiv:2103.08885
[77] M. Montero and J. Villarroel, PRE 94, 032132 (2016).
[78] L. DeVille, S. Dhople, A. D. Domínguez-García, and J. Zhang, SIAM J. Appl. Dyn. Syst., 15, 526 (2016)