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Large deviations at various levels for run-and-tumble processes with space-dependent velocities and space-dependent switching rates

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One-dimensional run-and-tumble processes may converge towards some localized non-equilibrium steady state when the two velocities and/or the two switching rates are space-dependent. A long dynamical trajectory can be then analyzed via the large deviations at Level 2.5 for the joint probability of the empirical densities, of the empirical spatial currents and of the empirical switching flows. The Level 2 for the empirical densities alone can be then derived via the optimization of the Level 2.5 over the empirical flows. More generally, the large deviations of any time-additive observable can be also obtained via contraction from the Level 2.5, or equivalently via the deformed generator method and the corresponding Doob conditioned process. Finally, the large deviations for the empirical intervals between consecutive switching events can be obtained via the introduction of the alternate Markov chain that governs the series of all the switching events of a long trajectory.

I. INTRODUCTION

Among the various stochastic intermittent dynamics that have attracted a lot of interest recently (see the reviews [1, 2] and references therein), run-and-tumble processes have played a major role and a lot of their properties have been studied in many different situations [3–32]. In the present paper, the goal is to analyze the large deviations properties of the one-dimensional run-and-tumble process on the infinite line, when the space-dependence of the two velocities and/or of the two switching rates produces a localized non-equilibrium steady state.

Within the theory of large deviations (see the reviews [33–35] and references therein), the traditional classification into three nested levels for dynamical trajectories over a large time-window T [33, 35], with the Level 1 for empirical time-averaged observables of the position, the Level 2 for the empirical time-averaged density of the position, and the Level 3 for the whole empirical process, has turned out to be inadequate for non-equilibrium steady states where steady currents play a major role, because the Level 2 is insufficient and not closed, while the Level 3 is too general. The introduction of the Level 2.5 concerning the joint probability distribution of the empirical time-averaged density of the position and of the empirical time-averaged flows has thus been a major achievement. Indeed, the rate functions at Level 2.5 are explicit for various types of Markov processes, including discrete-time Markov Chains [35–40], continuous-time Markov Jump processes [36, 39–54] and Diffusion processes described by Fokker-Planck equations [39, 40, 44, 45, 48, 55–57]. As a consequence, the explicit Level 2.5 plays a central role for any Markov process converging towards some non-equilibrium steady state, and many other large deviations properties can be derived from the Level 2.5 via contraction procedures. The first important example is of course the Level 2 for the empirical density alone that should be obtained via the optimization of the Level 2.5 over the empirical flows. More generally, the Level 2.5 can be contracted to obtain the large deviations properties of any time-additive observable of the dynamical trajectory involving both the position and the flows. The link with the studies of general time-additive observables via deformed Markov operators [48, 57–93] can be then understood via the corresponding ‘conditioned’ process obtained from the generalization of Doob’s h-transform. As in the field of stochastic resetting where the large deviations have been analyzed for excursions between resets [39, 94, 95], it will be interesting to analyze also the large deviations for the empirical intervals between consecutive switching events of the run-and-tumble process. In the present paper, this will be called the Level 2.75 in order to stress that it contains more information than the Level 2.5 that can be recovered via the contraction of the Level 2.75.

The paper is organized as follows. In section II, the one-dimensional run-and-tumble process with space-dependent velocities and/or switching rates is described, and the condition for the existence of a localized non-equilibrium steady state is given. The section III is devoted to the large deviations at Level 2.5 for the joint distribution of the empirical densities, of the empirical spatial currents and of the empirical switching flows. This Level 2.5 is then contracted to obtain the Level 2 for the empirical densities alone in section IV, and to characterize the large deviations of any time-additive observable in section V. In section VI, the large deviations at Level 2.75 for the empirical intervals between consecutive switching events is analyzed. Our conclusions are summarized in section VII. The seven appendices contain various complementary computations and discussions with respect to the main text.

II. NON-UNIFORM RUN-AND-TUMBLE PROCESSES WITH LOCALIZED STEADY-STATES

A. Run-and-tumble process with space-dependent velocities $v_{\pm}(x) > 0$ and switching rates $\gamma_{\pm}(x) \geq 0$

The particle is characterized by its position $x \in]-\infty, +\infty[$ and by its internal state $\sigma = \pm$. The dynamics can be summarized as follows :

- (i) when at position x and in the internal state $\sigma = +$, the particle moves with the positive velocity $v_+(x) > 0$ unless the switching towards the other internal state $\sigma = -$ occurs with the switching rate $\gamma_+(x) \geq 0$
- (ii) when at position x and in the internal state $\sigma = -$, the particle moves with the negative velocity $(-v_-(x)) < 0$ unless the switching towards the other internal state $\sigma = +$ occurs with the switching rate $\gamma_-(x) \geq 0$

So the probabilities $P_{\sigma}(x, t)$ to be at position x and in the internal state $\sigma = \pm$ at time t satisfy the dynamical system

$$\begin{aligned}\partial_t P_+(x, t) &= -\partial_x j_+(x, t) - Q_+(x, t) + Q_-(x, t) \\ \partial_t P_-(x, t) &= -\partial_x j_-(x, t) + Q_+(x, t) - Q_-(x, t)\end{aligned}\quad (1)$$

where the spatial currents $j_{\pm}(x, t)$ at position x when the internal state is $\sigma = \pm 1$ at time t involve the space-dependent velocities $v_{\pm}(x) > 0$

$$\begin{aligned}j_+(x, t) &\equiv v_+(x)P_+(x, t) \\ j_-(x, t) &\equiv -v_-(x)P_-(x, t)\end{aligned}\quad (2)$$

while the switching flows $Q_{\pm}(x, t)$ at position x out of the internal state $\sigma = \pm 1$ at time t (towards the opposite internal state $\sigma = \mp 1$ at the same position x) involve the space-dependent switching rates $\gamma_{\pm}(x) \geq 0$

$$\begin{aligned}Q_+(x, t) &\equiv \gamma_+(x)P_+(x, t) \\ Q_-(x, t) &\equiv \gamma_-(x)P_-(x, t)\end{aligned}\quad (3)$$

B. Condition on the velocities $v_{\pm}(x)$ and on the switching rates $\gamma_{\pm}(x)$ for a localized steady-state

Let us now consider the steady state of the dynamics of Eqs 1, with the two steady densities $P_{\pm}^*(x)$, the two steady spatial currents

$$\begin{aligned}j_+^*(x) &\equiv v_+(x)P_+^*(x) \\ j_-^*(x) &\equiv -v_-(x)P_-^*(x)\end{aligned}\quad (4)$$

and the two steady switching flows

$$\begin{aligned}Q_+^*(x) &\equiv \gamma_+(x)P_+^*(x) \\ Q_-^*(x) &\equiv \gamma_-(x)P_-^*(x)\end{aligned}\quad (5)$$

that should satisfy

$$\begin{aligned}0 &= -\frac{dj_+^*(x)}{dx} - Q_+^*(x) + Q_-^*(x) \\ 0 &= -\frac{dj_-^*(x)}{dx} + Q_+^*(x) - Q_-^*(x)\end{aligned}\quad (6)$$

The sum yields that the total steady-state spatial current

$$j^*(x) \equiv j_+^*(x) + j_-^*(x)\quad (7)$$

cannot depend on the position x , and should thus vanish since no current is possible at $x \rightarrow \pm\infty$

$$0 = j^*(x) = j_+^*(x) + j_-^*(x)\quad (8)$$

In order to maintain a symmetric treatment of the two internal states $\sigma = \pm 1$, it is more convenient to keep the spatial current $j_+^*(x) = -j_-^*(x)$ as the fundamental observable, from which the steady probabilities $P_\pm^*(x)$ can be obtained via Eqs 4

$$\begin{aligned} P_+^*(x) &= \frac{j_+^*(x)}{v_+(x)} \\ P_-^*(x) &= \frac{j_+^*(x)}{v_-(x)} \end{aligned} \quad (9)$$

and from which the steady switching flows can be obtained via Eqs 5

$$\begin{aligned} Q_+^*(x) &= \gamma_+(x)P_+^*(x) = \frac{\gamma_+(x)}{v_+(x)}j_+^*(x) \\ Q_-^*(x) &= \gamma_-(x)P_-^*(x) = \frac{\gamma_-(x)}{v_-(x)}j_+^*(x) \end{aligned} \quad (10)$$

Then Eq. 6 yields the following closed equation for the current $j_+^*(x)$

$$\frac{dj_+^*(x)}{dx} = \left(\frac{\gamma_-(x)}{v_-(x)} - \frac{\gamma_+(x)}{v_+(x)} \right) j_+^*(x) \quad (11)$$

The solution involving the integration constant $j_+^*(0)$

$$j_+^*(x) = j_+^*(0) e^{\int_0^x dy \left[\frac{\gamma_-(y)}{v_-(y)} - \frac{\gamma_+(y)}{v_+(y)} \right]} \quad (12)$$

will be valid only if the corresponding steady state $P_\pm^*(x)$ of Eq. 9 can be normalized

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} dx [P_+^*(x) + P_-^*(x)] = \int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+^*(x) \\ &= j_+^*(0) \int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] e^{\int_0^x dy \left[\frac{\gamma_-(y)}{v_-(y)} - \frac{\gamma_+(y)}{v_+(y)} \right]} \end{aligned} \quad (13)$$

in order to determine the finite integration constant $j_+^*(0)$ of Eq. 12.

In conclusion, the two velocities $v_\pm(x)$ and the two switching rates $\gamma_\pm(x)$ are able to produce a localized non-equilibrium steady-state only if the following integral is convergent

$$\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] e^{\int_0^x dy \left[\frac{\gamma_-(y)}{v_-(y)} - \frac{\gamma_+(y)}{v_+(y)} \right]} < +\infty \quad (14)$$

Appendix A contain some simple examples of localized non-equilibrium steady states, where only the switching rates or only the velocities are space-dependent.

III. LARGE DEVIATIONS AT LEVEL 2.5 FOR THE EMPIRICAL DENSITIES AND FLOWS

For a very long dynamical trajectory $[x(0 \leq t \leq T); \sigma(0 \leq t \leq T)]$ of the run-and-tumble process of Eq. 1 converging towards some localized non-equilibrium steady state (see the condition of Eq. 14), the large deviations at Level 2.5 characterize the joint distribution of the empirical time-averaged densities and empirical time-averaged flows as we now describe.

A. Empirical densities, empirical spatial currents and empirical switching flows with their constraints

The empirical densities $\rho_\sigma(x)$ measure the fraction of the time spent at position x and in the internal state $\sigma = \pm$

$$\rho_\sigma(x) \equiv \frac{1}{T} \int_0^T dt \delta_{\sigma(t), \sigma} \delta(x(t) - x) \quad (15)$$

with the normalization

$$\int_{-\infty}^{+\infty} dx [\rho_+(x) + \rho_-(x)] = 1 \quad (16)$$

The empirical spatial currents $j_\sigma(x)$ at position x while in the internal state σ

$$j_\sigma(x) \equiv \frac{1}{T} \int_0^T dt \frac{dx(t)}{dt} \delta(x(t) - x) \delta_{\sigma(t), \sigma} \quad (17)$$

are completely determined by the empirical densities $\rho_\sigma(x)$ as a consequence of the deterministic motion at velocities $[\sigma v_\sigma(x)]$ while in the internal state σ

$$\begin{aligned} j_+(x) &= v_+(x) \rho_+(x) \\ j_-(x) &= -v_-(x) \rho_-(x) \end{aligned} \quad (18)$$

The jumps between the two internal states $\sigma = \pm$ at position x are described by the two empirical switching flows

$$\begin{aligned} Q_+(x) &\equiv \frac{1}{T} \sum_{t \in [0, T]: \sigma(t^+) \neq \sigma(t^-)} \delta_{\sigma(t^-), +} \delta(x(t) - x) \\ Q_-(x) &\equiv \frac{1}{T} \sum_{t \in [0, T]: \sigma(t^+) \neq \sigma(t^-)} \delta_{\sigma(t^-), -} \delta(x(t) - x) \end{aligned} \quad (19)$$

where the sum is over the finite number of times $t \in [0, T]$ where the trajectory $\sigma(t)$ is discontinuous $\sigma(t^+) \neq \sigma(t^-)$, and jumps from the value $\sigma(t^-) = \sigma$ just before the jump at time t^- towards the opposite value $\sigma(t^+) = -\sigma$ just after the jump at time t^+ (see section VI for further details on the switching events and for the more explicit forms of Eqs 61 for the switching flows of Eqs 19).

The stationarity constraints read

$$\begin{aligned} 0 &= -\frac{dj_+(x)}{dx} - Q_+(x) + Q_-(x) \\ 0 &= -\frac{dj_-(x)}{dx} + Q_+(x) - Q_-(x) \end{aligned} \quad (20)$$

The sum yields that the total empirical spatial current

$$j(x) \equiv j_+(x) + j_-(x) = v_+(x) \rho_+(x) - v_-(x) \rho_-(x) \quad (21)$$

cannot depend on x , and should vanish as a consequence of the boundary conditions at $x \rightarrow \pm\infty$ with no flows

$$0 = j(x) = j_+(x) + j_-(x) \quad (22)$$

As for the steady state described in the previous section, it will be more convenient to maintain a symmetric treatment of the two internal states $\sigma = \pm 1$, and to keep the empirical spatial current $j_+(x) = -j_-(x)$ as the fundamental observable, from which the two empirical densities $\rho_\pm(x)$ can be computed

$$\begin{aligned} \rho_+(x) &= \frac{j_+(x)}{v_+(x)} \\ \rho_-(x) &= -\frac{j_-(x)}{v_-(x)} = \frac{j_+(x)}{v_-(x)} \end{aligned} \quad (23)$$

while the remaining stationary constraint of Eq. 20 reads

$$\frac{dj_+(x)}{dx} + Q_+(x) - Q_-(x) = 0 \quad (24)$$

The integral version of this stationary constraint reads

$$j_+(x) = \int_x^{+\infty} dy [Q_+(y) - Q_-(y)] = - \int_{-\infty}^x dy [Q_+(y) - Q_-(y)] \quad (25)$$

while the vanishing of the full integral

$$\int_{-\infty}^{+\infty} dy [Q_+(y) - Q_-(y)] = 0 \quad (26)$$

means that the total density of switching events out of the state $+$ and out of the state $-$ have to be equal.

B. Large deviations at Level 2.5 for the densities, the spatial currents and the switching flows

Taking into account the constitutive constraints of Eqs 22 and 23, one obtains that the joint distribution of the empirical densities $\rho_{\pm}(x)$, of the empirical spatial currents $j_{\pm}(x)$ and the empirical switching flows $Q_{\pm}(x)$ can be factorized into

$$P_T[\rho_{\pm}(\cdot), j_{\pm}(\cdot), Q_{\pm}(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \left[\prod_x \delta \left(\rho_+(x) - \frac{j_+(x)}{v_+(x)} \right) \delta \left(\rho_-(x) - \frac{j_-(x)}{v_-(x)} \right) \delta(j_-(x) + j_+(x)) \right] P_T[j_+(\cdot), Q_{\pm}(\cdot)] \quad (27)$$

where the joint distribution $P_T[j_+(\cdot), Q_{\pm}(\cdot)]$ of the three remaining variables satisfy the large deviation form

$$P_T[j_+(\cdot), Q_{\pm}(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \left[\prod_x \delta \left(Q_+(x) - Q_-(x) + \frac{dj_+(x)}{dx} \right) \right] e^{-TI_{2.5}[j_+(\cdot), Q_{\pm}(\cdot)]} \quad (28)$$

The constraints of the first line correspond to the normalization of Eq. 16 and to the stationarity constraint of Eq. 24, while the rate function at Level 2.5 follows the standard form for Markov jump processes [36, 39–54]

$$I_{2.5}[j_+, Q_{\pm}(\cdot)] = \int_{-\infty}^{+\infty} dx \left[Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) - Q_+(x) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] + \int_{-\infty}^{+\infty} dx \left[Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - Q_-(x) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \quad (29)$$

Appendix B contains the inference interpretation of the Level 2.5, while the two next sections describe how the Level 2.5 can be contracted to obtain other large deviations properties.

IV. FROM THE LEVEL 2.5 TOWARDS THE LEVEL 2 FOR THE EMPIRICAL DENSITIES ALONE

As recalled in the Introduction, the Level 2 for the empirical densities alone is not closed for non-equilibrium steady states involving steady currents, so that it can only be obtained via the optimization of the explicit Level 2.5 (described in the previous section) over the empirical flows. It is convenient to make this contraction in two steps via the introduction of the intermediate Level 2.25.

A. Level 2.25 for the distribution of the spatial current $j_+(x)$ and of the switching current $J(x)$

As explained in Appendix C, the explicit contraction over the switching activity $A(x) \equiv Q_+(x) + Q_-(x)$ yields that the joint distribution of the spatial current $j_+(x)$ and of the switching current $J(x) \equiv Q_+(x) - Q_-(x)$ follows the large deviation form

$$P_T[j_+(\cdot), J(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \left[\prod_x \delta \left(J(x) + \frac{dj_+(x)}{dx} \right) \right] e^{-TI_{2.25}[j_+(\cdot), J(\cdot)]} \quad (30)$$

where the rate function $I_{2.25}[j_+(\cdot), J(\cdot)]$ at Level 2.25 reads

$$I_{2.25}[j_+(\cdot), J(\cdot)] = \int_{-\infty}^{+\infty} dx J(x) \ln \left(\frac{\sqrt{J^2(x) + 4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} + J(x)}{2 \frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) + \int_{-\infty}^{+\infty} dx \left[\left(\frac{\gamma_+(x)}{v_+(x)} + \frac{\gamma_-(x)}{v_-(x)} \right) j_+(x) - \sqrt{J^2(x) + 4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} \right] \quad (31)$$

B. Large deviations at Level 2 for the empirical densities $\rho_{\pm}(x)$ alone

The distribution $P_T[j_+(\cdot)]$ of the spatial current $j_+(x)$ alone can be obtained from Eq. 30 via the elimination of the switching current $J(x)$ in terms of the spatial current $j_+(x)$

$$J(x) = -j'_+(x) \quad (32)$$

and reads

$$P_T[j_+(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) e^{-T I_2[j_+(\cdot)]} \quad (33)$$

with the rate function

$$\begin{aligned} I_2[j_+(\cdot)] &= - \int_{-\infty}^{+\infty} dx j'_+(x) \ln \left(\frac{\sqrt{[j'_+(x)]^2 + 4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} - j'_+(x)}{2 \frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) \\ &\quad + \int_{-\infty}^{+\infty} dx \left(\left[\frac{\gamma_+(x)}{v_+(x)} + \frac{\gamma_-(x)}{v_-(x)} \right] j_+(x) - \sqrt{[j'_+(x)]^2 + 4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} \right) \end{aligned} \quad (34)$$

Using Eq. 27, one obtains that the Level 2 for the joint distribution of the two densities $\rho_+(x) = \frac{j_+(x)}{v_+(x)}$ and $\rho_-(x) = \frac{j_-(x)}{v_-(x)}$ reads

$$P_T \left[\rho_+(\cdot) = \frac{j_+(\cdot)}{v_+(\cdot)}; \rho_-(\cdot) = \frac{j_-(\cdot)}{v_-(\cdot)} \right] \underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) e^{-T I_2[j_+(\cdot)]} \quad (35)$$

This Level 2 could be further contracted to obtain the Level 1 concerning time-additive observables of the position only. However, it is more interesting to consider the case of more general time-additive observables that involve also the switching flows as explained in the next section.

V. LARGE DEVIATIONS OF TIME-ADDITIVE OBSERVABLES VIA CONTRACTION OF LEVEL 2.5

In this section, the goal is to analyze the large deviations properties of the most general time-additive observable of the run-and-tumble trajectory $[x(0 \leq t \leq T); \sigma(0 \leq t \leq T)]$ that can a priori be parametrized by the six functions $(\alpha_{\pm}(x); \beta_{\pm}(x); \omega_{\pm}(x))$

$$A_T \equiv \frac{1}{T} \int_0^T dt [\alpha_{\sigma(t)}(x(t)) + \dot{x}(t) \omega_{\sigma(t)}(x(t))] + \frac{1}{T} \sum_{t \in [0, T]: \sigma(t^+) \neq \sigma(t^-)} \beta_{\sigma(t^-)}(x(t)) \quad (36)$$

A. Time-additive observable A_T in terms of the empirical densities and of the empirical flows

The additive observable of Eq. 36 can be rewritten in terms of the empirical densities $\rho_{\pm}(x)$ of Eq. 15, in terms of the empirical spatial currents $j_{\pm}(x)$ of Eq. 17 and in terms of the empirical switching flows $Q_{\pm}(x)$ of Eq. 19 as

$$A_T = \int_{-\infty}^{+\infty} dx \sum_{\sigma=\pm 1} [\alpha_{\sigma}(x) \rho_{\sigma}(x) + \omega_{\sigma}(x) j_{\sigma}(x) + \beta_{\sigma}(x) Q_{\sigma}(x)] \quad (37)$$

As discussed in detail in the subsection III A, these empirical observables are not all independent, so that the parametrization with six functions is actually very redundant. Rewriting $j_{\pm}(x)$ and $\rho_{\pm}(x)$ in terms of $j_+(x)$ via Eqs 22 and 23, one obtains that the additive observable of Eq. 37 can be parametrized in terms of three functions $(\omega(x), \beta_{\pm}(x))$

$$A_T = \int_{-\infty}^{+\infty} dx [\omega(x) j_+(x) + \beta_+(x) Q_+(x) + \beta_-(x) Q_-(x)] \quad (38)$$

where the new function $\omega(x)$ takes into account the four previous functions $(\omega_{\pm}(x), \alpha_{\pm}(x))$

$$\omega(x) \equiv \omega_+(x) - \omega_-(x) + \frac{\alpha_+(x)}{v_+(x)} + \frac{\alpha_-(x)}{v_-(x)} \quad (39)$$

B. Analysis of the generation function of A_T via the contraction of the Level 2.5

The rewriting of Eq. 38 yields that the generating function of the additive observable A_T

$$\begin{aligned} Z_T(k) &\equiv \langle e^{TkA_T} \rangle \\ &= \int \mathcal{D}j_+(\cdot) \int \mathcal{D}Q_\pm(\cdot) P_T[j_+(\cdot), Q_\pm(\cdot)] e^{Tk \int_{-\infty}^{+\infty} dx [\omega(x)j_+(x) + \beta_+(x)Q_+(x) + \beta_-(x)Q_-(x)]} \end{aligned} \quad (40)$$

can be evaluated from the joint distribution $P_T[j_+(\cdot), Q_\pm(\cdot)]$ of Eq. 28 and Eq. 29. As a consequence, the asymptotic behavior of the generating function of Eq. 40 for large T

$$\begin{aligned} Z_T(k) &\underset{T \rightarrow +\infty}{\simeq} \int \mathcal{D}j_+(\cdot) \int \mathcal{D}Q_\pm(\cdot) \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \left[\prod_x \delta \left(Q_+(x) - Q_-(x) + \frac{dj_+(x)}{dx} \right) \right] \\ &e^{T \left(-I_{2.5}[j_+(\cdot), Q_\pm(\cdot)] + k \int_{-\infty}^{+\infty} dx [\omega(x)j_+(x) + \beta_+(x)Q_+(x) + \beta_-(x)Q_-(x)] \right)} \end{aligned} \quad (41)$$

can be evaluated via the saddle-point method : one needs to optimize the functional appearing in the exponential of the second line in the presence of the constraints of the first line.

C. Optimization of the appropriate Lagrangian $\mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)]$

In order to solve the optimization problem of Eq. 41, let us introduce the following Lagrangian with the Lagrange multipliers $(\mu(k), \lambda_k(\cdot))$ to take into account the constraints

$$\begin{aligned} \mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)] &\equiv -I_{2.5}[j_+(\cdot), Q_\pm(\cdot)] + k \int_{-\infty}^{+\infty} dx [\omega(x)j_+(x) + \beta_+(x)Q_+(x) + \beta_-(x)Q_-(x)] \\ &- \mu(k) \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) - \int_{-\infty}^{+\infty} dx \lambda_k(x) \left(Q_+(x) - Q_-(x) + \frac{dj_+(x)}{dx} \right) \end{aligned} \quad (42)$$

Using the explicit form of the rate function of Eq. 29 and performing an integration by part of the last term involving the derivative $\frac{dj_+(x)}{dx}$, this Lagrangian can be rewritten more compactly as

$$\begin{aligned} \mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)] &\equiv \mu(k) + \int_{-\infty}^{+\infty} dx \left[-Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} e^{k\beta_+(x) - \lambda_k(x)} j_+(x)} \right) + Q_+(x) \right] \\ &+ \int_{-\infty}^{+\infty} dx \left[-Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} e^{k\beta_-(x) + \lambda_k(x)} j_+(x)} \right) + Q_-(x) \right] \\ &+ \int_{-\infty}^{+\infty} dx \left[k\omega(x) - \frac{\gamma_+(x) + \mu(k)}{v_+(x)} - \frac{\gamma_-(x) + \mu(k)}{v_-(x)} + \lambda'_k(x) \right] j_+(x) \end{aligned} \quad (43)$$

The optimization with respect to $Q_+(x)$, $Q_-(x)$ and $j_+(x)$ read

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)]}{\partial Q_+(x)} = -\ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} e^{k\beta_+(x) - \lambda_k(x)} j_+(x)} \right) \\ 0 &= \frac{\partial \mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)]}{\partial Q_-(x)} = -\ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} e^{k\beta_-(x) + \lambda_k(x)} j_+(x)} \right) \\ 0 &= \frac{\partial \mathcal{L}_k[j_+(\cdot), Q_\pm(\cdot)]}{\partial j_+(x)} = \frac{Q_+(x) + Q_-(x)}{j_+(x)} + k\omega(x) - \frac{\gamma_+(x) + \mu(k)}{v_+(x)} - \frac{\gamma_-(x) + \mu(k)}{v_-(x)} + \lambda'_k(x) \end{aligned} \quad (44)$$

Plugging the optimal solutions of the two first equations

$$\begin{aligned} Q_k^{+opt}(x) &= \frac{\gamma_+(x)}{v_+(x)} e^{k\beta_+(x)-\lambda_k(x)} j_k^{+opt}(x) \\ Q_k^{-opt}(x) &= \frac{\gamma_-(x)}{v_-(x)} e^{k\beta_-(x)+\lambda_k(x)} j_k^{+opt}(x) \end{aligned} \quad (45)$$

into the third equation yields the first-order non-linear differential equation for the Lagrange multiplier $\lambda_k(x)$

$$0 = \frac{\gamma_+(x)e^{k\beta_+(x)-\lambda_k(x)} - \gamma_+(x) - \mu(k)}{v_+(x)} + \frac{\gamma_-(x)e^{k\beta_-(x)+\lambda_k(x)} - \gamma_-(x) - \mu(k)}{v_-(x)} + k\omega(x) + \lambda'_k(x) \quad (46)$$

In addition, the optimal solutions of Eq. 45 should satisfy the constraints of the last line of Eq. 42

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_k^{+opt}(x) \\ \frac{dj_k^{+opt}(x)}{dx} &= -Q_k^{+opt}(x) + Q_k^{-opt}(x) = \left(-\frac{\gamma_+(x)}{v_+(x)} e^{k\beta_+(x)-\lambda_k(x)} + \frac{\gamma_-(x)}{v_-(x)} e^{k\beta_-(x)+\lambda_k(x)} \right) j_k^{+opt}(x) \end{aligned} \quad (47)$$

Finally, one needs to evaluate the optimal value of the Lagrangian of Eq. 43 and one obtains that it reduces to the Lagrange multiplier $\mu(k)$

$$\mathcal{L}_k^{opt} \equiv \mathcal{L}_k[j_k^{+opt}(\cdot), Q_k^{\pm opt}(\cdot)] = \mu(k) \quad (48)$$

So the asymptotic behavior of the generating function of Eq. 41

$$Z_T(k) \underset{T \rightarrow +\infty}{\simeq} e^{T\mathcal{L}_k^{opt}} = e^{T\mu(k)} \quad (49)$$

yields that the Lagrange multiplier $\mu(k)$ associated to the normalization constraint is the scaled cumulant generating function of the time-additive variable A_T as already found in other problems [39, 57].

D. Summary of the optimization procedure in three steps

The optimization procedure described above can be thus decomposed in three steps :

(1) the scaled cumulant generating function $\mu(k)$ and the Lagrange multiplier $\lambda_k(x)$ have to be found together from the differential equation for the Lagrange multiplier $\lambda_k(x)$ of Eq. 46 that can be rewritten as

$$\lambda'_k(x) + \left(\frac{\gamma_+(x)e^{k\beta_+(x)}}{v_+(x)} \right) e^{-\lambda_k(x)} + \left(\frac{\gamma_-(x)e^{k\beta_-(x)}}{v_-(x)} \right) e^{\lambda_k(x)} = \frac{\gamma_+(x) + \mu(k)}{v_+(x)} + \frac{\gamma_-(x) + \mu(k)}{v_-(x)} - k\omega(x) \quad (50)$$

If one is only interested into the scaled cumulant generating function $\mu(k)$, one can actually stop here, while if one is also interested into the saddle-point solution, one goes on with the two other steps.

(2) the optimal current $j_k^{+opt}(x)$ determined by Eqs 47 is then given by the analog of Eqs 12 and 13

$$j_k^{+opt}(x) = \frac{\int_0^x dy \left(\frac{\gamma_-(y)}{v_-(y)} e^{k\beta_-(y)+\lambda_k(y)} - \frac{\gamma_+(y)}{v_+(y)} e^{k\beta_+(y)-\lambda_k(y)} \right)}{\int_{-\infty}^{+\infty} dz \left[\frac{1}{v_+(z)} + \frac{1}{v_-(z)} \right] e^{\int_0^z dy \left(\frac{\gamma_-(y)}{v_-(y)} e^{k\beta_-(y)+\lambda_k(y)} - \frac{\gamma_+(y)}{v_+(y)} e^{k\beta_+(y)-\lambda_k(y)} \right)}} \quad (51)$$

that involves the Lagrange multiplier $\lambda_k(x)$ found in step (1)

(3) the optimal flows $Q_k^{\pm opt}(x)$ are then given by Eqs 45 using the Lagrange multiplier $\lambda_k(x)$ found in step (1) and the optimal current $j_k^{+opt}(x)$ found in step (2)

$$\begin{aligned} Q_k^{+opt}(x) &= \frac{\gamma_+(x)}{v_+(x)} e^{k\beta_+(x)-\lambda_k(x)} j_k^{+opt}(x) \\ Q_k^{-opt}(x) &= \frac{\gamma_-(x)}{v_-(x)} e^{k\beta_-(x)+\lambda_k(x)} j_k^{+opt}(x) \end{aligned} \quad (52)$$

Appendix E describes the perturbative solution in the parameter k of Eq. 50 in order to obtain the two first cumulants of any time-additive observable A_T . The equivalence of this optimization procedure with the standard deformed generator method to analyze the large deviations of additive observables is explained in Appendix F.

VI. LARGE DEVIATIONS AT LEVEL 2.75 FOR THE INTERVALS BETWEEN SWITCHING EVENTS

Up to now, we have described the Level 2.5 in section III and its contractions in sections IV and V. In the present section, we consider instead the higher Level 2.75 concerning the empirical intervals between consecutive switching events.

A. Decomposition of a long trajectory into its intervals between consecutive switching events

For a very long trajectory of initial internal state $\sigma(t=0) = -$, let us introduce the times t_i with $i = 1, \dots, 2N - 1$ where the internal state $\sigma(t)$ changes. It is convenient to add $t_0 = 0$ for the initial time and $t_{2N} = T$ for the end of the trajectory. During the odd intervals $t_{2i} < t < t_{2i+1}$, the internal state is negative $\sigma(t) = -$ and the spatial trajectory $x(t)$ is ballistic with the negative velocity ($-v_-(x) < 0$). During the even intervals $t_{2i-1} < t < t_{2i}$, the internal state is positive $\sigma(t) = +$ and the spatial trajectory $x(t)$ is ballistic with the positive velocity ($v_+(x) > 0$). Since the velocities $v_{\pm}(x)$ and the switching rates $\gamma_{\pm}(x)$ are space-dependent, it is convenient to characterize each interval by its left extremal position x_L and by its right extremal position x_R .

For an odd interval starting at position x_R , the probability to end at the position x_L reads

$$W_-(x_L|x_R) = \frac{\gamma_-(x_L)}{v_-(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_-(x)}{v_-(x)}} = \frac{\partial}{\partial x_L} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_-(x)}{v_-(x)}} \quad (53)$$

with the normalization

$$\int_{-\infty}^{x_R} dx_L W_-(x_L|x_R) = 1 \quad (54)$$

The time duration associated to an odd interval of extremal positions (x_L, x_R) involves the velocity $v_-(x)$ on this interval

$$\tau_-(x_L, x_R) = \int_{x_L}^{x_R} \frac{dx}{v_-(x)} \quad (55)$$

Similarly, for an even interval starting at position x_L , the probability to end at the position x_R reads

$$W_+(x_R|x_L) = \frac{\gamma_+(x_R)}{v_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_+(x)}{v_+(x)}} = -\frac{\partial}{\partial x_R} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_+(x)}{v_+(x)}} \quad (56)$$

with the normalization

$$\int_{x_L}^{+\infty} dx_R W_+(x_R|x_L) = 1 \quad (57)$$

The time duration associated to an even interval of extremal positions (x_L, x_R) involves the velocity $v_+(x)$ on this interval

$$\tau_+(x_L, x_R) = \int_{x_L}^{x_R} \frac{dx}{v_+(x)} \quad (58)$$

So if one considers only the spatial positions $x(t_j)$ at the times t_j where the internal state $\sigma(t)$ changes, the probabilities $P_{t_{2i}}(x(t_{2i}) = x_R)$ and $P_{t_{2i+1}}(x(t_{2i+1}) = x_L)$ follow the alternate Markov chain

$$\begin{aligned} P_{t_{2i}}(x_R) &= \int_{-\infty}^{x_R} dx_L W_+(x_R|x_L) P_{t_{2i-1}}(x_L) \delta(t_{2i} - t_{2i-1} - \tau_+(x_L, x_R)) \\ P_{t_{2i+1}}(x_L) &= \int_{x_L}^{+\infty} dx_R W_-(x_L|x_R) P_{t_{2i}}(x_R) \delta(t_{2i+1} - t_{2i} - \tau_-(x_L, x_R)) \end{aligned} \quad (59)$$

B. Empirical densities of intervals between switching events with their constraints

The empirical densities of intervals between consecutive switching events

$$\begin{aligned} q_+(x_R, x_L) &\equiv \frac{1}{T} \sum_{i=1}^N \delta(x_R - x(t_{2i})) \delta(x_L - x(t_{2i-1})) \\ q_-(x_L, x_R) &\equiv \frac{1}{T} \sum_{i=0}^{N-1} \delta(x_L - x(t_{2i+1})) \delta(x_R - x(t_{2i})) \end{aligned} \quad (60)$$

are defined for $x_L < x_R$ and contain the information of the empirical switching flows $Q_{\pm}(x)$ of Eq. 19 that can be rewritten as

$$\begin{aligned} Q_+(x_R) &= \frac{1}{T} \sum_{i=1}^N \delta(x_R - x(t_{2i})) = \int_{-\infty}^{x_R} dx_L q_+(x_R, x_L) = \int_{-\infty}^{x_R} dx_L q_-(x_L, x_R) \\ Q_-(x_L) &= \frac{1}{T} \sum_{i=1}^N \delta(x_L - x(t_{2i-1})) = \int_{x_L}^{+\infty} dx_R q_+(x_R, x_L) = \int_{x_L}^{+\infty} dx_R q_-(x_L, x_R) \end{aligned} \quad (61)$$

Their common normalization corresponds to the density $n = \frac{N}{T}$ of positive or negative intervals (Eq. 26)

$$\begin{aligned} n = \frac{N}{T} &= \int_{-\infty}^{+\infty} dx_R Q_+(x_R) = \int_{-\infty}^{+\infty} dx_L Q_-(x_L) \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R q_+(x_R, x_L) = \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R q_-(x_R, x_L) \end{aligned} \quad (62)$$

The empirical densities of intervals $q_{\pm}(\cdot, \cdot)$ of Eq. 60 also contain the information on the empirical densities of Eq. 15

$$\begin{aligned} \rho_+(x) &= \frac{1}{v_+(x)} \int_{-\infty}^x dx_L \int_x^{+\infty} dx_R q_+(x_R, x_L) \\ \rho_-(x) &= \frac{1}{v_-(x)} \int_{-\infty}^x dx_L \int_x^{+\infty} dx_R q_-(x_L, x_R) \end{aligned} \quad (63)$$

or equivalently on the empirical currents of Eq. 18

$$\begin{aligned} j_+(x) &= v_+(x) \rho_+(x) = \int_{-\infty}^x dx_L \int_x^{+\infty} dx_R q_+(x_R, x_L) \\ j_-(x) &= -v_-(x) \rho_-(x) = - \int_{-\infty}^x dx_L \int_x^{+\infty} dx_R q_-(x_L, x_R) \end{aligned} \quad (64)$$

whose derivatives can be rewritten in terms of Q_{\pm} using Eqs 61

$$\begin{aligned} j'_+(x) &= \int_x^{+\infty} dx_R q_+(x_R, x) - \int_{-\infty}^x dx_L q_+(x, x_L) = Q_-(x) - Q_+(x) \\ -j'_-(x) &= \int_x^{+\infty} dx_R q_-(x, x_R) - \int_{-\infty}^x dx_L q_-(x_L, x) = Q_-(x) - Q_+(x) \end{aligned} \quad (65)$$

so that one recovers the stationarity constraint of Eq. 24.

The normalization of Eq. 16 for the total empirical density using Eq. 63

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} dx [\rho_+(x) + \rho_-(x)] = \int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \int_{x_L}^{x_R} \frac{dx}{v_+(x)} + q_-(x_R, x_L) \int_{x_L}^{x_R} \frac{dx}{v_-(x)} \right] \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R [q_+(x_R, x_L) \tau_+(x_L, x_R) + q_-(x_R, x_L) \tau_-(x_L, x_R)] \end{aligned} \quad (66)$$

involves the time durations $\tau_{\pm}(x_L, x_R)$ of Eqs 55 and 58 of the two types of intervals. So Eq. 66 corresponds to the normalization of the total time T of the trajectory when decomposed into the durations of all empirical intervals.

C. Large deviations at Level 2.75 for the empirical densities of intervals between switching events

The joint distribution of the empirical current $j_+(\cdot)$, of the empirical switching flows $Q_\pm(\cdot)$ and of the empirical densities of intervals $q_\pm(\cdot, \cdot)$ follows the large deviation form

$$P_T[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)] \underset{T \rightarrow +\infty}{\simeq} C_{2.75}[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)] e^{-TI_{2.75}[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)]} \quad (67)$$

The constraints

$$\begin{aligned} C_{2.75}[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)] &= \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \left[\prod_x \delta \left(Q_+(x) - Q_-(x) + \frac{dj_+(x)}{dx} \right) \right] \\ &\left[\prod_{x_R} \delta \left(\int_{-\infty}^{x_R} dx_L q_+(x_R, x_L) - Q_+(x_R) \right) \right] \left[\prod_{x_R} \delta \left(\int_{-\infty}^{x_R} dx_L q_-(x_L, x_R) - Q_-(x_R) \right) \right] \\ &\left[\prod_{x_L} \delta \left(\int_{x_L}^{+\infty} dx_R q_+(x_R, x_L) - Q_-(x_L) \right) \right] \left[\prod_{x_L} \delta \left(\int_{x_L}^{+\infty} dx_R q_-(x_L, x_R) - Q_+(x_L) \right) \right] \end{aligned} \quad (68)$$

can be understood as follows : the first line contains the constraints already present at the Level 2.5 of Eq. 28, while the two last lines contain the definitions of $Q_\pm(\cdot)$ in terms of $q_\pm(\cdot, \cdot)$ discussed in Eqs 61. The rate function corresponding to the alternate Markov chain of Eq. 59 with the explicit kernels of Eqs 53 and 56 reads

$$\begin{aligned} I_{2.75}[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)] & \quad (69) \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \ln \left(\frac{q_+(x_R, x_L)}{W_+(x_R|x_L)Q_-(x_L)} \right) + q_-(x_L, x_R) \ln \left(\frac{q_-(x_L, x_R)}{W_-(x_L|x_R)Q_+(x_R)} \right) \right] \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \ln \left(\frac{q_+(x_R, x_L)}{\frac{\gamma_+(x_R)}{v_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_+(x)}{v_+(x)}} Q_-(x_L)} \right) + q_-(x_L, x_R) \ln \left(\frac{q_-(x_L, x_R)}{\frac{\gamma_-(x_L)}{v_-(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_-(x)}{v_-(x)}} Q_+(x_R)} \right) \right] \end{aligned}$$

and vanishes for the steady state values values

$$\begin{aligned} q_+^*(x_R, x_L) &= W_+(x_R|x_L)Q_-^*(x_L) = \frac{\gamma_+(x_R)}{v_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_+(x)}{v_+(x)}} \frac{\gamma_-(x_L)}{v_-(x_L)} j_+^*(x_L) \\ q_-^*(x_L, x_R) &= W_-(x_L|x_R)Q_+^*(x_R) = \frac{\gamma_-(x_L)}{v_-(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_-(x)}{v_-(x)}} \frac{\gamma_+(x_R)}{v_+(x_R)} j_-^*(x_R) \end{aligned} \quad (70)$$

The link between the Level 2.75 of Eq. 67 and the Level 2.5 of Eq. 28 is explained in Appendix G.

VII. CONCLUSIONS

In this paper, we have considered the one-dimensional run-and-tumble process on the infinite line, when the space-dependence of the two velocities $v_\pm(x)$ and/or of the two switching rates $\gamma_\pm(x)$ produces a localized non-equilibrium steady state. The goal was to analyze its large deviations properties at various levels.

As explained in the Introduction, for non-equilibrium steady states of Markov processes, one should start with the Level 2.5 that represents the lowest level where one can write the explicit rate function and the corresponding constraints. For the one-dimensional run-and-tumble process, we have explained that the Level 2.5 characterizes the joint probability of the two empirical densities $\rho_\pm(x)$, of the two empirical spatial currents $j_\pm(x)$ and of the two empirical switching flows $Q_\pm(x)$, that are related via many constitutive constraints. We have then described how the Level 2 for the empirical densities $\rho_\pm(x)$ alone can be obtained this Level 2.5 via contraction. More generally, we have explained how the Level 2.5 can be contracted to obtain the scaled cumulant generating function of any time-additive observable.

We have then analyzed the large deviations at Level 2.75 for the joint probability of the empirical intervals between consecutive switching events via the introduction of the alternate Markov chain that governs the series of all the switching events of a long trajectory. We have explained why this Level 2.75 contains more information than the Level

2.5 that can be recovered via contraction (see Appendix G). As explained in detail in the recent preprint [106], this property can be extended to other types of jump-drift processes, where the motion between jumps is also deterministic, while the analysis is different for jump-diffusion processes, where the motion between jumps is then stochastic.

As a final remark, let us mention that some large deviations properties of other active matter models are discussed in the recent preprint [107].

Appendix A: Examples of run-and-tumble processes with localized steady states

In this Appendix related to section II, we describe some simple examples of localized non-equilibrium steady states of one-dimensional run-and-tumble processes, where only the switching rates or only the velocities are space-dependent.

1. Steady states produced by uniform velocities $v_{\pm}(x) = v$ and space-dependent switching rates $\gamma_{\pm}(x)$

When the two velocities are uniform and equal $v_{\pm}(x) = v$, the steady-state of Eqs 9 and 12

$$P_{\pm}^*(x) = \frac{j_{\pm}^*(x)}{v} = \frac{j_{\pm}^*(0)}{v} e^{\frac{1}{v} \int_0^x dy [\gamma_{-}(y) - \gamma_{+}(y)]} \quad (\text{A1})$$

will be normalizable if the space-dependent switching rates $\gamma_{\pm}(x)$ ensure the convergence of the integral of Eq. 14

$$\int_{-\infty}^{+\infty} dx e^{\frac{1}{v} \int_0^x dy [\gamma_{-}(y) - \gamma_{+}(y)]} < +\infty \quad (\text{A2})$$

Among the various cases discussed in [31], let us mention :

(i) an example with a smooth variation of the switching rates $\gamma_{\pm}(x)$ over the characteristic length ξ between the minimum value γ_0 and the maximum value $\gamma > \gamma_0$

$$\gamma_{\pm}^{smooth}(x) = \frac{\gamma + \gamma_0}{2} \pm \frac{\gamma - \gamma_0}{2} \tanh \frac{x}{\xi} \quad (\text{A3})$$

with the corresponding localized steady state given by Eq. A1

$$P_{\pm}^*(x) = \frac{j_{\pm}^*(0)}{v} e^{-\frac{(\gamma - \gamma_0)}{v} \int_0^x dy \tanh \frac{y}{\xi}} = \frac{j_{\pm}^*(0)}{v} e^{-\frac{(\gamma - \gamma_0)\xi}{v} \ln \left(\cosh \frac{x}{\xi} \right)} = \frac{j_{\pm}^*(0)}{v \left[\cosh \frac{x}{\xi} \right]^{\frac{(\gamma - \gamma_0)\xi}{v}}} \quad (\text{A4})$$

(ii) an example with a step variation of the switching rates $\gamma_{\pm}(x)$ at the origin $x = 0$ (corresponding to the limit $\xi \rightarrow +\infty$ of Eq. A3)

$$\gamma_{\pm}^{step}(x) = \frac{\gamma + \gamma_0}{2} \pm \frac{\gamma - \gamma_0}{2} \text{sgn}(x) \quad (\text{A5})$$

or equivalently in terms of the Heaviside function $\theta(x) = \frac{1 + \text{sgn}(x)}{2}$

$$\begin{aligned} \gamma_{+}^{step}(x) &= \gamma_0 + (\gamma - \gamma_0)\theta(x) \\ \gamma_{-}^{step}(x) &= \gamma - (\gamma - \gamma_0)\theta(x) \end{aligned} \quad (\text{A6})$$

The corresponding steady state of Eq. A1 is then given by the simple symmetric exponential form

$$P_{\pm}^*(x) = \frac{j_{\pm}^*(0)}{v} e^{-\frac{(\gamma - \gamma_0)}{v} \int_0^x dy \text{sgn}(y)} = \frac{j_{\pm}^*(0)}{v} e^{-\frac{(\gamma - \gamma_0)}{v} |x|} \quad (\text{A7})$$

2. Steady states produced by space-dependent velocities $v_{\pm}(x)$ and uniform switching rates $\gamma_{\pm}(x) = \gamma_{\pm}$

When the two switching rates $\gamma_{\pm}(x) = \gamma_{\pm}$ do not depend on the position x , the dynamics for the internal variable $\sigma = \pm 1$ is of course closed : the two probabilities

$$P_{\sigma}(t) = \int_{-\infty}^{+\infty} dx P_{\sigma}(x, t) \quad (\text{A8})$$

then correspond to the so-called telegraph process

$$\begin{aligned} \partial_t P_+(t) &= -\gamma_+ P_+(t) + \gamma_- P_-(t) \\ \partial_t P_-(t) &= \gamma_+ P_+(t) - \gamma_- P_-(t) \end{aligned} \quad (\text{A9})$$

and converge towards the following steady state for the internal variable $\sigma = \pm$

$$\begin{aligned} P_+^* &= \frac{\gamma_-}{\gamma_+ + \gamma_-} \\ P_-^* &= \frac{\gamma_+}{\gamma_+ + \gamma_-} \end{aligned} \quad (\text{A10})$$

However, the dynamics for the position x will have a localized steady-state only if the two space-dependent velocities $v_{\pm}(x)$ satisfy the condition of Eq. 14

$$\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] e^{\int_0^x dy \left[\frac{\gamma_-}{v_-(y)} - \frac{\gamma_+}{v_+(y)} \right]} < +\infty \quad (\text{A11})$$

An interesting example is the exponential functional of the telegraph process $\sigma(t)$ studied previously in the context of anomalous diffusion in random media [96]

$$x(t) = \int_0^t dt' e^{\int_{t'}^t dt'' \sigma(t'')} \quad (\text{A12})$$

while analogous exponential functionals of Brownian motion have been also much studied [97–100].

The dynamics of $x(t)$ defined by Eq. A12 corresponds to the multiplicative stochastic process

$$\dot{x}(t) = 1 + \sigma(t)x(t) \quad (\text{A13})$$

described by Eqs 1 when the two velocities display the following linear dependences with respect to the position x

$$\begin{aligned} v_+(x) &= x + 1 \\ v_-(x) &= x - 1 \end{aligned} \quad (\text{A14})$$

While $v_+(x)$ is positive for any position $x \geq 0$, the velocity $v_-(x)$ is positive only for $x \geq 1$. As a consequence, the steady state $P_{\pm}^*(x)$ has for support $x \in]1, +\infty[$ (while the interval $0 \leq x < 1$ will be visited only in the transient regime starting from the initial condition $x(t=0) = 0$).

The solution of Eq. 6 needs to be adapted to the support $x \in]1, +\infty[$ in terms of the new integration constant K

$$j_+^*(x) = K e^{\gamma_- \ln(x-1) - \gamma_+ \ln(x+1)} = K \frac{(x-1)^{\gamma_-}}{(x+1)^{\gamma_+}} \quad (\text{A15})$$

The corresponding steady state $P_{\pm}^*(x)$ of Eq. 9

$$\begin{aligned} P_+^*(x) &= \frac{j_+^*(x)}{x+1} = K \frac{(x-1)^{\gamma_-}}{(x+1)^{\gamma_++1}} \\ P_-^*(x) &= \frac{j_+^*(x)}{x-1} = K \frac{(x-1)^{\gamma_- - 1}}{(x+1)^{\gamma_+}} \end{aligned} \quad (\text{A16})$$

is always normalizable at the boundary $x \rightarrow 1^+$, while their common power-law behavior for large x

$$P_{\pm}^*(x) \underset{x \rightarrow +\infty}{\simeq} \frac{K}{x^{\gamma_+ - \gamma_- + 1}} \quad (\text{A17})$$

is normalizable only if the two switching rates satisfy

$$\gamma_+ > \gamma_- \quad (\text{A18})$$

Then one can compute the normalizations of Eqs A16 in terms of the Gamma-function $\Gamma(\cdot)$

$$\begin{aligned} \int_1^{+\infty} dx P_+^*(x) &= K \int_1^{+\infty} dx \frac{(x-1)^{\gamma_-}}{(x+1)^{\gamma_++1}} = K \frac{\Gamma(\gamma_+ - \gamma_-)\Gamma(\gamma_-)}{2^{\gamma_+ - \gamma_-}\Gamma(\gamma_+)} \frac{\gamma_-}{\gamma_+} \\ \int_1^{+\infty} dx P_-^*(x) &= K \int_1^{+\infty} dx \frac{(x-1)^{\gamma_--1}}{(x+1)^{\gamma_+}} = K \frac{\Gamma(\gamma_+ - \gamma_-)\Gamma(\gamma_-)}{2^{\gamma_+ - \gamma_-}\Gamma(\gamma_+)} \end{aligned} \quad (\text{A19})$$

The compatibility with Eqs A10 yields that the normalization constant K reads

$$K = \frac{\gamma_+ \Gamma(\gamma_+) 2^{\gamma_+ - \gamma_-}}{(\gamma_+ + \gamma_-) \Gamma(\gamma_-) \Gamma(\gamma_+ - \gamma_-)} \quad (\text{A20})$$

Appendix B: Inference interpretation of the Level 2.5

In this Appendix related to section III, we describe another interpretation of the large deviations at Level 2.5 of Eq. 28 via the inverse problem of inference [40] : from the data of a long dynamical trajectory, one computes the empirical time-averaged observables described above, and one infers the best steady current $\hat{j}_+^*(x)$ and the best corresponding switching rates $\hat{\gamma}_\pm(x)$ of the model as follows.

(i) the best inferred steady current $\hat{j}_+^*(x)$ is simply the measured empirical current $j_+(x)$

$$\hat{j}_+^*(x) \equiv j_+(x) \quad (\text{B1})$$

(ii) the best inferred switching rates $\hat{\gamma}_\pm(x)$ are the rates that would make the switching flows typical with respect to the empirical densities or equivalently with respect to the empirical current $j_+(x)$

$$\hat{\gamma}_\pm(x) \equiv \frac{Q_\pm(x)}{\rho_\pm(x)} = \frac{Q_\pm(x)}{\frac{j_+(x)}{v_\pm(x)}} \quad (\text{B2})$$

Via this change of variables from $[j_+(x), Q_\pm(x)]$ towards $[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)]$, the large deviations at Level 2.5 of Eq. 28 translates into the joint probability to infer the two switching rates $\hat{\gamma}_\pm(x)$ and the corresponding steady current $\hat{j}_+^*(x)$ that they produce together

$$P_T^{Infer}[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)] \underset{T \rightarrow +\infty}{\simeq} C_{Infer}[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)] e^{-T I_{Infer}[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)]} \quad (\text{B3})$$

The rate function translated from Eq. 29 reads

$$\begin{aligned} I_{Infer}[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)] &= \int_{-\infty}^{+\infty} dx \frac{\hat{j}_+^*(x)}{v_+(x)} \left(\hat{\gamma}_+(x) \ln \left(\frac{\hat{\gamma}_+(x)}{\gamma_+(x)} \right) - \hat{\gamma}_+(x) + \gamma_+(x) \right) \\ &+ \int_{-\infty}^{+\infty} dx \frac{\hat{j}_+^*(x)}{v_-(x)} \left(\hat{\gamma}_-(x) \ln \left(\frac{\hat{\gamma}_-(x)}{\gamma_-(x)} \right) - \hat{\gamma}_-(x) + \gamma_-(x) \right) \end{aligned} \quad (\text{B4})$$

The constraints translated from Eq. 28

$$C_{Infer}[\hat{j}_+^*(x), \hat{\gamma}_\pm(x)] = \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \hat{j}_+^*(x) - 1 \right) \left[\prod_x \delta \left(\frac{d\hat{j}_+^*(x)}{dx} + \hat{j}_+^*(x) \left[\frac{\hat{\gamma}_+(x)}{v_+(x)} - \frac{\hat{\gamma}_-(x)}{v_-(x)} \right] \right) \right] \quad (\text{B5})$$

means that the current $j_+^*(x)$ is simply the steady current associated to the inferred rates $\hat{\gamma}_\pm(x)$: indeed, the second constraint is the analog of Eq. 11, while the first constraint is the analog of Eq. 13. As a consequence, the constraints can be fully solved to rewrite the current $j_+^*(x)$ in terms of the inferred switching rates $\hat{\gamma}_\pm(x)$ (analog of Eqs 12 and 13)

$$\hat{j}_+^*(x) = \frac{e^{\int_0^x dy \left[\frac{\hat{\gamma}_-(y)}{v_-(y)} - \frac{\hat{\gamma}_+(y)}{v_+(y)} \right]}}{\int_{-\infty}^{+\infty} dz \left[\frac{1}{v_+(z)} + \frac{1}{v_-(z)} \right] e^{\int_0^z dy \left[\frac{\hat{\gamma}_-(y)}{v_-(y)} - \frac{\hat{\gamma}_+(y)}{v_+(y)} \right]}} \quad (\text{B6})$$

Plugging this expression into the rate function of Eq. B4

$$I_{Infer}[\hat{\gamma}_{\pm}(x)] = \frac{\int_{-\infty}^{+\infty} dx \left[\frac{\hat{\gamma}_+(x) \ln \left(\frac{\hat{\gamma}_+(x)}{\gamma_+(x)} \right) - \hat{\gamma}_+(x) + \gamma_+(x)}{v_+(x)} + \frac{\hat{\gamma}_-(x) \ln \left(\frac{\hat{\gamma}_-(x)}{\gamma_-(x)} \right) - \hat{\gamma}_-(x) + \gamma_-(x)}{v_-(x)} \right] e^{\int_0^x dy \left[\frac{\hat{\gamma}_-(y)}{v_-(y)} - \frac{\hat{\gamma}_+(y)}{v_+(y)} \right]}}{\int_{-\infty}^{+\infty} dz \left[\frac{1}{v_+(z)} + \frac{1}{v_-(z)} \right] e^{\int_0^z dy \left[\frac{\hat{\gamma}_-(y)}{v_-(y)} - \frac{\hat{\gamma}_+(y)}{v_+(y)} \right]}} \quad (\text{B7})$$

one obtains that the joint probability to infer the two switching rates $\hat{\gamma}_{\pm}(x)$ instead of the true values $\gamma_{\pm}(x)$ reduces to

$$P_T^{Infer}[\hat{\gamma}_{\pm}(x)]_{T \rightarrow +\infty} \simeq e^{-T I_{Infer}[\hat{\gamma}_{\pm}(x)]} \quad (\text{B8})$$

Appendix C: Explicit contraction from the Level 2.5 to the Level 2.25

In this Appendix related to section IV, we describe how the Level 2.5 of Eq. 28 produces the Level 2.25 of Eq. 30 via contraction.

1. Large deviations at level 2.5 in terms of the switching activity $A(x)$ and of the switching current $J(x)$

In the Level 2.5 of Eq. 28, the two switching flows $Q_{\pm}(x)$ can be replaced by their symmetric and antisymmetric parts called the activity and the current

$$\begin{aligned} A(x) &\equiv Q_+(x) + Q_-(x) \\ J(x) &\equiv Q_+(x) - Q_-(x) \end{aligned} \quad (\text{C1})$$

i.e. reciprocally

$$\begin{aligned} Q_+(x) &= \frac{A(x) + J(x)}{2} \\ Q_-(x) &= \frac{A(x) - J(x)}{2} \end{aligned} \quad (\text{C2})$$

The large deviation form of Eq. 28 translates into

$$P_T[j_+(\cdot), J(\cdot), A(\cdot)]_{T \rightarrow +\infty} \simeq \frac{\delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \left[\prod_x \delta \left(J(x) + \frac{dj_+(x)}{dx} \right) \right]}{e^{-T I_{2.5}[j_+(\cdot), J(\cdot), A(\cdot)]}} \quad (\text{C3})$$

with the rate function translated from Eq. 29

$$\begin{aligned} I_{2.5}[j_+, J(\cdot), A(\cdot)] &= \int_{-\infty}^{+\infty} dx \left[\left(\frac{A(x) + J(x)}{2} \right) \ln \left(\frac{A(x) + J(x)}{2 \frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) \right] \\ &+ \int_{-\infty}^{+\infty} dx \left[\left(\frac{A(x) - J(x)}{2} \right) \ln \left(\frac{A(x) - J(x)}{2 \frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - A(x) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \end{aligned} \quad (\text{C4})$$

2. Explicit contraction over the switching activity $A(x)$

Since the switching activity $A(x)$ does not appear in the constraints of Eq. C3, one can optimize the rate function of Eq. C4 over the activity as in many other Markov jump processes [42, 49, 50, 52]

$$0 = \frac{\partial I_{2.5}[\rho_{\pm}(\cdot), J(\cdot), A(\cdot)]}{\partial A(x)} = \frac{1}{2} \ln \left(\frac{A^2(x) - J^2(x)}{4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} \right) \quad (\text{C5})$$

in order to obtain the optimal activity as a function of the spatial current $j_+(x)$ and of the switching current $J(x)$

$$A^{opt}[j_+(\cdot), J(\cdot)] = \sqrt{J^2(x) + 4 \frac{\gamma_+(x)\gamma_-(x)}{v_+(x)v_-(x)} j_+^2(x)} \quad (\text{C6})$$

The corresponding rate function obtained via this explicit contraction

$$I_{2.25}[j_+(\cdot), J(\cdot)] = I_{2.5}[j_+(\cdot), J(\cdot), A^{opt}[j_+(\cdot), J(\cdot)]] \quad (\text{C7})$$

is given in Eq. 31 of the main text.

Appendix D: Level 2.5 and Level 2 when the two switching rates $\gamma_{\pm}(x)$ have disjoint supports

In this Appendix, we describe how the Level 2.5 of section III and the Level 2 of section IV can be simplified when the two switching rates $\gamma_{\pm}(x)$ have disjoint supports. To be concrete, let us consider the case where the switching rates vanish when the particle is going back towards its home at the origin $x = 0$, i.e. $\gamma_+(x)$ vanishes in the whole region $x \in]-\infty, 0[$ and $\gamma_-(x)$ vanishes in the whole region $x \in]0, +\infty[$

$$\begin{aligned} \gamma_+(x < 0) &= 0 \\ \gamma_-(x > 0) &= 0 \end{aligned} \quad (\text{D1})$$

The condition of Eq. 14 for the existence of a localized steady state becomes

$$\int_{-\infty}^0 dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] e^{-\int_x^0 dy \frac{\gamma_-(y)}{v_-(y)}} + \int_0^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] e^{-\int_0^x dy \frac{\gamma_+(y)}{v_+(y)}} < +\infty \quad (\text{D2})$$

1. Simplifications for the Level 2.5

Since the empirical switching flow $Q_+(x)$ has for support $x \in]0, +\infty[$ and the empirical switching flow $Q_-(x)$ has for support $x \in]-\infty, 0[$, the integral version of the stationary constraint of Eq. 25 becomes

$$\begin{aligned} j_+(x) &= \int_x^{+\infty} dy Q_+(y) \quad \text{for } x \geq 0 \\ j_+(x) &= \int_{-\infty}^x dy Q_-(y) \quad \text{for } x \leq 0 \end{aligned} \quad (\text{D3})$$

while the consistency of the two expressions at the origin $x = 0$ ensures that the total density of switching events out of the state $+$ and out of the state $-$ are equal (Eq. 26).

The large deviations at Level 2.5 of Eqs 28 can be then factorized into the two regions $x > 0$ and $x < 0$ except for

the junction concerning the current $j_+(x=0)$ at the origin and the global normalization

$$\begin{aligned}
P_T[j_+(\cdot), Q_\pm(\cdot)]_{T \rightarrow +\infty} &\simeq \delta \left(\int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) - 1 \right) \\
&\left[\prod_{x \geq 0} \delta \left(j_+(x) - \int_x^{+\infty} dy Q_+(y) \right) \right] e^{-T \int_0^{+\infty} dx \left[Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) - Q_+(x) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right]} \\
&\left[\prod_{x \leq 0} \delta \left(j_+(x) - \int_{-\infty}^x dy Q_-(y) \right) \right] e^{-T \int_{-\infty}^0 dx \left[Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - Q_-(x) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right]} \quad (D4)
\end{aligned}$$

2. Simplifications for the Level 2

If one eliminates the empirical switching flows via

$$\begin{aligned}
Q_+(x) &= -j'_+(x) \quad \text{for } x \geq 0 \\
Q_-(x) &= j'_+(x) \quad \text{for } x \leq 0
\end{aligned} \quad (D5)$$

one obtains that the probability for the empirical spatial current $j_+(x)$ alone follows the large deviation form of Eq. 33 with the rate function

$$\begin{aligned}
I_2[j_+(\cdot)] &= \int_0^{+\infty} dx \left[-j'_+(x) \ln \left(\frac{(-j'_+(x))}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) + j'_+(x) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] + \int_{-\infty}^0 dx \left[j'_+(x) \ln \left(\frac{j'_+(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - j'_+(x) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \\
&= \int_0^{+\infty} dx \left[-j'_+(x) \ln \left(\frac{(-j'_+(x))}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) + \frac{d[j_+(x) \ln j_+(x)]}{dx} + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] + \int_{-\infty}^0 dx \left[j'_+(x) \ln \left(\frac{j'_+(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - \frac{d[j_+(x) \ln j_+(x)]}{dx} + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \\
&= -2j_+(0) \ln j_+(0) + \int_0^{+\infty} dx \left[-j'_+(x) \ln \left(\frac{(-j'_+(x))}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] + \int_{-\infty}^0 dx \left[j'_+(x) \ln \left(\frac{j'_+(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \\
&= \int_0^{+\infty} dx \left[-j'_+(x) \ln \left(\frac{(-j'_+(x))}{\frac{\gamma_+(x)}{v_+(x)} j_+(0)} \right) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] + \int_{-\infty}^0 dx \left[j'_+(x) \ln \left(\frac{j'_+(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(0)} \right) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \quad (D6)
\end{aligned}$$

3. Large deviations for the switching flows $Q_\pm(x)$ alone

If one wishes instead to eliminate the empirical spatial current $j_+(x)$ in terms of the two the empirical switching flows $Q_\pm(x)$ via Eqs D3, the normalization constraint becomes

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] j_+(x) \\
&= \int_{-\infty}^0 dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \int_{-\infty}^x dy Q_-(y) + \int_0^{+\infty} dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \int_x^{+\infty} dy Q_+(y) \\
&= \int_{-\infty}^0 dy Q_-(y) \int_y^0 dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] + \int_0^{+\infty} dy Q_+(y) \int_0^y dx \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \quad (D7)
\end{aligned}$$

while the total density n of the switching events of each kind is given by

$$n \equiv \int_0^{+\infty} dy Q_+(y) = \int_{-\infty}^0 dy Q_-(y) = j_+(0) \quad (D8)$$

and the rate function of Eq. D6 translates into

$$\begin{aligned}
& I_2[n, Q_{\pm}(\cdot)] \\
&= \int_0^{+\infty} dx Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} n} \right) + \int_0^{+\infty} dx \frac{\gamma_+(x)}{v_+(x)} \int_x^{+\infty} dy Q_+(y) + \int_{-\infty}^0 dx Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} n} \right) + \int_{-\infty}^0 dx \frac{\gamma_-(x)}{v_-(x)} \int_{-\infty}^x dy Q_-(y) \\
&= \int_0^{+\infty} dx Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} n} \right) + \int_0^{+\infty} dy Q_+(y) \int_0^y dx \frac{\gamma_+(x)}{v_+(x)} + \int_{-\infty}^0 dx Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} n} \right) + \int_{-\infty}^0 dy Q_-(y) \int_y^0 dx \frac{\gamma_-(x)}{v_-(x)} \\
&= \int_0^{+\infty} dx Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} e^{-\int_0^x dy \frac{\gamma_+(y)}{v_+(y)} n}} \right) + \int_{-\infty}^0 dx Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} e^{-\int_x^0 dy \frac{\gamma_-(y)}{v_-(y)} n}} \right) \tag{D9}
\end{aligned}$$

Putting everything together, one obtains that the joint distribution of the empirical switching flows $Q_{\pm}(x)$ and of the total density n of switching events of each kind follows the large deviation form

$$\begin{aligned}
& P_T[n, Q_{\pm}(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \delta \left(n - \int_0^{+\infty} dx Q_+(x) \right) \delta \left(n - \int_{-\infty}^0 dx Q_-(x) \right) \\
& \delta \left(\int_{-\infty}^0 dx Q_-(x) \int_x^0 dy \left[\frac{1}{v_+(y)} + \frac{1}{v_-(y)} \right] + \int_0^{+\infty} dx Q_+(x) \int_0^x dy \left[\frac{1}{v_+(y)} + \frac{1}{v_-(y)} \right] - 1 \right) \\
& e^{-T \int_0^{+\infty} dx Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} e^{-\int_0^x dy \frac{\gamma_+(y)}{v_+(y)} n}} \right) + \int_{-\infty}^0 dx Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} e^{-\int_x^0 dy \frac{\gamma_-(y)}{v_-(y)} n}} \right)} \tag{D10}
\end{aligned}$$

4. Physical interpretation in terms of alternate excursions

In Eq. D10, one recognizes the standard form of rate function for semi-Markov processes [36, 38, 39, 101–104]. The physical meaning in terms of the alternate excursions on the right and on the left of the origin $x = 0$ can be understood as follows : their common density n is given by the two constraints of the first line ; $Q_+(x)$ represents the empirical density of positive excursions ending at position x and consuming the total time (for the forward travel from 0 to $x > 0$ at the velocity $v_+(\cdot)$ and for the backward travel from x to 0 at the velocity $v_-(\cdot)$)

$$\tau_+(x) = \int_0^x dy \left[\frac{1}{v_+(y)} + \frac{1}{v_-(y)} \right] \tag{D11}$$

while $Q_-(x)$ represents the empirical density of negative excursions ending at position x and consuming the total time (for the travel from 0 to $x < 0$ at the velocity $v_-(\cdot)$ and for the travel from x to 0 at the velocity $v_+(\cdot)$)

$$\tau_-(x) = \int_x^0 dy \left[\frac{1}{v_+(y)} + \frac{1}{v_-(y)} \right] \tag{D12}$$

so that the constraint of the second line corresponds to the normalization of the total time consumed during these excursions.

For a positive excursion, the 'true' probability to end at position x appears in the rate function of Eq. D10

$$\mathcal{P}_+^{exc}(x) = \frac{\gamma_+(x)}{v_+(x)} e^{-\int_0^x dy \frac{\gamma_+(y)}{v_+(y)}} \tag{D13}$$

with the normalization

$$\int_0^{+\infty} dx \mathcal{P}_+^{exc}(x) = 1 \tag{D14}$$

Similarly for a negative excursion, the 'true' probability to end at position x appears in the rate function of Eq. D10

$$\mathcal{P}_-^{exc}(x) = \frac{\gamma_-(x)}{v_-(x)} e^{-\int_x^0 dy \frac{\gamma_-(y)}{v_-(y)}} \tag{D15}$$

with the normalization

$$\int_{-\infty}^0 dx \mathcal{P}_-^{exc}(x) = 1 \quad (\text{D16})$$

The similarity with the large deviations properties in some models of stochastic resetting [39] can be explained as follows. The present model of Eq. D1 can actually be re-interpreted as some kind of stochastic resetting where the reset events are not instantaneous but consume some return time towards the origin : when a switching event occurs at $x > 0$, one could say that there is a reset towards the origin $x = 0$ in the negative state $\sigma = -$ that consumes the return time duration $\int_0^x \frac{dy}{v_-(y)}$; similarly when a switching event occurs at $x < 0$, one could say that there is a reset towards the origin $x = 0$ in the positive state $\sigma = +$ that consumes the return time duration $\int_x^0 \frac{dy}{v_+(y)}$.

This analysis of excursions between returns to the origin is of course very specific to the present model of Eq. D1. For the general run-and-tumble process, the large deviations for the intervals between two consecutive switching events are analyzed in section VI.

Appendix E: Computation of the two first cumulants of any time-additive variable A_T

In this Appendix related to section V, we describe how the optimization procedure summarized in subsection V D can be implemented at the level of the perturbation theory in the parameter k up to order k^2 in order to obtain the two first cumulants of the time-additive variable A_T of Eq. 36. (If one is interested into scaled higher cumulants, one needs to add higher orders k^3, k^4, \dots into the perturbative framework described below).

Plugging the perturbative expansions of the scaled cumulant generating function

$$\mu(k) = k\mu^{(1)} + k^2\mu^{(2)} + O(k^3) \quad (\text{E1})$$

and of the Lagrange multiplier

$$\lambda_k(x) = k\lambda^{(1)}(x) + k^2\lambda^{(2)}(x) + O(k^3) \quad (\text{E2})$$

into Eq. 50 yields the following differential equations for $\lambda^{(1)}(x)$ and $\lambda^{(2)}(x)$ at order k and k^2 respectively

$$\begin{aligned} \frac{d\lambda^{(1)}(x)}{dx} + \left[\frac{\gamma_-(x)}{v_-(x)} - \frac{\gamma_+(x)}{v_+(x)} \right] \lambda^{(1)}(x) &= \Upsilon_1(x) \\ \frac{d\lambda^{(2)}(x)}{dx} + \left[\frac{\gamma_-(x)}{v_-(x)} - \frac{\gamma_+(x)}{v_+(x)} \right] \lambda^{(2)}(x) &= \Upsilon_2(x) \end{aligned} \quad (\text{E3})$$

with the inhomogeneous terms

$$\begin{aligned} \Upsilon_1(x) &\equiv \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \mu^{(1)} - \frac{\gamma_+(x)\beta_+(x)}{v_+(x)} - \frac{\gamma_-(x)\beta_-(x)}{v_-(x)} - \omega(x) \\ \Upsilon_2(x) &\equiv \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \mu^{(2)} - \frac{\gamma_+(x)[\beta_+(x) - \lambda^{(1)}(x)]^2}{2v_+(x)} - \frac{\gamma_-(x)[\beta_-(x) + \lambda^{(1)}(x)]^2}{2v_-(x)} \end{aligned} \quad (\text{E4})$$

The solutions of Eqs E3 that do not diverge exponentially at $x \rightarrow \pm\infty$ read

$$\begin{aligned} \lambda^{(1)}(x) &= e^{-\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \int_{-\infty}^x dy \Upsilon_1(y) e^{\int_0^y dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \\ \lambda^{(2)}(x) &= e^{-\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \int_{-\infty}^x dy \Upsilon_2(y) e^{\int_0^y dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \end{aligned} \quad (\text{E5})$$

and require the vanishing of the following integrals involving the inhomogeneous terms of Eq. E4

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} dx \Upsilon_1(x) e^{\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \\
&= \int_{-\infty}^{+\infty} dx e^{\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \left(\left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \mu^{(1)} - \frac{\gamma_+(x)\beta_+(x)}{v_+(x)} - \frac{\gamma_-(x)\beta_-(x)}{v_-(x)} - \omega(x) \right) \\
0 &= \int_{-\infty}^{+\infty} dx \Upsilon_2(x) e^{\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \\
&= \int_{-\infty}^{+\infty} dx e^{\int_0^x dz \left[\frac{\gamma_-(z)}{v_-(z)} - \frac{\gamma_+(z)}{v_+(z)} \right]} \left(\left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)} \right] \mu^{(2)} - \frac{\gamma_+(x)[\beta_+(x) - \lambda^{(1)}(x)]^2}{2v_+(x)} - \frac{\gamma_-(x)[\beta_-(x) + \lambda^{(1)}(x)]^2}{2v_-(x)} \right)
\end{aligned} \tag{E6}$$

that will determine the values of $\mu^{(1)}$ and $\mu^{(2)}$ respectively. Using the explicit forms of the steady state $P_{\pm}^*(\cdot)$ of Eqs 9 and of the steady state current $j_{\pm}^*(x)$ of Eq. 12, one obtains that the two first cumulants read

$$\begin{aligned}
\mu^{(1)} &= \int_{-\infty}^{+\infty} dx \left(\frac{\gamma_+(x)\beta_+(x)}{v_+(x)} + \frac{\gamma_-(x)\beta_-(x)}{v_-(x)} + \omega(x) \right) j_{\pm}^*(x) \\
\mu^{(2)} &= \int_{-\infty}^{+\infty} dx \left(\frac{\gamma_+(x)[\beta_+(x) - \lambda^{(1)}(x)]^2}{2v_+(x)} + \frac{\gamma_-(x)[\beta_-(x) + \lambda^{(1)}(x)]^2}{2v_-(x)} \right) j_{\pm}^*(x)
\end{aligned} \tag{E7}$$

The first cumulant $\mu^{(1)}$ depends only on the steady state as it should, while the scaled variance $\mu^{(2)}$ involves in addition the first-order solution $\lambda^{(1)}(x)$ of Eq. E5.

Appendix F: Link with the deformed generator method for additive observables

In this Appendix related to section V, we describe the standard deformed generator method to analyze the large deviations of additive observables in order to explain the equivalence with the contraction from the Level 2.5 described in section V.

1. The scaled cumulant generating function $\mu(k)$ from the deformed generator method

The generating function of the additive observable A_T of Eq. 36 as parametrized by the six functions $(\alpha_{\pm}(x); \beta_{\pm}(x); \omega_{\pm}(x))$

$$\langle e^{TkA_T} \rangle = \langle e^{k \left[\int_0^T dt [\alpha_{\sigma(t)}(x(t)) + \dot{x}(t)\omega_{\sigma(t)}(x(t))] + \sum_{t \in [0, T]: \sigma(t^+) \neq \sigma(t^-)} \beta_{\sigma(t^-)}(x(t)) \right]} \rangle \underset{T \rightarrow +\infty}{\simeq} e^{T\mu(k)} \tag{F1}$$

can be analyzed via the following non-conserved deformed generator

$$\begin{aligned}
\frac{\partial \tilde{P}_+(x, t)}{\partial t} &= -[\partial_x - k\omega_+(x)] \left[v_+(x)\tilde{P}_+(x, t) \right] + [k\alpha_+(x) - \gamma_+(x)] \tilde{P}_+(x, t) + \gamma_-(x)e^{k\beta_-(x)} \tilde{P}_-(x, t) \\
\frac{\partial \tilde{P}_-(x, t)}{\partial t} &= [\partial_x - k\omega_-(x)] \left[v_-(x)\tilde{P}_-(x, t) \right] + \gamma_+(x)e^{k\beta_+(x)} \tilde{P}_+(x, t) + [k\alpha_-(x) - \gamma_-(x)] \tilde{P}_-(x, t)
\end{aligned} \tag{F2}$$

The scaled cumulant generating function $\mu(k)$ governing the asymptotic behavior of Eq. F1 for large T corresponds to the highest eigenvalue of this deformed generator with its right positive eigenvector $\tilde{r}_k^{\pm}(x)$

$$\begin{aligned}
\mu(k)\tilde{r}_k^+(x) &= -\frac{d}{dx} \left[v_+(x)\tilde{r}_k^+(x) \right] + [k\alpha_+(x) + k\omega_+(x)v_+(x) - \gamma_+(x)] \tilde{r}_k^+(x) + \gamma_-(x)e^{k\beta_-(x)} \tilde{r}_k^-(x) \\
\mu(k)\tilde{r}_k^-(x) &= \frac{d}{dx} \left[v_-(x)\tilde{r}_k^-(x) \right] + \gamma_+(x)e^{k\beta_+(x)} \tilde{r}_k^+(x) + [k\alpha_-(x) - k\omega_-(x)v_-(x) - \gamma_-(x)] \tilde{r}_k^-(x)
\end{aligned} \tag{F3}$$

and its left positive eigenvector $\tilde{l}_k^\pm(x)$

$$\begin{aligned}\mu(k)\tilde{l}_k^+(x) &= v_+(x)\frac{d}{dx}\tilde{l}_k^+(x) + [k\alpha_+(x) + k\omega_+(x)v_+(x) - \gamma_+(x)]\tilde{l}_k^+(x) + \gamma_+(x)e^{k\beta_+(x)}\tilde{l}_k^-(x) \\ \mu(k)\tilde{l}_k^-(x) &= -v_-(x)\frac{d}{dx}\tilde{l}_k^-(x) + \gamma_-(x)e^{k\beta_-(x)}\tilde{l}_k^+(x) + [k\alpha_-(x) - k\omega_-(x)v_-(x) - \gamma_-(x)]\tilde{l}_k^-(x)\end{aligned}\quad (\text{F4})$$

with the normalization

$$\int_{-\infty}^{+\infty} dx \left[\tilde{l}_k^+(x)\tilde{r}_k^+(x) + \tilde{l}_k^-(x)\tilde{r}_k^-(x) \right] = 1 \quad (\text{F5})$$

2. The corresponding conditioned process obtained via the generalization of Doob's h-transform

The Doob conditioned process is conserved and very similar to the initial dynamics of Eq. 1

$$\begin{aligned}\frac{\partial \tilde{\tilde{P}}_+(x,t)}{\partial t} &= -\partial_x \left[v_+(x)\tilde{\tilde{P}}_+(x,t) \right] - \tilde{\tilde{\gamma}}_k^+(x)\tilde{\tilde{P}}_+(x,t) + \tilde{\tilde{\gamma}}_k^-(x)\tilde{\tilde{P}}_-(x,t) \\ \frac{\partial \tilde{\tilde{P}}_-(x,t)}{\partial t} &= \partial_x \left[v_-(x)\tilde{\tilde{P}}_-(x,t) \right] + \tilde{\tilde{\gamma}}_k^+(x)\tilde{\tilde{P}}_+(x,t) - \tilde{\tilde{\gamma}}_k^-(x)\tilde{\tilde{P}}_-(x,t)\end{aligned}\quad (\text{F6})$$

since the only difference is that the two effective switching rates depend on the two functions $\beta_\pm(x)$ and on the left eigenvector $\tilde{l}_k^\pm(x)$ of Eq. F4

$$\begin{aligned}\tilde{\tilde{\gamma}}_k^+(x) &\equiv \gamma_+(x)e^{k\beta_+(x)}\frac{\tilde{l}_k^-(x)}{\tilde{l}_k^+(x)} \\ \tilde{\tilde{\gamma}}_k^-(x) &\equiv \gamma_-(x)e^{k\beta_-(x)}\frac{\tilde{l}_k^+(x)}{\tilde{l}_k^-(x)}\end{aligned}\quad (\text{F7})$$

The steady state of the conditioned process of Eq. F6 involve the left and the right eigenvectors of Eq. F3 and Eq. F4

$$\begin{aligned}\tilde{\tilde{\rho}}_k^{+*}(x) &= \tilde{l}_k^+(x)\tilde{r}_k^+(x) \\ \tilde{\tilde{\rho}}_k^{-*}(x) &= \tilde{l}_k^-(x)\tilde{r}_k^-(x)\end{aligned}\quad (\text{F8})$$

with the corresponding steady spatial currents

$$\begin{aligned}\tilde{j}_k^{+*}(x) &= v_+(x)\tilde{\tilde{\rho}}_k^{+*}(x) = v_+(x)\tilde{l}_k^+(x)\tilde{r}_k^+(x) \\ \tilde{j}_k^{-*}(x) &= -v_-(x)\tilde{\tilde{\rho}}_k^{-*}(x) = -v_-(x)\tilde{l}_k^-(x)\tilde{r}_k^-(x) = -\tilde{j}_k^{+*}(x)\end{aligned}\quad (\text{F9})$$

and the corresponding steady switching flows

$$\begin{aligned}\tilde{Q}_k^{+*}(x) &= \tilde{\tilde{\gamma}}_k^+(x)\tilde{\tilde{\rho}}_k^{+*}(x) = \frac{\tilde{\tilde{\gamma}}_k^+(x)}{v_+(x)}\tilde{j}_k^{+*}(x) \\ \tilde{Q}_k^{-*}(x) &= \tilde{\tilde{\gamma}}_k^-(x)\tilde{\tilde{\rho}}_k^{-*}(x) = \frac{\tilde{\tilde{\gamma}}_k^-(x)}{v_-(x)}\tilde{j}_k^{+*}(x)\end{aligned}\quad (\text{F10})$$

3. Equivalence with the optimization from the Level 2.5

The consistency between the approach based on the optimization of the Level 2.5 (see section V) and the alternative approach based on the deformed generator with the corresponding Doob conditioned process yields that the steady state observables of the conditioned Doob process described above should coincide with the optimal solution found via the contraction of the Level 2.5 in section V. In the present case, this means that the optimal current $j_k^{+opt}(x)$

and the optimal switching flows $Q_k^{\pm opt}(x)$ of section V correspond to the Doob steady current of Eq. F9 and to the Doob steady switching flows of Eqs F10

$$\begin{aligned} j_k^{+opt}(x) &= \tilde{j}_k^{+*}(x) \\ Q_k^{\pm opt}(x) &= \tilde{Q}_k^{\pm*}(x) \end{aligned} \quad (\text{F11})$$

The identification between the Doob switching rates of Eq. F7 and the effective switching rates of the optimal solution of Eq. 52 yields that the Lagrange multiplier $\lambda_k(x)$ corresponds to the logarithm of the ratio of the two components $\tilde{l}_k^{\pm}(x)$ of the left eigenvector of Eq. F4

$$\lambda_k(x) = \ln \left(\frac{\tilde{l}_k^+(x)}{\tilde{l}_k^-(x)} \right) \quad (\text{F12})$$

Using Eq. F4, one obtains that the ratio of Eq. F12 satisfies the non-linear differential equation

$$\begin{aligned} \frac{d\lambda_k(x)}{dx} &= \frac{\frac{d\tilde{l}_k^+(x)}{dx}}{\tilde{l}_k^+(x)} - \frac{\frac{d\tilde{l}_k^-(x)}{dx}}{\tilde{l}_k^-(x)} \\ &= -\frac{\gamma_+(x)e^{k\beta_+(x)}}{v_+(x)} e^{-\lambda_k(x)} - \frac{\gamma_-(x)e^{k\beta_-(x)}}{v_-(x)} e^{\lambda_k(x)} + \frac{\gamma_+(x) + \mu(k)}{v_+(x)} + \frac{\gamma_-(x) + \mu(k)}{v_-(x)} - k \left[\omega_+(x) - \omega_-(x) + \frac{\alpha_+(x)}{v_+(x)} + \frac{\alpha_-(x)}{v_-(x)} \right] \end{aligned} \quad (\text{F13})$$

that indeed coincides with Eq. 50 in terms of the notation $\omega(x)$ introduced in Eq. 39.

Appendix G: Link between the Level 2.75 and the Level 2.5

In this Appendix related to section VI, we explain the link between the Level 2.75 and the Level 2.5. The comparison between the Level 2.5 of Eq. 28 and the Level 2.75 of Eq. 67 yields that the conditional probability to see the empirical densities of intervals $q_{\pm}(\cdot, \cdot)$ once the other empirical observables $[j_+(\cdot), Q_{\pm}(\cdot)]$ are given reads

$$\begin{aligned} P_T^{conditional}[q_{\pm}(\cdot, \cdot)|j_+(\cdot), Q_{\pm}(\cdot)] &\equiv \frac{P_T[j_+(\cdot), Q_{\pm}(\cdot), q_{\pm}(\cdot, \cdot)]}{P_T[j_+(\cdot), Q_{\pm}(\cdot)]} \underset{T \rightarrow +\infty}{\simeq} e^{-TI^{conditional}[j_+(\cdot), Q_{\pm}(\cdot), q_{\pm}(\cdot, \cdot)]} \quad (\text{G1}) \\ &\left[\prod_{x_R} \delta \left(\int_{-\infty}^{x_R} dx_L q_+(x_R, x_L) - Q_+(x_R) \right) \right] \left[\prod_{x_R} \delta \left(\int_{-\infty}^{x_R} dx_L q_-(x_L, x_R) - Q_+(x_R) \right) \right] \\ &\left[\prod_{x_L} \delta \left(\int_{x_L}^{+\infty} dx_R q_+(x_R, x_L) - Q_-(x_L) \right) \right] \left[\prod_{x_L} \delta \left(\int_{x_L}^{+\infty} dx_R q_-(x_L, x_R) - Q_-(x_L) \right) \right] \end{aligned}$$

The conditional rate function reads

$$\begin{aligned} I^{conditional}[j_+(\cdot), Q_{\pm}(\cdot), q_{\pm}(\cdot, \cdot)] &= I_{2.75}[j_+(\cdot), Q_{\pm}(\cdot), q_{\pm}(\cdot, \cdot)] - I_{2.5}[j_+, Q_{\pm}(\cdot)] \quad (\text{G2}) \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \ln \left(\frac{q_+(x_R, x_L)}{\frac{\gamma_+(x_R)}{v_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_+(x)}{v_+(x)}} Q_-(x_L)} \right) + q_-(x_L, x_R) \ln \left(\frac{q_-(x_L, x_R)}{\frac{\gamma_-(x_L)}{v_-(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{\gamma_-(x)}{v_-(x)}} Q_+(x_R)} \right) \right] \\ &- \int_{-\infty}^{+\infty} dx \left[Q_+(x) \ln \left(\frac{Q_+(x)}{\frac{\gamma_+(x)}{v_+(x)} j_+(x)} \right) - Q_+(x) + \frac{\gamma_+(x)}{v_+(x)} j_+(x) \right] - \int_{-\infty}^{+\infty} dx \left[Q_-(x) \ln \left(\frac{Q_-(x)}{\frac{\gamma_-(x)}{v_-(x)} j_+(x)} \right) - Q_-(x) + \frac{\gamma_-(x)}{v_-(x)} j_+(x) \right] \end{aligned}$$

The explicit form of the kernel $W_{\pm}(\cdot, \cdot)$ of Eqs 53 and 56, can be translated for the effective kernels $\hat{W}_{\pm}(\cdot, \cdot)$ associated to the effective switching rates $\hat{\gamma}_{\pm}(x) = v_{\pm}(x) \frac{Q_{\pm}(x)}{j_{\pm}(x)}$ of Eq. B2

$$\begin{aligned} \hat{W}_-(x_L|x_R) &= \frac{\hat{\gamma}_-(x_L)}{v_-(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{\hat{\gamma}_-(x)}{v_-(x)}} = \frac{Q_-(x_L)}{j_+(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{Q_-(x)}{j_+(x)}} \\ \hat{W}_+(x_R|x_L) &= \frac{\hat{\gamma}_+(x_R)}{v_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{\hat{\gamma}_+(x)}{v_+(x)}} = \frac{Q_+(x_R)}{j_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{Q_+(x)}{j_+(x)}} \end{aligned} \quad (\text{G3})$$

These effective kernels $\hat{W}_\pm(\cdot, \cdot)$ are useful together with the constraints to rewrite the conditional rate function of Eq. G2 as

$$\begin{aligned} & \mathcal{I}^{conditional}[j_+(\cdot), Q_\pm(\cdot), q_\pm(\cdot, \cdot)] \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \ln \left(\frac{q_+(x_R, x_L)}{\hat{W}_+(x_R|x_L)Q_-(x_L)} \right) + q_-(x_L, x_R) \ln \left(\frac{q_-(x_L, x_R)}{\hat{W}_-(x_L|x_R)Q_+(x_R)} \right) \right] \\ &= \int_{-\infty}^{+\infty} dx_L \int_{x_L}^{+\infty} dx_R \left[q_+(x_R, x_L) \ln \left(\frac{q_+(x_R, x_L)}{\frac{Q_+(x_R)}{j_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{Q_+(x)}{j_+(x)}} Q_-(x_L)} \right) + q_-(x_L, x_R) \ln \left(\frac{q_-(x_L, x_R)}{\frac{Q_-(x_L)}{j_+(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{Q_-(x)}{j_+(x)}} Q_+(x_R)} \right) \right] \end{aligned} \quad (\text{G4})$$

This factorized form shows that this conditional rate function vanishes for the optimal values of the empirical density of intervals

$$\begin{aligned} q_+^{opt}(x_R, x_L) &= \frac{Q_+(x_R)}{j_+(x_R)} e^{-\int_{x_L}^{x_R} dx \frac{Q_+(x)}{j_+(x)}} Q_-(x_L) \\ q_-^{opt}(x_L, x_R) &= \frac{Q_-(x_L)}{j_+(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{Q_-(x)}{j_+(x)}} Q_+(x_R) \end{aligned} \quad (\text{G5})$$

once the other one-position empirical observables $[j_+(\cdot), Q_\pm(\cdot)]$ are given. Taking into account the stationarity constraint of Eq. 24 to replace $Q_+(x) = -\frac{dj_+(x)}{dx} + Q_-(x)$ in the exponential of the first Eq G5

$$\begin{aligned} q_+^{opt}(x_R, x_L) &= \frac{Q_+(x_R)Q_-(x_L)}{j_+(x_R)} e^{\int_{x_L}^{x_R} dx \frac{d \ln j_+(x)}{dx}} - \int_{x_L}^{x_R} dx \frac{Q_-(x)}{j_+(x)} \\ &= \frac{Q_+(x_R)Q_-(x_L)}{j_+(x_L)} e^{-\int_{x_L}^{x_R} dx \frac{Q_-(x)}{j_+(x)}} = q_-^{opt}(x_L, x_R) \end{aligned} \quad (\text{G6})$$

one obtains that the two optimal values of Eq. G5 actually coincide. Since they satisfy the constraints of Eq. G1, they correspond to the optimal solutions for the contraction of the Level 2.75 to recover the Level 2.5.

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