



HAL
open science

Large deviations for Markov processes with stochastic resetting: analysis via the empirical density and flows or via excursions between resets

Cecile Monthus

► **To cite this version:**

Cecile Monthus. Large deviations for Markov processes with stochastic resetting: analysis via the empirical density and flows or via excursions between resets. *Journal of Statistical Mechanics: Theory and Experiment*, 2021, 2021 (3), pp.033201. 10.1088/1742-5468/abdeaf . hal-03367003

HAL Id: hal-03367003

<https://hal.science/hal-03367003>

Submitted on 11 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Large deviations for Markov processes with stochastic resetting : analysis via the empirical density and flows or via excursions between resets

Cécile Monthus

Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA, 91191 Gif-sur-Yvette, France

Markov processes with stochastic resetting towards the origin generically converge towards non-equilibrium steady-states. Long dynamical trajectories can be thus analyzed via the large deviations at Level 2.5 for the joint probability of the empirical density and the empirical flows, or via the large deviations of semi-Markov processes for the empirical density of excursions between consecutive resets. The large deviations properties of general time-additive observables involving the position and the increments of the dynamical trajectory are then analyzed in terms of the appropriate Markov tilted processes and of the corresponding conditioned processes obtained via the generalization of Doob's h-transform. This general formalism is described in detail for the three possible frameworks, namely discrete-time/discrete-space Markov chains, continuous-time/discrete-space Markov jump processes and continuous-time/continuous-space diffusion processes, and is illustrated with explicit results for the Sisyphus Random Walk and its variants, when the reset probabilities or reset rates are space-dependent.

I. INTRODUCTION

The recent interest into stochastic resetting (see the review [1] and references therein) can be explained via its many applications : within the field of intermittent search strategies (see the review [2] and references therein), the resetting procedure is clearly the simplest one; jump-diffusions processes have been also much studied in mathematical finance [3, 4]), in biology for integrate-and-fire neuronal models [5, 6] and in ecology to describe fires [7] or soil moisture [8, 9]; finally, for open quantum systems, the unravelling of the Lindblad dynamics in terms of quantum trajectories involve quantum jumps that are analogous to resetting procedures [10], and it is thus interesting to better understand the similarities and differences with resetting in classical stochastic models [10–12].

Besides all the possible applications, another motivation to study stochastic resetting is that the resetting procedure towards the origin will generically produce a non-equilibrium steady state localized around the origin, even if the process without resetting does not converge towards a stationary state [1, 13]. In the present paper, our goal will be thus to characterize Markov processes with stochastic resetting via the recent progresses made in the field of large deviations for non-equilibrium steady-states (see the reviews with different scopes [14–22], the PhD Theses [23–26] and the HDR Thesis [27]). The large deviations at Level 2.5 has emerged in order to characterize the non-equilibrium steady-states via the joint distribution of the empirical measure and of the empirical flows. While the theory of large deviations has a long history (see the reviews [28–30] and references therein), the explicit forms of rate functions at Level 2.5 have been obtained more recently for discrete-time/discrete-space Markov chains [23, 30–32], for continuous-time/discrete-space Markov jump processes [23, 26, 27, 33–43] and for continuous-time/continuous-space diffusion processes [26, 27, 36, 44–46], while the generalization to the Lindblad dynamics [10, 11] offers new possibilities to analyze open quantum systems. This Level 2.5 formulation plays an essential role, because any time-additive observable of the dynamical trajectory can be reconstructed via its decomposition in terms of the empirical density and of the empirical flows. It is thus closely related to the studies focusing on the generating functions of time-additive observables via deformed Markov operators, that have attracted a lot of interest in various models [10, 11, 14, 19–22, 24, 27, 46–71], with the formulation of the corresponding 'conditioned' process via the generalization of Doob's h-transform.

In the field of stochastic resetting, large deviations have been already considered for time-additive observables of various processes [1, 72–76]. In particular, for the special case of time-additive observables involving only the position in one-dimensional diffusion models with a uniform resetting rate, the studies [72, 75] have used the decomposition of the dynamical trajectory into excursions between resets in order to relate the generating functions with and without resetting. As a consequence, besides the analysis of large deviations at Level 2.5 for the empirical density and of the empirical flows mentioned above, we will also consider the large deviations for the empirical excursions between two consecutive resets. These two points of view will be then used to analyze the large deviations of the most general time-additive observables involving both the position and the increments of the dynamical trajectory, when the reset probabilities or reset rates are not uniform anymore but space-dependent, since this is relevant in many applications [6, 7, 9, 77–80].

Let us now describe in more detail the organization of the paper and mention the equations numbers where the main results can be found.

A. Discrete-time/discrete-space Markov chains in sections II and III and in Appendix A

In section II, Markov Chains in discrete time and discrete space with space-dependent resetting probabilities (Eq. 2) are analyzed via the large deviation at Level 2.5 for the empirical density and flows (Eq. 8) and via the large deviations for the empirical excursions between two consecutive resets (Eq. 18). The generating function of general time-additive observables (Eq. 27) is analyzed via the tilted dynamics (Eq. 30), via the generator of the conditioned process obtained as the generalization of Doob's h-transform (Eq. 51) and via the probability of excursions between resets of this conditioned process (Eq. 62).

In section III, this general formalism is applied to the example of the Sisyphus Random Walk [77] on the semi-infinite lattice $x = 0, 1, 2, \dots$, where Sisyphus at position x at time t (Eq 65) can either return to the origin with probability R_x or move forward to the next position ($x + 1$) with the complementary probability $(1 - R_x)$. Explicit expressions are given for the existence of a non-equilibrium steady-state (Eq. 71), for the large deviations at Level 2.5 (Eq. 78) and at Level 2 for the empirical density alone (Eq. 82), as well as for the large deviations of excursions (Eq. 86). The generating function of general time-additive observables (Eq. 89) is analyzed via the tilted dynamics and via the tilted excursions to obtain the same explicit result for the scaled cumulant generating function (Eqs 95 and 105).

Appendix A contains another application of the general formalism of section II to the Sisyphus Random Walk on the Cayley Tree of branching ratio b , that generalizes the results of the one-dimensional case $b = 1$ described in section III.

B. Continuous-time/discrete-space Markov jump processes in sections IV and V

In section IV, Markov Jump processes in continuous time and discrete space with space-dependent resetting rates (Eq. 113) are analyzed via the large deviation at Level 2.5 for the empirical density and flows (Eq. 120) and via the large deviations of empirical excursions between two consecutive resets (Eq. 127). The generating function of general time-additive observables (Eq. 133) is analyzed via the tilted dynamics (Eqs 136 and 137), via the generator of the conditioned process obtained as the generalization of Doob's h-transform (Eq. 142) and via the tilted excursions (Eq. 146) with the corresponding conditioned excursion probabilities (Eq 149).

In section V, this general formalism is applied to the example of the Sisyphus Jump process on the semi-infinite lattice $x = 0, 1, 2, \dots$, where Sisyphus at position x at time t (Eq 153) can return to the origin with rate r_x , move forward to the next position ($x + 1$) with rate w or remain at x . Explicit expressions are given for the existence of a non-equilibrium steady-state (Eq. 157), for the large deviations at Level 2.5 (Eq. 162) and for the large deviations of excursions (Eq. 172). The generating function of general time-additive observables (Eq. 173) is analyzed via the tilted dynamics and via the tilted excursions to obtain the same explicit result for the scaled cumulant generating function (Eqs 180 and 192).

C. Continuous-time/continuous-space diffusion processes in sections VI and VII

In section VI, diffusion processes in a force field in dimension d with space-dependent resetting rates (Eq. 193) are analyzed via the large deviation at Level 2.5 for the empirical density and currents (Eq. 200) and via the large deviations of empirical excursions between two consecutive resets (Eq. 209). The generating function of general time-additive observables (Eq. 215) is analyzed via the tilted dynamics (Eq 218), via the generator of the conditioned process obtained as the generalization of Doob's h-transform (Eqs 221 and 225) and via the tilted excursions (Eq. 233) with the corresponding conditioned excursion probabilities (Eq 235).

In section VII, this general formalism is applied to the example of the continuous-time Sisyphus process (Eq 238) on the semi-infinite line $x \in [0, +\infty[$. Explicit expressions are given for the existence of a non-equilibrium steady-state (Eq. 242), for the large deviations at Level 2.5 (Eq. 244) and at Level 2 for the empirical density alone (Eq. 246), as well as for the large deviations of excursions (Eq. 251). The generating function of general time-additive observables (Eq. 254) is analyzed via the tilted dynamics and via the tilted excursions to obtain the same explicit result for the scaled cumulant generating function (Eqs 259 and 269).

II. MARKOV CHAIN IN DISCRETE TIME AND DISCRETE SPACE WITH RESETTING

A. Models and notations

In this section, we focus on the Markov Chain dynamics for the probability $P_y(t)$ to be at position y at time t

$$P_x(t+1) = \sum_y W_{x,y}^{reset} P_y(t) \quad (1)$$

where the matrix

$$W_{x,y}^{reset} \equiv W_{x,y}(1 - R_y) + \delta_{x,0}R_y \quad (2)$$

has the following physical meaning. When the particle is at position y at time t , two types of moves are possible to obtain the new position x at time $t+1$:

(i) with probability $R_y \in [0, 1]$, there is a reset to the origin $x = 0$, i.e. the particle makes a non-local spatial move from y to 0.

(ii) with the complementary probability $(1 - R_y)$, the particle follows the spatially-local Markov Chain described by the matrix $W_{x,y}$, i.e. it jumps towards the some neighboring position x of y with probability $W_{x,y}$, with the normalization

$$\sum_x W_{x,y} = 1 \quad (3)$$

Although the case of uniform resetting probabilities $R_y = R$ is the most studied in the literature [1], it is also interesting to consider the case of space-dependent resetting probabilities, where R_y depends on the position y either deterministically or randomly [6, 7, 9, 77–80] since this is very relevant in many applications [6]. In the whole paper, we will thus consider that the resetting probability is space-dependent.

As stressed in the Introduction, the stochastic resetting towards the origin will generically produce a non-equilibrium steady state localized around the origin even if the process without resetting does not converge towards a stationary state [1, 13]. In this section, we will thus assume that the steady-state solution P_x^* of Eq. 1

$$P_x^* = \sum_y W_{x,y}^{reset} P_y^* \quad (4)$$

exists in order to apply the large deviations at Level 2.5 for non-equilibrium steady-states.

B. Large deviations at level 2.5 for the empirical density and the empirical flows

The empirical 2-point density characterizes the flows between two consecutive positions in a very long trajectory $x(0 \leq t \leq T)$

$$\rho_{x,y}^{(2)} \equiv \frac{1}{T} \sum_{t=1}^T \delta_{x(t),x} \delta_{x(t-1),y} \quad (5)$$

It contains the information on the empirical 1-point density that can be obtained via the sum over the first or the second position (up to a boundary term of order $1/T$ that is negligible for large duration $T \rightarrow +\infty$)

$$\rho_x \equiv \frac{1}{T} \sum_{t=1}^T \delta_{x(t),x} = \sum_y \rho_{x,y}^{(2)} = \sum_y \rho_{y,x}^{(2)} \quad (6)$$

with the normalization

$$\sum_x \rho_x = 1 \quad (7)$$

For this discrete-time/discrete-space Markov chain framework, the joint probability to see the empirical 2-point and 1-point densities follows the large deviation form at Level 2.5 [23, 30–32]

$$P_T(\rho_{\dots}^{(2)}; \rho_{\dots}) \underset{T \rightarrow +\infty}{\simeq} C(\rho_{\dots}^{(2)}; \rho_{\dots}) e^{-TI(\rho_{\dots}^{(2)}; \rho_{\dots})} \quad (8)$$

with the constraints discussed in Eqs 6 and 7

$$C(\rho_{\cdot,\cdot}^{(2)}; \rho_{\cdot}) = \delta \left(\sum_x \rho_x - 1 \right) \prod_x \left[\delta \left(\sum_y \rho_{x,y}^{(2)} - \rho_x \right) \delta \left(\sum_y \rho_{y,x}^{(2)} - \rho_x \right) \right] \quad (9)$$

while the rate function involves the kernel $W_{x,y}^{reset}$ of Eq. 2

$$I(\rho_{\cdot,\cdot}^{(2)}; \rho_{\cdot}) = \sum_x \sum_y \rho_{x,y}^{(2)} \ln \left(\frac{\rho_{x,y}^{(2)}}{W_{x,y}^{reset} \rho_y} \right) = \sum_x \sum_y \rho_{x,y}^{(2)} \ln \left(\frac{\rho_{x,y}^{(2)}}{[W_{x,y}(1 - R_y) + \delta_{x,0} R_y] \rho_y} \right) \quad (10)$$

Then one needs to discuss the possible empirical flows $\rho_{x,y}^{(2)}$ due to the Markov chain $W_{x,y}$ and to the resetting procedure in order to obtain more explicit expressions, as explained on the specific examples of section III and Appendix A.

C. Large deviations for the empirical excursions between two consecutive resets

1. Empirical density of excursions between two consecutive resets

A very long dynamical trajectory $x(0 \leq t \leq T)$ starting at $x_0 = 0$ can also be analyzed via its decomposition into excursions between resets. Let us note t_l with $l = 0, 1, \dots, N - 1$ the times where a reset towards the origin occurs $x(t_l) = 0$, with $t_0 = 0$. It is convenient to introduce the durations of these excursions

$$\tau_l \equiv t_{l+1} - t_l \quad (11)$$

and the corresponding internal trajectory for $s = 1, \dots, \tau_l - 1$

$$y^{(l)}(s) \equiv x(t_l + s) \quad (12)$$

between $y^{(l)}(s = 0) = x(t_l) = 0$ and $y^{(l)}(\tau_l) = x(t_l + \tau_l) = x(t_{l+1}) = 0$.

For large time T , we assume that the density $n = \frac{N}{T}$ of resets will be finite, so the total number $N = nT$ of excursions will be large, and we may neglect the fact that the last excursion does not finish exactly at time T , in order to analyze the empirical density of excursions of duration τ and internal trajectory $y(1 \leq s \leq \tau - 1)$

$$n[\tau; y(\cdot)] \equiv n[\tau; y(1 \leq s \leq \tau - 1)] \equiv \frac{1}{T} \sum_{l=0}^{N-1} \delta_{\tau_l, \tau} \left(\prod_{s=1}^{\tau-1} \delta_{y^{(l)}(s), y(s)} \right) \quad (13)$$

The total density of excursions is then

$$n \equiv \frac{N}{T} = \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \quad (14)$$

while the normalization is given by the total duration of the excursions

$$\sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} n[\tau; y(\cdot)] = \frac{1}{T} \sum_{l=0}^{N-1} \tau_l = 1 \quad (15)$$

Note that if we had not neglected the fact that the last excursion does not finish exactly at time T , there would be a correction of order $1/T$ in Eq. 15 that would be the analog of the boundary correction of order $1/T$ that one neglects in Eq. 6.

2. Large deviations for the empirical density of excursions between two consecutive resets

For the dynamics of Eq. 2, the probability to have an excursion of duration τ and of internal trajectory $y(1 \leq s \leq \tau - 1)$ reads

$$\begin{aligned} P^{exc}[\tau; y(\cdot)] &\equiv P^{exc}[\tau; y(1 \leq s \leq \tau - 1)] = R_{y(\tau-1)} W_{y(\tau-1), y(\tau-2)} (1 - R_{y(\tau-2)}) \dots W_{y(1), 0} (1 - R_0) \\ &= R_{y(\tau-1)} \left[\prod_{s=1}^{\tau-1} W_{y(s), y(s-1)} (1 - R_{y(s-1)}) \right] \end{aligned} \quad (16)$$

For $\tau = 1$ corresponding to a consecutive reset towards the origin, there is no-internal trajectory $y(1 \leq s \leq \tau - 1)$ and the probability reduces to

$$P^{exc}[\tau = 1] = R_0 \quad (17)$$

The framework of semi-Markov processes [23, 32, 81–84] allows to write the large deviations properties of empirical intervals as follows. The probability to see the empirical density $n[\tau; y(1 \leq s \leq \tau - 1)]$ of excursions between resets and the total density n follows the large deviation form

$$P_T(n[:, :], n) \underset{T \rightarrow +\infty}{\simeq} C(n[:, :], n) e^{-TI(n[:, :], n)} \quad (18)$$

with the constraints of Eqs 13 and 15

$$C(n[:, :], n) = \delta \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] - n \right] \delta \left[\sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} n[\tau; y(\cdot)] - 1 \right] \quad (19)$$

and the rate function that involves the probability of excursions (Eq. 16)

$$I(n[:, :], n) = \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{P^{exc}[\tau; y(\cdot)]n} \right) \quad (20)$$

3. Simplifications for uniform reset probability $R_y = R$

When the reset probability is uniform $R_y = R$, the probability of excursions of Eq. 16 becomes factorized

$$P_{uniform}^{exc}[\tau; y(\cdot)] = p^{geo}(\tau) p_{\tau}^{config}[y(\cdot)] \quad (21)$$

into the geometric distribution that describes the probability of the duration $\tau = 1, 2, \dots$ of the excursion independently of the internal trajectory $y(1 \leq s \leq \tau - 1)$

$$p^{geo}(\tau) = R(1 - R)^{\tau-1} \quad (22)$$

while

$$p_{\tau}^{config}[y(\cdot)] = \prod_{s=1}^{\tau-1} W_{y(s), y(s-1)} \quad (23)$$

represents the probability of the internal trajectory $y(1 \leq s \leq \tau - 1)$ once the duration τ is fixed, and coincides with the probability of the unperturbed Markov Chain of kernel $W_{x,y}$ starting at $y(0) = 0$. This decoupling is at the origin of most results obtained for the case of uniform resetting. In particular, the statistical properties of the durations of the intervals between resets is completely independent from the configurational part. For instance, the probability of the total density $n = \frac{N}{T}$ of excursions simply follows the binomial distribution with its corresponding large deviation form

$$P_T^{Binomial}(n) = \frac{T!}{(nT)!((1-n)T)!} R^{nT} (1-R)^{(1-n)T} \underset{T \rightarrow +\infty}{\propto} e^{-T \left[n \ln \left(\frac{n}{R} \right) + (1-n) \ln \left(\frac{1-n}{1-R} \right) \right]} \quad (24)$$

On the contrary, whenever the reset probability R_y depends on the position y , the duration τ and the internal trajectory $y(1 \leq s \leq \tau - 1)$ are coupled via Eq. 16.

D. Large deviations for time-additive observables via three points of view

In this section, we focus on time-additive observables of the trajectory $x(0 \leq t \leq T)$ that involves some function α_x

$$A_T \equiv \frac{1}{T} \sum_{t=1}^T \alpha_x(t) \quad (25)$$

or that involves some function $\beta_{x,y}$

$$B_T \equiv \frac{1}{T} \sum_{t=1}^T \beta_{x(t),x(t-1)} \quad (26)$$

In the following, we consider the observable $(A_T + B_T)$ and analyze its large deviations properties via the scaled cumulant generating function $\mu(k)$ appearing in the asymptotic behavior

$$Z_T(k) \equiv \langle e^{Tk(A_T+B_T)} \rangle \underset{T \rightarrow +\infty}{\simeq} e^{T\mu(k)} \quad (27)$$

1. Analysis via the tilted dynamics

As recalled in the Introduction, the standard method to analyze the generating function of time-additive observables is based on the spectral analysis of deformed Markov operators [10, 11, 14, 19–22, 24, 27, 46–71]. Let us recall how this method works within the present Markov Chain framework. The probabilities of the whole trajectories $x(0 \leq t \leq T)$

$$\mathcal{P}[x(T), x(T-1), \dots, x(1), x(0) = 0] = \delta_{x(0),0} \prod_{t=1}^T W_{x(t),x(t-1)}^{reset} \quad (28)$$

allow to compute the generating function of Eq. 27

$$\begin{aligned} Z_T(k) = \langle e^{k \sum_{t=1}^T (\alpha_{x(t)} + \beta_{x(t),x(t-1)})} \rangle &= \delta_{x(0),0} \sum_{x(1 \leq t \leq T)} \prod_{t=1}^T \left[W_{x(t),x(t-1)}^{reset} e^{k(\alpha_{x(t)} + \beta_{x(t),x(t-1)})} \right] \\ &\equiv \delta_{x(0),0} \sum_{x(1 \leq t \leq T)} \prod_{t=1}^T \left[\tilde{W}_{x(t),x(t-1)}^{[k]} \right] \end{aligned} \quad (29)$$

via the tilted matrix

$$\tilde{W}_{x,y}^{[k]} \equiv W_{x,y}^{reset} e^{k(\alpha_x + \beta_{x,y})} = [W_{x,y}(1 - R_y) + \delta_{x,0} R_y] e^{k(\alpha_x + \beta_{x,y})} \quad (30)$$

The scaled cumulant generating function $\mu(k)$ of Eq. 27 will then correspond to the logarithm of the highest eigenvalue $e^{\mu(k)}$ of the tilted matrix of Eq. 30 that will dominate the propagator

$$\langle x | \left(W^{[k]} \right)^T | y \rangle \underset{T \rightarrow +\infty}{\simeq} e^{T\mu(k)} \tilde{r}_x^{[k]} \tilde{l}_y^{[k]} \quad (31)$$

where $\tilde{r}_x^{[k]}$ is the corresponding positive right eigenvector

$$e^{\mu(k)} \tilde{r}_x^{[k]} = \sum_y \tilde{W}_{x,y}^{[k]} \tilde{r}_y^{[k]} = \sum_y W_{x,y} (1 - R_y) e^{k(\alpha_x + \beta_{x,y})} \tilde{r}_y^{[k]} + \delta_{x,0} \sum_y R_y e^{k(\alpha_0 + \beta_{0,y})} \tilde{r}_y^{[k]} \quad (32)$$

and where $\tilde{l}_y^{[k]}$ is the corresponding positive left eigenvector

$$e^{\mu(k)} \tilde{l}_y^{[k]} = \sum_x \tilde{l}_x^{[k]} \tilde{W}_{x,y}^{[k]} = \sum_x \tilde{l}_x^{[k]} W_{x,y} (1 - R_y) e^{k(\alpha_x + \beta_{x,y})} + \tilde{l}_0^{[k]} R_y e^{k(\alpha_0 + \beta_{0,y})} \quad (33)$$

with the normalization

$$\sum_x \tilde{l}_x^{[k]} \tilde{r}_x^{[k]} = 1 \quad (34)$$

2. Analysis via the empirical density and empirical flows

The empirical 1-point density ρ_x of Eq. 6 allows to reconstruct any time-additive observable A_T of the position (Eq. 25)

$$A_T = \sum_x \alpha_x \rho_x \quad (35)$$

while the empirical 2-point density $\rho_{x,y}^{(2)}$ of Eq. 5 allows to reconstruct any time-additive observable B_T of the increments (Eq. 26)

$$B_T = \sum_x \sum_y \beta_{x,y} \rho_{x,y}^{(2)} \quad (36)$$

As a consequence, the generating function of Eq. 27 can be evaluated from the joint probability of Eqs 8 9 10

$$\begin{aligned} Z_T(k) &= \int \mathcal{D}\rho_{\dots}^{(2)} \int \mathcal{D}\rho, P_T(\rho_{\dots}^{(2)}; \rho) e^{Tk \left(\sum_x \alpha_x \rho_x + \sum_x \sum_y \beta_{x,y} \rho_{x,y}^{(2)} \right)} \\ &\underset{T \rightarrow +\infty}{\simeq} \int \mathcal{D}\rho_{\dots}^{(2)} \int \mathcal{D}\rho, \delta \left(\sum_x \rho_x - 1 \right) \prod_x \left[\delta \left(\sum_y \rho_{x,y}^{(2)} - \rho_x \right) \delta \left(\sum_x \rho_{x,y}^{(2)} - \rho_y \right) \right] \\ &e^T \left[- \sum_x \sum_y \rho_{x,y}^{(2)} \ln \left(\frac{\rho_{x,y}^{(2)}}{\tilde{W}_{x,y}^{reset} \rho_y} \right) + k \sum_x \alpha_x \rho_x + k \sum_x \sum_y \beta_{x,y} \rho_{x,y}^{(2)} \right] \end{aligned} \quad (37)$$

Using the constraints, one can rewrite the functional in the exponential in a more compact form involving the tilted generator of Eq. 30

$$Z_T(k) \underset{T \rightarrow +\infty}{\simeq} \int \mathcal{D}\rho_{\dots}^{(2)} \int \mathcal{D}\rho, \delta \left(\sum_x \rho_x - 1 \right) \prod_x \left[\delta \left(\sum_y \rho_{x,y}^{(2)} - \rho_x \right) \delta \left(\sum_x \rho_{x,y}^{(2)} - \rho_y \right) \right] e^{-T \sum_x \sum_y \rho_{x,y}^{(2)} \ln \left(\frac{\rho_{x,y}^{(2)}}{\tilde{W}_{x,y}^{[k]} \rho_y} \right)} \quad (38)$$

It is convenient to make the change of variables from the 2-point density $\rho_{x,y}^{(2)}$ to the effective Markov-Chain kernel that would make $\rho_{\dots}^{(2)}$ and ρ typical

$$\tilde{W}_{x,y} \equiv \frac{\rho_{x,y}^{(2)}}{\rho_y} \quad (39)$$

to obtain

$$\begin{aligned} Z_T(k) &\underset{T \rightarrow +\infty}{\simeq} \int \mathcal{D}\tilde{W}_{\dots} \int \mathcal{D}\rho, \delta \left(\sum_x \rho_x - 1 \right) \prod_x \left[\delta \left(\sum_y \tilde{W}_{x,y} \rho_y - \rho_x \right) \delta \left(\sum_x \tilde{W}_{x,y} - 1 \right) \right] \\ &e^{-T \sum_y \rho_y \sum_x \tilde{W}_{x,y} \ln \left(\frac{\tilde{W}_{x,y}}{\tilde{W}_{x,y}^{[k]}} \right)} \end{aligned} \quad (40)$$

The two last constraints mean that $\tilde{r}_x = \rho_x$ and $\tilde{l}_x = 1$ are the right and the left eigenvectors of the matrix \tilde{W} associated to the eigenvalue unity. In order to optimize the functional in the exponential in the presence of the constraints, it is convenient to introduce the following Lagrangian with the Lagrange multipliers $(\omega^{[k]}, \nu_x^{[k]}, \lambda_y^{[k]})$

$$\begin{aligned} \mathcal{L}(\tilde{W}_{\dots}; \rho) &= - \sum_y \rho_y \sum_x \tilde{W}_{x,y} \ln \left(\frac{\tilde{W}_{x,y}}{\tilde{W}_{x,y}^{[k]}} \right) - \omega^{[k]} \left(\sum_x \rho_x - 1 \right) + \sum_x \nu_x^{[k]} \left(\sum_y \tilde{W}_{x,y} \rho_y - \rho_x \right) + \sum_y \lambda_y^{[k]} \left(\sum_x \tilde{W}_{x,y} - 1 \right) \\ &= \omega^{[k]} + \sum_y \rho_y \left[- \sum_x \tilde{W}_{x,y} \ln \left(\frac{\tilde{W}_{x,y}}{\tilde{W}_{x,y}^{[k]}} \right) - \omega^{[k]} + \sum_x \nu_x^{[k]} \tilde{W}_{x,y} - \nu_y^{[k]} \right] + \sum_y \lambda_y^{[k]} \left(\sum_x \tilde{W}_{x,y} - 1 \right) \end{aligned} \quad (41)$$

Again it is useful to use the constraints to obtain the more compact form

$$\mathcal{L}(\tilde{W}_{\dots}; \rho_{\cdot}) = \omega^{[k]} + \sum_y \rho_y \left[- \sum_x \tilde{W}_{x,y} \ln \left(\frac{\tilde{W}_{x,y}}{\tilde{W}_{x,y}^{[k]} e^{-\omega^{[k]} + \nu_x^{[k]} - \nu_y^{[k]}}} \right) \right] + \sum_y \lambda_y^{[k]} \left(\sum_x \tilde{W}_{x,y} - 1 \right) \quad (42)$$

The optimization with respect to the density ρ_y

$$0 = \frac{\partial \mathcal{L}(\tilde{W}_{\dots}; \rho_{\cdot})}{\partial \rho_y} = - \sum_x \tilde{W}_{x,y} \ln \left(\frac{\tilde{W}_{x,y}}{\tilde{W}_{x,y}^{[k]} e^{-\omega^{[k]} + \nu_x^{[k]} - \nu_y^{[k]}}} \right) \quad (43)$$

yields directly that the optimal value of the Lagrangian will simply be given by the Lagrange multiplier $\omega^{[k]}$ that should thus coincide with the scaled cumulant generating function $\mu(k)$ of Eq. 27

$$\mathcal{L}^{opt} = \omega^{[k]} = \mu(k) \quad (44)$$

In addition, the solution of Eq. 43 reads using Eq. 44

$$\tilde{W}_{x,y} = \tilde{W}_{x,y}^{[k]} e^{-\mu(k) + \nu_x^{[k]} - \nu_y^{[k]}} \quad (45)$$

The two last constraints of Eq. 38

$$\begin{aligned} \rho_x &= \sum_y \tilde{W}_{x,y} \rho_y = e^{-\mu(k) + \nu_x^{[k]}} \sum_y \tilde{W}_{x,y}^{[k]} e^{-\nu_y^{[k]}} \rho_y \\ 1 &= \sum_x \tilde{W}_{x,y} = e^{-\mu(k) - \nu_y^{[k]}} \sum_x e^{\nu_x^{[k]}} \tilde{W}_{x,y}^{[k]} \end{aligned} \quad (46)$$

can be rewritten as eigenvalues equations for the tilted operator $\tilde{W}_{x,y}^{[k]}$

$$\begin{aligned} e^{\mu(k)} \left(e^{-\nu_x^{[k]}} \rho_x \right) &= \sum_y \tilde{W}_{x,y}^{[k]} \left(e^{-\nu_y^{[k]}} \rho_y \right) \\ e^{\mu(k)} e^{\nu_y^{[k]}} &= \sum_x e^{\nu_x^{[k]}} \tilde{W}_{x,y}^{[k]} \end{aligned} \quad (47)$$

where $e^{\mu(k)}$ is the eigenvalue associated to the left eigenvector

$$\tilde{l}_y^{[k]} = e^{\nu_y^{[k]}} \quad (48)$$

and to the right eigenvector

$$\tilde{r}_x^{[k]} = e^{-\nu_x^{[k]}} \rho_x = \frac{\rho_x}{\tilde{l}_x^{[k]}} \quad (49)$$

while the remaining constraint of Eq 38 concerning the normalization of the density ρ_x reads

$$1 = \sum_x \rho_x = \sum_x \tilde{l}_x^{[k]} \tilde{r}_x^{[k]} \quad (50)$$

So the equivalence with Eqs 32 33 34 is complete, and the matrix of Eq. 45

$$\tilde{W}_{x,y} = e^{-\mu(k)} \tilde{l}_x^{[k]} \tilde{W}_{x,y}^{[k]} \frac{1}{\tilde{l}_y^{[k]}} \quad (51)$$

corresponds to the generator of the conditioned process obtained via the generalization of Doob's h-transform.

3. Analysis via the empirical density of excursions between two consecutive resets

Any time-additive observable of the position (Eq. 25) can be rewritten in terms of the empirical density of excursions of Eq. 13 as

$$A_T = \frac{1}{T} \sum_{l=0}^{N-1} \sum_{t=t_l+1}^{t=t_{l+1}} \alpha_{x(t)} = \frac{1}{T} \sum_{l=0}^{N-1} \sum_{s=1}^{\tau_l} \alpha_{y^{(l)}(s)} = \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\sum_{s=1}^{\tau-1} \alpha_{y(s)} + \alpha_0 \right] \quad (52)$$

Similarly, any time-additive observable of the increments (Eq. 26) can be rewritten as

$$\begin{aligned} B_T &= \frac{1}{T} \sum_{l=0}^{N-1} \left(\sum_{t=t_l+1}^{t=t_{l+1}-1} \beta_{x(t), x(t-1)} + \beta_{x(t_{l+1}), x(t_{l+1}-1)} \right) = \frac{1}{T} \sum_{l=0}^{N-1} \left(\sum_{s=1}^{\tau_l-1} \beta_{y^{(l)}(s), y^{(l)}(s-1)} + \beta_{0, y^{(l)}(\tau_l-1)} \right) \\ &= \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\sum_{s=1}^{\tau-1} \beta_{y(s), y(s-1)} + \beta_{0, y(\tau-1)} \right] \end{aligned} \quad (53)$$

So the generating function of Eq. 27 can be evaluated from the joint probability of Eqs 18 19 20 as

$$\begin{aligned} Z_T(k) &= \int dn \int \mathcal{D}n[:, \dots] P_T(n[:, \dots], n) e^{Tk \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\sum_{s=1}^{\tau-1} (\alpha_{y(s)} + \beta_{y(s), y(s-1)}) + \alpha_0 + \beta_{0, y(\tau-1)} \right]} \\ &\stackrel{T \rightarrow +\infty}{\simeq} \int dn \int \mathcal{D}n[:, \dots] \delta \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] - n \right] \delta \left[\sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} n[\tau; y(\cdot)] - 1 \right] \\ &e^T \left[- \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{P^{exc}[\tau; y(\cdot)] n} \right) + k \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \left(\sum_{s=1}^{\tau-1} (\alpha_{y(s)} + \beta_{y(s), y(s-1)}) + \alpha_0 + \beta_{0, y(\tau-1)} \right) \right] \end{aligned} \quad (54)$$

One can now proceed exactly as in the previous subsection. In terms of the tilted non-conserved quantity

$$\tilde{P}_k^{exc}[\tau; y(\cdot)] \equiv P^{exc}[\tau; y(\cdot)] e^{k \left[\sum_{s=1}^{\tau-1} (\alpha_{y(s)} + \beta_{y(s), y(s-1)}) + \alpha_0 + \beta_{0, y(\tau-1)} \right]} \quad (55)$$

the functional in the exponential of Eq. 54 can be written in the more compact form

$$\begin{aligned} Z_T(k) &\stackrel{T \rightarrow +\infty}{\simeq} \int dn \int \mathcal{D}n[:, \dots] \delta \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] - n \right] \delta \left[\sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} n[\tau; y(\cdot)] - 1 \right] \\ &e^{-T \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{\tilde{P}_k^{exc}[\tau; y(\cdot)] n} \right)} \end{aligned} \quad (56)$$

One can then make the change of variables from the excursion density $n[\tau; y(\cdot)]$ to the excursion probability that would make $n[\tau; y(\cdot)]$ and n typical

$$\tilde{P}^{exc}[\tau; y(\cdot)] \equiv \frac{n[\tau; y(\cdot)]}{n} \quad (57)$$

to obtain the new expression

$$\begin{aligned} Z_T(k) &\stackrel{T \rightarrow +\infty}{\simeq} \int dn \int \mathcal{D}\tilde{P}^{exc}[:, \dots] \delta \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] \delta \left[n \sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] \\ &e^{-Tn \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \ln \left(\frac{\tilde{P}^{exc}[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)]} \right)} \end{aligned} \quad (58)$$

The physical meaning of the two constraints is that $\tilde{P}^{exc}[\tau; y(\cdot)]$ should be normalized, while n should correspond to the inverse of the first moment of the duration τ of excursions.

In order to optimize the functional in the exponential in the presence of the constraints, it is convenient to introduce the following Lagrangian with the Lagrange multipliers $(\theta^{[k]}, \zeta^{[k]})$

$$\begin{aligned} \mathcal{L}(\tilde{P}^{exc}[:, \cdot], n) &= -n \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \ln \left(\frac{\tilde{P}^{exc}[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)]} \right) \\ &\quad - \theta^{[k]} \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] - \zeta^{[k]} \left[n \sum_{\tau=1}^{+\infty} \tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] \\ &= \zeta^{[k]} - n \left[\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \ln \left(\frac{\tilde{P}^{exc}[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)] e^{-\tau \zeta^{[k]}}} \right) \right] - \theta^{[k]} \left(\sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right) \end{aligned} \quad (59)$$

The optimization with respect to the total density n of excursions

$$0 = -\frac{\partial \mathcal{L}(\tilde{P}^{exc}[:, \cdot], n)}{\partial n} = \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \ln \left(\frac{\tilde{P}^{exc}[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)] e^{-\tau \zeta^{[k]}}} \right) \quad (60)$$

yields directly that the optimal value of the Lagrangian will simply be given by the Lagrange multiplier $\zeta^{[k]}$ that should thus coincide with the scaled cumulant generating function $\mu(k)$ of Eq. 27

$$\mathcal{L}^{opt} = \zeta^{[k]} = \mu(k) \quad (61)$$

In addition, the solution of Eq. 60 reads using Eq. 61

$$\tilde{P}^{exc}[\tau; y(\cdot)] = \tilde{P}^{exc}[\tau; y(\cdot)] e^{-\tau \mu(k)} \quad (62)$$

The two constraints of Eq. 58 read in terms of the tilted non-conserved quantity of Eq. 55

$$\begin{aligned} 1 &= \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] = \sum_{\tau=1}^{+\infty} e^{-\tau \mu(k)} \left(\sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \right) \\ \frac{1}{n} &= \sum_{\tau=1}^{+\infty} \sum_{y(\cdot)} \tau \tilde{P}^{exc}[\tau; y(\cdot)] = \sum_{\tau=1}^{+\infty} \tau e^{-\tau \mu(k)} \left(\sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \right) \end{aligned} \quad (63)$$

So the first condition determines the scaled cumulant generation function $\mu(k)$, while the second condition determines the optimal density n of excursions.

By consistency with the previous subsection, $\tilde{P}^{exc}[\tau; y(\cdot)]$ represents the probability of excursions in the conditioned process generated by the matrix \tilde{W} of Eq. 51. It is now interesting to analyze a specific example in the next section.

III. APPLICATION TO THE SISYPHUS RANDOM WALK ON THE HALF LINE

The Sisyphus Random Walk [77] on the semi-infinite lattice $x = 0, 1, 2, \dots$ corresponds to the matrices (Eq. 2)

$$\begin{aligned} W_{x,y} &= \delta_{x,y+1} \\ W_{x,y}^{reset} &= W_{x,y}(1 - R_y) + \delta_{x,0} R_y \end{aligned} \quad (64)$$

so the dynamics of Eq. 1 reads

$$P_x(t+1) = (1 - R_{x-1})P_{x-1}(t) + \delta_{x,0} \left(\sum_{y=0}^{+\infty} R_y P_y(t) \right) \quad (65)$$

In order to apply the general formalism described in the previous section, one should first check that the hypothesis concerning the existence of a steady-state (Eq. 4) is satisfied.

A. Condition on the reset probabilities R_x for the existence of a non-equilibrium steady-state

The stationary state P_x^* of the dynamics of Eq. 65

$$P_x^* = (1 - R_{x-1})P_{x-1}^* + \delta_{x,0} \sum_{y=0}^{+\infty} R_y P_y^* \quad (66)$$

follows the simple recurrence for $x \geq 1$

$$P_x^* = (1 - R_{x-1})P_{x-1}^* = (1 - R_{x-1})(1 - R_{x-2})P_{x-2}^* = \dots = \left[\prod_{y=0}^{x-1} (1 - R_y) \right] P_0^* \quad (67)$$

while for $x = 0$ Eq. 2 becomes

$$P_0^* = \sum_{x=0}^{+\infty} R_x P_x^* = R_0 P_0^* + \sum_{x=1}^{+\infty} R_x \left[\prod_{y=0}^{x-1} (1 - R_y) \right] P_0^* \quad (68)$$

so P_0^* disappears if it is not vanishing, and the remaining equation is just the normalization concerning the probability of the first reset

$$1 = R_0 + \sum_{x=1}^{+\infty} R_x \left[\prod_{y=0}^{x-1} (1 - R_y) \right] = R_0 + R_1(1 - R_0) + R_2(1 - R_1)(1 - R_0) + \dots \quad (69)$$

The steady-state P_0^* at the origin $x = 0$ is thus determined by the normalization

$$1 = \sum_{x=0}^{+\infty} P_x^* = P_0^* \left(1 + \sum_{x=1}^{+\infty} \left[\prod_{y=0}^{x-1} (1 - R_y) \right] \right) \quad (70)$$

The condition $P_0^* > 0$ to produce a non-equilibrium steady-state localized around the origin corresponds to the requirement of convergence for the series involving the reset probabilities R_y

$$\sum_{x=1}^{+\infty} \left[\prod_{y=0}^{x-1} (1 - R_y) \right] < +\infty \quad (71)$$

In the uniform case $R_y = R$, this series converges for any value $0 < R \leq 1$, i.e. the only case of divergence corresponds to $R = 0$, where the model without any reset is of course transient (Eq. 64). In many applications, R_y depends on the position y either deterministically or randomly [6, 7, 9, 77–80] and it is thus interesting to discuss the criterion of Eq. 71 for these cases :

(i) Random cases : if the reset probabilities $R_y \in [0, 1]$ are independent random variables, the specific structure of Eq. 71 corresponds to the well-known class of Kesten random variables that appear in many disordered systems [85–93]. The condition of convergence for Eq. 71 is $\ln(1 - R_y) < 0$ and will be thus satisfied, since the random variable R_y represents a positive reset probability $0 \leq R_y \leq 1$.

(ii) Deterministic cases : the series of Eq. 71 will generically converge, except when R_y decays towards zero too rapidly for $y \rightarrow +\infty$, for instance for the choice $R_y = \frac{1}{y+1}$ that leads to the logarithmic divergence of Eq. 71.

In the following, we assume the convergence of Eq. 71 in order to have a non-equilibrium steady-state that can be analyzed via the large deviations at Level 2.5.

B. Large deviations at level 2.5 for the joint probability of the empirical density and the empirical currents

The empirical 2-point density of Eq. 5 contains two types of contributions, namely the non-local reset currents from any point $x \geq 0$ towards the origin

$$J_x \equiv \rho_{0,x}^{(2)} \quad (72)$$

and the local currents arriving at any point $x \geq 1$ from its left neighbor ($x - 1$)

$$j_x \equiv \rho_{x,x-1}^{(2)} \quad (73)$$

As a consequence, the consistency constraints of Eq. 6 involving the 1-point density become for the origin

$$\rho_0 = J_0 + \sum_{x=1}^{+\infty} J_x = J_0 + j_1 \quad (74)$$

and for the other points $x \geq 1$

$$\rho_x = j_x = J_x + j_{x+1} \quad (75)$$

The large deviations at level 2.5 of Eqs 8 9 10 yield that the joint probability to see the empirical density ρ , and the empirical currents j , and J , read

$$\begin{aligned} P_T(\rho, j, J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) \delta \left(\sum_{x=0}^{+\infty} J_x - \rho_0 \right) \delta (J_0 + j_1 - \rho_0) \prod_{x=1}^{+\infty} [\delta (j_x - \rho_x) \delta (J_x + j_{x+1} - \rho_x)] \\ &e^{-T \left[\sum_{x=1}^{+\infty} j_x \ln \left(\frac{j_x}{(1 - R_{x-1}) \rho_{x-1}} \right) + \sum_{x=0}^{+\infty} J_x \ln \left(\frac{J_x}{R_x \rho_x} \right) \right]} \end{aligned} \quad (76)$$

The fourth constraint on the first line shows that all the local currents j_x for $x \geq 1$ can be eliminated in terms of the density

$$j_x = \rho_x \quad (77)$$

so that Eq. 76 becomes for the joint distribution of the density ρ , and the non-local reset currents J ,

$$\begin{aligned} P_T(\rho, J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) \delta \left(\sum_{x=0}^{+\infty} J_x - \rho_0 \right) \prod_{x=0}^{+\infty} \delta (J_x + \rho_{x+1} - \rho_x) \\ &e^{-T \left[\sum_{x=1}^{+\infty} \rho_x \ln \left(\frac{\rho_x}{(1 - R_{x-1}) \rho_{x-1}} \right) + \sum_{x=0}^{+\infty} J_x \ln \left(\frac{J_x}{R_x \rho_x} \right) \right]} \end{aligned} \quad (78)$$

One can now further use the constraints to obtain the large deviations for the empirical density alone or for the non-local reset currents alone, as described in the following two subsections.

C. Large deviations at Level 2 for the empirical density ρ , alone

The last constraint on the first line of Eq. 78 can be used to eliminate the non-local reset currents J_x in terms of the empirical density

$$J_x = \rho_x - \rho_{x+1} \quad (79)$$

Then the second constraint on the first line of Eq. 78 is automatically satisfied

$$\sum_{x=0}^{+\infty} J_x = \sum_{x=0}^{+\infty} (\rho_x - \rho_{x+1}) = \rho_0 \quad (80)$$

so that Eq. 78 yields for the large deviations for the empirical density ρ , alone

$$P_T(\rho) \underset{T \rightarrow +\infty}{\simeq} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) e^{-T \left[\sum_{x=1}^{+\infty} \rho_x \ln \left(\frac{\rho_x}{(1 - R_{x-1}) \rho_{x-1}} \right) + \sum_{x=0}^{+\infty} (\rho_x - \rho_{x+1}) \ln \left(\frac{(\rho_x - \rho_{x+1})}{R_x \rho_x} \right) \right]} \quad (81)$$

One may further simplify the rate function in the exponential to obtain

$$P_T(\rho.) \underset{T \rightarrow +\infty}{\simeq} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) e^{-T \left[-\rho_0 \ln(\rho_0) - \sum_{x=1}^{+\infty} \rho_x \ln(1 - R_{x-1}) + \sum_{x=0}^{+\infty} (\rho_x - \rho_{x+1}) \ln \left(\frac{\rho_x - \rho_{x+1}}{R_x} \right) \right]} \quad (82)$$

The fact that the Large deviations at Level 2 for the empirical density ρ , alone can be written in closed form in the present model is not generic : indeed in most non-equilibrium models, it is not possible to use the constraints of the large deviations at Level 2.5 to eliminate completely the flows in terms of the density alone.

D. Large deviations for the empirical reset currents J , alone

If one wishes instead to eliminate the empirical density in terms of the empirical reset currents J , via

$$\rho_x = \sum_{y=x}^{+\infty} J_y \quad (83)$$

Eq. 82 yields for the large deviations for the empirical reset currents J , alone

$$\begin{aligned} P_T(J.) \underset{T \rightarrow +\infty}{\simeq} & \delta \left(\sum_{x=0}^{+\infty} \sum_{y=x}^{+\infty} J_y - 1 \right) \delta \left(\sum_{x=0}^{+\infty} J_x - \rho_0 \right) e^{-T \left[-\rho_0 \ln(\rho_0) - \sum_{x=1}^{+\infty} \left(\sum_{y=x}^{+\infty} J_y \right) \ln(1 - R_{x-1}) + \sum_{x=0}^{+\infty} J_x \ln \left(\frac{J_x}{R_x} \right) \right]} \\ & \underset{T \rightarrow +\infty}{\simeq} \delta \left(\sum_{y=0}^{+\infty} (y+1) J_y - 1 \right) \delta \left(\sum_{x=0}^{+\infty} J_x - \rho_0 \right) e^{-T \left[J_0 \ln \left(\frac{J_0}{R_0 \rho_0} \right) + \sum_{x=1}^{+\infty} J_x \ln \left(\frac{J_x}{R_x \left(\prod_{y=0}^{x-1} (1 - R_y) \right) \rho_0} \right) \right]} \end{aligned} \quad (84)$$

This expression allows to make the link with the large deviations for excursions between two consecutive resets as explained in next subsection.

E. Large deviations for the empirical density of excursions between two consecutive resets

Since the internal trajectory $y(s) = s$ of an excursion is deterministic, the probability of excursion of Eq. 16 only involves its duration τ

$$P^{exc}[\tau] = R_{\tau-1}(1 - R_{\tau-2}) \dots (1 - R_1)(1 - R_0) = R_{\tau-1} \left[\prod_{s=0}^{\tau-2} (1 - R_s) \right] \quad (85)$$

As a consequence, the probability to see the empirical density $n[\tau]$ of excursions between resets and the total density n follows the large deviation form of Eqs 18 20 19

$$P_T(n[.], n) \underset{T \rightarrow +\infty}{\simeq} \delta \left[\sum_{\tau=1}^{+\infty} \tau n[\tau] - 1 \right] \delta \left[\sum_{\tau=1}^{+\infty} n[\tau] - n \right] e^{-T \sum_{\tau=1}^{+\infty} n[\tau] \ln \left(\frac{n[\tau]}{P^{exc}[\tau] n} \right)} \quad (86)$$

and coincides with Eq 84 with the dictionary

$$\begin{aligned} \rho_0 &= n \\ J_x &= n[\tau = x + 1] \end{aligned} \quad (87)$$

F. Large deviations of general time-additive observables

In this section, we analyze the large deviations of time-additive observables of the form $(A_T + B_T)$ (Eqs 25 26 35 36) via their generating function (Eq 27)

$$Z_T(k) \equiv \langle e^{Tk(A_T + B_T)} \rangle = \langle e^{k \sum_{t=1}^T [\alpha_{x(t)} + \beta_{x(t),x(t-1)}]} \rangle = \langle e^{Tk \left(\sum_x \alpha_x \rho_x + \sum_x \sum_y \beta_{x,y} \rho_{x,y}^{(2)} \right)} \rangle \quad (88)$$

In the present model where the empirical 2-point density $\rho_{x,y}^{(2)}$ involves the two contributions of Eqs 72 and 73, the generating function of Eq. 88 reads more explicitly in terms of the local currents $j_x = \rho_{x,x-1}^{(2)}$ and of the non-local reset currents $J_x = \rho_{0,x}^{(2)}$

$$Z_T(k) = \langle e^{Tk \left(\sum_{x=0}^{+\infty} \alpha_x \rho_x + \sum_{x=1}^{+\infty} \beta_{x,x-1} j_x + \sum_{x=0}^{+\infty} \beta_{0,x} J_x \right)} \rangle \quad (89)$$

As explained in detail above, the constraints existing in the present model between the density ρ_x , the local currents j_x and the non-local reset currents J_x could be used to eliminate some variables in terms of others via various choices. This means that there will be some redundancy between the three functions $(\alpha_x, \beta_{x,x-1}, \beta_{0,x})$. However, in order to understand more clearly how the general formalism of the previous section works, it is more pedagogical to keep these redundant notations and to see what combinations of the three functions $(\alpha_x, \beta_{x,x-1}, \beta_{0,x})$ naturally emerge in the solutions.

1. Analysis via the tilted matrix

As explained in detail in section IID, the two first points of views both lead to the analysis of the tilted matrix of Eq. 30 for the model of Eq. 64

$$\tilde{W}_{x,y}^{[k]} = [\delta_{x,y+1}(1 - R_y) + \delta_{x,0}R_y] e^{k(\alpha_x + \beta_{x,y})} = \delta_{x,y+1}(1 - R_y) e^{k(\alpha_x + \beta_{x,y})} + \delta_{x,0}R_y e^{k(\alpha_0 + \beta_{0,y})} \quad (90)$$

The eigenvalues Eqs 32 and 33 for the right eigenvector $\tilde{r}_x^{[k]}$ and the left eigenvector $\tilde{l}_y^{[k]}$ read

$$e^{\mu(k)} \tilde{r}_x^{[k]} = \sum_y \tilde{W}_{x,y}^{[k]} \tilde{r}_y^{[k]} = (1 - R_{x-1}) e^{k(\alpha_x + \beta_{x,x-1})} \tilde{r}_{x-1}^{[k]} + \delta_{x,0} \sum_y R_y e^{k(\alpha_0 + \beta_{0,y})} \tilde{r}_y^{[k]} \quad (91)$$

and

$$e^{\mu(k)} \tilde{l}_y^{[k]} = \sum_x \tilde{l}_x^{[k]} \tilde{W}_{x,y}^{[k]} = \tilde{l}_{y+1}^{[k]} (1 - R_y) e^{k(\alpha_{y+1} + \beta_{y+1,y})} + \tilde{l}_0^{[k]} R_y e^{k(\alpha_0 + \beta_{0,y})} \quad (92)$$

For $x > 0$, Eq. 91 is a simple recurrence with the solution

$$\begin{aligned} \tilde{r}_x^{[k]} &= (1 - R_{x-1}) e^{-\mu(k) + k(\alpha_x + \beta_{x,x-1})} \tilde{r}_{x-1}^{[k]} = \left[\prod_{z=0}^{x-1} (1 - R_z) e^{-\mu(k) + k(\alpha_{z+1} + \beta_{z+1,z})} \right] \tilde{r}_0^{[k]} \\ &= e^{-x\mu(k) + k \sum_{z=1}^x (\alpha_z + \beta_{z,z-1})} \left[\prod_{z=0}^{x-1} (1 - R_z) \right] \tilde{r}_0^{[k]} \end{aligned} \quad (93)$$

while Eq. 32 for the origin yields

$$\begin{aligned} e^{\mu(k)} \tilde{r}_0^{[k]} &= \sum_{x=0}^{+\infty} R_x e^{k(\alpha_0 + \beta_{0,x})} \tilde{r}_x^{[k]} \\ &= R_0 e^{k(\alpha_0 + \beta_{0,0})} \tilde{r}_0^{[k]} + \sum_{x=1}^{+\infty} e^{-x\mu(k) + k(\alpha_0 + \beta_{0,x}) + k \sum_{z=1}^x (\alpha_z + \beta_{z,z-1})} \left[R_x \prod_{z=0}^{x-1} (1 - R_z) \right] \tilde{r}_0^{[k]} \end{aligned} \quad (94)$$

So the amplitude $\tilde{r}_0^{[k]}$ of the right eigenvector at the origin disappears and the remaining equation determines the eigenvalue $e^{\mu(k)}$ in terms of the resetting probabilities R_y and in terms of the functions α_\cdot and β_\cdot , that parametrize the time-additive observable under study

$$1 = R_0 e^{-\mu(k)+k(\alpha_0+\beta_{0,0})} + \sum_{x=1}^{+\infty} e^{-(x+1)\mu(k)+k(\alpha_0+\beta_{0,x})+k\sum_{z=1}^x(\alpha_z+\beta_{z,z-1})} R_x \prod_{z=0}^{x-1} (1-R_z) \quad (95)$$

The solution of Eq. 92 for the left eigenvector reads

$$\tilde{l}_y^{[k]} = \tilde{l}_0^{[k]} \left[R_y e^{-\mu(k)+k(\alpha_0+\beta_{0,y})} + \sum_{x=y+1}^{+\infty} e^{-(x+1-y)\mu(k)+k(\alpha_0+\beta_{0,x})+k\sum_{z=y+1}^x(\alpha_z+\beta_{z,z-1})} R_x \prod_{z=y}^{x-1} (1-R_z) \right] \quad (96)$$

where the consistency for $y = 0$ reproduces the equation 95 for the eigenvalue $\mu(k)$. The matrix generating the conditioned process obtained via the generalization of Doob's h-transform (Eq. 51) involves the same non-vanishing matrix elements as the initial matrix $W_{x,y}^{reset}$ of Eq. 64

$$\tilde{W}_{x,y} = e^{-\mu(k)} \tilde{l}_x^{[k]} \tilde{W}_{x,y}^{[k]} \frac{1}{\tilde{l}_y^{[k]}} = \delta_{x,y+1} \tilde{W}_{x,x-1} + \delta_{x,0} \tilde{W}_{0,y} \quad (97)$$

where the probability to jump from $(x-1)$ to x has changed from its initial value $(1-R_y)$ of Eq. 64 to its new value

$$\tilde{W}_{x,x-1} = (1-R_{x-1}) e^{-\mu(k)+k(\alpha_x+\beta_{x,x-1})} \frac{\tilde{l}_x^{[k]}}{\tilde{l}_{x-1}^{[k]}} \quad (98)$$

while the probability to have a reset from y to the origin has changed from its initial value R_y of Eq. 64 to its new value

$$\tilde{W}_{0,y} = R_y e^{-\mu(k)+k(\alpha_0+\beta_{0,y})} \frac{\tilde{l}_0^{[k]}}{\tilde{l}_y^{[k]}} \quad (99)$$

Both involve explicitly the eigenvalue $e^{\mu(k)}$ and the left eigenvector $\tilde{l}^{[k]}$.

The stationary density (Eq. 49) of this conditioned process reads

$$\tilde{\rho}_x = \tilde{l}_x^{[k]} \tilde{r}_x^{[k]} = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]} \left[\sum_{y=x}^{+\infty} e^{-(y+1)\mu(k)+k(\alpha_0+\beta_{0,y})+k\sum_{z=1}^y(\alpha_z+\beta_{z,z-1})} R_y \prod_{z=0}^{y-1} (1-R_z) \right] \quad (100)$$

where the normalization determines the value of the density at the origin $\tilde{\rho}_0 = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]}$

$$\begin{aligned} 1 = \sum_{x=0}^{+\infty} \tilde{\rho}_x &= \tilde{l}_0^{[k]} \tilde{r}_0^{[k]} \left[\sum_{x=0}^{+\infty} \sum_{y=x}^{+\infty} e^{-(y+1)\mu(k)+k(\alpha_0+\beta_{0,y})+k\sum_{z=1}^y(\alpha_z+\beta_{z,z-1})} R_y \prod_{z=0}^{y-1} (1-R_z) \right] \\ &= \tilde{l}_0^{[k]} \tilde{r}_0^{[k]} \left[\sum_{y=0}^{+\infty} (y+1) e^{-(y+1)\mu(k)+k(\alpha_0+\beta_{0,y})+k\sum_{z=1}^y(\alpha_z+\beta_{z,z-1})} R_y \prod_{z=0}^{y-1} (1-R_z) \right] \end{aligned} \quad (101)$$

The stationary local currents and reset currents of this conditioned process are then given by

$$\begin{aligned} \tilde{j}_x &= \tilde{W}_{x,x-1} \tilde{\rho}_{x-1} = (1-R_{x-1}) e^{-\mu(k)+k(\alpha_x+\beta_{x,x-1})} \tilde{l}_x^{[k]} \tilde{r}_{x-1}^{[k]} \\ \tilde{J}_y &= \tilde{W}_{0,y} \tilde{\rho}_y = R_y e^{-\mu(k)+k(\alpha_0+\beta_{0,y})} \tilde{l}_0^{[k]} \tilde{r}_y^{[k]} \end{aligned} \quad (102)$$

2. Analysis via the tilted excursions

Here one needs to consider the tilted quantity of Eq. 55 for the present model where the internal trajectory $y(s) = s$ of an excursion is deterministic (Eq. 85)

$$\tilde{P}_k^{exc}[\tau] = P^{exc}[\tau] e^{k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} = \left[R_{\tau-1} \prod_{s=0}^{\tau-2} (1 - R_s) \right] e^{k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} \quad (103)$$

The conditioned excursion probability of Eq. 62 reads

$$\begin{aligned} \tilde{P}^{exc}[\tau] &= \tilde{P}^{exc}[\tau] e^{-\tau\mu(k)} = P^{exc}[\tau] e^{-\tau\mu(k) + k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} \\ &= \left[R_{\tau-1} \prod_{s=0}^{\tau-2} (1 - R_s) \right] e^{-\tau\mu(k) + k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} \end{aligned} \quad (104)$$

Its normalization (Eq. 63) determines the scaled cumulant generating function $\mu(k)$

$$\begin{aligned} 1 &= \sum_{\tau=1}^{+\infty} \tilde{P}^{exc}[\tau] \\ &= R_0 e^{-\mu(k)k[\alpha_0 + \beta_{0,\tau-1}]} + \sum_{\tau=2}^{+\infty} e^{-\tau\mu(k) + k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} \left[R_{\tau-1} \prod_{s=0}^{\tau-2} (1 - R_s) \right] \end{aligned} \quad (105)$$

which is equivalent to Eq. 95 via the correspondence $x = \tau - 1$, while its first moment (Eq. 63) determines the inverse of the density n of excursions of the conditioned process

$$\begin{aligned} \frac{1}{n} &= \sum_{\tau=1}^{+\infty} \tau \tilde{P}^{exc}[\tau] \\ &= R_0 e^{-\mu(k)k[\alpha_0 + \beta_{0,\tau-1}]} + \sum_{\tau=2}^{+\infty} \tau e^{-\tau\mu(k) + k \left[\sum_{s=1}^{\tau-1} (\alpha_s + \beta_{s,s-1}) + \alpha_0 + \beta_{0,\tau-1} \right]} \left[R_{\tau-1} \prod_{s=0}^{\tau-2} (1 - R_s) \right] \end{aligned} \quad (106)$$

This equation is equivalent to Eq. 101 for the density at the origin $\tilde{\rho}_0 = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]}$ in agreement with the dictionary $\tilde{\rho}_0 = n$ and $\tau = y + 1$ of Eq. 87.

IV. MARKOV JUMP PROCESS IN CONTINUOUS TIME AND DISCRETE SPACE WITH RESETS

A. Models and notations

In this section, we consider the continuous-time dynamics in discrete space defined by the Master Equation

$$\frac{\partial P_x(t)}{\partial t} = \sum_y w_{x,y}^{reset} P_y(t) \quad (107)$$

where the off-diagonal matrix elements $x \neq y$ represent the transitions rates from y to x

$$w_{x,y}^{reset} = w_{x,y} + \delta_{x,0} r_y \quad \text{for } x \neq y \quad (108)$$

The physical meaning is that the resetting procedure takes place with the resetting rate r_y from y towards the origin 0, while the Markov Jump Process that would exist without resetting is described by the off-diagonal matrix elements $w_{x,y}$ and by the corresponding diagonal elements that are fixed by the conservation of probability to be

$$w_{y,y} = - \sum_{x \neq y} w_{x,y} \equiv -w_y^{out} \quad (109)$$

where the notation w_y^{out} represents the total escape rate out of the position y . The diagonal matrix elements of the matrix w^{reset} of Eq. 108 are also fixed by the conservation of probability in terms of the off-diagonal elements

$$w_{y,y}^{reset} = - \sum_{x \neq y} w_{x,y}^{reset} = - \sum_{x \neq y} w_{x,y} - r_y + r_0 \delta_{y,0} = -(w_y^{out} + r_y) + r_0 \delta_{y,0} \quad (110)$$

Note that the reset rate r_0 from the origin to itself disappears from the diagonal matrix element

$$w_{0,0}^{reset} = -(w_0^{out} + r_0) + r_0 = -w_0^{out} \quad (111)$$

since it does not produce a change of position. However the rate r_0 is important for the book-keeping of the number of resets. For instance, if one chooses $r_0 = 0$ in order to suppress the possibility of resets from the origin to itself, the whole statistics of resets will be coupled to the time spent at the origin. As a consequence, it is useful to keep the possibility of $r_0 > 0$ to remain more general, as we have also considered the possibility of positive reset probability $R_0 > 0$ from the origin to itself in the Markov Chain framework described in the previous sections. As a consequence, we will consider that the diagonal term of Eq. 111 contains the two contributions 'out' and 'in'

$$\begin{aligned} w_{0,0}^{reset} &= -w_0^{reset(out)} + w_0^{reset(in)} \\ w_0^{reset(out)} &\equiv w_0^{out} + r_0 \\ w_{0,0}^{reset(in)} &\equiv r_0 \end{aligned} \quad (112)$$

In summary, the Master Equation 107 can be rewritten more explicitly as

$$\frac{\partial P_x(t)}{\partial t} = -[w_x^{out} + r_x] P_x(t) + \sum_{y \neq x} [w_{x,y} + \delta_{x,0} r_y] P_y(t) + \delta_{x,0} r_0 P_0(t) \quad (113)$$

Again as in Eq. 4, we will assume that the steady-state solution P_x^* of Eq. 107

$$0 = \sum_y w_{x,y}^{reset} P_y^* \quad (114)$$

exists in order to apply the large deviations at Level 2.5 for non-equilibrium steady-states.

B. Large deviations at level 2.5 for the empirical density and the empirical flows

For this continuous-time jump process, the empirical density reads

$$\rho_x \equiv \frac{1}{T} \int_0^T dt \delta_{x(t),x} \quad (115)$$

and satisfies the normalization

$$\sum_x \rho_x = 1 \quad (116)$$

while the jump density from y to $x \neq y$

$$q_{x,y} \equiv \frac{1}{T} \sum_{t:x(t^-) \neq x(t^+)} \delta_{x(t^+),x} \delta_{x(t^-),y} \quad (117)$$

satisfies the following stationarity constraint (for any x , the total incoming flow should be equal to the total outgoing flow)

$$\sum_{y \neq x} q_{x,y} = \sum_{y \neq x} q_{y,x} \quad (118)$$

As explained around Eq. 111, we will also need to introduce the density of resets from the origin to itself

$$q_{0,0} = \frac{\text{Number of resets from 0 to 0 during } [0, T]}{T} \quad (119)$$

that will also lead to Eq 118 for $x = 0$ since $q_{0,0}$ will appear on both sides and thus disappear.

The joint probability distribution of the empirical density ρ and flows $q_{..}$ satisfy the following large deviation form at level 2.5 [23, 26, 27, 33–43]

$$P_T[\rho.; q_{..}] \underset{T \rightarrow +\infty}{\propto} C[\rho.; q_{..}] e^{-TI[\rho.; q_{..}]} \quad (120)$$

with the constraints discussed in Eqs 116 and 118

$$C[\rho.; q_{..}] = \delta \left(\sum_x \rho_x - 1 \right) \prod_x \left[\sum_{y \neq x} q_{x,y} - \sum_{y \neq x} q_{y,x} \right] \quad (121)$$

while the rate function involves the off-diagonal rates $w_{x,y}^{reset} = w_{x,y} + \delta_{x,0} r_y$ of Eq. 108 as well as the special contribution from the resets from the origin to itself with the rate r_0 (Eq 119)

$$I[\rho.; q_{..}] = \sum_y \sum_{x \neq y} \left[q_{x,y} \ln \left(\frac{q_{x,y}}{w_{x,y}^{reset} \rho_y} \right) - q_{x,y} + w_{x,y}^{reset} \rho_y \right] + \left[q_{0,0} \ln \left(\frac{q_{0,0}}{r_0 \rho_0} \right) - q_{0,0} + r_0 \rho_0 \right] \quad (122)$$

Again one needs to discuss the possible empirical flows $q_{x,y}$ due to the local rates $w_{x,y}$ and to the resetting procedure in order to obtain more explicit expressions, as explained on the specific example of section V.

C. Large deviations for the empirical density of excursions between two consecutive resets

1. Probabilities of excursions between two consecutive resets

For the dynamics of Eq. 107, the probability to have an excursion of duration τ and of internal trajectory $y(0 \leq s \leq \tau)$ with the initial position $y(0^+) = 0$ fixed by the reset and the final position $y(\tau)$ being the last position before the reset jump to the origin $y(\tau^+) = 0$ reads

$$P^{exc}[\tau; y(\cdot)] = r_{y(\tau)} e^{-\int_0^\tau ds [w_{y(s)}^{out} + r_{y(s)}]} \prod_{s: y(s^-) \neq y(s^+)} w_{y(s^+), y(s^-)} \quad (123)$$

To be more precise, the internal trajectory $y(0 \leq s \leq \tau)$ will contain a certain number M of jumps $m = 1, \dots, M$ occurring at times $s_0 = 0 < s_1 < \dots < s_M < \tau = s_{M+1}$ between the successive positions $(y(0 \leq s < s_1) = 0; y(s_1 \leq s < s_2) = z_1; \dots; y(s_M \leq s < \tau) = z_M)$ with $z_{m+1} \neq z_m$ that are visited between these jumps. The probability of this trajectory reads

$$P^{exc}[\tau; 0; s_1; z_1, s_2; \dots; z_{M-1}; s_M; z_M] = r_{z_M} e^{-\sum_{m=0}^M (s_{m+1} - s_m) [w_{z_m}^{out} + r_{z_m}]} \prod_{m=1}^M w_{z_m, z_{m-1}} \quad (124)$$

while the special case $M = 0$ where the reset occurs while the particle is still at the origin reads (see the discussion around Eq. 111)

$$P^{exc}[\tau; 0] = r_0 e^{-\tau [w_0^{out} + r_0]} \quad (125)$$

The normalization over all possible excursions reads, first in compact form and then in more explicit form

$$\begin{aligned} 1 &= \int_0^{+\infty} d\tau \sum_{y(\cdot)} P^{exc}[\tau; y(\cdot)] \\ &= \int_0^{+\infty} d\tau \left[P^{exc}[\tau; 0] + \sum_{M=1}^{+\infty} \int_0^\tau ds_M \int_0^{s_M} ds_{M-1} \dots \int_0^{s_2} ds_1 \sum_{z_M \neq z_{M-1}} \dots \sum_{z_2 \neq z_1} \sum_{z_1 \neq 0} P^{exc}[\tau; 0; s_1; z_1; \dots; s_M; z_M] \right] \end{aligned} \quad (126)$$

2. Large deviations for the excursions density

The probability to see the empirical density $n[\tau; y(0 \leq s \leq \tau)]$ of excursions between resets and the total density n follows the large deviation form analog to Eq. 18

$$P_T(n[.; \dots], n) \underset{T \rightarrow +\infty}{\simeq} \delta \left[\int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] - n \right] \delta \left[\int_0^{+\infty} d\tau \sum_{y(\cdot)} \tau n[\tau; y(\cdot)] - 1 \right] e^{-T \int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{P^{exc}[\tau; y(\cdot)]n} \right)} \quad (127)$$

where the sums over trajectories can be written in more explicit form as in Eq. 126.

3. Simplifications for uniform resetting rate $r_y = r$

When the resetting rate $r_y = r$ does not depend on the position y , one obtains the factorization analogous to Eq. 21

$$P_{uniform}^{exc}[\tau; (y(\cdot))] = p^{exp}(\tau) p_{\tau}^{config}[(y(\cdot))] \quad (128)$$

The exponential distribution

$$p^{exp}(\tau) = r e^{-r\tau} \quad (129)$$

describes the probability of the duration τ of the excursion independently of the internal trajectory (and replaces the geometric distribution of Eq. 22 of the discrete-time framework), while the probability $p_{\tau}^{config}[y(\cdot)]$ of the internal trajectory once its duration τ is given characterizes the process without resetting. In particular, the probability of the total density n of excursions simply follows the Poisson distribution with its corresponding large deviation form

$$P_T^{Poisson}(n) = \frac{(rT)^{nT}}{(nT)!} e^{-rT} \underset{T \rightarrow +\infty}{\propto} e^{-T \left[n \ln \left(\frac{n}{r} \right) - n + r \right]} \quad (130)$$

On the contrary, whenever the reset rates r_y depends on the position y , the duration τ and the internal trajectory $y(\cdot)$ are coupled via Eq. 123.

D. Large deviations for general time-additive observables

The empirical density of Eq. 115 allows to reconstruct any time-additive observable that involves some function α_x of the position $x(t)$

$$A_T \equiv \frac{1}{T} \int_0^T dt \alpha_{x(t)} = \sum_x \alpha_x \rho_x \quad (131)$$

while the jump density of Eq. 117 allows to reconstruct any time-additive observable that involves some function $\beta_{x,y}$ of the jumps ($x(t^+) \neq x(t^-)$) and of the resets from the origin to itself

$$\begin{aligned} B_T &\equiv \frac{1}{T} \sum_{t: x(t^-) \neq x(t^+)} \beta_{x(t^+), x(t^-)} + \beta_{0,0} \left(\frac{\text{Number of resets from 0 to 0 during } [0, T]}{T} \right) \\ &= \sum_y \sum_{x \neq y} \beta_{x,y} q_{x,y} + \beta_{0,0} q_{0,0} \end{aligned} \quad (132)$$

The large deviations of the sum ($A_T + B_T$) can be analyzed via the scaled cumulant generating function $\mu(k)$ appearing in the asymptotic behavior

$$Z_T(k) \equiv \langle e^{Tk(A_T + B_T)} \rangle \underset{T \rightarrow +\infty}{\simeq} e^{T\mu(k)} \quad (133)$$

1. *Analysis via the tilted dynamics*

The probability of a trajectory for the process of Eq. 113 can be written only in terms of its empirical observables

$$P_T^{Traj} \underset{T \rightarrow +\infty}{\simeq} e^{T \left[- \sum_x \rho_x [w_x^{out} + r_x] + \sum_y \sum_{x \neq y} q_{x,y} \ln [w_{x,y} + \delta_{x,0} r_y] + q_{0,0} \ln(r_0) \right]} \quad (134)$$

As a consequence, the generating function of Eq. 133 can be analyzed by tilting each term of the initial Master Eq. 113 to produce the non-conserved dynamics

$$\begin{aligned} \frac{\partial \tilde{P}_x(t)}{\partial t} &= - [w_x^{out} + r_x - k\alpha_x] \tilde{P}_x(t) + \sum_{y \neq x} [w_{x,y} + \delta_{x,0} r_y] e^{k\beta_{x,y}} \tilde{P}_y(t) + \delta_{x,0} r_0 e^{k\beta_{0,0}} \tilde{P}_0(t) \\ &\equiv \sum_y \tilde{w}_{x,y}^{[k]} \tilde{P}_y(t) \end{aligned} \quad (135)$$

where the tilted matrix involves the off-diagonal elements $x \neq y$

$$\tilde{w}_{x,y}^{[k]} \equiv [w_{x,y} + \delta_{x,0} r_y] e^{k\beta_{x,y}} \quad \text{for } x \neq y \quad (136)$$

and the diagonal elements

$$\tilde{w}_{x,x}^{[k]} \equiv -w_x^{out} - r_x + k\alpha_x + \delta_{x,0} r_0 e^{k\beta_{0,0}} \quad (137)$$

The scaled cumulant generating function $\mu(k)$ of Eq. 27 will then correspond to the highest eigenvalue $\mu(k)$ of the tilted matrix $\tilde{w}_{..}$, that will dominate the propagator

$$\tilde{P}_{x,y}(T) \underset{T \rightarrow +\infty}{\simeq} e^{T\mu(k)} \tilde{r}_x^{[k]} \tilde{l}_y^{[k]} \quad (138)$$

where $\tilde{r}_x^{[k]}$ is the corresponding positive right eigenvector

$$\mu(k) \tilde{r}_x^{[k]} = \sum_y \tilde{w}_{x,y}^{[k]} \tilde{r}_y^{[k]} = [-w_x^{out} - r_x + k\alpha_x] \tilde{r}_x^{[k]} + \sum_{y \neq x} w_{x,y} e^{k\beta_{x,y}} \tilde{r}_y^{[k]} + \delta_{x,0} \sum_y r_y e^{k\beta_{0,y}} \tilde{r}_y^{[k]} \quad (139)$$

and where $\tilde{l}_y^{[k]}$ is the corresponding positive left eigenvector

$$\mu(k) \tilde{l}_y^{[k]} = \sum_x \tilde{l}_x^{[k]} \tilde{w}_{x,y}^{[k]} = \tilde{l}_y^{[k]} [-w_y^{out} - r_y + k\alpha_y] + \sum_{x \neq y} \tilde{l}_x^{[k]} w_{x,y} e^{k\beta_{x,y}} + \tilde{l}_0^{[k]} r_y e^{k\beta_{0,y}} \quad (140)$$

with the normalization

$$\sum_x \tilde{l}_x^{[k]} \tilde{r}_x^{[k]} = 1 \quad (141)$$

The generator of the conditioned process obtained via the generalization of Doob's h-transform reads

$$\tilde{w}_{x,y} = \tilde{l}_x^{[k]} \tilde{w}_{x,y}^{[k]} \frac{1}{\tilde{r}_y^{[k]}} - \mu(k) \delta_{x,y} \quad (142)$$

2. *Analysis via the tilted excursions*

As in the discrete time context (Eq. 52), one can use the empirical density of excursions to write any time-additive observable of the position (Eq. 131)

$$A_T = \int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\int_0^\tau ds \alpha_{y(s)} \right] \quad (143)$$

and any time-additive observable of the jumps (Eq. 132)

$$B_T = \int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\sum_{0 < s < \tau: y(s^-) \neq y(s^+)} \beta_{y(s^+), y(s^-)} + \beta_{0, y(\tau^-)} \right] \quad (144)$$

where the last term contains the contribution of $\beta_{0,0}$ when the excursion has never left the origin $y(\tau^-) = 0$.
So the generating function of Eq. 27 can be evaluated from the joint probability of Eq. 127 as

$$\begin{aligned} Z_T(k) &= \int dn \int \mathcal{D}n[; \dots] P_T(n[; \dots], n) e^{Tk \int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[\int_0^\tau ds \alpha_{y(s)} + \sum_{0 < s < \tau: y(s^-) \neq y(s^+)} \beta_{y(s^+), y(s^-)} + \beta_{0, y(\tau^-)} \right]} \\ &\underset{T \rightarrow +\infty}{\simeq} \int dn \int \mathcal{D}n[; \dots] \delta \left[\int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] - n \right] \delta \left[\int_0^{+\infty} d\tau \sum_{y(\cdot)} \tau n[\tau; y(\cdot)] - 1 \right] \\ &e^{T \int_0^{+\infty} d\tau \sum_{y(\cdot)} n[\tau; y(\cdot)] \left[-\ln \left(\frac{n[\tau; y(\cdot)]}{P^{exc}[\tau; y(\cdot)]n} \right) + k \int_0^\tau ds \alpha_{y(s)} + k \sum_{0 < s < \tau: y(s^-) \neq y(s^+)} \beta_{y(s^+), y(s^-)} + k \beta_{0, y(\tau^-)} \right]} \end{aligned} \quad (145)$$

It is thus convenient to introduce tilted non-conserved quantity

$$\tilde{P}_k^{exc}[\tau; y(\cdot)] \equiv P^{exc}[\tau; y(\cdot)] e^{k \left[\int_0^\tau ds \alpha_{y(s)} + \sum_{0 < s < \tau: y(s^-) \neq y(s^+)} \beta_{y(s^+), y(s^-)} + \beta_{0, y(\tau^-)} \right]} \quad (146)$$

and to make the change of variable from $n[\tau; y(\cdot)]$ to the probability distribution that would make $n[\tau; y(\cdot)]$ and n typical

$$\tilde{P}^{exc}[\tau; y(\cdot)] \equiv \frac{n[\tau; y(\cdot)]}{n} \quad (147)$$

to rewrite Eq. 145 as

$$\begin{aligned} Z_T(k) &\underset{T \rightarrow +\infty}{\simeq} \int \mathcal{D}\tilde{P}^{exc}[; \dots] \int dn \delta \left[\int_0^{+\infty} d\tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] \delta \left[n \int_0^{+\infty} d\tau \tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] - 1 \right] \\ &e^{-Tn \int_0^{+\infty} d\tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \ln \left(\frac{\tilde{P}^{exc}[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)]n} \right)} \end{aligned} \quad (148)$$

Then the analysis is exactly as for 58 with the same output : the conditioned distribution of excursions is given by (Eq 62)

$$\tilde{P}^{exc}[\tau; y(\cdot)] = \tilde{P}^{exc}[\tau; y(\cdot)] e^{-\tau \mu(k)} \quad (149)$$

where $\mu(k)$ is fixed by its normalization (Eq 63)

$$1 = \int_0^{+\infty} d\tau \sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] = \int_0^{+\infty} d\tau e^{-\tau \mu(k)} \left(\sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \right) \quad (150)$$

while the density n of the conditioned process is fixed by the first moment of the duration τ (Eq 63)

$$\frac{1}{n} = \int_0^{+\infty} d\tau \sum_{y(\cdot)} \tau \tilde{P}^{exc}[\tau; y(\cdot)] = \int_0^{+\infty} d\tau \tau e^{-\tau \mu(k)} \left(\sum_{y(\cdot)} \tilde{P}^{exc}[\tau; y(\cdot)] \right) \quad (151)$$

V. SISYPHUS MARKOV JUMP PROCESS IN CONTINUOUS TIME ON THE HALF LINE

The Sisyphus Markov Jump process defined on the half-line $x = 0, 1, 2, \dots$ corresponds to the off-diagonal matrix elements $x \neq y$ (Eq. 108)

$$\begin{aligned} w_{x,y} &= w\delta_{x,y+1} \\ w_{x,y}^{reset} &= w\delta_{x,y+1} + \delta_{x,0}r_y \end{aligned} \quad (152)$$

so the dynamics of Eq. 113 reads

$$\begin{aligned} \frac{\partial P_x(t)}{\partial t} &= -[w + r_x]P_x(t) + wP_{x-1}(t) \quad \text{for } x \geq 1 \\ \frac{\partial P_0(t)}{\partial t} &= -[w + r_0]P_0(t) + \sum_{x=0}^{+\infty} r_x P_x(t) = -wP_0(t) + \sum_{x=1}^{+\infty} r_x P_x(t) \end{aligned} \quad (153)$$

In order to apply the general formalism described in the previous section, one should first check that the hypothesis concerning the existence of a steady-state (Eq. 114) is satisfied.

A. Condition on the reset rates r for the existence of a non-equilibrium steady-state

The stationary solution P_x^* of Eq. 153 satisfies a simple recurrence for $x > 0$

$$P_x^* = \frac{w}{w + r_x} P_{x-1}^* = \dots = \left[\prod_{y=1}^x \frac{w}{w + r_y} \right] P_0^* \quad (154)$$

while Eq. 153 reads for $x = 0$

$$wP_0^* = \sum_{x=1}^{+\infty} r_x P_x^* = \sum_{x=1}^{+\infty} r_x \left[\prod_{y=1}^x \frac{w}{w + r_y} \right] P_0^* \quad (155)$$

so P_0^* disappears if it is not vanishing, and the remaining equation is just an identity. So P_0^* is determined by the normalization

$$1 = \sum_{x=0}^{+\infty} P_x^* = P_0^* \left(1 + \sum_{x=1}^{+\infty} \left[\prod_{y=1}^x \frac{w}{w + r_y} \right] \right) \quad (156)$$

The condition $P_0^* > 0$ to produce a non-equilibrium steady-state localized around the origin corresponds to the requirement of convergence for the series involving the reset rates r_y

$$\sum_{x=1}^{+\infty} \left[\prod_{y=1}^x \frac{w}{w + r_y} \right] < +\infty \quad (157)$$

This criterion has the same form as Eq. 71 if one introduces the notation $\hat{R}_y \equiv \frac{r_y}{w+r_y}$ so that the discussion following Eq. 71 concerning the uniform case $r_y = r$, the random cases or the deterministic cases can be directly translated and will not be repeated here.

In the following, we assume the convergence of Eq. 157 in order to have a non-equilibrium steady-state that can be analyzed via the large deviations at Level 2.5.

B. Large deviations at level 2.5 for the empirical density and the empirical currents

The empirical jump density of Eq. 117 contains two types of contributions, namely the non-local reset currents from any point x towards the origin (even $x = 0$ as discussed around Eq. 119)

$$J_x \equiv q_{0,x} \quad (158)$$

and the local currents arriving at any point $x \geq 1$ from its left neighbor $(x - 1)$

$$j_x \equiv q_{x,x-1} \quad (159)$$

For any position $x \geq 1$ the constraint of Eq. 118 reads

$$j_x = J_x + j_{x+1} \quad (160)$$

For the origin $x = 0$, the constraint of Eq. 118

$$\sum_{x=1}^{+\infty} J_x = j_1 \quad (161)$$

can be recovered by summing Eq. 160 over $x = 1, 2, \dots$ so that it is sufficient to impose Eq. 160. As a consequence, the large deviation at level 2.5 for the empirical density and the empirical currents of Eqs 120 122 121 become

$$P_T[\rho., j., J.] \underset{T \rightarrow +\infty}{\propto} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) \prod_{x=1}^{+\infty} \delta (J_x + j_{x+1} - j_x) \\ e^{-T \left(\sum_{x=1}^{+\infty} \left[j_x \ln \left(\frac{j_x}{w\rho_{x-1}} \right) - j_x + w\rho_{x-1} \right] + \sum_{x=0}^{+\infty} \left[J_x \ln \left(\frac{J_x}{r_x\rho_x} \right) - J_x + r_x\rho_x \right] \right)} \quad (162)$$

One can use the constraints to eliminate either the non-local reset currents J , or the local currents j , as described in the following two subsections, but it is not possible to obtain in closed form the Large deviations at Level 2 for the density ρ , alone, in contrast to the case of the Sisyphus Random Walk (Eq 82).

C. Large deviations for the empirical density ρ , and the local currents j .

The last constraints on the first line of Eq. 162 can be used to eliminate the non-local reset currents J_x for $x \geq 1$ in terms of the local currents

$$J_x = j_x - j_{x+1} \quad (163)$$

while the reset current J_0 from the origin to itself remains. So Eq. 162 yields the following large deviations for the empirical density ρ , and the local currents j .

$$P_T[\rho., j., J_0] \underset{T \rightarrow +\infty}{\propto} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) e^{-T \left[J_0 \ln \left(\frac{J_0}{r_0\rho_0} \right) - J_0 + r_0\rho_0 \right]} \\ e^{-T \left(\sum_{x=1}^{+\infty} \left[j_x \ln \left(\frac{j_x}{w\rho_{x-1}} \right) - j_x + w\rho_{x-1} \right] + \sum_{x=1}^{+\infty} \left[(j_x - j_{x+1}) \ln \left(\frac{(j_x - j_{x+1})}{r_x\rho_x} \right) - (j_x - j_{x+1}) + r_x\rho_x \right] \right)} \quad (164)$$

D. Large deviations for the empirical density ρ , and the non-local reset currents J .

If one wishes instead to eliminate the local currents in terms of the reset currents J , via

$$j_x = \sum_{y=x}^{+\infty} J_y \quad (165)$$

one obtains the following large deviations for the empirical density ρ , and the reset currents J .

$$P_T[\rho., J.] \underset{T \rightarrow +\infty}{\propto} \delta \left(\sum_{x=0}^{+\infty} \rho_x - 1 \right) \\ e^{-T \left(\sum_{x=1}^{+\infty} \left[\left(\sum_{y=x}^{+\infty} J_y \right) \ln \left(\frac{\left(\sum_{y=x}^{+\infty} J_y \right)}{w\rho_{x-1}} \right) - \left(\sum_{y=x}^{+\infty} J_y \right) + w\rho_{x-1} \right] + \sum_{x=0}^{+\infty} \left[J_x \ln \left(\frac{J_x}{r_x\rho_x} \right) - J_x + r_x\rho_x \right] \right)} \quad (166)$$

E. Large deviations for excursions between two consecutive resets

For the dynamics of Eq. 107, the simplifications in Eq. 124 are that the rate w is uniform and the internal trajectory $y(0 \leq s \leq \tau)$ can only involve the positions $z_m = m$. As a consequence, it will be clearer to denote the trajectory by the times $0 \leq s_1 \leq s_2 \dots \leq s_Y \leq Y$, where s_y denotes the time where the particle jumps from $(y-1)$ to y , while Y represents the last position before the reset towards $y=0$ at time τ . So the probability of Eq. 124 reads (with $s_0 = 0$ and $s_{Y+1} = \tau$)

$$\begin{aligned} P^{exc}[\tau; 0; s_{y=1, \dots, Y}] &= r_Y w^Y e^{-\sum_{y=0}^Y (s_{y+1} - s_y)(w + r_y)} \\ &= w^Y e^{-w\tau} r_Y e^{-s_1 r_0 - (s_2 - s_1)r_1 - \dots - (s_Y - s_{Y-1})r_{Y-1} - (\tau - s_Y)r_Y} \end{aligned} \quad (167)$$

while the special case $Y = 0$ where the reset occurs while the particle is still at the origin reads

$$P^{exc}[\tau; 0] = r_0 e^{-\tau(w+r_0)} \quad (168)$$

The normalization over all possible excursions of Eq. 126 becomes

$$1 = \int_0^{+\infty} d\tau \left[P^{exc}[\tau; 0] + \sum_{Y=1}^{+\infty} \int_{0 \leq s_1 \dots \leq s_Y \leq \tau} ds_1 \dots ds_Y P^{exc}[\tau; 0; s_{y=1, \dots, Y}] \right] \quad (169)$$

In order to make the link with the steady-state $P^*(x)$ of Eq. 154, it is useful to introduce to introduce the probability $P_{end}^{exc}(Y)$ of the end-position Y of an excursion after integration over the duration τ

$$P_{end}^{exc}(Y) \equiv \int_0^{+\infty} d\tau P^{exc}[\tau; 0; s_{y=1, \dots, Y}] = \frac{r_Y}{w + r_0} \prod_{y=1}^Y \frac{w}{w + r_y} = \frac{r_Y}{w + r_0} \left(\frac{P^*(Y)}{P^*(0)} \right) \quad (170)$$

where the normalization is satisfied using Eq. 155

$$\sum_{Y=0}^{+\infty} P_{end}^{exc}(Y) = \frac{r_0}{w + r_0} + \frac{1}{(w + r_0)P^*(0)} \sum_{Y=1}^{+\infty} r_Y P^*(Y) = \frac{r_0}{w + r_0} + \frac{w}{w + r_0} = 1 \quad (171)$$

The empirical density $n[\tau; 0; s_{y=1, \dots, Y}]$ of excursions between resets and the total density n follows the large deviations of Eq. 127 in the more explicit form

$$\begin{aligned} P_T(n[\cdot; \cdot], n) &\underset{T \rightarrow +\infty}{\simeq} \delta \left[\int_0^{+\infty} d\tau \left[n[\tau; 0] + \sum_{Y=1}^{+\infty} \int_{0 \leq s_1 \dots \leq s_Y \leq \tau} ds_1 \dots ds_Y n[\tau; 0; s_{y=1, \dots, Y}] \right] - n \right] \\ &\delta \left[\int_0^{+\infty} d\tau \tau \left[n[\tau; 0] + \sum_{Y=1}^{+\infty} \int_{0 \leq s_1 \dots \leq s_Y \leq \tau} ds_1 \dots ds_Y n[\tau; 0; s_{y=1, \dots, Y}] \right] - 1 \right] \\ &e^{-T \int_0^{+\infty} d\tau \left[n[\tau; 0] \ln \left(\frac{n[\tau; 0]}{P^{exc}[\tau; 0]n} \right) + \sum_{Y=1}^{+\infty} \int_{0 \leq s_1 \dots \leq s_Y \leq \tau} ds_1 \dots ds_Y n[\tau; 0; s_{y=1, \dots, Y}] \ln \left(\frac{n[\tau; 0; s_{y=1, \dots, Y}]}{P^{exc}[\tau; 0; s_{y=1, \dots, Y}]n} \right) \right]} \end{aligned} \quad (172)$$

F. Large deviations of general time-additive observables

In this section, we consider time-additive observables of the form $(A_T + B_T)$ (Eqs 131-132). Using Eqs 158 and 159, their generating function of Eq. 133 reads more explicitly in terms of the density ρ_x , the local current j_x and the non-local reset currents J_x

$$Z_T(k) \equiv \langle e^{Tk(A_T + B_T)} \rangle = \langle e^{Tk \left(\sum_{x=0}^{+\infty} \alpha_x \rho_x + \sum_{x=1}^{+\infty} \beta_{x, x-1} j_x + \sum_{x=0}^{+\infty} \beta_{0, x} J_x \right)} \rangle \quad (173)$$

1. Analysis via the tilted matrix

For the present model (Eq. 152), the tilted matrix involves the off-diagonal elements $x \neq y$ (Eq. 136)

$$\tilde{w}_{x,y}^{[k]} \equiv [w\delta_{x,y+1} + \delta_{x,0}r_y]e^{k\beta_{x,y}} \quad \text{for } x \neq y \quad (174)$$

and the diagonal elements (Eq. 137)

$$\tilde{w}_{x,x}^{[k]} \equiv -w - r_x + k\alpha_x + \delta_{x,0}r_0e^{k\beta_{0,0}} \quad (175)$$

So the eigenvalue Eqs 139 and 140 read

$$[w + r_x - k\alpha_x + \mu(k)]\tilde{r}_x^{[k]} = we^{k\beta_{x,x-1}}\tilde{r}_{x-1}^{[k]} + \delta_{x,0}\left[\sum_{y=0}^{+\infty}r_ye^{k\beta_{0,y}}\tilde{r}_y^{[k]}\right] \quad (176)$$

and

$$[w + r_y - k\alpha_y + \mu(k)]\tilde{l}_y^{[k]} = \tilde{l}_{y+1}^{[k]}we^{k\beta_{y+1,y}} + \tilde{l}_0^{[k]}r_ye^{k\beta_{0,y}} \quad (177)$$

For $x \geq 1$, Eq. 176 corresponds to the simple recurrence

$$\tilde{r}_x^{[k]} = \left[\frac{we^{k\beta_{x,x-1}}}{w + r_x - k\alpha_x + \mu(k)}\right]\tilde{r}_{x-1}^{[k]} = \left[\prod_{z=1}^x\frac{we^{k\beta_{z,z-1}}}{w + r_z - k\alpha_z + \mu(k)}\right]\tilde{r}_0^{[k]} \quad (178)$$

while Eq. 176 for $x = 0$ yields

$$[w + r_0 - k\alpha_0 + \mu(k)]\tilde{r}_0^{[k]} = \sum_{x=0}^{+\infty}r_xe^{k\beta_{0,x}}\tilde{r}_x^{[k]} = r_0e^{k\beta_{0,0}}\tilde{r}_0^{[k]} + \sum_{x=1}^{+\infty}r_xe^{k\beta_{0,x}}\left[\prod_{z=1}^x\frac{we^{k\beta_{z,z-1}}}{w + r_z - k\alpha_z + \mu(k)}\right]\tilde{r}_0^{[k]} \quad (179)$$

So $\tilde{r}_0^{[k]}$ disappears and the remaining equation determines the scaled cumulant generating function $\mu(k)$

$$1 = \frac{r_0e^{k\beta_{0,0}}}{w + r_0 - k\alpha_0 + \mu(k)} + \sum_{x=1}^{+\infty}\frac{r_xe^{k\beta_{0,x}}}{w + r_0 - k\alpha_0 + \mu(k)}\left[\prod_{z=1}^x\frac{we^{k\beta_{z,z-1}}}{w + r_z - k\alpha_z + \mu(k)}\right] \quad (180)$$

The solution of Eq. 177 for the left eigenvector reads

$$\tilde{l}_y^{[k]} = \tilde{l}_0^{[k]}\left[\frac{r_ye^{k\beta_{0,y}}}{w + r_y - k\alpha_y + \mu(k)} + \sum_{x=y+1}^{+\infty}\frac{r_xe^{k\beta_{0,x}}}{w + r_x - k\alpha_x + \mu(k)}\prod_{z=y}^{x-1}\frac{we^{k\beta_{z+1,z}}}{w + r_z - k\alpha_z + \mu(k)}\right] \quad (181)$$

where the consistency for $y = 0$ reproduces the equation 180 for the eigenvalue $\mu(k)$ after some minimal rewriting.

The matrix generating the conditioned process obtained via the generalization of Doob's h-transform (Eq. 142) involves the same non-vanishing matrix elements as the initial matrix $w_{x,y}^{reset}$ of Eq. 152 : the off-diagonal elements $x \neq y$ read (Eq. 174)

$$\tilde{\tilde{w}}_{x,y} = \tilde{l}_x^{[k]}\tilde{w}_{x,y}^{[k]}\frac{1}{\tilde{l}_y^{[k]}} = \delta_{x,y+1}\tilde{\tilde{w}}_{x,x-1} + \delta_{x,0}\tilde{\tilde{w}}_{0,y} \quad (182)$$

where the rate to jump from $(x - 1)$ to x has changed from its initial value w of Eq. 152 to its new value

$$\tilde{\tilde{w}}_{x,x-1} = we^{k\beta_{x,x-1}}\frac{\tilde{l}_x^{[k]}}{\tilde{l}_{x-1}^{[k]}} \quad (183)$$

while the rate for a reset from y to the origin has changed from its initial value r_y of Eq. 64 to its new value

$$\tilde{\tilde{w}}_{0,y} = r_ye^{k\beta_{0,y}}\frac{\tilde{l}_0^{[k]}}{\tilde{l}_y^{[k]}} \quad (184)$$

Both involve explicitly the left eigenvector $\tilde{l}^{[k]}$. The diagonal elements (Eq. 175)

$$\tilde{w}_{x,x} = -w - r_x + k\alpha_x + \delta_{x,0}r_0 e^{k\beta_{0,0}} - \mu(k) \quad (185)$$

involve the eigenvalue $\mu(k)$. The stationary density of this conditioned process reads

$$\tilde{\rho}_x = \tilde{l}_x^{[k]} \tilde{r}_x^{[k]} = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]} \left[\sum_{y=x}^{+\infty} \frac{r_y e^{k\beta_{0,y}}}{w + r_x - k\alpha_x + \mu(k)} \prod_{z=1}^y \frac{w e^{k\beta_{z,z-1}}}{w + r_z - k\alpha_z + \mu(k)} \right] \quad (186)$$

where the normalization determines the value of the density at the origin $\tilde{\rho}_0 = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]}$

$$1 = \sum_{x=0}^{+\infty} \tilde{\rho}_x = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]} \left[\sum_{x=0}^{+\infty} \sum_{y=x}^{+\infty} \frac{r_y e^{k\beta_{0,y}}}{w + r_x - k\alpha_x + \mu(k)} \prod_{z=1}^y \frac{w e^{k\beta_{z,z-1}}}{w + r_z - k\alpha_z + \mu(k)} \right] \quad (187)$$

Finally, the stationary local currents and reset currents of this conditioned process are given by

$$\begin{aligned} \tilde{j}_x &= \tilde{w}_{x,x-1} \tilde{\rho}_{x-1} = w e^{k\beta_{x,x-1}} \tilde{l}_x^{[k]} \tilde{r}_{x-1}^{[k]} \\ \tilde{J}_y &= \tilde{w}_{0,y} \tilde{\rho}_y = r_y e^{k\beta_{0,y}} \tilde{l}_0^{[k]} \tilde{r}_y^{[k]} \end{aligned} \quad (188)$$

2. Analysis via the tilted excursions

For the present model where the probabilities of excursions are given by Eqs 167 and 168 the non-conserved tilted quantities of Eq. 146 read for $Y \geq 1$

$$\begin{aligned} \tilde{P}_k^{exc}[\tau; 0; s_{y=1,\dots,Y}] &\equiv P^{exc}[\tau; 0; s_{y=1,\dots,Y}] e^{k \left[\sum_{y=0}^Y (s_{y+1} - s_y) \alpha_y + \sum_{y=1}^Y \beta_{y,y-1} + \beta_{0,Y} \right]} \\ &= (r_Y e^{k\beta_{0,Y}}) \left[\prod_{l_y=1}^Y (w e^{k\beta_{y,y-1}}) \right] \left[\prod_{l_y=0}^Y e^{-(s_{y+1} - s_y)(w + r_y - k\alpha_y)} \right] \end{aligned} \quad (189)$$

and for $Y = 0$

$$\tilde{P}_k^{exc}[\tau; 0] \equiv P^{exc}[\tau; 0] e^{k\tau\alpha_0 + k\beta_{0,0}} = (r_0 e^{k\beta_{0,0}}) e^{-\tau(w + r_0 - k\alpha_0)} \quad (190)$$

The conditioned excursion probability of Eq. 149 reads

$$\tilde{P}^{exc}[\tau; 0; s_{y=1,\dots,Y}] = \tilde{P}^{exc}[\tau; 0; s_{y=1,\dots,Y}] e^{-\tau\mu(k)} \quad (191)$$

where the scaled cumulant generating function $\mu(k)$ is fixed by its normalization (Eq. 150)

$$\begin{aligned} 1 &= \int_0^{+\infty} d\tau e^{-\tau\mu(k)} \left[\tilde{P}^{exc}[\tau; 0] + \sum_{Y=1}^{+\infty} \int_{0 \leq s_1 \dots \leq s_Y \leq \tau} ds_1 \dots ds_Y \tilde{P}^{exc}[\tau; 0; s_{y=1,\dots,Y}] \right] \\ &= \frac{r_0 e^{k\beta_{0,0}}}{w + r_0 - k\alpha_0 + \mu(k)} + \sum_{Y=1}^{+\infty} \frac{r_Y e^{k\beta_{0,Y}}}{w + r_0 - k\alpha_0 + \mu(k)} \prod_{y=1}^Y \frac{w e^{k\beta_{y,y-1}}}{w + r_y - k\alpha_y + \mu(k)} \end{aligned} \quad (192)$$

that coincides with Eq. 180 as it should for consistency between the two approaches.

VI. DIFFUSION PROCESSES IN A FORCE FIELD IN DIMENSION d WITH RESETS

A. Models and notations

In this section, we turn to the continuous time/continuous space framework. We focus on the dynamics described by the Fokker-Planck equation in the force field $\vec{F}(\vec{x})$ in dimension d , with space-independent diffusion coefficient D and where the reset from \vec{y} towards the origin $\vec{0}$ is governed by the rate $r(\vec{y})$

$$\frac{\partial P_t(\vec{x})}{\partial t} = -\vec{\nabla} \cdot [P_t(\vec{x}) \vec{F}(\vec{x}) - D \vec{\nabla} P_t(\vec{x})] - r(\vec{x}) P_t(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r(\vec{y}) P_t(\vec{y}) \quad (193)$$

Again as in Eqs 4 and 114, we will assume that the steady-state solution $P^*(\vec{x})$ of Eq. 193

$$0 = -\vec{\nabla} \cdot \left[P^*(\vec{x}) \vec{F}(\vec{x}) - D \vec{\nabla} P^*(\vec{x}) \right] - r(\vec{x}) P^*(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r(\vec{y}) P^*(\vec{y}) \quad (194)$$

exists in order to apply the large deviations at Level 2.5 for non-equilibrium steady-states.

B. Large deviations at level 2.5 for the empirical density and the empirical currents

1. Empirical density and the empirical currents with their constraints

The empirical density

$$\rho(\vec{x}) \equiv \frac{1}{T} \int_0^T dt \delta^{(d)}(\vec{x}(t) - \vec{x}) \quad (195)$$

satisfies the normalization

$$\int d^d \vec{x} \rho(\vec{x}) = 1 \quad (196)$$

The non-local reset currents $J(\vec{x}) \geq 0$ measure the non-local jumps from $\vec{x} \neq \vec{0}$ towards the origin $\vec{0}$

$$J(\vec{x}) \equiv \frac{1}{T} \sum_{t: \vec{x}(t^+) = \vec{0} \neq \vec{x}(t^-)} \delta_{\vec{x}(t^-), \vec{x}} \quad (197)$$

while the local current field $\vec{j}(\vec{x})$ characterizes the diffusion process in the force field

$$\vec{j}(\vec{x}) \equiv \frac{1}{T} \int_0^T dt \frac{d\vec{x}(t)}{dt} \delta^{(d)}(\vec{x}(t) - \vec{x}) \quad (198)$$

Here the stationarity constraint for all positions $\vec{x} \neq \vec{0}$ yields that the non-local reset currents $J(\vec{x})$ are related to the divergence of the local current field $\vec{j}(\vec{x})$

$$J(\vec{x}) = -\vec{\nabla} \cdot \vec{j}(\vec{x}) \quad (199)$$

that replaces the standard divergence-free condition for diffusion processes without resetting.

2. Large deviations at level 2.5 for the density, the local currents and the non-local reset currents

The joint distribution of the empirical density $\rho(\cdot)$, the empirical local currents $\vec{j}(\vec{x})$ and the empirical non-local reset currents $J(\vec{x})$ satisfy the large deviation form

$$P_T[\rho(\cdot), \vec{j}(\cdot), J(\cdot)]_{T \rightarrow +\infty} \simeq \delta \left(\int d^d \vec{x} \rho(\vec{x}) - 1 \right) \left[\prod_{\vec{x} \neq \vec{0}} \delta \left(J(\vec{x}) + \vec{\nabla} \cdot \vec{j}(\vec{x}) \right) \right] e^{-T \left(\frac{1}{4D} \int \frac{d^d \vec{x}}{\rho(\vec{x})} \left[\vec{j}(\vec{x}) - \rho(\vec{x}) \vec{F}(\vec{x}) + D \vec{\nabla} \rho(\vec{x}) \right]^2 + \int d^d \vec{x} \left[J(\vec{x}) \ln \left(\frac{J(\vec{x})}{r(\vec{x}) \rho(\vec{x})} \right) - J(\vec{x}) + r(\vec{x}) \rho(\vec{x}) \right] \right)} \quad (200)$$

with the constraints discussed in Eqs 196 and 199, while the rate function contains two contributions : the first contribution involving the local current field $\vec{j}(\vec{x})$ corresponds to the usual rate function for continuous-time/continuous-space diffusion processes [26, 27, 36, 44–46], while the second contribution involving the non-local reset current $J(\vec{x})$ from \vec{x} to the origin $\vec{0}$ corresponds to the usual rate function for jump processes, that we have already seen in Eq. 122.

3. Large deviations for the empirical density $\rho(\cdot)$ and the empirical local currents $\vec{j}(\cdot)$

The constraints of Eq. 199 can be used to eliminate all the non-local reset currents to obtain the large deviations for the empirical density $\rho(\cdot)$ and the empirical local currents $\vec{j}(\cdot)$

$$P_T[\rho(\cdot), \vec{j}(\cdot)]_{T \rightarrow +\infty} \simeq \delta \left(\int d^d \vec{x} \rho(\vec{x}) - 1 \right) \quad (201)$$

$$e^{-T \left[\frac{1}{4D} \int \frac{d^d \vec{x}}{\rho(\vec{x})} \left(\vec{j}(\vec{x}) - \rho(\vec{x}) \vec{F}(\vec{x}) + D \vec{\nabla} \rho(\vec{x}) \right)^2 + \int d^d \vec{x} \left[\left(-\vec{\nabla} \cdot \vec{j}(\vec{x}) \right) \ln \left(\frac{(-\vec{\nabla} \cdot \vec{j}(\vec{x}))}{r(\vec{x}) \rho(\vec{x})} \right) + \vec{\nabla} \cdot \vec{j}(\vec{x}) + r(\vec{x}) \rho(\vec{x}) \right] \right]}$$

4. Large deviations for the empirical density $\rho(\cdot)$ and the empirical reset currents $J(\cdot)$ in dimension $d = 1$

In arbitrary dimension $d > 1$, the constraints of Eq. 199 cannot be explicitly inverted to obtain the local current field in terms of the non-local reset currents. However in dimension $d = 1$, the constraints of Eq. 199 become for all $x \neq 0$

$$J(x) = -\frac{dj(x)}{dx} \quad (202)$$

i.e. the positive non-local reset current $J(x) \geq 0$ determines the gradient of the local current $j(x)$, that can be thus reconstructed as follows. In the region $x > 0$, the local current is positive $j(x) > 0$ and given by

$$j(x) = \int_x^{+\infty} dy J(y) \quad (203)$$

while in the region $x < 0$, the local current is negative $j(x) < 0$ and given by

$$j(x) = -\int_{-\infty}^x dy J(y) \quad (204)$$

In particular, the discontinuity of the local current at the origin will correspond to the integral of all the reset currents that are re-injected at the origin

$$j(0^+) - j(0^-) = \int_{0^+}^{+\infty} dy J(y) + \int_{-\infty}^{0^-} dy J(y) = \int_{-\infty}^{+\infty} dy J(y) > 0 \quad (205)$$

Using Eqs 203 and 204, the local currents can be eliminated from Eq. 200 to obtain the following large deviations for the empirical density $\rho(\cdot)$ and the empirical reset currents $J(\cdot)$

$$P_T^{(d=1)}[\rho(\cdot), J]_{T \rightarrow +\infty} \simeq \delta \left(\int_{-\infty}^{+\infty} dx \rho(x) - 1 \right) e^{-T \int_{-\infty}^{+\infty} dx \left[J(x) \ln \left(\frac{J(x)}{r(x) \rho(x)} \right) - J(x) + r(x) \rho(x) \right]} \quad (206)$$

$$e^{-\frac{T}{4D} \left[\int_{-\infty}^0 \frac{dx}{\rho(x)} \left(-\int_{-\infty}^x dy J(y) - \rho(x) F(x) + D \rho'(x) \right)^2 + \int_0^{+\infty} \frac{dx}{\rho(x)} \left(\int_x^{+\infty} dy J(y) - \rho(x) F(x) + D \rho'(x) \right)^2 \right]}$$

C. Large deviations for excursions between two consecutive resets

1. Probabilities of excursions between two consecutive resets

The probability to have an excursion of duration τ and of internal trajectory $\vec{y}(0 \leq s \leq \tau)$ with the initial condition $\vec{y}(s=0) = \vec{0}$ fixed by the reset and the end-position $\vec{y}(s=\tau)$ being the last position before the next reset leading to $\vec{y}(\tau^+) = \vec{0}$ reads

$$P^{exc}[\tau; \vec{y}(\cdot)] = r(\vec{y}(\tau)) e^{-\int_0^\tau ds r(\vec{y}(s)) - \frac{1}{4D} \int_0^\tau ds \left(\frac{d\vec{y}(s)}{ds} - \vec{F}(\vec{y}(s)) \right)^2 - \frac{1}{2} \int_0^\tau ds \vec{\nabla} \cdot \vec{F}(\vec{y}(s))} \quad (207)$$

The normalization over all possible excursions involves the integration over the duration τ and the path-integral over the internal trajectory $y(0 \leq s \leq \tau)$

$$1 = \int_0^{+\infty} d\tau \int_{y(0)=0} \mathcal{D}y(\cdot) P^{exc}[\tau; y(\cdot)] \quad (208)$$

2. Large deviations for the excursions density

The probability to see the empirical density $n[\tau; y(\cdot)]$ of excursions between resets and the total density n follows the large deviation form analogous to Eq. 127

$$P_T(n[\cdot; \cdot], n) \underset{T \rightarrow +\infty}{\simeq} \delta \left[\int_0^{+\infty} d\tau \int_{y(0)=0} \mathcal{D}y(\cdot) n[\tau; y(\cdot)] - n \right] \delta \left[\int_0^{+\infty} d\tau \int_{y(0)=0} \mathcal{D}y(\cdot) \tau n[\tau; y(\cdot)] - 1 \right] e^{-T \int_0^{+\infty} d\tau \int_{y(0)=0} \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{P^{exc}[\tau; y(\cdot)] n} \right)} \quad (209)$$

3. Simplifications for uniform resetting rate $r_y = r$

When the resetting rate $r_y = r$ does not depend on the position y , the excursion probability of Eq. 207 becomes factorized as in Eq. 128

$$P_{uniform}^{exc}[\tau; (y(\cdot))] = p^{exp}(\tau) p_{\tau}^{config}[(y(\cdot))] \quad (210)$$

into the exponential distribution of Eq. 129 that describes the probability of the duration τ of the excursion independently of the internal trajectory and the probability $p_{\tau}^{config}[(y(\cdot))]$ of the internal trajectory once its duration τ is given that characterizes the process without resetting

$$p_{\tau}^{config}[(y(\cdot))] = e^{-\frac{1}{4D} \int_0^{\tau} ds \left(\frac{d\vec{y}(s)}{ds} - \vec{F}(\vec{y}(s)) \right)^2 - \frac{1}{2} \int_0^{\tau} ds \vec{\nabla} \cdot \vec{F}(\vec{y}(s))} \quad (211)$$

In particular, the probability of the total density n of excursions simply follows the Poisson distribution given in Eq. 130 with its corresponding large deviation form. On the contrary, whenever the reset rates $r(\vec{y})$ depends on the position \vec{y} , the duration τ and the internal trajectory $\vec{y}(\cdot)$ are coupled via Eq. 207.

D. Large deviations for general time-additive observables

The empirical density of Eq. 195 allows to reconstruct any time-additive observable that involves some function $\alpha(\vec{x})$ of the position $\vec{x}(t)$

$$A_T \equiv \frac{1}{T} \int_0^T dt \alpha(\vec{x}(t)) = \int d^d \vec{x} \alpha(\vec{x}) \rho(\vec{x}) \quad (212)$$

The non-local reset currents $J(\vec{x})$ of Eq. 197 allows to reconstruct any time-additive observable that involves some function $\beta(\vec{x})$ of the reset jumps from \vec{x} towards the origin $\vec{0}$

$$B_T \equiv \frac{1}{T} \sum_{t: \vec{x}(t^+) = \vec{0} \neq \vec{x}(t^-)} \beta(\vec{x}(t^-)) = \int d^d \vec{x} \beta(\vec{x}) J(\vec{x}) \quad (213)$$

Finally the local current field $\vec{j}(\vec{x})$ of Eq. 198 allows to reconstruct any time-additive observable involving the increments $\frac{d\vec{x}(t)}{dt}$ of the diffusion process parametrized by some vector field $\vec{\gamma}(\vec{x})$

$$\Gamma_T \equiv \frac{1}{T} \int_0^T dt \frac{d\vec{x}(t)}{dt} \cdot \vec{\gamma}(\vec{x}(t)) = \int d^d \vec{x} \vec{j}(\vec{x}) \cdot \vec{\gamma}(\vec{x}) \quad (214)$$

The large deviations of the sum $(A_T + B_T + \Gamma_T)$ can be analyzed via the generating function

$$Z_T(k) \equiv \langle e^{Tk(A_T + B_T + \Gamma_T)} \rangle = \langle e^{Tk \int d^d \vec{x} \left[\alpha(\vec{x}) \rho(\vec{x}) + \beta(\vec{x}) J(\vec{x}) + \vec{\gamma}(\vec{x}) \cdot \vec{j}(\vec{x}) \right]} \rangle \quad (215)$$

1. Analysis via the tilted dynamics

The probability of a trajectory for the process of Eq. 193 can be written in terms of its empirical observables apart from the Wiener measure on the first line

$$P_T^{Traj} \underset{T \rightarrow +\infty}{\simeq} e^{-\frac{1}{4D} \int_0^T ds \left(\frac{d\vec{y}(s)}{ds} \right)^2} e^{T \int d^d \vec{x} \left[-\rho(\vec{x}) \left(r(\vec{x}) + \frac{(\vec{F}(\vec{x}))^2}{4D} + \frac{\vec{\nabla} \cdot \vec{F}(\vec{x})}{2} \right) + J(\vec{x}) \ln(r(\vec{x})) + \vec{j}(\vec{x}) \cdot \frac{\vec{F}(\vec{x})}{2D} \right]} \quad (216)$$

As a consequence, the generating function of Eq. 215 reads

$$Z_T(k) = e^{-\frac{1}{4D} \int_0^T ds \left(\frac{d\vec{y}(s)}{ds} \right)^2} e^{T \int d^d \vec{x} \left[-\rho(\vec{x}) \left(r(\vec{x}) + \frac{(\vec{F}(\vec{x}))^2}{4D} + \frac{\vec{\nabla} \cdot \vec{F}(\vec{x})}{2} - k\alpha(\vec{x}) \right) + J(\vec{x}) \ln(r(\vec{x})e^{k\beta(\vec{x})}) + \vec{j}(\vec{x}) \cdot \frac{\vec{F}(\vec{x}) + 2D\vec{\gamma}(\vec{x})}{2D} \right]} \quad (217)$$

and can be analyzed by tilting each term of the Fokker-Planck Eq. 193 with resets to produce the non-conserved dynamics

$$\begin{aligned} \frac{\partial \tilde{P}_t(\vec{x})}{\partial t} &= \tilde{\mathcal{F}}_k \tilde{P}_t(\cdot) \\ &= -\left(\vec{\nabla} - k\vec{\gamma}(\vec{x}) \right) \cdot \left[\vec{F}(\vec{x}) P_t(\vec{x}) - D \left(\vec{\nabla} - k\vec{\gamma}(\vec{x}) \right) P_t(\vec{x}) \right] - [r(\vec{x}) - k\alpha(\vec{x})] P_t(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r(\vec{y}) e^{k\beta(\vec{y})} P_t(\vec{y}) \end{aligned} \quad (218)$$

The scaled cumulant generating function $\mu(k)$ corresponds to the highest eigenvalue of the tilted operator $\tilde{\mathcal{F}}_k$, with its right eigenvector $\tilde{r}^{[k]}(\vec{x})$

$$\begin{aligned} \mu(k) \tilde{r}^{[k]}(\vec{x}) &= \tilde{\mathcal{F}}_k \tilde{r}^{[k]}(\cdot) \\ &= -\left(\vec{\nabla} - k\vec{\gamma}(\vec{x}) \right) \cdot \left[\vec{F}(\vec{x}) - D \left(\vec{\nabla} - k\vec{\gamma}(\vec{x}) \right) \right] \tilde{r}^{[k]}(\vec{x}) - [r(\vec{x}) - k\alpha(\vec{x})] \tilde{r}^{[k]}(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r(\vec{y}) e^{k\beta(\vec{y})} \tilde{r}^{[k]}(\vec{y}) \\ &= D\Delta \tilde{r}^{[k]}(\vec{x}) - \left(\vec{F}(\vec{x}) + 2Dk\vec{\gamma}(\vec{x}) \right) \cdot \vec{\nabla} \tilde{r}^{[k]}(\vec{x}) \\ &+ \left[-\vec{\nabla} \cdot \vec{F}(\vec{x}) - Dk\vec{\nabla} \cdot \vec{\gamma}(\vec{x}) + k\vec{F}(\vec{x}) \cdot \vec{\gamma}(\vec{x}) + Dk^2\vec{\gamma}^2(\vec{x}) - r(\vec{x}) + k\alpha(\vec{x}) \right] \tilde{r}^{[k]}(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r(\vec{y}) e^{k\beta(\vec{y})} \tilde{r}^{[k]}(\vec{y}) \end{aligned} \quad (219)$$

and its left eigenvector $\tilde{l}^{[k]}(\vec{y})$

$$\begin{aligned} \mu(k) \tilde{l}^{[k]}(\vec{y}) &= \tilde{\mathcal{F}}_k^\dagger \tilde{l}^{[k]}(\cdot) \\ &= \left[\vec{F}(\vec{y}) + D \left(\vec{\nabla} + k\vec{\gamma}(\vec{y}) \right) \right] \left(\vec{\nabla} + k\vec{\gamma}(\vec{y}) \right) \tilde{l}^{[k]}(\vec{y}) - [r(\vec{y}) - k\alpha(\vec{y})] \tilde{l}^{[k]}(\vec{y}) + r(\vec{y}) e^{k\beta(\vec{y})} \tilde{l}^{[k]}(\vec{0}) \\ &= D\Delta \tilde{l}^{[k]}(\vec{y}) + \left(\vec{F}(\vec{y}) + 2Dk\vec{\gamma}(\vec{y}) \right) \cdot \vec{\nabla} \tilde{l}^{[k]}(\vec{y}) \\ &+ \left[Dk\vec{\nabla} \cdot \vec{\gamma}(\vec{y}) + k\vec{F}(\vec{y}) \cdot \vec{\gamma}(\vec{y}) + Dk^2\vec{\gamma}^2(\vec{y}) - r(\vec{y}) + k\alpha(\vec{y}) \right] \tilde{l}^{[k]}(\vec{y}) + \tilde{l}^{[k]}(\vec{0}) r(\vec{y}) e^{k\beta(\vec{y})} \end{aligned} \quad (220)$$

In order to compute explicitly the generator of the conditioned process obtained via the generalization of Doob's h-transform

$$\tilde{\mathcal{F}}_k \equiv \tilde{l}^{[k]}(\cdot) \tilde{\mathcal{F}}_k \frac{1}{\tilde{l}^{[k]}(\cdot)} - \mu(k) \quad (221)$$

it is technically simpler to compute the action of its adjoint operator $\tilde{\mathcal{F}}_k^\dagger$ on some test function $\phi(\cdot)$

$$\begin{aligned}\tilde{\mathcal{F}}_k^\dagger \phi(\cdot) &= \frac{1}{\tilde{l}^{[k]}(\cdot)} \tilde{\mathcal{F}}^\dagger \left[\tilde{l}^{[k]}(\cdot) \phi(\cdot) \right] - \mu(k) \phi(\cdot) \\ &= \frac{1}{\tilde{l}^{[k]}(\vec{y})} D\Delta \left[\tilde{l}^{[k]}(\vec{y}) \phi(\vec{y}) \right] + \frac{1}{\tilde{l}^{[k]}(\vec{y})} \left(\vec{F}(\vec{y}) + 2Dk\vec{\gamma}(\vec{y}) \right) \cdot \vec{\nabla} \left[\tilde{l}^{[k]}(\vec{y}) \phi(\vec{y}) \right] + \frac{r(\vec{y}) e^{k\beta(\vec{y})}}{\tilde{l}^{[k]}(\vec{y})} \tilde{l}^{[k]}(\vec{0}) \phi(\vec{0}) \\ &+ \left[Dk\vec{\nabla} \cdot \vec{\gamma}(\vec{y}) + k\vec{F}(\vec{y}) \cdot \vec{\gamma}(\vec{y}) + Dk^2\vec{\gamma}^2(\vec{y}) - r(\vec{y}) + k\alpha(\vec{y}) - \mu(k) \right] \phi(\vec{y})\end{aligned}\quad (222)$$

One can then use the eigenvalue Eq. 220 for the left eigenvector $\tilde{l}^{[k]}(\cdot)$ to rewrite the coefficient of $\phi(\vec{y})$ in the last line of Eq. 222 as

$$\begin{aligned}& \left[Dk\vec{\nabla} \cdot \vec{\gamma}(\vec{y}) + k\vec{F}(\vec{y}) \cdot \vec{\gamma}(\vec{y}) + Dk^2\vec{\gamma}^2(\vec{y}) - r(\vec{y}) + k\alpha(\vec{y}) - \mu(k) \right] \\ &= -\frac{1}{\tilde{l}^{[k]}(\vec{y})} D\Delta \tilde{l}^{[k]}(\vec{y}) - \frac{1}{\tilde{l}^{[k]}(\vec{y})} \left(\vec{F}(\vec{y}) + 2Dk\vec{\gamma}(\vec{y}) \right) \cdot \vec{\nabla} \tilde{l}^{[k]}(\vec{y}) - \frac{\tilde{l}^{[k]}(\vec{0}) r(\vec{y}) e^{k\beta(\vec{y})}}{\tilde{l}^{[k]}(\vec{y})}\end{aligned}\quad (223)$$

so that Eq. 222 reduces to the form

$$\tilde{\mathcal{F}}_k^\dagger \phi(\cdot) = D\Delta \phi(\vec{y}) + \left[\vec{F}(\vec{y}) + 2Dk\vec{\gamma}(\vec{y}) + 2D \frac{(\vec{\nabla} \tilde{l}^{[k]}(\vec{y}))}{\tilde{l}^{[k]}(\vec{y})} \right] \cdot \vec{\nabla} \phi(\vec{y}) + \frac{\tilde{l}^{[k]}(\vec{0}) r(\vec{y}) e^{k\beta(\vec{y})}}{\tilde{l}^{[k]}(\vec{y})} \left[\phi(\vec{y}) - \phi(\vec{0}) \right] \quad (224)$$

where it is explicit that the constant $\tilde{l}^{[k]}(\vec{y}) = 1$ is eigenvector of $\tilde{\mathcal{F}}_k^\dagger$ associated to the eigenvalue zero.

So the operator $\tilde{\mathcal{F}}_k = (\tilde{\mathcal{F}}_k^\dagger)^\dagger$ generates the following conservative dynamics analogous to the initial dynamics of Eq. 193

$$\frac{\partial \tilde{P}_t(\vec{x})}{\partial t} = \tilde{\mathcal{F}}_k \tilde{P}_t(\cdot) = -\vec{\nabla} \cdot \left(\vec{F}^{eff}(\vec{x}) \tilde{P}_t(\vec{x}) - D\vec{\nabla} \tilde{P}_t(\vec{x}) \right) - r^{eff}(\vec{x}) \tilde{P}_t(\vec{x}) + \delta^{(d)}(\vec{x}) \int d^d \vec{y} r^{eff}(\vec{y}) \tilde{P}_t(\vec{y}) \quad (225)$$

where the effective force that replaces the initial value $\vec{F}(\vec{x})$ of Eq. 193

$$\vec{F}^{eff}(\vec{x}) \equiv \vec{F}(\vec{x}) + 2Dk\vec{\gamma}(\vec{x}) + 2D \frac{(\vec{\nabla} \tilde{l}^{[k]}(\vec{x}))}{\tilde{l}^{[k]}(\vec{x})} \quad (226)$$

and where the effective resetting rate from \vec{x} to the origin $\vec{0}$ that replaces the initial value $r(\vec{x})$ of Eq. 193

$$r^{eff}(\vec{x}) \equiv r(\vec{x}) e^{k\beta(\vec{x})} \frac{\tilde{l}^{[k]}(\vec{0})}{\tilde{l}^{[k]}(\vec{x})} \quad (227)$$

involve both explicitly the left eigenvector $\tilde{l}^{[k]}(\cdot)$. The right eigenvector $\tilde{r}^{[k]}(\cdot)$ will appear in the corresponding stationary density

$$\tilde{\rho}(\vec{x}) = \tilde{l}^{[k]}(\vec{x}) \tilde{r}^{[k]}(\vec{x}) \quad (228)$$

in the corresponding local currents

$$\begin{aligned}\tilde{j}(\vec{x}) &= \vec{F}^{eff}(\vec{x}) \tilde{\rho}(\vec{x}) - D\vec{\nabla} \tilde{\rho}(\vec{x}) = \left[\vec{F}(\vec{x}) + 2Dk\vec{\gamma}(\vec{x}) + 2D \frac{(\vec{\nabla} \tilde{l}^{[k]}(\vec{x}))}{\tilde{l}^{[k]}(\vec{x})} \right] \tilde{l}^{[k]}(\vec{x}) \tilde{r}^{[k]}(\vec{x}) - D\vec{\nabla} \left(\tilde{l}^{[k]}(\vec{x}) \tilde{r}^{[k]}(\vec{x}) \right) \\ &= \left[\vec{F}(\vec{x}) + 2Dk\vec{\gamma}(\vec{x}) \right] \tilde{l}^{[k]}(\vec{x}) \tilde{r}^{[k]}(\vec{x}) + D \left[(\vec{\nabla} \tilde{l}^{[k]}(\vec{x})) \tilde{r}^{[k]}(\vec{x}) - \tilde{l}^{[k]}(\vec{x}) \vec{\nabla} \tilde{r}^{[k]}(\vec{x}) \right]\end{aligned}\quad (229)$$

and in the corresponding non-local reset currents

$$\tilde{J}(\vec{x}) = r^{eff}(\vec{x}) \tilde{\rho}(\vec{x}) = r(\vec{x}) e^{k\beta(\vec{x})} \tilde{l}^{[k]}(\vec{0}) \tilde{r}^{[k]}(\vec{x}) \quad (230)$$

2. *Analysis via the tilted excursions*

The empirical density $n[\tau; y(\cdot)]$ of excursions can be used to write any time-additive observable of the forms of Eqs 212 213 and 214

$$\begin{aligned} A_T &= \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \left[\int_0^\tau ds \alpha(y(s)) \right] \\ B_T &= \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \beta(y(\tau^-)) \\ \Gamma_T &= \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \left[\int_0^\tau ds \frac{d\vec{y}(s)}{ds} \cdot \vec{\gamma}(\vec{y}(s)) \right] \end{aligned} \quad (231)$$

So the generating function of Eq. 215 can be evaluated from the joint probability of Eq. 209 as

$$\begin{aligned} Z_T(k) &= \int dn \int \mathcal{D}n[:, :] P_T(n[:, :], n) e^{Tk} \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \left[\int_0^\tau ds \left(\alpha(y(s)) + \frac{d\vec{y}(s)}{ds} \cdot \vec{\gamma}(\vec{y}(s)) \right) + \beta(y(\tau^-)) \right] \\ &\underset{\tau \rightarrow +\infty}{\simeq} \int dn \int \mathcal{D}n[:, :] \delta \left[\int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] - n \right] \delta \left[\int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) \tau n[\tau; y(\cdot)] - 1 \right] \\ &\quad e^{-T} \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) n[\tau; y(\cdot)] \ln \left(\frac{n[\tau; y(\cdot)]}{\tilde{P}^{exc}[\tau; y(\cdot)] n} \right) \end{aligned} \quad (232)$$

in terms of the tilted non-conserved quantity

$$\tilde{P}_k^{exc}[\tau; y(\cdot)] \equiv P^{exc}[\tau; y(\cdot)] e^{k \int_0^\tau ds \left(\alpha(y(s)) + \frac{d\vec{y}(s)}{ds} \cdot \vec{\gamma}(\vec{y}(s)) \right) + k\beta(y(\tau^-))} \quad (233)$$

Again the change of variable from $n[\tau; y(\cdot)]$ to the probability of excursions that will make $n[\tau; y(\cdot)]$ and n typical

$$\tilde{P}^{exc}[\tau; y(\cdot)] \equiv \frac{n[\tau; y(\cdot)]}{n} \quad (234)$$

allows to analyze Eq. 233 exactly as Eq. 148 with the same output : the conditioned distribution of excursions is given by (Eq 149)

$$\tilde{P}^{exc}[\tau; y(\cdot)] = \tilde{P}^{exc}[\tau; y(\cdot)] e^{-\tau\mu(k)} \quad (235)$$

where the scaled cumulant generating function $\mu(k)$ is fixed by its normalization (Eq 150)

$$1 = \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) \tilde{P}^{exc}[\tau; y(\cdot)] = \int_0^{+\infty} d\tau e^{-\tau\mu(k)} \left(\int \mathcal{D}y(\cdot) \tilde{P}^{exc}[\tau; y(\cdot)] \right) \quad (236)$$

while the inverse of the density n of the conditioned process is given by the first moment of the duration τ (Eq 151)

$$\frac{1}{n} = \int_0^{+\infty} d\tau \int \mathcal{D}y(\cdot) \tau \tilde{P}^{exc}[\tau; y(\cdot)] = \int_0^{+\infty} d\tau \tau e^{-\tau\mu(k)} \left(\int \mathcal{D}y(\cdot) \tilde{P}^{exc}[\tau; y(\cdot)] \right) \quad (237)$$

VII. SISYPHUS PROCESS IN CONTINUOUS TIME AND CONTINUOUS SPACE

In this section, we consider the dynamics of Eq. 193 in dimension $d = 1$, in the absence of diffusion $D = 0$ and in the constant force field of value unity $F(x) = 1$: this Sisyphus process taking place on the half-line $x \in [0, +\infty[$

$$\frac{\partial P_t(x)}{\partial t} = -\frac{\partial P_t(x)}{\partial x} - r(x)P_t(x) + \delta(x) \int_0^{+\infty} dy r(y)P_t(y) \quad (238)$$

is the continuous time/continuous space analog of the one-dimension Sisyphus Random Walk [77] discussed in section III. It also corresponds to the diffusionless version of the neuronal integrate-and-fire model studied in Ref [6].

In order to apply the general formalism described in the previous section, one should first check that the hypothesis concerning the existence of a steady-state (Eq. 194) is satisfied.

A. Condition on the reset rates $r(\cdot)$ for the existence of a non-equilibrium steady-state

The stationary solution $P^*(x)$ of Eq. 238

$$0 = -\frac{dP^*(x)}{dx} - r(x)P^*(x) + \delta(x) \int_0^{+\infty} dy r(y)P^*(y) \quad (239)$$

reads in terms of the heaviside step function $\theta(x)$

$$P^*(x) = \theta(x)P^*(0)e^{-\int_0^x dy r(y)} \quad (240)$$

where the steady-state $P^*(0)$ at the origin is fixed by the normalization

$$1 = \int_0^{+\infty} dx P^*(x) = P^*(0) \int_0^{+\infty} dx e^{-\int_0^x dy r(y)} \quad (241)$$

The condition on the reset rates $r(\cdot)$ to have a non-equilibrium steady-state localized around the origin $P^*(0) > 0$ is thus given by the convergence of the integral

$$\int_0^{+\infty} dx e^{-\int_0^x dy r(y)} < +\infty \quad (242)$$

that represents the continuous analog of Eq. 71. In the uniform case $r(y) = r$, this integral converges for any value $r > 0$, i.e. the only case of divergence corresponds to $r = 0$, where the model without any reset is of course transient (Eq. 238). In the random case, the specific structure of Eq. 242 corresponds to the continuous version of Kesten random variables [85–93] so that the condition of convergence $\overline{r_y} > 0$ will be satisfied since the random variable r_y represents a positive reset rate $r_y \geq 0$.

Finally for the case where the reset rate $r(y)$ is a deterministic positive function of the position y , the integral of Eq. 242 will generically converge, except when $r(y)$ decays towards zero too rapidly for $y \rightarrow +\infty$, for instance for the choice $r(y \rightarrow +\infty) \simeq 1/y$ that leads to the logarithmic divergence of Eq. 242.

In the following, we assume the convergence of Eq. 242 in order to have a non-equilibrium steady-state that can be analyzed via the large deviations at Level 2.5.

B. Large deviations at Level 2.5 for the density $\rho(\cdot)$ and the non-local reset currents $J(\cdot)$

In the limit of vanishing diffusion $D \rightarrow 0$, the first contribution of the rate function of Eq. 200 becomes the constraint

$$j(x) = \rho(x) \quad (243)$$

between the local current $j(x)$ and the density $\rho(x)$, as it should since the motion is deterministic and ballistic between resets (Eq. 243 is the analog of Eq. 77). As a consequence, the large deviations at Level 2.5 for the joint distribution of the density $\rho(\cdot)$ and the non-local reset currents $J(\cdot)$ read (Eq. 200)

$$P_T[\rho(\cdot), J(\cdot)] \underset{T \rightarrow +\infty}{\simeq} \delta\left(\int_0^{+\infty} dx \rho(x) - 1\right) \prod_{x>0} \delta(J(x) + \rho'(x)) e^{-T \int_0^{+\infty} dx \left[J(x) \ln\left(\frac{J(x)}{r(x)\rho(x)}\right) - J(x) + r(x)\rho(x) \right]} \quad (244)$$

C. Large deviations at Level 2 for the density $\rho(\cdot)$ alone

One may use the last constraints of the first line of Eq. 244 to eliminate all the non-local reset currents

$$J(x) = -\rho'(x) \quad (245)$$

and one obtains the large deviations at Level 2 for the density $\rho(\cdot)$ alone

$$\begin{aligned}
P_T[\rho(\cdot)] &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx \rho(x) - 1 \right) e^{-T \int_0^{+\infty} dx \left[-\rho'(x) \ln \left(\frac{(-\rho'(x))}{r(x)\rho(x)} \right) + \rho'(x) + r(x)\rho(x) \right]} \\
&\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx \rho(x) - 1 \right) e^{-T \left[\int_0^{+\infty} dx (-\rho'(x)) \ln \left(\frac{(-\rho'(x))}{r(x)} \right) + \int_0^{+\infty} dx \frac{d}{dx} [\rho(x) \ln(\rho(x))] + \int_0^{+\infty} dx r(x)\rho(x) \right]} \\
&\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx \rho(x) - 1 \right) e^{-T \left[\int_0^{+\infty} dx (-\rho'(x)) \ln \left(\frac{(-\rho'(x))}{r(x)} \right) - \rho(0) \ln(\rho(0)) + \int_0^{+\infty} dx r(x)\rho(x) \right]} \quad (246)
\end{aligned}$$

D. Large deviations for the non-local reset currents $J(\cdot)$ alone

One may instead eliminate the empirical density

$$\rho(x) = \int_x^{+\infty} dy J(y) \quad (247)$$

to translate Eq. 246 into the large deviations for the non-local reset currents $J(\cdot)$ alone

$$\begin{aligned}
P_T[\rho(\cdot)] &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx x J(x) - 1 \right) \delta \left(\int_0^{+\infty} dx J(x) - \rho(0) \right) \\
&\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx x J(x) - 1 \right) \delta \left(\int_0^{+\infty} dx J(x) - \rho(0) \right) e^{-T \left[\int_0^{+\infty} dx J(x) \ln \left(\frac{J(x)}{r(x)} \right) - \rho(0) \ln(\rho(0)) + \int_0^{+\infty} dx J(x) \int_0^x dy r(y) \right]} \\
&\underset{T \rightarrow +\infty}{\simeq} \delta \left(\int_0^{+\infty} dx x J(x) - 1 \right) \delta \left(\int_0^{+\infty} dx J(x) - \rho(0) \right) e^{-T \int_0^{+\infty} dx J(x) \ln \left(\frac{J(x)}{r(x) e^{-\int_0^x dy r(y)} \rho(0)} \right)} \quad (248)
\end{aligned}$$

The compact form of the last line allows to make the link with the large deviations for the excursions between two consecutive resets as explained below.

E. Large deviations for the excursions between two consecutive resets

Here the internal trajectory of an excursion between to resets is deterministic and ballistic

$$y(0 \leq s \leq \tau) = s \quad (249)$$

so that the probability for an excursion of Eq. 207 can be labelled by its duration τ only and reduces to

$$P^{exc}(\tau) = P^{exc}[\tau; y(0 \leq s \leq \tau) = s] = r(\tau) e^{-\int_0^\tau ds r(s)} = -\frac{d}{d\tau} e^{-\int_0^\tau ds r(s)} \quad (250)$$

The large deviation form for the empirical density $n(\tau)$ of excursions of duration τ and the total density n of excursions

$$\begin{aligned}
P_T(n(\cdot), n) &\underset{T \rightarrow +\infty}{\simeq} \delta \left[\int_0^{+\infty} d\tau n(\tau) - n \right] \delta \left[\int_0^{+\infty} d\tau \tau n(\tau) - 1 \right] e^{-T \int_0^{+\infty} d\tau n(\tau) \ln \left(\frac{n(\tau)}{P^{exc}(\tau)n} \right)} \\
&\underset{T \rightarrow +\infty}{\simeq} \delta \left[\int_0^{+\infty} d\tau n(\tau) - n \right] \delta \left[\int_0^{+\infty} d\tau \tau n(\tau) - 1 \right] e^{-T \int_0^{+\infty} d\tau n(\tau) \ln \left(\frac{n(\tau)}{r(\tau) e^{-\int_0^\tau ds r(s)} n} \right)} \quad (251)
\end{aligned}$$

thus coincides with Eq. 248 with the following dictionary analogous to Eqs A24 and A25 : the empirical density at the origin $\rho(0)$ corresponds to the empirical density n of excursions

$$\rho(0) = n \quad (252)$$

while the non-local reset current $J(x)$ corresponds to the empirical density of excursions of duration $\tau = x$

$$J(x) = n(\tau = x) \quad (253)$$

F. Large deviations for general time-additive observables

In this section, we focus on the generating function of time-additive observables of the form $(A_T + B_T + \Gamma_T)$ (Eqs 212 213 214) involving a priori three function $\alpha(x), \beta(x), \gamma(x)$

$$\begin{aligned} Z_T(k) &\equiv \langle e^{Tk(A_T + \Gamma_T + B_T)} \rangle = \langle e^{k \left[\int_0^T dt [\alpha(x(t)) + \dot{x}(t)\gamma(x(t))] + \sum_{t:x(t^+) = 0 \neq x(t^-)} \beta(x(t^-)) \right]} \rangle \\ &= \langle e^{Tk \int_0^{+\infty} dx [\alpha(x)\rho(x) + j(x)\gamma(x) + J(x)\beta(x)]} \rangle \end{aligned} \quad (254)$$

In the present model, the local current $j(x)$ coincides with the density $\rho(x)$ (Eq. 243), so the function $\gamma(x)$ is redundant with respect to $\alpha(x)$, but as in the previous applications, it is more pedagogical to keep them both to see more clearly how the general formalism of the previous section works.

1. Analysis via the tilted dynamics

In the present one-dimensional model (Eq. 238) where the diffusion coefficient vanishes $D = 0$ and where the force is constant of value unity $F(x) = 1$, the eigenvalue equations for the right eigenvector $\tilde{r}^{[k]}(x)$ (Eq. 219) and for the left eigenvector $\tilde{l}^{[k]}(x)$ (Eq. 220) become

$$\mu(k)\tilde{r}^{[k]}(x) = -\frac{d\tilde{r}^{[k]}(x)}{dx} + [k\alpha(x) + k\gamma(x) - r(x)]\tilde{r}^{[k]}(x) + \delta(x) \int_0^{+\infty} dy r(y)e^{k\beta(y)}\tilde{r}^{[k]}(y) \quad (255)$$

and

$$\mu(k)\tilde{l}^{[k]}(y) = \frac{d\tilde{l}^{[k]}(y)}{dy} + [k\alpha(y) + k\gamma(y) - r(y)]\tilde{l}^{[k]}(y) + r(y)e^{k\beta(y)}\tilde{l}^{[k]}(0) \quad (256)$$

The solution of Eq. 255 for the right eigenvector reads

$$\tilde{r}^{[k]}(x) = \theta(x)\tilde{r}^{[k]}(0)e^{-x\mu(k) + \int_0^x dx' [k\alpha(x') + k\gamma(x') - r(x')]} \quad (257)$$

with the condition at the origin

$$\tilde{r}^{[k]}(0) = \int_0^{+\infty} dx r(x)e^{k\beta(x)}\tilde{r}^{[k]}(x) = \int_0^{+\infty} dx r(x)e^{k\beta(x)}\tilde{r}^{[k]}(0)e^{-x\mu(k) + \int_0^x dx' [k\alpha(x') + k\gamma(x') - r(x')]} \quad (258)$$

so $\tilde{r}^{[k]}(0)$ disappears and the remaining equation determines the scaled cumulant generating function $\mu(k)$

$$1 = \int_0^{+\infty} dx r(x)e^{-x\mu(k) + k\beta(x) + \int_0^x dx' [k\alpha(x') + k\gamma(x') - r(x')]} \quad (259)$$

The solution of Eq. 256 for the left eigenvector reads

$$\tilde{l}^{[k]}(y) = \tilde{l}^{[k]}(0) \int_y^{+\infty} dx r(x)e^{+k\beta(x) + \int_y^x dx' [-\mu(k) + k\alpha(x') + k\gamma(x') - r(x')]} \quad (260)$$

where the consistency at $y = 0$ reproduces Eq. 259 for the eigenvalue $\mu(k)$.

The operator $\tilde{\mathcal{F}}_k$ that generates the conditioned process (Eq. 225) has the same form as the initial dynamics with a constant force $F(x) = 1$ and no diffusion $D = 0$

$$\frac{\partial \tilde{P}_t(x)}{\partial t} = \tilde{\mathcal{F}}_k \tilde{P}_t(\cdot) = -\frac{\partial \tilde{P}_t(x)}{\partial x} - r^{eff}(x) \tilde{P}_t(x) + \delta(x) \int_0^{+\infty} dy r^{eff}(y) \tilde{P}_t(y) \quad (261)$$

where the effective resetting rate from x to the origin 0 has changed from its initial value $r(x)$ of Eq. 238 to its new value of Eq. 227 that involves the left eigenvector $\tilde{l}^{[k]}(\cdot)$

$$r^{eff}(x) \equiv r(x) e^{k\beta(x)} \frac{\tilde{l}^{[k]}(0)}{\tilde{l}^{[k]}(x)} \quad (262)$$

The corresponding stationary density reads

$$\tilde{\rho}(x) = \tilde{l}^{[k]}(x) \tilde{r}^{[k]}(x) = \tilde{l}^{[k]}(0) \tilde{r}^{[k]}(0) \int_x^{+\infty} dz r(z) e^{-z\mu(k) + k\beta(z) + \int_0^z dx' [k\alpha(x') + k\gamma(x') - r(x')]} \quad (263)$$

where the normalization determines the value of the density at the origin $\tilde{\rho}(0) = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]}$

$$\begin{aligned} 1 = \int_0^{+\infty} dx \tilde{\rho}(x) &= \tilde{l}^{[k]}(0) \tilde{r}^{[k]}(0) \int_0^{+\infty} dx \int_x^{+\infty} dz r(z) e^{-z\mu(k) + k\beta(z) + \int_0^z dx' [k\alpha(x') + k\gamma(x') - r(x')]} \\ &= \tilde{l}^{[k]}(0) \tilde{r}^{[k]}(0) \int_0^{+\infty} dz z r(z) e^{-z\mu(k) + k\beta(z) + \int_0^z dx' [k\alpha(x') + k\gamma(x') - r(x')]} \end{aligned} \quad (264)$$

The corresponding local currents coincide with the density (Eq. 229)

$$\tilde{j}(x) = \tilde{\rho}(x) \quad (265)$$

while the corresponding non-local reset currents read (Eq. 230)

$$\tilde{J}(x) = r(x) e^{k\beta(x)} \tilde{l}^{[k]}(0) \tilde{r}^{[k]}(x) \quad (266)$$

2. Analysis via the tilted excursions

In the present model where the probability of excursions is given by Eq. 250, the tilted non-conserved quantity of Eq. 233 reads

$$\tilde{P}_k^{exc}[\tau] = P^{exc}[\tau] e^{+k\beta(\tau) + k \int_0^\tau ds [\alpha(s) + \gamma(s)]} = r(\tau) e^{k\beta(\tau) + \int_0^\tau ds [-r(s) + k\alpha(s) + k\gamma(s)]} \quad (267)$$

The conditioned distribution of excursions is given by (Eq 235)

$$\tilde{P}^{exc}[\tau] = \tilde{P}^{exc}[\tau] e^{-\tau\mu(k)} = r(\tau) e^{-\tau\mu(k) + k\beta(\tau) + \int_0^\tau ds [-r(s) + k\alpha(s) + k\gamma(s)]} \quad (268)$$

where $\mu(k)$ is fixed by its normalization (Eq 236)

$$1 = \int_0^{+\infty} d\tau \tilde{P}^{exc}[\tau] = \int_0^{+\infty} d\tau r(\tau) e^{-\tau\mu(k) + k\beta(\tau) + \int_0^\tau ds [-r(s) + k\alpha(s) + k\gamma(s)]} \quad (269)$$

that coincides with Eq. 259 as it should by consistency between the two points of view. Finally the inverse of the density n of the conditioned process is fixed by the first moment of the duration τ (Eq 237)

$$\frac{1}{n} = \int_0^{+\infty} d\tau \tau \tilde{P}^{exc}[\tau] = \int_0^{+\infty} d\tau \tau r(\tau) e^{-\tau\mu(k) + k\beta(\tau) + \int_0^\tau ds [-r(s) + k\alpha(s) + k\gamma(s)]} \quad (270)$$

This equation is equivalent to Eq. 264 for the density at the origin $\tilde{\rho}_0 = \tilde{l}_0^{[k]} \tilde{r}_0^{[k]}$ in agreement with the dictionary Eqs 252 and 253.

VIII. CONCLUSIONS

In order to characterize the non-equilibrium steady-states of Markov processes with stochastic resetting towards the origin, we have described the large deviations at level 2.5 for the joint probability of the empirical density and the empirical flows, the large deviations for the empirical excursions between consecutive resets, and the large deviations of general time-additive observables within the three possible frameworks : discrete-time/discrete-space Markov chains, continuous-time/discrete-space Markov jump processes, and continuous-time/continuous-space diffusion processes. In each case, we have illustrated the general formalism via simple examples based on the Sisyphus Random Walk and its variants. Since the equations containing the main results have already been pointed out in the subsections I-A,I-B,I-C of the Introduction, we shall not repeat them here. We hope that these explicit formula will be useful to analyze the various applications mentioned in the Introduction, where the resetting probabilities or the resetting rates depend on the position either deterministically or randomly.

Appendix A: Application to the Sisyphus Random Walk on the Cayley tree

In this section, we focus on the generalization of the one-dimensional Sisyphus Random Walk discussed in section III for the geometry of the Cayley tree of branching ratio b . (The one-dimensional model of section III corresponds to the special case $b = 1$ with no branching).

1. Model and notations

We consider a tree of branching parameter b , starting at the root (0), with b sites labelled by $i_1 = 1, 2, \dots, b$ at the first generation $g = 1$, b^2 sites labelled by (i_1, i_2) at the second generation $g = 2$, i.e. the b^g sites of generation g will be labelled by the g indices (i_1, i_2, \dots, i_g) containing the information on the whole line of ancestors.

In the absence of resetting, a particle on site (i_1, \dots, i_g) at time t can only jump towards one of its b children, so one can use the simplified notation for the corresponding probabilities $W_{(i_1, i_2, \dots, i_g, i_{g+1}), (i_1, i_2, \dots, i_g)} \equiv W_{i_1, i_2, \dots, i_g, i_{g+1}}$ with the normalization

$$1 = \sum_{i_{g+1}=1}^b W_{i_1, \dots, i_g, i_{g+1}} \quad (\text{A1})$$

The simplest example corresponds to the uniform case, where each of the b children is chosen with equal probability $1/b$ at each step

$$W_{i_1, \dots, i_g, i_{g+1}}^{uniform} = \frac{1}{b} \quad (\text{A2})$$

but other choices are also interesting, so we will continue the analysis for the general case.

In the presence of resetting towards the origin, the dynamics of Eq. 2 reads for the root and for the sites of other generations $g \geq 1$ respectively

$$\begin{aligned} P_0(t+1) &= R_0 P_0(t) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} R_{i_1, \dots, i_g} P_{i_1, \dots, i_g}(t) \\ P_{i_1, \dots, i_g}(t+1) &= W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) P_{i_1, \dots, i_{g-1}}(t) \end{aligned} \quad (\text{A3})$$

In order to apply the general formalism described in section II, one should first check that the hypothesis concerning the existence of a steady-state (Eq. 4) is satisfied.

2. Condition on the reset probabilities R_{\dots} for the existence of a non-equilibrium steady-state

The equations for the stationary state P^* of the dynamics of Eq. A3

$$\begin{aligned} P_0^* &= R_0 P_0^* + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} R_{i_1, \dots, i_g} P_{i_1, \dots, i_g}^* \\ P_{i_1, \dots, i_g}^* &= W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) P_{i_1, \dots, i_{g-1}}^* \end{aligned} \quad (\text{A4})$$

can be directly solved by recurrence for all the sites of generation $g \geq 1$ in terms of P_0^*

$$P_{i_1, \dots, i_g}^* = W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) \dots W_{i_1} (1 - R_0) P_0^* = \left[\prod_{g'=1}^g W_{i_1, \dots, i_{g'}} (1 - R_{i_1, \dots, i_{g'-1}}) \right] P_0^* \quad (\text{A5})$$

while P_0^* is then determined by the normalization

$$1 = P_0^* + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} P_{i_1, \dots, i_g}^* = P_0^* \left(1 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \left[\prod_{g'=1}^g W(i_1, \dots, i_{g'}) (1 - R_{i_1, \dots, i_{g'-1}}) \right] \right) \quad (\text{A6})$$

The condition $P_0^* > 0$ to produce a non-equilibrium steady-state localized around the origin corresponds to the requirement of convergence for the following series

$$\sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \left[\prod_{g'=1}^g W(i_1, \dots, i_{g'}) (1 - R_{i_1, \dots, i_{g'-1}}) \right] < +\infty \quad (\text{A7})$$

In the following, we assume the convergence of Eq. A7 in order to have a non-equilibrium steady-state that can be analyzed via the large deviations at Level 2.5.

3. Empirical density and empirical currents with their constraints

The empirical 2-point density of Eq. 5 contains two types of contributions, namely the non-local reset currents from any point (i_1, \dots, i_g) of any generation $g \geq 0$ towards the origin

$$\begin{aligned} J_{i_1, \dots, i_g} &\equiv \rho_{0, (i_1, \dots, i_g)}^{(2)} \\ J_0 &\equiv \rho_{0, 0}^{(2)} \end{aligned} \quad (\text{A8})$$

and the local currents arriving at any point (i_1, \dots, i_g) of any generation $g \geq 1$ from its ancestor (i_1, \dots, i_{g-1})

$$\begin{aligned} \hat{j}_{i_1, \dots, i_g} &\equiv \rho_{(i_1, \dots, i_g), (i_1, \dots, i_{g-1})}^{(2)} \\ \hat{j}_{i_1} &\equiv \rho_{i_1, 0}^{(2)} \end{aligned} \quad (\text{A9})$$

As a consequence, the consistency constraints of Eq. 6 involving the 1-point density become for the origin

$$\rho_0 = J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} = J_0 + \sum_{i_1} \hat{j}_{i_1} \quad (\text{A10})$$

and for the other points of generation $g \geq 1$

$$\rho_{i_1, \dots, i_g} = \hat{j}_{i_1, \dots, i_g} = J_{i_1, \dots, i_g} + \sum_{i_{g+1}} \hat{j}_{i_1, \dots, i_g, i_{g+1}} \quad (\text{A11})$$

4. Large deviations at level 2.5 for the joint probability of the empirical density and the empirical currents

The large deviations at level 2.5 of Eqs 8 9 10 yield that the joint probability to see the empirical density ρ , and the empirical currents j , and J , read

$$\begin{aligned}
P_T(\rho, j, J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\rho_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} - 1 \right) \delta \left(J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - \rho_0 \right) \delta \left(J_0 + \sum_{i_1} j_{i_1} - \rho_0 \right) \\
&\prod_{g=1}^{+\infty} \prod_{i_1, \dots, i_g} \left[\delta \left(j_{i_1, \dots, i_g} - \rho_{i_1, \dots, i_g} \right) \delta \left(J_{i_1, \dots, i_g} + \sum_{i_{g+1}} j_{i_1, \dots, i_g, i_{g+1}} - \rho_{i_1, \dots, i_g} \right) \right] \\
&\underset{e}{-T} \left[\sum_{i_1} j_{i_1} \ln \left(\frac{j_{i_1}}{W_{i_1} (1 - R_0) \rho_0} \right) + \sum_{g=2}^{+\infty} \sum_{i_1, \dots, i_g} j_{i_1, \dots, i_g} \ln \left(\frac{j_{i_1, \dots, i_g}}{W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) \rho_{i_1, \dots, i_{g-1}}} \right) \right] \\
&\underset{e}{-T} \left[J_0 \ln \left(\frac{J_0}{R_0 \rho_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} \ln \left(\frac{J_{i_1, \dots, i_g}}{R_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g}} \right) \right]
\end{aligned} \tag{A12}$$

The first constraints on the second line shows that all the local currents j , can be eliminated in terms of the density

$$j_{i_1, \dots, i_g} = \rho_{i_1, \dots, i_g} \tag{A13}$$

so that Eq. A12 becomes for the joint distribution of the density ρ , and the non-local reset currents J .

$$\begin{aligned}
P_T(\rho, J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\rho_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} - 1 \right) \delta \left(J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - \rho_0 \right) \delta \left(J_0 + \sum_{i_1} \rho_{i_1} - \rho_0 \right) \\
&\prod_{g=1}^{+\infty} \prod_{i_1, \dots, i_g} \delta \left(J_{i_1, \dots, i_g} + \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}} - \rho_{i_1, \dots, i_g} \right) \\
&\underset{e}{-T} \left[\sum_{i_1} \rho_{i_1} \ln \left(\frac{\rho_{i_1}}{W_{i_1} (1 - R_0) \rho_0} \right) + \sum_{g=2}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} \ln \left(\frac{\rho_{i_1, \dots, i_g}}{W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) \rho_{i_1, \dots, i_{g-1}}} \right) \right] \\
&\underset{e}{-T} \left[J_0 \ln \left(\frac{J_0}{R_0 \rho_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} \ln \left(\frac{J_{i_1, \dots, i_g}}{R_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g}} \right) \right]
\end{aligned} \tag{A14}$$

One can now further use the constraints to obtain the large deviations for the empirical density alone or for the non-local reset currents alone, as described in the following two subsections.

5. Large deviations at Level 2 for the empirical density ρ , alone

The constraints on the second line of Eq. A14 can be used to eliminate the non-local reset currents associated to generations $g \geq 1$ in terms of the empirical density

$$J_{i_1, \dots, i_g} = \rho_{i_1, \dots, i_g} - \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}} \tag{A15}$$

while the third constraint on the first line of Eq. A14 gives the similar equation for the origin

$$J_0 = \rho_0 - \sum_{i_1} \rho_{i_1} \tag{A16}$$

Then the second constraint on the first line of Eq. A14 is automatically satisfied, so that Eq. A14 yields for the large deviations for the empirical density ρ , alone

$$\begin{aligned}
P_T(\rho) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\rho_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} - 1 \right) \\
&e^{-T \left[\sum_{i_1} \rho_{i_1} \ln \left(\frac{\rho_{i_1}}{W_{i_1}(1-R_0)\rho_0} \right) + \sum_{g=2}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} \ln \left(\frac{\rho_{i_1, \dots, i_g}}{W_{i_1, \dots, i_g}(1-R_{i_1, \dots, i_{g-1}})\rho_{i_1, \dots, i_{g-1}}} \right) \right]} \\
&e^{-T \left[\left(\rho_0 - \sum_{i_1} \rho_{i_1} \right) \ln \left(\frac{\rho_0 - \sum_{i_1} \rho_{i_1}}{R_0 \rho_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \left(\rho_{i_1, \dots, i_g} - \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}} \right) \ln \left(\frac{\rho_{i_1, \dots, i_g} - \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}}}{R_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g}} \right) \right]}
\end{aligned} \tag{A17}$$

One may further simplify the rate function of the two last lines to obtain

$$\begin{aligned}
P_T(\rho) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(\rho_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} - 1 \right) \\
&e^{-T \left[-\rho_0 \ln(\rho_0) - \sum_{i_1} \rho_{i_1} \ln(W_{i_1}(1-R_0)) - \sum_{g=2}^{+\infty} \sum_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} \ln(W_{i_1, \dots, i_g}(1-R_{i_1, \dots, i_{g-1}})) \right]} \\
&e^{-T \left[\left(\rho_0 - \sum_{i_1} \rho_{i_1} \right) \ln \left(\frac{\rho_0 - \sum_{i_1} \rho_{i_1}}{R_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} \left(\rho_{i_1, \dots, i_g} - \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}} \right) \ln \left(\frac{\rho_{i_1, \dots, i_g} - \sum_{i_{g+1}} \rho_{i_1, \dots, i_g, i_{g+1}}}{R_{i_1, \dots, i_g}} \right) \right]}
\end{aligned} \tag{A18}$$

6. Large deviations for the empirical reset currents J , alone

If one wishes instead to eliminate the empirical density in terms of the empirical reset currents J , via

$$\begin{aligned}
\rho_{i_1, \dots, i_g} &= J_{i_1, \dots, i_g} + \sum_{g'=g+1}^{+\infty} \sum_{i_{g+1}, \dots, i_{g'}} J_{i_1, \dots, i_g, i_{g+1}, \dots, i_{g'}} \\
\rho_0 &= J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g}
\end{aligned} \tag{A19}$$

Eq. A18 yields for the large deviations for the empirical reset currents J , alone

$$\begin{aligned}
P_T(J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(J_0 + \sum_{g=1}^{+\infty} (g+1) \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - 1 \right) \delta \left(J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - \rho_0 \right) \\
&e^{-T \left[-\rho_0 \ln(\rho_0) - \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} \ln \left(\prod_{g'=1}^g W_{i_1, \dots, i_{g'}}(1-R_{i_1, \dots, i_{g'-1}}) \right) \right]} \\
&e^{-T \left[J_0 \ln \left(\frac{J_0}{R_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} \ln \left(\frac{J_{i_1, \dots, i_g}}{R_{i_1, \dots, i_g}} \right) \right]}
\end{aligned} \tag{A20}$$

One may further simplify the rate function of the two last lines to obtain the more compact form

$$\begin{aligned}
P_T(J) &\underset{T \rightarrow +\infty}{\simeq} \delta \left(J_0 + \sum_{g=1}^{+\infty} (g+1) \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - 1 \right) \delta \left(J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} - \rho_0 \right) \\
&e^{-T \left[J_0 \ln \left(\frac{J_0}{R_0 \rho_0} \right) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} J_{i_1, \dots, i_g} \ln \left(\frac{J_{i_1, \dots, i_g}}{R_{i_1, \dots, i_g} \left(\prod_{g'=1}^g W_{i_1, \dots, i_{g'}}(1-R_{i_1, \dots, i_{g'-1}}) \right) \rho_0} \right) \right]}
\end{aligned} \tag{A21}$$

in order to make the link with the large deviations for excursions between two consecutive resets as explained in next subsection.

7. Large deviations for the empirical density of excursions between two consecutive resets

For the Sisyphus Random Walk on the Cayley tree, the simplification in Eq. 16 is that the internal trajectory $y(1 \leq s \leq \tau - 1)$ of an excursion is ballistic along one branch of the tree, and can be thus labelled only by its end-point $y(\tau - 1) = (i_1, \dots, i_{\tau-1})$, so that its probability of Eq. 16 becomes

$$\begin{aligned} P^{exc}[\tau; (i_1, \dots, i_{\tau-1})] &= R_{i_1, \dots, i_{\tau-1}} W_{i_1, \dots, i_{\tau-1}} (1 - R_{i_1, \dots, i_{\tau-2}}) \dots W_{i_1, i_2} (1 - R_{i_1}) W_{i_1} (1 - R_0) \\ &= R_{i_1, \dots, i_{\tau-1}} \left[\prod_{g'=1}^{\tau-1} W_{i_1, \dots, i_{g'}} (1 - R_{i_1, \dots, i_{g'-1}}) \right] \end{aligned} \quad (\text{A22})$$

As a consequence, the probability to see the empirical density $n[\tau; (i_1, \dots, i_{\tau-1})]$ of excursions between resets and the total density n follows the large deviation form of Eqs 18 20 19

$$\begin{aligned} P_T(n[.; \dots], n) &\underset{T \rightarrow +\infty}{\simeq} \delta \left[\sum_{\tau=1}^{+\infty} \tau \sum_{i_1, \dots, i_{\tau-1}} n[\tau; i_1, \dots, i_{\tau-1}] - 1 \right] \delta \left[\sum_{\tau=1}^{+\infty} \sum_{i_1, \dots, i_{\tau-1}} n[\tau; i_1, \dots, i_{\tau-1}] - n \right] \\ &\underset{e}{-T} \sum_{\tau=1}^{+\infty} \sum_{i_1, \dots, i_{\tau-1}} n[\tau; i_1, \dots, i_{\tau-1}] \ln \left(\frac{n[\tau; i_1, \dots, i_{\tau-1}]}{R_{i_1, \dots, i_{\tau-1}} \left[\prod_{g'=1}^{\tau-1} W_{i_1, \dots, i_{g'}} (1 - R_{i_1, \dots, i_{g'-1}}) \right] n} \right) \end{aligned} \quad (\text{A23})$$

The correspondence with Eq A21 is now obvious with the following dictionary : the empirical density at the origin ρ_0 corresponds to the empirical density n of excursions

$$\rho_0 = n \quad (\text{A24})$$

while the non-local reset current J_{i_1, \dots, i_g} corresponds to the empirical density of excursions of duration $\tau = g + 1$ with end-point $(i_1, \dots, i_{\tau-1}) = (i_1, \dots, i_g)$

$$J_{i_1, \dots, i_g} = n[g + 1; i_1, \dots, i_g] \quad (\text{A25})$$

8. Large deviations of general time-additive observables

In this section, we analyze the large deviations of time-additive observables of the form $(A_T + B_T)$ (Eqs 25 26 35 36) via their generating function (Eq 27) that reads in terms of the empirical density ρ , the empirical local currents j , and the empirical non-local reset currents J .

$$\begin{aligned} Z_T(k) &\equiv \langle e^{Tk(A_T + B_T)} \rangle \\ &= \langle e^{Tk \left(\alpha_0 \rho_0 + \beta_{0,0} J_0 + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} [\alpha_{i_1, \dots, i_g} \rho_{i_1, \dots, i_g} + \beta_{i_1, \dots, i_g} j_{i_1, \dots, i_g} + \beta_{0; i_1, \dots, i_g} J_{i_1, \dots, i_g}] \right)} \rangle \end{aligned} \quad (\text{A26})$$

The tilted dynamics of Eq. 30 reads for the model of Eq. A3 for the root and for the sites of other generations $g \geq 1$ respectively

$$\begin{aligned} \tilde{P}_0(t+1) &= R_0 e^{k(\alpha_0 + \beta_{0;0})} \tilde{P}_0(t) + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} R_{i_1, \dots, i_g} e^{k(\alpha_0 + \beta_{0; i_1, \dots, i_g})} \tilde{P}_{i_1, \dots, i_g}(t) \\ \tilde{P}_{i_1, \dots, i_g}(t+1) &= W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) e^{k(\alpha_{i_1, \dots, i_g} + \beta_{i_1, \dots, i_g})} \tilde{P}_{i_1, \dots, i_{g-1}}(t) \end{aligned} \quad (\text{A27})$$

The eigenvalue Eq. 32 for the right eigenvector $\tilde{r}_0^{[k]}$ reads

$$\begin{aligned} e^{\mu(k)} \tilde{r}_0^{[k]} &= R_0 e^{k(\alpha_0 + \beta_{0;0})} \tilde{r}_0^{[k]} + \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} R_{i_1, \dots, i_g} e^{k(\alpha_0 + \beta_{0; i_1, \dots, i_g})} \tilde{r}_{i_1, \dots, i_g}^{[k]} \\ e^{\mu(k)} \tilde{r}_{i_1, \dots, i_g}^{[k]} &= W_{i_1, \dots, i_g} (1 - R_{i_1, \dots, i_{g-1}}) e^{k(\alpha_{i_1, \dots, i_g} + \beta_{i_1, \dots, i_g})} \tilde{r}_{i_1, \dots, i_{g-1}}^{[k]} \end{aligned} \quad (\text{A28})$$

Plugging the solution for $g \geq 1$

$$\tilde{r}_{i_1, \dots, i_g}^{[k]} = \left[\prod_{g'=1}^g e^{-\mu(k) + k(\alpha_{i_1, \dots, i_{g'}} + \beta_{i_1, \dots, i_{g'}})} W_{i_1, \dots, i_{g'}} (1 - R_{i_1, \dots, i_{g'-1}}) \right] \tilde{r}_0^{[k]} \quad (\text{A29})$$

into Eq. A28 for the root yields the equation that determines the scaled cumulant generating function $\mu(k)$

$$\begin{aligned} 1 &= R_0 e^{-\mu(k) + k(\alpha_0 + \beta_{0;0})} \\ &+ \sum_{g=1}^{+\infty} \sum_{i_1, \dots, i_g} R_{i_1, \dots, i_g} e^{-(g+1)\mu(k) + k(\alpha_0 + \beta_{0; i_1, \dots, i_g})} \left[\prod_{g'=1}^g e^{k(\alpha_{i_1, \dots, i_{g'}} + \beta_{i_1, \dots, i_{g'}})} W_{i_1, \dots, i_{g'}} (1 - R_{i_1, \dots, i_{g'-1}}) \right] \end{aligned} \quad (\text{A30})$$

-
- [1] M. R. Evans, S. N. Majumdar, G. Schehr, J. Phys. A: Math. Theor. 53 193001 (2020).
[2] O. Bénichou, C. Loverdo, M. Moreau, R. Voituriez Reviews of Modern Physics 83, 81 (2011).
[3] R. Cont and P. Tankov, Financial Modelling with Jump Processes, Chapman and Hall / CRC Press, 2004.
[4] P. Tankov and E. Voltchkova, Banque et Marchés, vol. 99, 2009.
[5] G. Dumont, J. Henry, C. O. Tarniceriu, Journal of Mathematical Biology 73, 1413 (2016).
[6] C. E. Miles, J. P. Keener, J. Phys. A: Math. Theor. 50 425003 (2017)
[7] E. Daly and A. Porporato, PRE 74, 041112 (2006).
[8] E. Daly and A. Porporato, PRE 73, 026108 (2006).
[9] E. Daly and A. Porporato, PRE 75, 011119 (2007).
[10] F. Carollo, R. L. Jack, J. P. Garrahan, Phys. Rev. Lett. 122, 130605 (2019).
[11] F. Carollo, J. P. Garrahan, I. Lesanovsky, C. Perez-Espigares, Phys. Rev. A 98, 010103 (2018).
[12] D. C. Rose, H. Touchette, I. Lesanovsky, J. P. Garrahan, Phys. Rev. E 98, 022129 (2018)
[13] S. C. Manrubia and D. H. Zanette, Phys. Rev. E 59, 4945 (1999).
[14] B. Derrida, J. Stat. Mech. P07023 (2007).
[15] R. J. Harris and G. M. Schütz, J. Stat. Mech. P07020 (2007).
[16] E. M. Sevick, R. Prabhakar, S. R. Williams, D. J. Searles, Ann. Rev. of Phys. Chem. Vol 59, 603 (2008).
[17] H. Touchette and R. J. Harris, chapter "Large deviation approach to nonequilibrium systems" of the book "Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond", Wiley 2013.
[18] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim Rev. Mod. Phys. 87, 593 (2015).
[19] R. L. Jack, P. Sollich, The European Physical Journal Special Topics 224, 2351 (2015).
[20] A. Lazarescu, J. Phys. A: Math. Theor. 48 503001 (2015).
[21] A. Lazarescu, J. Phys. A: Math. Theor. 50 254004 (2017).
[22] R. L. Jack, Eur. Phys. J. B 93, 74 (2020)
[23] A. de La Fortelle, PhD (2000) "Contributions to the theory of large deviations and applications" INRIA Rocquencourt.
[24] V. Lecomte, PhD Thesis (2007) "Thermodynamique des histoires et fluctuations hors d'équilibre" Université Paris 7.
[25] R. Chétrite, PhD Thesis 2008 "Grandes déviations et relations de fluctuation dans certains modèles de systèmes hors d'équilibre" ENS Lyon
[26] B. Wynants, arXiv:1011.4210, PhD Thesis (2010), "Structures of Nonequilibrium Fluctuations", Catholic University of Leuven.
[27] R. Chétrite, HDR Thesis (2018) "Pérégrinations sur les phénomènes aléatoires dans la nature", Laboratoire J.A. Dieudonné, Université de Nice.
[28] Y. Oono, Progress of Theoretical Physics Supplement 99, 165 (1989).
[29] R.S. Ellis, Physica D 133, 106 (1999).
[30] H. Touchette, Phys. Rep. 478, 1 (2009).
[31] G. Fayolle and A. de La Fortelle, Problems of Information Transmission 38, 354 (2002).
[32] C. Monthus, Eur. Phys. J. B 92, 149 (2019) in the topical issue "Recent Advances in the Theory of Disordered Systems" edited by F. Igloi and H. Rieger.
[33] A. de La Fortelle, Problems of Information Transmission 37, 120 (2001).
[34] C. Maes and K. Netocny, Europhys. Lett. 82, 30003 (2008)

- [35] C. Maes, K. Netocny and B. Wynants, *Markov Proc. Rel. Fields.* 14, 445 (2008).
- [36] A. C. Barato and R. Chétrite, *J. Stat. Phys.* 160, 1154 (2015).
- [37] L. Bertini, A. Faggionato and D. Gabrielli, *Ann. Inst. Henri Poincare Prob. and Stat.* 51, 867 (2015).
- [38] L. Bertini, A. Faggionato and D. Gabrielli, *Stoch. Process. Appli.* 125, 2786 (2015).
- [39] C. Monthus, *J. Stat. Mech.* (2019) 023206
- [40] C. Monthus, *J. Phys. A: Math. Theor.* 52, 135003 (2019)
- [41] C. Monthus, *J. Phys. A: Math. Theor.* 52, 025001 (2019)
- [42] A. C. Barato, R. Chétrite, *J. Stat. Mech.* (2018) 053207
- [43] L. Chabane, R. Chétrite, G. Verley, *J. Stat. Mech.* (2020) 033208.
- [44] C. Maes, K. Netocny and B. Wynants *Physica A* 387, 2675 (2008).
- [45] J. Hoppenau, D. Nickelsen and A. Engel, *New J. Phys.* 18 083010 (2016).
- [46] C. Monthus, arxiv 2010.14994
- [47] V. Lecomte, C. Appert-Rolland and F. van Wijland, *Phys. Rev. Lett.* 95 010601 (2005).
- [48] V. Lecomte, C. Appert-Rolland and F. van Wijland, *J. Stat. Phys.* 127 51-106 (2007).
- [49] V. Lecomte, C. Appert-Rolland and F. van Wijland, *Comptes Rendus Physique* 8, 609 (2007).
- [50] J.P. Garrahan, R.L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk, F. van Wijland, *Phys. Rev. Lett.* 98, 195702 (2007).
- [51] J.P. Garrahan, R.L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk and F. van Wijland, *J. Phys. A* 42, 075007 (2009).
- [52] K. van Duijvendijk, R.L. Jack and F. van Wijland, *Phys. Rev. E* 81, 011110 (2010).
- [53] R. L. Jack, P. Sollich, *Prog. Theor. Phys. Supp.* 184, 304 (2010)
- [54] D. Simon, *J. Stat. Mech.* (2009) P07017
- [55] V. Popkov, G. M. Schuetz, D. Simon, *J. Stat. Mech.* P10007 (2010).
- [56] D. Simon, *J. Stat. Phys.* 142, 931 (2011)
- [57] V. Popkov, G. M. Schuetz, *J. Stat. Phys* 142, 627 (2011)
- [58] V. Belitsky, G. M. Schuetz, *J. Stat. Phys.* 152, 93 (2013)
- [59] O. Hirschberg, D. Mukamel, G. M. Schuetz, *J. Stat. Mech.* P11023 (2015).
- [60] G. M. Schuetz, *From Particle Systems to Partial Differential Equations II*, Springer Proceedings in Mathematics and Statistics Volume 129, pp 371-393, P. Gonçalves and A.J. Soares (Eds.), (Springer, Cham, 2015).
- [61] R. Chétrite and H. Touchette, *Phys. Rev. Lett.* 111, 120601 (2013).
- [62] R. Chétrite and H. Touchette *Ann. Henri Poincare* 16, 2005 (2015).
- [63] R. Chétrite, H. Touchette, *J. Stat. Mech.* P12001 (2015).
- [64] P. T. Nyawo, H. Touchette, *Phys. Rev. E* 94, 032101 (2016)
- [65] H. Touchette, *Physica A* 504, 5 (2018).
- [66] F. Angeletti, H. Touchette, *Journal of Mathematical Physics* 57, 023303 (2016).
- [67] P. T. Nyawo, H. Touchette, *Europhys. Lett.* 116, 50009 (2016);
P. T. Nyawo, H. Touchette, *Phys. Rev. E* 98, 052103 (2018).
- [68] B. Derrida and T. Sadhu, *Journal of Statistical Physics* 176, 773 (2019);
B. Derrida and T. Sadhu, *Journal of Statistical Physics* 177, 151 (2019).
- [69] K. Proesmans, B. Derrida, *J. Stat. Mech.* (2019) 023201.
- [70] N. Tizon-Escamilla, V. Lecomte and E. Bertin, *J. Stat. Mech.* (2019) 013201.
- [71] J. du Buisson, H. Touchette, *Phys. Rev. E* 102, 012148 (2020)
- [72] J.M. Meylahn, S. Sabhapandit, H. Touchette, *Phys. Rev. E* 92, 062148 (2015).
- [73] R. J. Harris, H. Touchette, *J. Phys. A: Math. Theor.* 50 10LT01 (2017)
- [74] A. Pal, R. Chatterjee, S. Reuveni, A. Kundu, *J. Phys. A: Math. Theor.* 52, 264002 (2019).
- [75] F. den Hollander, S.N. Majumdar, J.M. Meylahn, H. Touchette, *J. Phys. A: Math. Theor.* 52, 175001 (2019).
- [76] F. Coghi, R. J. Harris, *J. Stat. Phys.* 179,131 (2020)
- [77] M. Montero and J. Villarroel, *PRE* 94, 032132 (2016).
- [78] M.R. Evans, S.N. Majumdar, *J. Phys. A: Math. Theor.* 44, 435001 (2011).
- [79] R.G. Pinsky, *Stochastic Processes and their Applications* Volume 130, Issue 5, May 2020, Pages 2954-2973.
- [80] E. Roldan, S. Gupta, *Phys. Rev. E* 96 022130 (2017)
- [81] D. Andrieux and P. Gaspard, *JSTAT* P11007 (2008).
- [82] C. Maes, K.l Netocny and B. Wynants, *J. Phys. A: Math. Theor.* 42 (2009) 365002
- [83] M. Mariani and L. Zambotti, *Adv. Appl. Prob.* 48, 648 (2016).
- [84] A. Faggionato, arXiv:1709.05653
- [85] H. Kesten, *Acta Math.* 131, 208 (1973); H. Kesten et al. , *Compositio Math* 30, 145 (1975).
- [86] B. Derrida and Y. Pomeau, *Phys. Rev. Lett.* 48 , 627 (1982).
- [87] J. P. Bouchaud and A. Georges, *Phys. Rep.* 195, 127 (1990).
- [88] B. Derrida and H. Hilhorst, *J. Phys. A* 16, 2641 (1983).
- [89] C. de Callan, J.M. Luck, Th. Nieuwenhuizen and D. Petritis, *J. Phys. A* 18, 501 (1985).
- [90] F. Igloi and C. Monthus, *Phys. Rep.* 412, 277 (2005).
- [91] C. Monthus, *Phys. Rev. B* 69, 054431 (2004).
- [92] C. Monthus, *J. Stat. Mech.* P06036 (2015).
- [93] C. Monthus, *J. Stat. Mech.* 123304 (2017).