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# VESPo: Verified Evaluation of Secret Polynomials

(with application to dynamic proofs of retrievability)\*

Jean-Guillaume Dumas<sup>†</sup>   Aude Maignan<sup>†</sup>   Clément Pernet<sup>†</sup>   Daniel S. Roche<sup>‡</sup>

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## Abstract

We consider the problem of efficiently evaluating a secret polynomial at a given public point, when the polynomial is stored on an untrusted server. The server performs the evaluation and returns a certificate, and the client can efficiently check that the evaluation is correct using some pre-computed keys. Our protocols support two important features: the polynomial itself can be encrypted on the server, and it can be dynamically updated by changing individual coefficients cheaply without redoing the entire setup. As an important application, we show how these new techniques can be used to instantiate a Dynamic Proof of Retrievability (DPoR) for arbitrary outsourced data storage that achieves both low server storage size and audit complexity. Our methods rely on linearly homomorphic encryption and pairings, and preliminary timing results indicate reasonable performance for polynomials with millions of coefficients, and efficient DPoR with terabytes of databases.

## 1 Introduction

### Verifiable computing.

Verifiable computing, first formalized by [24], consists in delegating the computation of some function to an untrusted server, who must return the result as well as a proof of its correctness. Generally, verifying a result should be much less expensive than computing it directly, and result in a provably low probability that the result is incorrect. While certified and verified computation protocols date back decades, the practical need for efficient methods is especially evident in cloud computing, wherein a low-powered device such as a mobile phone may wish to outsource expensive and critical computations to an untrusted, shared-resource commercial cloud provider.

The extensive literature on verifiable computation protocols can be divided into general-purpose computations — of an arbitrary algebraic circuit — and more limited and (hopefully) efficient special-purpose computations of certain functions. In the latter category, one important problem is Verifiable Polynomial Evaluation (VPE), where a client wishes to outsource the evaluation of a univariate polynomial  $P$  at a given point  $x$  and efficiently verify the result.

### Verifiable Polynomial Evaluation.

A VPE scheme is conventionally composed of three algorithms. First, a client runs  $\text{Setup}(P)$  to compute some public representation of  $P$  (which may be stored on the server) as well as some private information which will be used to verify later evaluations. This step may be somewhat expensive, but only needs to be performed once.

The second algorithm,  $\text{Eval}(x, \alpha)$ , is run by the server using a public evaluation point  $x$ , as well as possibly some additional information  $\alpha$  provided by the client. The server produces the evaluation  $y = P(x)$  as well as some proof or certificate  $\beta$  that this evaluation is correct.

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Finally, the third algorithm,  $\text{Verify}(y, \beta)$ , is run by the client to check the correctness of the evaluation. This verification should be *always correct* and *probabilistically sound*, meaning that an honest server can always produce a result  $y$  and proof  $\beta$  that will pass the verification, whereas an incorrect evaluation  $y$  will always fail the verification with high probability. Furthermore, the  $\text{Verify}$  algorithm should be efficient, ideally much cheaper in time and/or space than the computation itself.

### Additional protocol features.

In the simplest case, the considered polynomial  $P$  is static and stored in cleartext by both the server and the client. But constraints can then be added to this framework.

- **Polynomial outsourcing** When the client device has limited storage, or to facilitate multiple clients, the polynomial and its computation must be externalized. This can always be trivially achieved by storing all client secrets on the server via symmetric encryption and a saved cryptographic hash digest; the challenge is to do so while minimizing the communication costs required for the client to verify an evaluation.
- **Secret polynomial.** To guarantee data privacy, the polynomial could be hidden from the server, or the client, or both. Typically, the polynomial will be stored under a fully- or partially-homomorphic encryption scheme, in such a way that the server can still compute the (necessarily encrypted) evaluation and certificate for verification. This setting has been extensively studied in the literature, with both general-purpose protocols as well as some specific for verified polynomial evaluation.
- **Dynamic updates.** The initial  $\text{Setup}$  protocol requires knowledge of the entire polynomial and generally is much more costly than running  $\text{Verify}$ . This creates a challenge when the client wishes to update only a few of the coefficients of the polynomial. A *dynamic* VPE protocol allows for such updates efficiently. Namely, the client and server storing polynomial  $P$  for verified evaluation can engage in an additional  $\text{Update}(\delta, i)$  protocol, which effectively updates  $P(x)$  to  $P(x) + \delta x^i$  for future evaluations, along with any secret and/or public verification information. To the best of our knowledge, no prior work in the literature discusses dynamic updates for verified polynomial evaluation, which is especially challenging when the polynomial (as well as any update) needs to be hidden from the server. The difficulty is in general to preserve the security properties while allowing those partial updates. The importance of allowing efficient updates is motivated by our application to verifiable data storage, which we explain next.
- **Private/public verification** The verification protocol is said to be *private* when only a party which holds the secrets derived during  $\text{Setup}$  can verify evaluations. That is, any potential verifiers (sometimes called *readers*) must be trusted not to divulge secret information to the untrusted server. Sometimes, it is desirable also to have untrusted verifiers, who can check the result of an evaluation without knowing any secrets. In this *public verification* setting, the client at setup time publishes some additional information, distributed reliably but insecurely to any verifiers, which may be used to check evaluations and proofs issued by the server.

### Proofs of Retrievability.

One important application of VC in general, and VPE in particular, is to *Proofs of Retrievability* (PoR), somewhat overlapping with the problem of *Provable Data Possession* (PDP) [26, 6]. In these settings, a client wishes to store her data on an untrusted server, then verify (without full retrieval) that the server still stores the data intact. The crucial protocol is an  $\text{Audit}$ , wherein the client issues some challenge to the server, then verifies the response using some pre-computed information to prove that the original data is still recoverable in its entirety.

A variety of tools have been employed to develop efficient PoR and PDP protocols, and some of these are based on verifiable computing, so that a PoR audit consists of some verified computation over the stored data. Retrievability is proven when any sequence of successful audits can, with high probability, be used to recover the original data, e.g., by polynomial interpolation; thus any server with a good chance to pass a random audit must hold the entire data intact. Note that this recovery mechanism is not actually crucial except to *prove* the soundness of the audit protocol; the important feature is how cheaply the audits can be performed by a server and resource-constrained client.

## 1.1 Our contributions

Our contributions are the following:

- An (unencrypted) Verifiable Polynomial Evaluation (VPE) scheme with public verification, supporting *secured dynamic updates*, meaning that updating only a few coefficients of  $P$  does not require performing the whole setup again (Section 4 and Table 3). The polynomial is stored in cleartext on the server, and the technique used to provide a correct and sound protocol uses both Merkle trees and pairings. A Horner-like evaluation scheme is used to optimize the evaluation of the difference polynomial for the proof, and no secrets are required to perform the verification.
- A novel *encrypted, dynamic and private* VPE protocol (Section 5 and Table 10). That is, the polynomial is stored encrypted on the server, and efficient updates to individual coefficients can be performed. This is achieved by combining a linearly homomorphic cryptosystem with techniques from the first scheme. Note however, this scheme does not support public verification as this verification now requires some secrets from the client.
- A new Dynamic Proof of Retrievability (DPoR) scheme that is the first to simultaneously support small server storage, dynamic updates, and efficient audits (Section 6 and Table 8), based on our novel encrypted, dynamic VPE protocol. Previous work either had poly-logarithmic audits and linear extra storage, or sub-linear extra storage and polynomial audits; ours is the first to achieve both sub-linear extra storage and optimal  $\mathcal{O}(\log n)$  client time for updates and audits. This could be beneficial especially in blockchain settings such as FileCoin where the proof and verification must be done on-chain.
- Experimental timings based on our encrypted VPE and dynamic PoR protocols that indicate VPE up to millions of coefficients and DPoR up to terabytes of data, both with client cost less than a few milliseconds (Tables 6 and 9).

A complete security definition of verifiable polynomial evaluation can be found in Section 2. This definition follows previous results, with the novel inclusion of an `Update` protocol. Then Section 3 introduces the tools for verification of polynomial evaluation. A motivating example is presented in the form of a direct extension of the bilinear pairing scheme of [28], now supporting an encrypted input polynomial (Section 3 and Table 2). Since the privacy of this protocol is not proven and it does not support neither public verifiability nor dynamic updates, it motivates the more involved contributions of Section 4 (for public verifiability and dynamicity, but on an unciphered polynomial) and of Section 5 (for dynamicity on a ciphered polynomial, but without public verifiability).

The efficiency of our protocols is measured by the computational complexity of the server-side `Eval` algorithm, the volume of persistent client storage, and the amount of communication and client-side complexity to perform a `Verify` or `Audit`. Improving on previously-known results, our protocols all have  $\mathcal{O}(d)$  (parallelizable) server-side computation,  $\mathcal{O}(\log d)$  communication and client-side computation time, and  $\mathcal{O}(1)$  client-side persistent storage. We include some practical timings in Sections 5.2 and 6.3 and Appendix D.

In addition, our new dynamic PoR scheme requires only  $o(d)$  extra server space. This improves on [40] in terms of server storage and on [5] in terms of communication and client computation complexity for `Audit`. For instance on a 1TB size database, with a server extra storage lower than 0.08%, and a client persistent storage less than one KB, our client can check in less than 7ms that their entire outsourced data is fully recoverable from the cloud server.

## 1.2 Related work

While ours is the first work we are aware of which considers verifiable polynomial computation while hiding the polynomial from the server and allowing efficient dynamic updates, there have been a number of prior works on different settings of the VPE problem.

One line of work considers *commitment schemes* for polynomial evaluation [16, 14, 33, 22, 42, 11, 36, 20, 32]. There, the polynomial  $P$  is known to the server, who publishes a binding commitment.

The verifier then confirms that a given evaluation is consistent with the pre-published commitment. By contrast, our protocols aim to *hide the polynomial  $P$  from the server*.

Another line of work considers polynomial evaluation as an encrypted function, which can be evaluated at any chosen point. Function-hiding inner product encryption (IPE) [10, 30, 2] can be used to perform polynomial evaluation without revealing the polynomial  $P$ , but this inherently requires linear-time for the client, who must compute the first  $d$  powers of the desired evaluation point  $x$ . Similarly, protocols using a Private Polynomial Evaluation (PPE) scheme have been developed in [13]. This primitive, based on an ElGamal scheme, ensures that the polynomial is protected and that the user is able to verify the result given by the server. Here the aim of the protocol is not to outsource the polynomial evaluation, but to obtain  $P(x)$  and a proof without knowing anything about the polynomial. To check the proof, as with IPE the client has to produce a computation which is linear in the degree of  $P$ .

A third and more general approach which can be applied to the VPE problem is that of secure evaluation of arithmetic circuits. These protocols make use of fully homomorphic encryption (FHE) to outsource the evaluation of an arbitrary arithmetic circuit without revealing the circuit itself to the server. The VC Scheme of [24] is based on Yao’s label construction.  $P$  is first transformed into an arithmetic circuit. The circuit is garbled once in a setup phase and sent to the server. To later perform a verified evaluation, the client sends an encryption of  $x$ , the server computes  $P(x)$  through the garbled circuit, and the client can verify the result in time proportional to the circuit depth, which for us is  $\mathcal{O}(\log d)$

Using similar techniques, Fiore et al. and Elkhyaoui et al. [8, 19, 17] propose high-degree polynomial evaluations with a fully secure public verification solution. More recently, Fiore et al. [20] propose a new protocol for more general circuits, using SNARKs over a quotient polynomial ring. In contrast to our work, these protocols use more expensive cryptographic primitives, and they do not consider the possibility of efficiently updating the polynomial – while preserving the security properties.

Then, Proof of retrievability (PoR) and Provable data possession (PDP) protocols also have an extensive literature. PDPs generally optimize server storage and efficiency at the cost of soundness; a PDP audit may succeed even when a constant fraction of the data is unrecoverable. PoRs have stronger soundness guarantees, but at the expense of larger and more complicated server storage, often based on erasure codes and/or ORAM techniques.

State-of-the-art PoR protocols either incur a constant-factor blowup in server storage with polylogarithmic audit cost [15, 40], or use negligible extra server storage space but require polynomial-time for audits on the client and server [39, 5]. A lower bound argument from [5] proves that some time/space tradeoff is inherent, although the proof does not distinguish between server and client computation time during audits.

## 2 Security properties and assumptions

A verifiable dynamic polynomial evaluation (VDPE) scheme consists of three algorithms: **Setup**, **Update**, **Eval**, between a client  $\mathcal{C}$  with state  $st_{\mathcal{C}}$ , a server  $\mathcal{S}$  with state  $st_{\mathcal{S}}$  and a verifier  $\mathcal{V}$  with (potentially public) state  $st_{\mathcal{V}}$ .

- $(st_{\mathcal{C}}, st_{\mathcal{V}}, st_{\mathcal{S}}) \leftarrow \text{Setup}(1^\kappa, P)$ : On input of the security parameters and the polynomial  $P$  of degree  $d$ , outputs the client state  $st_{\mathcal{C}}$ , the verifier  $st_{\mathcal{V}}$  and the server state  $st_{\mathcal{S}}$ .
- $\{(st'_{\mathcal{C}}, st'_{\mathcal{V}}, st'_{\mathcal{S}}), \text{reject}\} \leftarrow \text{Update}(i, \delta, st_{\mathcal{C}}, st_{\mathcal{V}}, st_{\mathcal{S}})$ : On input of an index  $i \in 0..d$ , data  $\delta$ , the client/verifier/server states  $st_{\mathcal{C}}/st_{\mathcal{V}}/st_{\mathcal{S}}$ , outputs new client/verifier/server states  $st'_{\mathcal{C}}/st'_{\mathcal{V}}/st'_{\mathcal{S}}$ , representing the polynomial  $P + \delta X^i$ , or **reject**.
- $\{z, \text{reject}\} \leftarrow \text{Eval}(st_{\mathcal{V}}, st_{\mathcal{S}}, r)$ : On input of the verifier state  $st_{\mathcal{C}}$ , the server state  $st_{\mathcal{S}}$  and an evaluation point  $r$ , outputs a successful evaluation  $z = P(r)$  or **reject**.

The client may use random coins for any algorithm. This is the general setting for *public verification*, the idea being that for a *private verification*, the client will play the role of the verifier too and their states will be identical:  $st_{\mathcal{V}} = st_{\mathcal{C}}$ .

Adapted from [28], in order to take into account dynamicity, we propose the following security properties:

**Definition 1.**  $(\text{Setup}, \text{Update}, \text{Eval})$  is a secure publicly verifiable polynomial evaluation scheme if it satisfies the following properties:

**Correctness.** Let  $d \in \mathbb{N}$ ,  $(a_0, \dots, a_d)$  in a ring  $\mathcal{R}$  and  $P(X) = \sum_{i=0}^d a_i X^i$ , then:  $\text{Eval}(\text{Setup}(1^\kappa, P), r) = P(r)$  and for any  $\delta \in \mathcal{R}$  and  $0 \leq i \leq d$ :

$$\text{Eval}(\text{Update}(i, \delta, st_C, st_V, st_S), r) = \text{Eval}(st_V, st_S, r) + \delta r^i$$

or **reject** has been returned by one of the protocols.

**Soundness.** The soundness requirement stipulates that the client can always detect (except with negligible probability) if any message sent by the server deviates from honest behavior<sup>1</sup>. We use the following game between two observers  $\mathcal{O}_1$  &  $\mathcal{O}_2$  (respectively playing the roles of the client and the verifier), a potentially malicious server  $\mathcal{A}$  and an honest server  $\mathcal{S}$ , with the game:

1.  $\mathcal{A}$  chooses an initial polynomial  $P$ .  $\mathcal{O}_1$  runs **Setup** and sends the initial server part,  $st_S$ , of the memory layout to both  $\mathcal{A}$  and  $\mathcal{S}$ ; and the verifier part to  $\mathcal{O}_2$ .
2. For a polynomial number of steps  $t = 1, 2, \dots, \text{poly}(\kappa)$ ,  $\mathcal{A}$  picks an operation  $op_t$  where operation  $op_t$  is either **Update** or **Eval**.  $\mathcal{O}_1$  executes the **Update** operations with both  $\mathcal{A}$  and  $\mathcal{S}$ , while  $\mathcal{O}_2$  executes the **Eval** operations with also both  $\mathcal{A}$  and  $\mathcal{S}$ .
3.  $\mathcal{A}$  is said to win the game, if any message sent by  $\mathcal{A}$  differs from that of  $\mathcal{S}$  and neither  $\mathcal{O}_1$  nor  $\mathcal{O}_2$  did output **reject**.

A VDPE scheme is sound, if no polynomial-time adversary has more than negligible probability in winning the above security game.

**Privacy.** A VDPE scheme is private, if no polynomial-time adversary has more than negligible probability in obtaining any coefficient of  $P$ , given access to the transcript of all exchanged messages for any number of runs of **Setup**, **Update** or **Eval**, and the associated server parts  $st_S$ , of the memory layout.

**Definition 2.**  $(\text{Setup}, \text{Update}, \text{Eval})$  is a secure privately verifiable polynomial evaluation scheme if it verifies the **Correctness**, **Soundness** and **Privacy** requirements of [Definition 1](#), where the verifier state  $st_V$  is included in the client state  $st_C$  and no polynomial-time adversary  $\mathcal{A}$  has more than negligible probability in winning the soundness security game when  $\mathcal{O}_1$  also plays the role of  $\mathcal{O}_2$ .

In [Section 5](#) we apply our new verifiable protocols to the development of a new Dynamic Proof of Retrievability (DPoR) scheme, provably achieving correctness, soundness, and retrievability for DPoR. We follow the exact same security definition for DPoR as in [\[5\]](#), adapted from [\[40\]](#), which we will not restate here for the sake of brevity.

To prove the security of our protocols we rely on classical discrete logarithm and Diffie-Hellman like assumptions, all related to polynomial computations. The first assumption, a decisional one, is the distinct leading monomials assumption: informally it states that polynomial evaluations “in the exponents” where the polynomials have distinct leading monomials are merely indistinguishable from randomness. The formal version is recalled in [Definition 5](#). Then we need computational assumptions, including the hardness to compute discrete logarithms, in [Definition 3](#), and polynomial extensions of the hardness to produce Diffie-Hellman-like secrets even with bilinear pairings, in [Definition 4](#).

**Definition 3** (Discrete Logarithm, DLOG, hardness assumption [\[29, Def. 9.63\]](#)). A discrete-logarithm problem is hard relative a group  $\mathbb{G}$  of group order  $p \geq 2^{2\kappa}$ , a generator  $g$  and a randomly sampled element  $h$  of the group, if for any probabilistic polynomial-time algorithms  $\mathcal{A}$ , there exists a negligible function  $\text{negl}$  such that  $\Pr [\mathcal{A}_{DLOG}(\mathbb{G}, g, h) = x \text{ s.t. } h = g^x] \leq \text{negl}(\kappa)$ .

In the following, We use the notation  $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$  to denote a bilinear pairing in groups of the same prime order. Our constructions can work with any pairing types, 1, 2 or 3. If such a pairing exists then  $\mathbb{G}_1$  and  $\mathbb{G}_2$  are denoted as bilinear groups.

**Definition 4** (t-Bilinear Strong Diffie-Hellman, t-BSDH, assumption, from [\[25, 28\]](#)). Let  $\alpha \in \mathbb{Z}_p^*$ , with  $p \geq 2^{2\kappa}$ , and  $j \in \{1, 2\}$ . Given as input a  $(t+1)$ -tuple  $\langle g_j, g_j^\alpha, g_j^{\alpha^2}, \dots, g_j^{\alpha^t} \rangle \in \mathbb{G}_j^{t+1}$ , in a bilinear group

<sup>1</sup>One might also ask for *knowledge soundness* on the coefficients of the outsourced polynomial, but this is easily achieved for any VDPE scheme by definition: the client can simply interpolate from the evaluations

$\mathbb{G}_j$  of order  $p$  with a bilinear pairing  $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$ , for every adversary  $\mathcal{A}_{t-BSDH}$  and for any value of  $c \in \mathbb{Z}_p \setminus \{-\alpha\}$ , we have the probability:

$$\Pr \left[ \mathcal{A}_{t-BSDH}(g_1, g_2, g_j^\alpha, g_j^{\alpha^2}, \dots, g_j^{\alpha^t}) = \left\langle c, e(g_1; g_2)^{\frac{1}{\alpha+c}} \right\rangle \right] \leq \text{negl}(\kappa)$$

In the following we often use groups of prime order, in order to be able to easily compute with exponents. In particular, thanks to the homomorphic property of exponentiation, we will perform some linear algebra over the group and need some notations for this. For a matrix  $A$ ,  $g^A$  denotes the coefficient-wise exponentiation of a generator  $g$  to each entry in  $A$ . Similarly, for a matrix  $W$  of group elements and a matrix  $B$  of scalars,  $W^B$  denotes the extension of matrix multiplication using the group action. If we have  $W = g^A$ , then  $W^B = (g^A)^B$ . Further, this quantity can actually be computed if needed by working in the exponents first, i.e., it is equal to  $g^{(AB)}$ . For example:

$$\left( g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{\begin{pmatrix} e \\ f \end{pmatrix}} = \begin{pmatrix} g^a & g^b \\ g^c & g^d \end{pmatrix}^{\begin{pmatrix} e \\ f \end{pmatrix}} = \begin{pmatrix} g^{ae+bf} \\ g^{ce+df} \end{pmatrix} = g \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right). \quad (1)$$

For the sake of simplicity, when there is no ambiguity, we also use the associated notation shortcuts like:  $e(g_1^{\begin{pmatrix} a \\ b \end{pmatrix}}; g_2^{\begin{pmatrix} c \\ d \end{pmatrix}}) = e(g_1; g_2)^{\begin{pmatrix} ca \\ db \end{pmatrix}}$ .

Next is the DLM assumption that states that polynomial evaluations “in the exponents” where the polynomials have distinct leading monomials are merely indistinguishable from randomness. In [1] the assumption is given for  $n$ -multivariate polynomials with matrices of dimension  $k \times k$  and projections of dimension  $k \times m$  for  $k \geq 2$  and  $m \geq 1$ . Here We will only use univariate polynomials,  $n = 1$ , and dimensions  $k = 2$ ,  $m = 1$ . We therefore recall the assumption only for this particular case.

**Definition 5** (Distinct Leading Monomial, DLM, assumption [1, Theorem 6]). *Let  $\mathbb{G} = \langle g \rangle$  be a bilinear group of prime order  $p$ . The advantage of an adversary  $\mathcal{A}$  against the  $(2, 1, d)$ -DLM security of  $\mathbb{G}$ , denoted  $\text{Adv}_{\mathbb{G}}^{(2,1,d)\text{-DLM}}(\mathcal{A})$ , is the probability of success in the game defined in Table 1 and is negligible, with  $\mathcal{A}$  being restricted to make queries  $P \in \mathbb{Z}_p[T]$  such that for any challenge  $P$ , the maximum degree in one indeterminate in  $P$  is at most  $d$ , and for any sequence  $(P_1, \dots, P_q)$  of queries, there exists an invertible matrix  $M \in \mathbb{Z}_p^{q \times q}$  such that the leading monomials of  $M \cdot [P_1, \dots, P_q]^T$  are distinct.*

Table 1:  $(2, 1, d)$ -DLM security game for a bilinear group  $\mathbb{G}$  [1]

Init	Challenge( $P$ )	Response( $b'$ )
$r \xleftarrow{\$} \mathbb{Z}_p^{2 \times 2}$	If $b == 0$	
$\beta \xleftarrow{\$} \mathbb{Z}_p^2$	Then Return $y \leftarrow g^{P(r) \cdot \beta}$	Return $b' == b$
$b \xleftarrow{\$} \{0, 1\}$	Else Return $y \xleftarrow{\$} \mathbb{G}^2$	

In fact, the DLM security can also be reduced to the Matrix Diffie-Hellman assumption (MDDH) [1, Theorem 5], a generalization of the widely used decision linear assumption [23, 35, 3, 4, 7].

We will also use a public-key partially homomorphic encryption scheme where both addition and multiplication are considered. We need the following properties on the linearly homomorphic encryption function  $E$  (according to the context, we use  $E_{pk}$  or just  $E$  to denote the encryption function, similarly for the decryption function,  $D$  or  $D_{sk}$ ): computing several modular additions on ciphered messages and modular multiplications but only between a ciphered message and a cleartext.

$$D(E(m_1)E(m_2)) = m_1 + m_2 \quad \text{AND} \quad D(E(m_1)^{m_2}) = m_1 m_2 \quad (2)$$

**Remark 6.** *For instance, Paillier-like cryptosystems [37, 9, 21] can satisfy these requirements, via multiplication in the ground ring, for addition of enciphered messages, and via exponentiation for ciphered multiplication.*

*Note though that an implementation with Paillier cryptosystem of the evaluation  $P(r)$ , in a modular ring  $\mathbb{Z}_m$ , providing the functionalities of Equation (2), requires some care: indeed these equations are usually satisfied modulo an RSA composite number  $N$ , not equal to  $m$ . More precisely, Paillier*

cryptosystem will provide  $D(E(P(r))) \equiv (\sum_{i=0}^d p_i r^i) \pmod{N}$ . Thus a possibility to recover the correct value, is to precompute  $r^i \pmod{m}$  and require that:  $(d+1)(m-1)^2 < N$ . This way one can actually homomorphically compute over  $\mathbb{Z}$  and use the modulo  $m$  only after decryption. See [Appendix C](#) for more details.

Finally, we will use a Merkle hash tree to allow verifications of updates and therefore need to use a cryptographic hash function with *collision resistance*.

Overall, since we consider the semantic security of the cryptosystem, we assume that adversaries are probabilistic polynomial time machines. More precisely we consider *Malicious adversaries*: a corrupted server controls the network and stops, forges or listens to messages in order to gain information or fool the client.

### 3 Tools for the verification of a polynomial evaluation

Our first step is to define a verification protocol for polynomial evaluation that supports a ciphered input polynomial over a finite ring  $\mathbb{Z}_p$ . For this we propose an adaptation and combination of both [28, 19] and thus first need to define a difference polynomial that we will use to check consistency.

**Definition 7.** For a polynomial  $P(X) \in \mathbb{Z}_p[X] = \sum_{i=0}^d p_i X^i$  of degree  $d$ , let its subset polynomials be:  $T_{k,P}(X) = \sum_{i=k+1}^d p_i X^{i-k-1} = \sum_{j=0}^{d-1-k} p_{j+k-1} X^j$ .

**Lemma 8.** Let  $Q_P(Y, X) = \frac{P(Y) - P(X)}{Y - X}$  be the difference polynomial of a polynomial  $P$ ; then:

$$Q_P(Y, X) = \sum_{i=1}^d p_i \sum_{k=0}^{i-1} Y^{i-k-1} X^k = \sum_{k=0}^{d-1} T_{k,P}(Y) X^k \quad (3)$$

*Proof.* As  $Y^i - X^i = (Y - X)(\sum_{k=0}^{i-1} Y^{i-k-1} X^k)$ , we obtain that  $Q_P(Y, X) = \sum_{i=1}^d p_i \sum_{k=0}^{i-1} Y^{i-k-1} X^k$ . This is also  $Q_P(Y, X) = \sum_{k=0}^{d-1} X^k \left( \sum_{i=k+1}^d p_i Y^{i-k-1} \right)$ .  $\square$

This identity relates two evaluations of  $P$ :  $P(Y) = P(X) + (Y - X)Q_P(Y, X)$ . This equation allows one to verify  $z \stackrel{?}{=} P(r)$  by checking, for a secret  $s$ , that:

$$P(s) = z + (s - r)Q_P(s, r) \quad (4)$$

For this, let  $E, D$  be the encryption and decryption functions of a partially homomorphic cryptosystem, supporting addition of two ciphertexts and multiplication of ciphertext by a cleartext, as in [Equation \(2\)](#). Therefore it is possible to evaluate a ciphered polynomial at a clear evaluation point, using powers of the evaluation point: for  $x = [1, r, r^2, \dots, r^d]^\top$ , denote by  $E(P)^\top \square x = \prod_{i=0}^d E(p_i) r^i = E(P(r))$ , the homomorphic polynomial evaluation.

Similarly, if  $H = [h_i] = [g^{a_i}]$ , denote by  $H \odot x = \prod_{i=0}^d h_i^{x_i} = g^{\sum a_i x_i}$  the dot-product in the exponents. Then [Table 2](#) shows how the server produces the evaluation via the partially homomorphic cipher and the subset polynomials. Then this evaluation is bound to be correct by the consistency check in the exponents.

Table 2: Verifiable Ciphered Polynomial Evaluation

	Server	Communications	Client
<b>Setup</b>		$\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ groups of order $p$ pairing $e$ generators $g_1, g_2, g_T = e(g_1; g_2)$	$P \in \mathbb{Z}_p[X], 1 \leq d^\circ(P) \leq d$ $s \xleftarrow{\$} \mathbb{Z}_p, W \leftarrow E(P), \mathcal{K} \leftarrow g_T^{P(s)}$ $H \leftarrow [g_1^{T_{k,P}(s)}]_{k=0..d-1}$
		$\xleftarrow{W, H}$	Discard $P, W, H$
<b>Eval</b>	$x \leftarrow [1, r, r^2, \dots, r^d]^\top$	$\leftarrow r$	$r \xleftarrow{\$} \mathbb{Z}_p$
	$\zeta = W^\top \square x; \xi = H^\top \odot x_{0..d-1}$	$\xleftarrow{\zeta, \xi}$	$e(\xi; g_2^{s-r}) g_T^{D(\zeta)} \stackrel{?}{=} \mathcal{K}$



**Proposition 9** (A proof is given in [Appendix B](#)). *The protocol of [Table 2](#) is correct and sound.*

From this proof, one can see that using a decipherable partially homomorphic function for the coefficients of  $P$  is required for the soundness (otherwise one could not compute the exponentiation on  $\xi/\xi'$ ). Several issues remain with this protocol: first it is not dynamic. Indeed, for a dynamic version, the problem is that updating only one coefficient of  $P$  requires to update up to  $d - 1$  coefficients of  $H$ . This work would be of the same order of magnitude as recomputing the whole setup. Second it is not fully hiding the coefficients of  $P$  as they are just put in the exponents without any masking, and we do not prove the privacy requirement<sup>2</sup>. Third, the protocol is not fully publicly verifiable since the decryption key of the partially homomorphic system is required. We incrementally solve the first two issues in the sequel of this paper and obtain a thus fully secure private protocol. We also are able to provide a dynamic protocol, publicly verifiable, but for an unciphered polynomial. Combining all three properties, that is, designing a publicly verifiable dynamic protocol for ciphered polynomials, preserving a good efficiency while still being secure, remains an open question to us (usually when adapting a static protocol, either dynamicity involves too much recomputation or the security is compromised by the updates).

## 4 Outsourced dynamic verification of the evaluation

In order to be able to deal with updates, we introduce Merkle trees that are updated along with the polynomial parts. Checking the root of the Merkle tree allows for logarithmic verifications and updates of any coefficient of the polynomial. Modifications of the polynomial coefficients are also included in the Client state so that old polynomials cannot be used for the verification of `Eval`.

### 4.1 Merkle trees for logarithmic client storage

Thus, to avoid storing the polynomial coefficients on the client side, we use a Merkle hash tree [[34](#), [31](#), [5](#)]. Then it is sufficient to store the root of the Merkle tree. For our purpose, an implementation of such trees must just provide the following algorithms:

- $T \leftarrow \text{MTTree}(X)$  creates a Merkle hash tree from a database  $X$ .
- $r \leftarrow \text{MTRoot}(X)$  computes from scratch the root of the Merkle hash tree of the whole database  $X$ .
- $(a, L) \leftarrow \text{MTLeafPath}(i, X, T)$  is an algorithm providing the client with the requested leaf element  $a$ , together with the corresponding list  $L$  of Merkle tree uncles.
- $r \leftarrow \text{MTpathRoot}(i, a, L)$  computes the root of the Merkle hash tree from a leaf element  $a$  and the associated path of uncles  $L$ .
- $T' \leftarrow \text{MTupdLeaf}(i, a, T)$  updates the whole Merkle tree  $T$  by changing the  $i$ -th leaf to be  $a$ .

The requirements are thus that:

$$\forall i, X, \text{MTRoot}(X) = \text{MTpathRoot}(i, \text{MTLeafPath}(i, X, \text{MTTree}(X))) \quad (5)$$

$$\begin{aligned} \forall i, a, X, \text{ Let } (b, L) \leftarrow \text{MTLeafPath}(i, X, \text{MTTree}(X)), \\ \text{ and let } X' \leftarrow X \setminus \{(i, b)\} \cup \{(i, a)\}, \\ \text{ then } \text{MTupdLeaf}(i, a, \text{MTTree}(X)) = \text{MTTree}(X') \end{aligned} \quad (6)$$

### 4.2 Public Dynamic unciphered Polynomial Evaluation

Thanks to these additional Merkle-tree operations, we can now give a protocol for the public verification of the evaluation of a dynamic polynomial  $P$ . It consists in three algorithms (`Setup`, `Update`, `Eval`) detailed in [Table 3](#) and it requires, for now, a *symmetric* pairing.

<sup>2</sup>Efficient updates in similar schemes are considered, e.g., in [[42](#)] but to a protocol that verifies coefficients known to the server, *not* its evaluation at hidden coefficients

Table 3: Public and Dynamic unciphered polynomial evaluation

	Server	Communications	Client
<b>Setup</b>	$T_P \leftarrow \text{MTTree}(P)$ Store $P, T_P, S$	$\mathbb{G}, \mathbb{G}_T$ of order $p$ symm. pairing $e$ gen. $g, e(g; g)$ $\xleftrightarrow{P, S}$	$P \in \mathbb{Z}_p[X], 0 \leq d^\circ(P) \leq d$ Let $s \xleftarrow{\$} \mathbb{Z}_p$ , $\mathcal{K}_1 \leftarrow e(g^{P(s)}; g), S \leftarrow [g^{s^k}]_{k=0..d-1}$ $r_p \leftarrow \text{MTRoot}(P)$ Publish $\mathcal{K}_1, \mathcal{K}_2 \leftarrow g^s$ ; discard $P, S$ .
<b>Update</b>	$(p_i, L_i) \leftarrow \text{MTLeafPath}(i, P, T_P)$ $T_P \leftarrow \text{MTupdLeaf}(i, p_i + \delta, T_P)$	$\xleftrightarrow{i, \delta}$ $\xrightarrow{p_i, L_i}$	$r_P \stackrel{?}{=} \text{MTpathRoot}(i, p_i, L_i)$ $r_P \leftarrow \text{MTpathRoot}(i, p_i + \delta, L_i)$ $\mathcal{K}_1 \leftarrow \mathcal{K}_1 \cdot e(g^{s^i \delta}; g)$ ; publish $\mathcal{K}_1$
<b>Eval</b>	Form $x \leftarrow [1, r, r^2, \dots, r^d]^\top$ $\zeta \leftarrow P(r); \xi \leftarrow \prod_{i=1}^d \prod_{k=0}^{i-1} S_{i-k-1}^{p_i x_k}$	$\xleftarrow{r}$ $\xrightarrow{\zeta, \xi}$	$r \xleftarrow{\$} \mathbb{Z}_p$ $e(\xi; \mathcal{K}_2 / g^r) e(g^\zeta; g) \stackrel{?}{=} \mathcal{K}_1$

During the **Setup** algorithm, the Client sends the unciphered polynomial to the Server and deletes it to minimize its storage. The Client uses a random coin  $s$  to create some data to be published or to be sent to the server. We introduce a third part named the Verifier. The Verifier collects the published data and is authorized to run the **Read** and the **Eval** algorithms. But she is not authorized to run the **Setup** algorithm and  $s$  is not known by the Verifier.

**Proposition 10** (A proof is given in [Appendix B](#)). *The protocol of [Table 3](#) is correct and sound.*

One difficulty is to preserve a linear-time Server. We show next that this is indeed possible here.

### 4.3 Efficient linear-time evaluation

As a first approach to evaluate our protocols, we consider that the cardinality of the coefficient domain is a constant. Therefore, we count as arithmetic operations in the field not only the usual addition, subtraction, multiplication and inversion, but also the exponentiations that are independent of the degree of the polynomial. We thus express our asymptotic complexity bounds in [Table 4](#), only with respect to that degree  $d$ .

Table 4: Complexity bounds for the publicly verifiable dynamic and unciphered polynomial evaluation of [Table 3](#) for a degree  $d$  polynomial.

		Server	Communication	Client
Storage		$\mathcal{O}(d)$		$\mathcal{O}(1)$
Comput.	<b>Setup</b>	$\mathcal{O}(d)$	$\mathcal{O}(d)$	$\mathcal{O}(d)$
	<b>Update</b>	$\mathcal{O}(\log(d))$	$\mathcal{O}(\log(d))$	$\mathcal{O}(\log(d))$
	<b>Eval</b>	$\mathcal{O}(d)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$

**Proposition 11** (A proof is given in [Appendix B](#)). *In [Table 3](#), the setup protocol requires  $\mathcal{O}(d)$  arithmetic operations; the update protocol requires  $\mathcal{O}(\log(d))$  arithmetic operations; the verification protocol requires  $\mathcal{O}(1)$  communications and arithmetic operations for the Client, and  $\mathcal{O}(d)$  arithmetic operations for the Server.*

In the next Section, we then propose a novel fully private protocol, combining and formalizing the ideas from the encrypted one and the dynamic one.

## 5 Fully private, dynamic and ciphered protocol for polynomial evaluation

So far we have a polynomial evaluation verification, that allows efficient updates of its coefficients. We now propose a scheme which combine the polynomial evaluation with the externalization of the polynomial itself. For this, two more ingredients are added in [Section 5.1](#): an efficient masking in the exponents in order to fulfill the hiding security property and an outsourcing of the (ciphered) polynomial itself. This latter feature allows the client to not even store the polynomial and reduces her need for storage to a small constant number of field elements. For this we use Merkle hash trees presented in [Section 4.1](#). They ensure the authenticity of the coefficient updates, with the storage of only one hash. Finally note that the bilinear pairing need not be symmetric anymore, but need to be applied twice for the security hypothesis to hold.

### 5.1 Private, dynamic, ciphered protocol

Here we add a masking of the polynomial coefficients in order to make the protocol hiding. For this we use the security hypothesis of [Definition 5](#): indeed, DLM security states that in a group  $\mathbb{G}$  of prime order, the values  $(g^{P_1(A)\beta}, \dots, g^{P_d(A)\beta})$  are indistinguishable from a random tuple of the same size, when  $P_1, \dots, P_d$  have distinct leading monomials of bounded degree and  $A$  and  $\beta$  are the  $2 \times 2$  and  $2 \times 1$  secrets. Therefore, in our modified protocol, the coefficients  $g^{\Phi^i \beta}$  for a secret  $2 \times 2$  matrix  $\Phi$ , are indistinguishable from a random tuple  $(g^{\Gamma^i})$  since the polynomials  $X^i$ ,  $i = 1..d$  are just distinct monomials.

We start this section with linear algebra tools and an overview and then give a full formalization and the associated proofs of security. We end the section with experiments showing the efficiency of our approach.

#### 5.1.1 Linear algebra toolbox.

For the next protocol to hold, we need to adapt the difference polynomial to the matrix case. For instance [Lemma 8](#) holds in the matrix case provided that the, now matrices,  $Y$  and  $X$  commute and that  $Y - X$  is invertible. Let  $I_n$  be the  $n \times n$  identity matrix. Then, we will for instance use  $Y = sI_2$  and  $X = rI_2$  with  $s \neq r$ .

Also to speed-up things with the DLM masks, we need to efficiently compute geometric sums of matrices. Thanks to Fiduccia's algorithm [18], this is easily done with a number of operations logarithmic in the exponent, provided that one is not an eigenvalue of the matrix. Indeed, first, any matrix commutes with the identity so the geometric sum can be computed via one matrix exponentiation, one matrix inverse and one matrix multiplication:  $\sum_{i=0}^d A^i = (A^{d+1} - I_n)(A - I_n)^{-1}$ . Then, second, Fiduccia's algorithm computes the exponentiation modulo the characteristic polynomial, using the square and multiply fast recursive algorithm. This summarized in [Algorithms 1](#) and [2](#) and analyzed in [Appendix B](#).

---

#### Algorithm 1 Degree 2 modular monomial powers (2-MMP)

---

**Input:**  $d \in \mathbb{Z}$ ,  $d \geq 1$ ,  $P = p_0 + p_1Z + Z^2 \in \mathbb{Z}_p[Z]$  monic degree 2 polynomial.

**Output:**  $Z^d \bmod P$ .

```

1: if  $d == 1$  then return  $Z$ 
2:  $T \leftarrow$  2-MMP( $\lfloor d/2 \rfloor, P$ );
3:  $S \leftarrow (t_0^2 - t_1^2 p_0) + (2t_0 t_1 - t_1^2 p_1)Z$ ; { $T(Z)^2$  modulo  $P(Z)$ }
4: if  $d$  is odd then
5:   return  $(-s_1 p_0) + (s_0 - s_1 p_1)Z$ ; { $Z \cdot S(Z)$  modulo  $P(Z)$ }
6: else
7:   return  $S$ .
8: end if

```

---

#### 5.1.2 Formalization of the protocol.

The dynamic externalized polynomial evaluation scheme consist of the following algorithms **Setup**, **Update** and **Eval** between a client  $\mathcal{C}$  with state  $st_{\mathcal{C}}$  and the server  $\mathcal{S}$  of state  $st_{\mathcal{S}}$ . The exchanges

---

**Algorithm 2** Projected matrix geometric sum (PMGS)

---

**Input:**  $k \in \mathbb{Z}$ ,  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^{2 \times 2}$ , s.t.  $A - I_2$  is invertible,  $\beta \in \mathbb{Z}_p^2$ .

**Output:**  $\sum_{i=0}^k A^i \beta$ .

- 1: Let  $\pi(Z) = (ad - bc) - (a + d)Z + Z^2$ ; {The characteristic polynomial of  $A$ }
  - 2: Let  $F(Z) = f_0 + f_1 Z = 2\text{-MMP}(k + 1, \pi)$ ; { $Z^{k+1} \bmod \pi(Z)$ , using [Algorithm 1](#)}
  - 3: **return**  $(f_1 A + (f_0 - 1)I_2)(A - I_2)^{-1} \beta$ . { $(A^{k+1} - I_2)(A - I_2)^{-1} \beta$ }
- 

are summarized in [Table 10](#).

- $(st_C, st_S) \leftarrow \text{Setup}(1^\kappa, P)$ : on input of the security parameters and the polynomial  $P$ , outputs the client state  $st_C$  and the server state  $st_S$ , as detailed in [Algorithm 3](#).
- $\{(st'_C, st'_S), \text{reject}\} \leftarrow \text{Update}(i, \delta, st_C, st_S)$ : on input of an index  $i \in 0..d$ , the difference data  $\delta$ , the client state  $st_C$  and the server state  $st_S$ , outputs a new client state  $st'_C$  and a new server state  $st'_S$  (such that now the new  $i$ -th coefficient of the polynomial is  $P'_i = P_i + \delta$ , for  $P_i$  the previous  $i$ -th coefficient), or **reject**, as detailed in [Algorithm 4](#).
- $\{z, \text{reject}\} \leftarrow \text{Eval}(st_C, st_S, r)$ : on input of the client state  $st_C$ , the server state  $st_S$  and an evaluation point  $r$ , outputs a successful evaluation  $z = P(r)$  or **reject**, as detailed in [Algorithm 5](#).

---

**Algorithm 3** Setup( $1^\kappa, P$ )

---

**Input:**  $1^\kappa$ ;  $p \in \mathbb{P}$ ,  $P = \sum_{i=0}^d p_i X^i \in \mathbb{Z}_p[X]$ ;

**Input:** a partially homomorphic cryptosystem  $E/D$  satisfying [Equation \(2\)](#), for any dot-product of size  $d + 1$ , modulo  $p$ .

**Output:**  $st_S, st_C$ .

- 1: Client: generates order  $p$  groups  $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$  with non-degenerate pairing  $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$  and generators  $g_1, g_2, g_T = e(g_1; g_2)$ ;
  - 2: Client: generates a public/private key pair  $(pk, sk)$  for  $E/D$ ;
  - 3: Client: randomly selects  $s \xleftarrow{\$} \mathbb{Z}_p \setminus \{0, 1\}$ ,  $\alpha, \beta \xleftarrow{\$} \mathbb{Z}_p^2$ ,  $\Phi \xleftarrow{\$} \mathbb{Z}_p^{2 \times 2}$ , s. t.  $s\Phi - I_2$  is invertible;
  - 4: Client: computes  $\bar{P}(X) = \sum_{i=0}^d X^i (p_i \alpha + \Phi^i \beta)$ ,  $W = E_{pk}(P)$ ,  $\bar{H} = [g_1^{\bar{p}_i}]_{i=1..d} \in \mathbb{G}_1^{2 \times d}$ ,  $\bar{K} = g_T^{\bar{P}(s)} \in \mathbb{G}_T^2$  and  $S = [g_2^{\bar{s}_k}]_{k=0..d-1} \in \mathbb{G}_2^d$ ;
  - 5: Client:  $r_W = \text{MTRoot}(W)$ ; {root of the Merkle tree}
  - 6: Client: sends  $pk, \mathbb{G}_1, \mathbb{G}_2, g_1, g_2, \mathbb{G}_T, e, W, \bar{H}, S$  to the Server;
  - 7: Client: **return**  $st_C \leftarrow (pk, sk, \mathbb{G}_1, \mathbb{G}_2, g_1, g_2, \mathbb{G}_T, e, s, \alpha, \beta, \Phi, \bar{K}, r_W)$ ;
  - 8: Server:  $T_W \leftarrow \text{MTTree}(W)$ ; {the Merkle tree}
  - 9: Server: **return**  $st_S \leftarrow (pk, \mathbb{G}_1, \mathbb{G}_2, g_1, g_2, \mathbb{G}_T, e, W, T_W, \bar{H}, S)$ .
- 

We have now the complete result for the Dynamic Verified Evaluation of Secret Polynomials, summarized in [Appendix A](#) (a parallelization of the server part of [Algorithm 5](#) is given in [Appendix D](#)).

**Theorem 12.** *Under the security assumptions of [Section 2](#), the protocol composed of [Algorithms 3 to 5](#) (summarized in [Table 10](#)) is a fully secure verifiable polynomial evaluation scheme, as defined in [Definition 1](#) and the complexity bounds of its algorithms are given in [Table 5](#).*

For the complexity bounds we still consider the cardinality of the coefficient domain to be a constant (so that, again, even exponentiations not involving the degree are considered constant) and we also consider that one encryption/decryption with the linearly homomorphic cryptosystem requires a number of arithmetic operations constant with respect to the degree.

*Proof. Correctness.* We use the left hand side of [Lemma 8](#) and [Equation \(4\)](#). Applying this to  $\bar{P}$ , we directly obtain that:  $\bar{\xi} = \prod_{i=1}^d \prod_{k=0}^{i-1} e(\bar{H}_i; S_{i-k-1})^{x^k} = \prod_{i=1}^d \prod_{k=0}^{i-1} e(g_1^{\bar{p}_i}; g_2^{s^{i-k-1}}) r^k$  so that  $\bar{\xi} = e(g_1; g_2)^{Q_{\bar{P}}(s \cdot I_2, r \cdot I_2)}$ . Denote by  $G(Z) = \frac{Z^{d+1}-1}{Z-1}$ . Now  $\bar{P}(X) = P(X)\alpha + G(X\Phi)\beta$ , then  $c = G(r\Phi)\beta = G(r \cdot I_2\Phi)\beta$  and thus  $\bar{P}(r \cdot I_2) = D(\zeta)\alpha + c = P(r)\alpha + c$ . Therefore the **Eval** verification

---

**Algorithm 4** Update( $i, \delta, st_C, st_S$ )

---

**Input:**  $i \in [0..d]$ ,  $\delta \in \mathbb{Z}_p^*$ ,  $(st_C, st_S) = \text{Setup}(1^\kappa, P)$ .**Output:**  $(st'_C, st'_S) = \text{Setup}(1^\kappa, P + \delta X^i)$  or **reject**.

- 1: Client: computes  $e_\delta = E_{pk}(\delta)$ ,  $\Delta = g_1^{\delta\alpha}$ ;
  - 2: Client: sends  $i, e_\delta, \Delta$  to the Server;
  - 3: Server:  $(w_i, L_i) \leftarrow \text{MTLeafPath}(i, W, T_W)$ ; {gets  $w_i$  and its uncles from the tree}
  - 4: Server:  $T'_W \leftarrow \text{MTupLeaf}(i, w_i \cdot e_\delta, T_W)$ ; {updates the Merkle tree}
  - 5: Server: sends  $w_i, L_i$  to the Client;
  - 6: Server: **return**  $st'_S \leftarrow st_S \setminus \{T_W, w_i, \bar{h}_i\} \cup \{T'_W, w_i \cdot e_\delta\} \cup_{j=1}^2 \{\bar{h}_i[j] \cdot \Delta[j]\}$ ;
  - 7: **if**  $r_W = \text{MTPathRoot}(i, w_i, L_i)$  **then**
  - 8:   Client: computes  $\bar{\mathcal{K}}'[j] \leftarrow e(\Delta[j]^{s^i}; g_2) \cdot \bar{\mathcal{K}}[j]$  for  $j = 1..2$ ;
  - 9:   Client: computes  $r'_W = \text{MTPathRoot}(i, w_i \cdot e_\delta, L_i)$ ;
  - 10:   Client: **return**  $st'_C \leftarrow st_C \setminus \{\bar{\mathcal{K}}, r_W\} \cup \{\bar{\mathcal{K}}', r'_W\}$ .
  - 11: **else** {the stored root does not match the received element and uncles}
  - 12:   Client: **return reject**.
  - 13: **end if**
- 

---

**Algorithm 5** Eval( $st_C, st_S, r$ )

---

**Input:**  $st_C, st_S$  and  $r \in \mathbb{Z}_p$ ;**Output:**  $z = P(r)$  or **reject**.

- 1: Client: computes  $r\Phi$  and  $c \leftarrow ((r\Phi)^{d+1} - I_2) \cdot (r\Phi - I_2)^{-1} \cdot \beta$  {via Algorithm 2}
  - 2: Client: sends  $r$  to the Server;
  - 3: Server: homomorphically computes  $\zeta = W^\top \square x = \prod_{i=0}^d w_i^{(r^i \bmod p)}$   
{via Equation (2), see also, e.g., Remark 6 and Algorithm 7}
  - 4: Server:  $\bar{\xi} = [1_{\mathbb{G}_T}, 1_{\mathbb{G}_T}]^\top \in \mathbb{G}_T^2$ ;  $t = 1_{\mathbb{G}_2}$ ;
  - 5: **for**  $i = 1$  **to**  $d$  **do** {Following the ideas of Algorithm 6}
  - 6:   Server:  $t \leftarrow S_{i-1} \cdot t^r$ ;
  - 7:   Server:  $\bar{\xi}[j] \leftarrow \bar{\xi}[j] \cdot e(\bar{H}_i[j]; t)$  for  $j = 1..2$ ;
  - 8: **end for**
  - 9: Server: sends  $\zeta, \bar{\xi}$  to the Client;
  - 10: Client: computes  $z = D_{sk}(\zeta) \bmod p$ ;
  - 11: **if**  $\bar{\xi}[j]^{s-r} g_T^{z\alpha[j]+c[j]} = \bar{\mathcal{K}}[j]$  for  $j = 1..2$  **then**
  - 12:   Client: **return**  $z$ .
  - 13: **else**
  - 14:   Client: **return reject**.
  - 15: **end if**
- 

Table 5: Complexity bounds for verifiable dynamic and ciphered polynomial evaluation (for groups and prime fields of supposed constant order/cardinality, the asymptotics are here function of the degree  $d$  of the evaluated polynomial: storage units are given in number of group/field elements, computational operations are given in number of group/prime field arithmetic operations).

		Server	Communication	Client
Storage		$\mathcal{O}(d)$		$\mathcal{O}(1)$
Comput.	Setup	$\mathcal{O}(d)$	$\mathcal{O}(d)$	$\mathcal{O}(d)$
	Update	$\mathcal{O}(\log(d))$	$\mathcal{O}(\log(d))$	$\mathcal{O}(\log(d))$
	Eval	$\mathcal{O}(d)$	$\mathcal{O}(1)$	$\mathcal{O}(\log(d))$

is indeed that  $g_T^{Q_{\bar{P}(s \cdot I_2, r \cdot I_2)}(s-r) + \bar{P}(r \cdot I_2)} \stackrel{?}{=} g_T^{\bar{P}(s \cdot I_2)} = g_T^{\bar{P}(s)}$ . Now for the **Update** operation,  $P'_i = P_i + \delta$  so that  $\bar{P}'(s) = \bar{P}'(s \cdot I_2) = \delta s^i \alpha + \bar{P}(s \cdot I_2)$  and  $e(g_1^{\bar{P}'(s \cdot I_2)[j]}; g_2) = e(g_1^{s^i \delta \alpha[j]}; g_2) e(g_1^{\bar{P}(s \cdot I_2)[j]}; g_2) = e(\Delta[j]^{s^i}; g_2) \cdot g_T^{\bar{P}(s)[j]} = e(\Delta[j]^{s^i}; g_2) \cdot \bar{\mathcal{K}}[j]$  for  $j = 1..2$ .

**Complexity bounds.** In terms of storage, apart from the public/private key pair and the groups,

the client just has to store nine elements mod  $p$ , that is  $s$ ,  $\alpha \neq [0, 0]$ ,  $\beta$ , and  $\Phi$ , together with two group elements,  $\bar{\mathcal{K}}$ ; the server has to store the polynomial ciphered thrice, the ciphered powers of  $s$  and the Merkle tree for the ciphered polynomial: all this is  $O(d)$ . In terms of communications, during the **Update** phase the client sends one index and three group elements, while receiving one group element and the list of its  $\log(d)$  uncles. During the **Eval** phase, only four elements are exchanged. Finally, in terms of computations, the server performs  $O(d)$  operations for the Merkle tree generation at **Setup**; fetches  $O(\log(d))$  uncles at **Update**; and  $O(d)$  (homomorphic) operations at **Eval**, thanks to [Algorithm 6](#). For the client, **Update** requires  $O(\log(d))$  arithmetic operations to check the uncles and to compute the exponentiation  $s^i$ , together with a constant number of other arithmetic operations, independent of the degree. Similarly, computing  $(r\Phi)^{d+1}$  also requires  $O(\log(d))$  classical arithmetic operations thanks to [Algorithm 2](#) and the rest is a constant number of operations that are independent of the degree.

**Soundness.** Let  $\langle g_2, g_2^s, g_2^{s^2}, \dots, g_2^{s^t} \rangle \in \mathbb{G}_2^{t+1}$  be a t-BSDH instance. For the setup phase, randomly select  $\alpha, \beta, \Phi$  and  $[p_0, \dots, p_t]$ . Then compute  $W = E(P)$ ,  $\bar{H} = g^{\bar{P}}$ , and let  $S = \langle \mathbb{G}, g, g^s, g^{s^2}, \dots, g^{s^t} \rangle$ . Finally homomorphically compute:

$$\bar{\mathcal{K}} = e \left( g_1; \langle g_2, g_2^s, g_2^{s^2}, \dots, g_2^{s^t} \rangle \odot [\bar{p}_0, \dots, \bar{p}_t] \right).$$

These inputs are indistinguishable from random inputs to the protocol of [Table 10](#). For any number of update phases, randomly select  $\delta$  and compute  $e_\delta = E(\delta)$  and  $\Delta = g_1^{\delta\alpha}$ . Also compute  $\mathcal{K}' = e(g_1; S_i^{\delta\alpha}) \cdot \mathcal{K}$ . Finally, select a random evaluation point  $r$ , compute  $(\zeta, \xi)$  and call an attacker of the **Eval** part of the protocol to get  $(\zeta', \xi')$  such that  $(D(\zeta'), \xi') \neq (D(\zeta), \xi)$ , even though both are passing the verification. This means, again, that if, on the one hand,  $D(\zeta') = D(\zeta)$ , then  $\xi^{(s-r)} = \xi'^{(s-r)}$  with  $\xi \neq \xi'$ . Therefore  $s = r$  and the secret is exposed. If, on the other hand,  $D(\zeta') \neq D(\zeta)$  then, as  $\alpha \neq [0, 0]$ , set  $j \in \{1, 2\}$  such that  $\alpha[j] \neq 0$  and we have again:  $\left( \frac{\xi[j]}{\xi'[j]} \right)^{\frac{1}{\alpha[j](D(\zeta') - D(\zeta))}} = e(g_1; g_2)^{\frac{1}{s-r}}$ . This proves that the adversary would solve the t-BSDH  $\langle -r, e(g_1; g_2)^{\frac{1}{s-r}} \rangle$  challenge.

**Privacy.** We show that the protocol is hiding both  $p_i$  and  $\bar{p}_i$ .

For  $\bar{p}_i$  first. Let  $B = g_1^b$  be a DLOG instance. For the setup phase, randomly select  $s, \alpha, \Phi, d, [p_0, \dots, p_d]$  and two non-zero elements  $b_1, b_2 \in \mathbb{Z}_p^*$ . Then compute  $W = E(P)$ ,  $\bar{h}_i = g_1^{\alpha p_i} B^{\Phi^i [b_1, b_2]^T}$ ,  $S = \langle g_2, g_2^s, g_2^{s^2}, \dots, g_2^{s^t} \rangle$ , and  $\bar{\mathcal{K}} = e(g_1^{\alpha P(s)} B^{G(s\Phi)[b_1, b_2]^T}; g_2)$ . These inputs are indistinguishable from random inputs to the protocol of [Table 10](#). For any update phase, randomly select  $\delta$  and compute  $e_\delta = E(\delta)$  and  $\Delta = g_1^{\delta\alpha}$ . Also compute  $\bar{\mathcal{K}}'[j] = e(\Delta[j]^{s^i}; g_2) \cdot \bar{\mathcal{K}}[j]$  for  $j = 1..2$ . Such updates are indistinguishable from random updates to the protocol of [Table 10](#). Randomly select any number of evaluation points  $r$  and run the associated **Eval** phases, randomly alternated with **Update** phases. Now, if an attacker can find from this transcript one coefficient  $\bar{p}_i[j]$  for  $j \in \{1, 2\}$ , then compute  $b = (\bar{p}_i[j] - p_i \alpha[j]) / (\Phi^i [b_1, b_2]^T)[j]$  and the DLOG is revealed.

For  $p_i$ , we proceed with a sequence of two indistinguishable games. Under DLM security, cf. [Definition 5](#), the parameter  $\bar{h}_i$ , or more precisely, the pair  $(E(p_i), g^{p_i \alpha + \Phi^i \beta})$ , is indistinguishable from  $(E(p_i), g^{p_i \alpha + \Gamma_i})$  for some random 2-dimensional vectors  $\Gamma_i$ . Therefore the protocol of [Table 10](#) is indistinguishable, as a whole, from the same protocol where  $\Phi^i \beta$  is everywhere replaced by  $\Gamma_i$ , and  $c$  is (now inefficiently) computed as  $\sum r^i \Gamma_i$ . Now we prove that the latter is hiding. Let  $Z = E(\omega)$  be the cipher of a secret  $\omega$ . Randomly select  $d$  and  $[u_0, \dots, u_d] \xleftarrow{\$} \mathbb{Z}_p^{d+1}$ . Compute  $W_i = Z \cdot E(u_i) = E(\omega + u_i)$ . Randomly select  $\alpha$  and  $h_i$  (so that  $\Gamma_i = \log_{g_1}(h_i) - (\omega + u_i)\alpha \in \mathbb{Z}_p^2$  exists, but remains unknown) for  $i = 1..d$ . Randomly select  $s$  and compute  $\bar{\mathcal{K}} = e(H \odot [1, s, \dots, s^d]; g_2)$ . For any number of updates, randomly select  $\delta$ , compute  $e_\delta = E(\delta)$ , so that  $\delta = p'_i - p_i = (\omega + u'_i) - (\omega + u_i) = u'_i - u_i$ . Thus update  $u'_i \leftarrow \delta + u_i$  and, therefore, compute  $\Delta = g_1^{\delta\alpha}$  and  $\bar{\mathcal{K}}'[j] = e(\Delta[j]^{s^i}; g_2) \cdot \bar{\mathcal{K}}[j]$  for  $j = 1..2$ . Alternatively run such updates with random **Eval** phases; all this is indistinguishable from a normal transcript of the protocol. Now if from this transcript an attacker could find one  $p_j$ , then compute  $\omega = p_j - u_j$  and the encrypted value would be revealed.  $\square$

## 5.2 Experiments

To assess the efficiency of our protocol, we implemented Table 10 using the following libraries<sup>3</sup>: `gmp-6.2.1` for modular operations, `fflas-ffpack-2.4.3` for linear algebra, `libsark` for baseline polynomial evaluation verification, `relic-0.6.0` for Paillier’s cryptosystem and pairings (we used a “bn-p254” pairing).

To observe the effect of the chosen homomorphic systems (Paillier and the pairing), we ran the experiments, on a single core of an intel Gold 6126 2.6GHz for the Client and Horner computations and on one or four cores for the Server, with a RSA modulus size of 2048 bits.

In Table 6, we thus compare the Server time to the Client time of our protocol, to that of a simple (witness) polynomial evaluation (Horner-like) in this group and of an unciphered static polynomial evaluation with a SNARK (a ciphered evaluation with these SNARK would require to arithmetize the Homomorphic cryptosystem and seems still out of reach).

Table 6: Comparative behaviors of pairings and Paillier system on the Server and Client sides with a 254-bits group size for the protocol of Table 10 (on the client side, column ‘pows’ is the time to perform the left hand-side exponentiations (by  $s - r$  and by  $D(\zeta)\alpha[j] + c[j]$ ); column ‘c’ is the time to perform the matrix geometric sum (the clients’s only part non constant in  $d$ ); and column ‘D’ is the time to perform the single 2048 bits Paillier’s deciphering; below are some baseline comparisons: ‘Horner’ is a witness simple polynomial evaluation in that group, ‘libsark’ is an unciphered and static polynomial evaluation verification. Each experiment was performed 11 times and we report the median value, with a maximum variance lower than 16.4% between runs).

Degree	Server Certif.				Client Verif.		
	$\zeta$	$\xi$	$\zeta$	$\xi$	pows	c	D
	1 core		4 cores		1 core		
256	0.12s	0.09s	0.04s	0.03s	0.7ms	<0.1ms	0.9ms
512	0.24s	0.17s	0.07s	0.06s			
1024	0.46s	0.34s	0.14s	0.12s			
2048	0.93s	0.68s	0.25s	0.23s			
4096	1.81s	1.38s	0.49s	0.45s			
8192	3.68s	2.75s	1.00s	0.90s			
16384	7.17s	5.46s	1.96s	1.78s			
32768	14.48s	10.97s	3.97s	3.26s			
65536	29.44s	22.12s	7.83s	6.38s			
131072	59.13s	43.94s	15.51s	12.92s			

  

	Client 1 core	Server (1 core)					Proof size
		$d^o$	256	1024	8192	131072	
Horner (no verif., no crypt.)		<0.1ms	0.2ms	1.6ms	32.0ms		-
libsark (no crypt.)	3.8ms	0.06s	0.20s	1.32s	18.90s		287B
Here (v. & c. & dyn.)	1.6ms	0.21s	0.80s	6.43s	103.07s		320B

First of all, of course, the Server time, using homomorphic arithmetic, can be several orders of magnitude slower than the simple polynomial evaluation, while indeed being clearly linear. Second, for the protocol itself, we see that both homomorphic evaluations of the Server are quite similar, even if the Paillier cryptosystem is more expensive for large modulus. Then, on the Client side and for the considered degrees, the dominant computation is that of a single Paillier’s deciphering (and that the only part non-constant in the degree is by far the most negligible). Third our Client is even faster than an unciphered one (we use less pairings than `libsark`) and for a large enough degree, we can observe the logarithmic Client time to win over the linear time pure polynomial evaluation. Also, our ciphered server slowdown remains within a factor close to four (or only two without Paillier) when compared to the static and unciphered one.

<sup>3</sup><https://gmplib.org>, <https://linbox-team.github.io/fflas-ffpack>, <https://github.com/scipr-lab/libsark>.  
[git, https://github.com/relic-toolkit/relic](https://github.com/relic-toolkit/relic).

## 6 Low server storage dynamic proof of retrievability

Recall that Proofs of Retrievability (PoR) allow a client with limited storage, who has outsourced her data to an untrusted server, to confirm via an efficient Audit protocol that the data is still being stored in its entirety.

The lower bound of [5, Theorem 4] proves that a tradeoff is inevitable between low/high audit cost and high/low storage overhead. Roughly speaking, for any PoR on an  $N$ -bit database, the product of persistent storage overhead times audit computational complexity must be at least  $N$ .

The dynamic PoR schemes of [15, 40] optimize for fast audits. They incur a large  $\mathcal{O}(N)$  storage overhead on the server, but can perform audits with only  $(\log N)^{\mathcal{O}(1)}$  communication and computation for the client and server.

Instead, [5] optimizes for small storage; their scheme has only sub-linear storage overhead of  $\mathcal{O}(N/\log N)$ , but a higher audit cost of  $\mathcal{O}(N)$  on the server, and  $\mathcal{O}(\sqrt{N})$  client time and communication. The authors demonstrate that, for reasonable deployment scenarios on commercial cloud platforms, the higher audit cost is more than offset by the greatly reduced costs of extra persistent storage, especially if audits are only performed a few times per day.

We here further improve on the low storage overhead approach of [5], by our scheme with a small  $o(N)$  storage overhead, but only  $\mathcal{O}(\log N)$  communication and client computation cost for audits. That is, our new protocol still benefits from small storage overhead, while effectively pushing the higher computational cost of audits (which is inevitable from the lower bound) entirely off the client and onto the server. These savings are highlighted in Table 7.

An easy argument demonstrates that in fact our  $\mathcal{O}(\log N)$  client cost for audits is a optimal. If each audit has  $o(\log N)$  cost (and therefore transcript size) for audits, then the total number of possible transcripts is  $o(N)$ , which is a contradiction with the definition of retrievability; not every  $N$ -bit database could be recoverable via independent audit transcripts.

Table 7: Attributes of some selected DPoR schemes

Protocol	Server		Audit Comm.	Client	
	Extra Storage	Audit Comput.		Storage	Audit Comput.
[40]	$\mathcal{O}(N)$	$\mathcal{O}(\log N)$	$\mathcal{O}(\log N)$	$\mathcal{O}(1)$	$\mathcal{O}(\log N)$
[5]	$o(N)$	$N + o(N)$	$\mathcal{O}(\sqrt{N})$	$\mathcal{O}(1)$	$\mathcal{O}(\sqrt{N})$
Here	$o(N)$	$N + o(N)$	$\mathcal{O}(\log N)$	$\mathcal{O}(1)$	$\mathcal{O}(\log N)$

### 6.1 Matrix based approach for audits

Here we summarize the DPoR presented in [5] upon which our new scheme is based. The basic premise is to treat the data, consisting of  $N$  bits organized in machine words, as a matrix  $M \in \mathbb{Z}_p^{m \times n}$ , where  $\mathbb{Z}_p$  is a suitable finite field of size  $p$ . Crucially, the choice of ring  $\mathbb{Z}_p$  does not require any modification to the raw data itself; that is, any element of the matrix  $M$  can be retrieved in  $\mathcal{O}(1)$  time from the underlying raw data storage. The scheme is based on the commutativity of matrix-vector products. During the Setup phase, the client chooses a secret vector  $u$  of dimension  $m$  and computes  $v^\top = u^\top M$ ; both vectors  $u$  and  $v$  are then stored by the client for later use, while the server stores the original data and hence the matrix  $M$  in the clear. Reading or updating individual entries in  $M$  can be performed efficiently with the use of Merkle hash trees and from the observation that changing one element of  $M$  only requires changing one entry in the client's secret control vector  $v$ . To perform an audit, the client and server engage in a 1-round protocol:

1. Client chooses a random vector  $x$  of dimension  $n$ , and sends  $x$  to Server.
2. Server computes  $y = Mx$  and sends the dimension- $m$  vector  $y$  back to Client.
3. Client computes two dot products  $u^\top y$  and  $v^\top x$ , and checks that they are equal.



The proof of retrievability relies on the fact that observing several successful audits allows, with high probability, recovery of the correct matrix  $M$ , and therefore of the entire database.

The communication costs are  $\mathcal{O}(n)$  and  $\mathcal{O}(m)$  in steps 1 and 2 respectively, and the client computation in step 3 is  $\mathcal{O}(m+n)$ , resulting in  $\mathcal{O}(\sqrt{N})$  total communication and client computation when optimizing the matrix dimensions to roughly  $m = n = \sqrt{N}$ . While this square-matrix setup is the basic protocol presented by [5], the authors also discuss a potential improvement in communication complexity. Instead of  $x$  being uniformly random over  $\mathbb{Z}_p^n$ , it can instead be a *structured* vector formed from a single random element  $r \in \mathbb{Z}_p$  as  $x = [r^i]_{i=1..n}$ . Then the communication on step 1 is reduced to constant, and hence the total communication depends only on the row dimension  $\mathcal{O}(m)$ . By choosing a rectangular matrix  $M$  with few rows and many columns, the communication can be made arbitrarily small.

The tradeoff for this reduction in communication complexity is higher client storage of the control vector  $v$  as well as higher client computation cost for the  $n$ -dimensional dot product  $v^\top x$ . In [5], the authors found that the savings in communication were not worth the higher client storage and computation, and their experimental evaluation was based on the square matrix version with overhead  $\mathcal{O}(\sqrt{N})$ .

## 6.2 Bootstrapping part of the client computation via ciphered and dynamic polynomial evaluation

Now we show how to modify the reduced communication version of the DPoR protocol of [5] just presented in order to eliminate the costly client storage of  $v \in \mathbb{Z}_p^n$  and computation of  $v^\top x$  during audits. Our improved protocol is based on the observation that, when the audit challenge vector  $x$  is structured as  $x = [r^i]$ , then the expensive client dot product computation of  $v^\top x$  is actually a polynomial evaluation: if the entire of  $v$  are the coefficients of a polynomial  $P$ , then  $v^\top x$  is simply  $P(r)$ . We therefore eliminate the  $\mathcal{O}(n)$  client persistent storage and computation cost during audits by outsourcing the (encrypted) storage of vector  $v$  and computation of  $v^\top x = P(r)$  with our novel protocol for dynamic, encrypted, verifiable polynomial evaluation scheme of Table 10. The obtained private-verification DPoR protocol, combining the DPoR of [5] with our ciphered polynomial evaluation in Section 5, is presented in Table 8.

Table 8: Private verifiable Client/server DPoR protocol with low storage server

	Server	Communications	Client
<b>Setup</b>	$T_M \leftarrow \text{MTTree}(M)$ $T_w \leftarrow \text{MTTree}(w)$ Store $M, T_M, w, T_w, \bar{H}, S$	$N = mn \log_2 p$ $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ of ord. $p$ pairing $e$ to $\mathbb{G}_T$ gen. $g_1, g_2, g_T$ $M_{\overleftarrow{\langle w, \bar{H}, S \rangle}}$	$\gamma, s \xleftarrow{\$} \mathbb{Z}_p^*, \alpha, \beta \xleftarrow{\$} (\mathbb{Z}_p^*)^2, \Phi \xleftarrow{\$} \mathbb{Z}_p^{2 \times 2}$ form $u^\top \leftarrow [\gamma^i]_{i=0..m-1} \in \mathbb{Z}_p^n$ $v^\top \leftarrow u^\top M \in \mathbb{Z}_p^n$ $\bar{v} \leftarrow [(v_k \alpha + \Phi^k \beta)]_{k=0..n-1} \in \mathbb{Z}_p^{2 \times n}$ $w \leftarrow E(v), \bar{H} \leftarrow [g_1^{v_k}]_{k=0..n-1} \in \mathbb{Z}_p^{2 \times n}$ $\sigma \leftarrow [s^k]_{k=0..n-1}, \bar{K} = g_T^{\bar{v}^\top \sigma}, S \leftarrow g_2^\sigma$ $r_M \leftarrow \text{MTRoot}(M), r_w \leftarrow \text{MTRoot}(w)$ Discard $M, v, \bar{v}, w, \bar{H}, \sigma, S$ . Store $\gamma, r_M, \alpha, \beta, s, \Phi, \bar{K}, r_w$
<b>Update</b>	$(M_{ik}, L_{M_{ik}}) \leftarrow \text{MTLeafPath}(k + i \cdot n, M, T_M)$ $(w_k, L_{w_k}) \leftarrow \text{MTLeafPath}(k, w, T_w)$ $M_{ik} \leftarrow M'_{ik}, w_k \leftarrow w_k \cdot e_\delta$ $\bar{H}_k[j] \leftarrow \Delta[j] \bar{H}_k[j]$ for $j = 1..2$ $T_w \leftarrow \text{MTupdLeaf}(k, w_k, T_w)$ $T_M \leftarrow \text{MTupdLeaf}(k + i \cdot n, M'_{ik}, T_M)$	$\overleftarrow{\langle i, k \rangle}$ $M_{ik}, L_{M_{ik}}, w_k, L_{w_k}$ $M'_{ik}, e_\delta, \Delta$	$r_w \stackrel{?}{=} \text{MTPathRoot}(k, w_k, L_{w_k})$ $r_M \stackrel{?}{=} \text{MTPathRoot}(k + i \cdot n, M_{ik}, L_{M_{ik}})$ $\delta \leftarrow \gamma^i (M'_{ik} - M_{ik}), e_\delta \leftarrow E(\delta), \Delta \leftarrow g_1^{\delta \alpha}$ $\bar{K}[j] \leftarrow e(\Delta[j]^{s^k}; g_2) \cdot \bar{K}[j]$ for $j = 1..2$ $r_w \leftarrow \text{MTPathRoot}(k, w_k e_\delta, L_{w_k})$ $r_M \leftarrow \text{MTPathRoot}(k + i \cdot n, M'_{ik}, L_{M_{ik}})$
<b>Audit</b>	form $x \leftarrow [r^k]_{k=0..n-1}^\top \in \mathbb{Z}_p^n$ $y \leftarrow Mx, \zeta = w^\top \square x$ $\bar{\xi} = \prod_{i=1}^{n-1} \prod_{k=0}^{i-1} e(\bar{H}_i; S_{i-k-1})^{x_k}$	$\overleftarrow{\langle r \rangle}$ $y, \zeta, \bar{\xi}$	$r \xleftarrow{\$} \mathbb{Z}_p^*$ s.t. $(r\Phi - I_2) \in GL_2(\mathbb{Z}_p)$ $c \leftarrow ((r\Phi)^{d+1} - I_2)(r\Phi - I_2)^{-1} \beta$ $\bar{\xi}[j]^{s-r} g_T^{D(\zeta)\alpha[j]+c[j]} \stackrel{?}{=} \bar{K}[j]$ for $j = 1..2$ $u^\top y \stackrel{?}{=} D(\zeta)$

**Theorem 13.** *The protocol of Table 8 is correct and sound.*

*Proof.* For the sake of simplicity, we here only consider the case  $t = 1$ , that is a single control vector.

**Correctness.** Assume that all the parties are honest. After each update phase, thanks to the correctness of the Merkle hash tree algorithms  $w^\top = E(u^\top M)$  and  $\bar{\mathcal{K}} = e(g_1^{\bar{v}^\sigma}; g_2)$ . To see this, suppose a modification of the database at indices  $i$  and  $k$ , and let  $M' = M + (M'_{ik} - M_{ik})\mathcal{E}_{ik}$  where  $\mathcal{E}_{ik}$  is the single entry matrix with 1 at position  $(i, k)$ . We have  $u^\top M' = u^\top M + u^\top (M'_{ik} - M_{ik})\mathcal{E}_{ik} = u^\top M + \gamma^i e_k (M'_{ik} - M_{ik})$  where  $e_k$  is the  $k$ -th canonical vector. Thus,  $v' = v + \gamma^i (M'_{ik} - M_{ik})e_k = v + \delta e_k$  satisfies  $u^\top M' = v'^\top$ . Only the  $k$ -th coefficients are different in  $v$  and  $v'$ , and in  $w$  and  $w'$  as well. For the latter,  $w'_k = E(v'_k) = E(v_k + \delta) = E(v_k)E(\delta) = w_k E(\delta)$ . The server thus computes  $w'$  such that  $w' = E(u^\top M')$ . Moreover, for  $j = 1..2$ ,  $\bar{v}'[j] = \bar{v}[j] + \delta \alpha[j] e_k$ , so that, similarly,  $\bar{H}'_k[j] = \Delta[j] \bar{H}_k[j]$  with  $\Delta = g_1^{\delta \alpha}$ , and  $\bar{\mathcal{K}}'[j] = e(g_1^{\bar{v}'[j]\sigma}; g_2) = e(g_1^{\bar{v}[j]\sigma} g_1^{\delta \alpha[j] e_k \sigma}; g_2) = \bar{\mathcal{K}}[j] \cdot e(g_1^{\delta \alpha[j] s^k}; g_2) = \bar{\mathcal{K}}[j] \cdot e(\Delta[j] s^k; g_2)$ . Now, concerning the audit phase. Since we consider the polynomial evaluation as a dotproduct, the application of Proposition 8 to our notations gives:  $(s - r) \left( \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \bar{v}_i s^{i-k-1} r^k \right) + \sum_{i=0}^{n-1} \bar{v}_i r^i = \sum_{i=0}^{n-1} \bar{v}_i s^i$ . Thus we have:  $\bar{\xi} = \prod_{i=1}^{n-1} \prod_{k=0}^{i-1} e(\bar{H}_i; S_{i-k-1})^{x_k} = \prod_{i=1}^{n-1} \prod_{k=0}^{i-1} e(g_1^{\bar{v}_i}; g_2^{s^{i-k-1}} r^k) = e(g_1; g_2)^{\sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \bar{v}_i s^{i-k-1} r^k}$ . Moreover,  $\alpha D(\zeta) + c = \alpha v x + ((r\Phi)^{d+1} - I_2)(r\Phi - I_2)^{-1} \beta = \alpha v x + \sum_{k=0}^{n-1} r^k \Phi^k \beta = \bar{v} x$ . Thus we have that  $\bar{\mathcal{K}}[j] = g_T^{\bar{v}[j]\sigma} = g_T^{(s-r)(\sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \bar{v}_i [j] s^{i-k-1} r^k) + \bar{v}[j] x}$ , that is  $\bar{\mathcal{K}}[j] = \bar{\xi}[j]^{s-r} g_T^{D(\zeta)\alpha[j] + c[j]}$  and, finally,  $u^\top y = u^\top M x = v^\top x$ .

**Soundness.** An attacker to the protocol must provide  $(y', \zeta', \xi')$  such that  $(y', \zeta', \xi') \neq (y, \zeta, \xi)$ , but still  $u^\top y' = D_{sk}(\zeta')$ , with a non negligible advantage  $\epsilon$ . There are two cases: if  $(D_{sk}(\zeta'), \xi') \neq (D_{sk}(\zeta), \xi)$  then the attacker had to break the polynomial evaluation; otherwise, it must be that  $u^\top y' = u^\top y$  with  $y' \neq y$ .

For the first case, Theorem 12 assesses the security of the polynomial evaluation. For the second case, we consider  $T = E_{pk}(t)$  the cipher of a secret  $t$  by the homomorphic scheme. Here, we use again the fact that the protocol of Table 8 is indistinguishable as a whole from the same protocol where, within the polynomial evaluation of,  $\Phi^i \beta$  is everywhere replaced by a random  $\Gamma_i$ . Further, this is indistinguishable from a third protocol where, at each Update of index  $i$ , a new  $\Gamma'_i$  is also randomly redrawn and replaces  $\Gamma_i$  in the client state. We thus continue the proof with this third game setting. Now, using  $e_\ell$  the  $\ell$ -th canonical vector of  $\mathbb{Z}_p^m$ , we can (abstractly) consider  $\tilde{u} = u + t e_\ell$  and  $\tilde{v}^\top = \tilde{u}^\top M = (u^\top + t e_\ell) M = v + t M_{\ell,*}$ . Then, for the Setup phase, we can randomly select  $m, n$  and  $\ell \leq m$ . Then also  $M \in \mathbb{Z}_p^{m \times n}$ ,  $u \in \mathbb{Z}_p^m$ , and compute  $v^\top = u^\top M$ . From this, compute  $w_k = E(v_k) T^{M_{\ell k}} = E(v_k + t M_{\ell k}) = E(\tilde{v}_k)$ . We also randomly select  $s, \alpha$  and  $\bar{h}_k$  (so that  $\Gamma_k = \log_{g_1}(\bar{h}_k) - \tilde{v}_k \alpha$  exists, but is unknown). For any Update phases, compute  $w'_k = w_k T^{M'_{\ell k} - M_{\ell k}}$  and select randomly a  $\Delta$  (so that  $\bar{h}'_k[j] = \bar{h}_k[j] \Delta[j]$  for  $j = 1..2$  now correspond to a new  $\Gamma_k = \log_{g_1}(h'_k) - \alpha \tilde{v}'_k$  still unknown).

Finally, the attacker provides a vector  $y'$  such that both  $\tilde{u}^\top (y' - y) = 0$  and  $y' \neq y \pmod p$ . Since  $\ell$  is randomly chosen from  $1..m$ , the probability that the vectors are distinct at index  $\ell$ , in other words that  $y'_\ell \neq y_\ell \pmod p$ , is at least  $1/m$ . If this is the case, then, denoting  $z = y' - y$ , we have that  $z_\ell \neq 0 \pmod p$ . Now,  $\tilde{u}^\top z = 0$  implies that  $u^\top z + t z_\ell = 0$  so that the secret can be computed as  $t \equiv -z_\ell^{-1} \cdot (u^\top z) \pmod p$  and the homomorphic cryptosystem is subject to an attack with advantage  $\epsilon/m$ .  $\square$

### 6.3 Experiments

We now compare our modification of the DPoR protocol with the one in [5], publicly available there: <https://github.com/dsroche/la-por>.

Table 9 has three blocks of experiments, each for four database sizes ranging from 1GB to 1TB. The first block of experiments is a run of the original statistically secure DPoR protocol with two dotproducts for the verification, considering the matrix as 56 bits elements modulo a 57-bits prime. The second block of experiments is our new modification, but still using close to square matrices. Subject now to computational security, we have to use a larger coefficient domain, namely here a 254-bits prime (with associated bilinear groups and a 2048-bits Paillier modulus, both estimated equivalent to a 112-bit computational security). We separate the timings of the Update phase in two phases, the remaining linear algebra phase and the new polynomial evaluation phase. In the third block of experiments we use a more rectangular matrix, trying to reduce communications while not increasing too much the Server computational effort.

Overall, we see first in Table 9, that changing the coefficient domain size increases the computational effort of the server in the linear algebra phase. Still, reducing the dimension of the dotproduct for the client, as shown in he third block, allows the client to be faster for databases larger than 100GB.

Table 9: Modification of the DPoR audit protocol, with 254-bits groups, 2048-bits Paillier, on a Gold 6126 2.6GHz & 10 GB/core (real time are median values for a single run; each experiment was performed 11 times; the maximum relative difference between the runs was at most 3.6%).

Database	1GB	10GB	100GB	1TB
Private-verified audit using 57-bits prime [5, Figure 1]				
Matrix view	12339×12432	39131×39200	123831×123872	396281×396368
Server extra storage	<0.01%	<0.01%	<0.01%	<0.01%
Client Storage	169KB	535KB	1 693KB	5 418KB
Server Audit (1/16 cores)	0.13s/0.02s	1.30s/0.17s	13.01s/1.62s	191.83s/12.24s
Communications	169KB	535KB	1 693KB	5 418KB
Client Audit (1 core)	0.2ms	0.7ms	1.9ms	8.5ms
Square Dynamic-ciphered delegated polynomial evaluation with 254-bits groups of Table 8				
Matrix view	5815×5816	18390×18390	58154×58154	186092×186093
Server extra storage	0.12%	0.04%	0.01%	<0.01%
Client storage	0.94KB	0.94KB	0.94KB	0.94KB
Server Audit (1/16 cores): matrix-vector step	1.1s/0.2s	11.3s/1.3s	113.4s/12.9s	1 152.5s/131.1s
Server Audit (1/16 cores): polynomial step	4.6s/0.7s	14.3s/1.8s	46.0s/5.3s	145.7s/16.4s
Communications	181KB	571KB	1 803KB	5 770KB
Client Audit (1 core): dotproduct step	3.2ms	8.4ms	13.1ms	37.9ms
Client Audit (1 core): polynomial step	1.7ms	1.7ms	1.7ms	1.7ms
Rectangular Dynamic-ciphered delegated polynomial evaluation with 254-bits groups of Table 8				
Matrix view	6599×5125	7265×46551	7929×426519	8600×4026778
Server extra storage	0.11%	0.10%	0.09%	0.08%
Client storage	0.94KB	0.94KB	0.94KB	0.94KB
Server Audit (1/16 cores): matrix-vector step	1.1s/0.2s	11.3s/1.3s	113.2s/12.8s	1 147.9s/130.7s
Server Audit (1/16 cores): polynomial step	4.0s/0.6s	36.8s/4.3s	335.7s/36.7s	3 125.9s/340.0s
Communications	205KB	226KB	246KB	267KB
Client Audit (1 core): dotproduct step	3.7ms	4.0ms	4.4ms	4.8ms
Client Audit (1 core): polynomial step	1.7ms	1.7ms	1.7ms	1.7ms

In any case, the client audit computational effort is never larger than a few milliseconds and thus the dominant part is most certainly communications. On this aspect, we see that our modification allows for large reductions in both the Client storage (even with square matrices) and the overall communications. Indeed, the client private state is the vector dimension, the Paillier’s private key, twelve group elements and two Merkle tree roots; while the communications are mostly one vector of modular integers in the smallest dimension.

The price to pay is from about a factor of four (large database) to an order of magnitude (tiny database) for the server computations (more limited losses in the more realistic case where the server can use multiple cores). In any case, the persistent client storage is going from dozens of MB to less than one KB, and the communication volume can be decreased by more than two orders of magnitude.

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## 7 Conclusion

We have presented a protocol verifying publicly a dynamic unciphered polynomial evaluation and then a protocol verifying privately a dynamic ciphered polynomial evaluation. Now, combining efficient and proven dynamicity for ciphered polynomial with public verifiability raises security issues and reminds an open problem. Still, we have also presented a protocol verifying the outsourced evaluation of secret polynomials. Client verification is of the order of a few milliseconds and is faster than direct polynomial evaluation over a small finite field, as soon as the degree of the polynomial is larger than a few thousand.

This enables us in turn to reduce by several orders of magnitude the communications, Client storage and Client computations for state-of-the-art low server-storage dynamic proofs of retrievability.

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## A Overview of VESPo

The exchanges of Algorithms 3 to 5 are summarized in Table 10.

## B Intermediate Proofs

Now, we give a proof on the logarithmic complexity bound for the client verification. Then we give the proofs of the propositions in Sections 3 and 4.

Table 10: Private, Dynamic, Ciphred and logarithmic polynomial evaluation

	Server	Communications	Client
<b>Setup</b>	$T_W \leftarrow \text{MTTree}(W)$ Store $W, T_W, \bar{H}, S$	$\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ groups of order $p$ pairing $e$ to $\mathbb{G}_T$ , gen. $g_1, g_2, g_T = e(g_1; g_2)$	$P \in \mathbb{Z}_p[X], 1 \leq d^\circ(P) \leq d$ $s \xleftarrow{\$} \mathbb{Z}_p \setminus \{0, 1\}, \alpha, \beta \xleftarrow{\$} \mathbb{Z}_p^2, \Phi \xleftarrow{\$} \mathbb{Z}_p^{2 \times 2}$ , s.t. $(s\Phi - I_2) \in GL_2(\mathbb{Z}_p)$ Let $\bar{P}(X) \leftarrow \sum_{i=0}^d X^i (p_i \alpha + \Phi^i \beta)$ $W \leftarrow E(P), \bar{H} \leftarrow [g_1^{\bar{P}^i}]_{i=1..d} \in \mathbb{G}_1^{2 \times d}$ $\bar{\mathcal{K}} \leftarrow g_T^{\bar{P}(s)} \in \mathbb{G}_T^2, S \leftarrow [g_2^s]_{k=0..d-1}$ $r_W \leftarrow \text{MTRoot}(W)$ discard $P, \bar{P}, W, \bar{H}, S$ .
<b>Update</b>	$\bar{h}'_i[j] \leftarrow \Delta[j] \cdot \bar{h}_i[j]$ for $j = 1..2$ $(w_i, L_i) \leftarrow \text{MTLeafPath}(i, W, T_W)$ $w'_i \leftarrow w_i \cdot e_\delta$ $T_W \leftarrow \text{MTupdLeaf}(i, w'_i, T_W)$	$\xleftarrow{i, e_\delta, \Delta}$ $\xrightarrow{w_i, L_i}$	$e_\delta \leftarrow E(\delta), \Delta \leftarrow g_1^{\delta \alpha}$ $r_W \stackrel{?}{=} \text{MTpathRoot}(i, w_i, L_i)$ $w'_i \leftarrow w_i \cdot e_\delta, \bar{\mathcal{K}}[j] \leftarrow e(\Delta[j]^{s^i}; g_2) \cdot \bar{\mathcal{K}}[j]$ $r_W \leftarrow \text{MTpathRoot}(i, w'_i, L_i)$
<b>Audit</b>	Form $x \leftarrow [1, r, r^2, \dots, r^d]^\top$ $\zeta \leftarrow W^\top \square x$ $\bar{\xi} \leftarrow \prod_{i=1}^d \prod_{k=0}^{i-1} e(\bar{H}_i; S_{i-k-1})^{x_k}$	$\xleftarrow{r}$ $\xrightarrow{\zeta, \bar{\xi}}$	For $r \in \mathbb{Z}_p$ s.t. $(r\Phi - I_2) \in GL_2(\mathbb{Z}_p)$ $c \leftarrow ((r\Phi)^{d+1} - I_2)(r\Phi - I_2)^{-1} \beta$ $\bar{\xi}[j]^{s-r} g_T^{D(\zeta)\alpha[j]+c[j]} \stackrel{?}{=} \bar{\mathcal{K}}[j]$ for $j = 1..2$

**Lemma 14.** *Algorithm 2, computing the matrix geometric sum, requires between  $40 + 8\lceil \log_2(d+1) \rceil$  and  $40 + 11\lceil \log_2(d+1) \rceil$  arithmetic operations.*

*Proof.* Counting only (modular) field operations, Algorithm 1 requires between 8 and 11 times  $\lceil \log_2(d) \rceil$  additions and multiplications depending on the binary decomposition of  $d$ . Then we have 5 operations for the matrix inverse, twice 6 operations for the matrix-vector multiplications and 18 operations for the matrix polynomial evaluation. Plus 5 operations for the characteristic polynomial.  $\square$

**Proposition 9** (From page 8). *The protocol of Table 2 is correct and sound.*

*Proof. Correctness.* First,  $\zeta = W^\top \square x = \prod_{i=0}^d E(p_i)^{(r^i)} = E(P(r))$ . Then, second,  $\xi = H^\top \odot x = \prod_{k=0}^{d-1} g_1^{T_{k,P}(s)r^k} = g_1^{Q_P(s,r)}$ , by Lemma 8. Therefore, the verification is that  $g_T^{Q_P(s,r)(s-r)+P(r)} \stackrel{?}{=} g_T^{P(s)}$  and this is guaranteed by Equation (4).

**Soundness.** Let  $\langle g_1, g_1^s, g_1^{s^2}, \dots, g_1^{s^t} \rangle \in \mathbb{G}_1^{t+1}$  be a t-BSDH instance and suppose that there exists an attack to the Audit protocol.

Let  $[p_0, \dots, p_t] \xleftarrow{\$} \mathbb{Z}_p^{t+1}$  for a degree  $t$  polynomial and  $d = t$ . Then compute  $W = E(P)$ ,  $T_{k,P} = \sum_{i=k+1}^t p_i Y^{i-k-1} = \sum_{j=0}^{t-1-k} t_{k,j} Y^j$  and homomorphically compute

$$\mathcal{K} = e \left( \left\langle g_1, g_1^s, g_1^{s^2}, \dots, g_1^{s^t} \right\rangle \odot [p_0, \dots, p_t]; g_2 \right)$$

together with  $H = [h_k]$ , where  $h_k = \left\langle g_1, g_1^s, g_1^{s^2}, \dots, g_1^{s^{t-1-k}} \right\rangle \odot [t_{k,0}, \dots, t_{k,t-1-k}]$ . These inputs are indistinguishable from a generic setup of the protocol of Table 2 and can thus be given to its attacker.

Finally, select a random evaluation point  $r$  and compute  $(\zeta, \xi)$ . The supposition is that an attacker of the Audit part of the protocol can get  $(\zeta', \xi')$ , with some advantage, such that  $(D(\zeta'), \xi') \neq (D(\zeta), \xi)$ , even though both would be passing the verification. Now, on the one hand, if  $D(\zeta') = D(\zeta)$ , then  $\xi \neq \xi'$  and it must be that  $e(\xi; g_2^{s-r}) g_T^{D(\zeta)} = \mathcal{K}$  and  $e(\xi'; g_2^{s-r}) g_T^{D(\zeta)} = \mathcal{K}$ . Therefore, if  $r \neq s$ , then  $e(\xi; g_2^{s-r})^{\frac{1}{s-r}} = e(\xi'; g_2^{s-r})^{\frac{1}{s-r}}$ . But this contradicts the fact that  $\xi \neq \xi'$ , so  $r = s$ , and the secret can be exposed. On the other hand, if  $D(\zeta') \neq D(\zeta)$ , then it means that we must have the equality  $(e(\xi; g_2) / (e(\xi'; g_2))^{s-r}) = g_T^{D(\zeta') - D(\zeta)}$  and therefore:  $\left( \frac{e(\xi; g_2)}{e(\xi'; g_2)} \right)^{\frac{1}{D(\zeta') - D(\zeta)}} = g_T^{\frac{1}{s-r}}$ . This proves that the adversary would solve the t-BSDH  $\langle -r, e(g_1; g_2)^{\frac{1}{s-r}} \rangle$  challenge with the same advantage.  $\square$

**Proposition 10** (From page 9). *The protocol of Table 3 is correct and sound.*



*Proof. Correctness.* For the update, the new polynomial is  $P'(s) = P(s) + \delta s^i$ , so that the key is updated as  $\mathcal{K}'_1 = \mathcal{K}_1 \cdot e(g^{\delta s^i}; g)$ . Now for the evaluation, first,  $\xi = \prod_{i=1}^d \prod_{k=0}^{i-1} S_{i-k-1}^{p_i x_k} = g^{\sum \sum s^{i-k-1} p_i x_k} = g^{Q_{P'(r,s)}}$  and, second, we have that:

$$e(\xi; \mathcal{K}_2/g^r) e(g; g)^\zeta = e(\xi; g^{s-r}) e(g; g)^{P(r)} = e(g; g)^{Q_{P'(r,s)(s-r)+P(r)}} = e(g; g)^{P(s)}.$$

Hence we see that  $e(\xi; \mathcal{K}_2/g^r) e(g; g)^\zeta = \mathcal{K}_1$  and, therefore, the protocol is correct.

**Soundness.** Let  $\langle g, g^s, g^{s^2}, \dots, g^{s^t} \rangle \in \mathbb{G}^{t+1}$  be a t-BSDH instance. For the setup phase, set  $d = t$  and randomly select  $[p_0, \dots, p_t] \xleftarrow{\$} \mathbb{Z}_p^{t+1}$ . Then set  $S = \langle \mathbb{G}, g, g^s, g^{s^2}, \dots, g^{s^t} \rangle$  and

$$\mathcal{K}_1 = e \left( \langle g, g^s, g^{s^2}, \dots, g^{s^t} \rangle \odot [p_0, \dots, p_t]; g \right).$$

These inputs are indistinguishable from generic inputs to the protocol of Table 3. For any number of update phase, randomly select  $\delta$ , receive  $p_i$  and  $L_i$  from the Server, compute  $\mathcal{K}'_1 = \mathcal{K}_1 e(S_i^\delta; g)$  and refresh  $r_p$ . Finally, select a random evaluation point  $r$ , compute  $(\zeta, \xi)$  and call an attacker of the PVDUeval part of the protocol to get  $(\zeta', \xi')$  such that  $(\zeta', \xi') \neq (\zeta, \xi)$ , even though both are passing the verification. If  $\zeta' = \zeta$ , then as  $\xi \neq \xi'$  it must be that  $r = s$  and the secret is revealed; otherwise,  $\zeta' \neq \zeta$  and we have both  $e(\xi'; \mathcal{K}_2/g^r) e(g; g)^{\zeta'} = \mathcal{K}_1$ , on the one hand, and  $\mathcal{K}_1 = e(\xi; \mathcal{K}_2/g^r) e(g; g)^\zeta$ , on the other hand. This gives  $e(\frac{\xi'}{\xi}; g^{s-r}) = e(g^{\zeta-\zeta'}; g)$  and thus  $e\left(\left(\frac{\xi'}{\xi}\right)^{s-r}; g\right) = e(g^{\zeta-\zeta'}; g)$ . Finally, we have that:

$$e\left(\frac{\xi'}{\xi}; g\right)^{\frac{1}{\zeta'-\zeta}} = e(g; g)^{\frac{1}{s-r}}. \text{ This proves that the adversary would solve the t-BSDH } \left\langle -r, e(g; g)^{\frac{1}{s-r}} \right\rangle \text{ challenge with the same advantage. } \square$$

**Proposition 11** (From page 9). *In Table 3, the setup protocol requires  $\mathcal{O}(d)$  arithmetic operations; the update protocol requires  $\mathcal{O}(\log(d))$  arithmetic operations; the verification protocol requires  $\mathcal{O}(1)$  communications and arithmetic operations for the Client, and  $\mathcal{O}(d)$  arithmetic operations for the Server.*

*Proof.* For the update phase, the client computes the root of the Merkle tree from the new value  $p_i + \delta$  and the path  $L_i$  given by the server in  $\mathcal{O}(\log(d))$ . She also has to compute an exponentiation and a product in  $\mathbb{Z}_p[X]$ , this is in  $\mathcal{O}(1)$ .

For the verification phase, communications are just 3 group elements. The client work is only 2 pairing and 2 exponentiations and 1 product.

Now for the server. First, computing  $\zeta$  is  $d + 1$  homomorphic multiplications and  $d$  additions. Second, the server has to compute  $\xi = \prod_{i=1}^d \prod_{k=0}^{i-1} S_{i-k-1}^{p_i x_k} = \prod_{i=1}^d \left( \prod_{k=0}^{i-1} S_{i-k-1}^{r^k} \right)^{p_i}$ . Therefore, one can use a Horner-like prefix computation [27]: consider  $t_0 = 1$ , and  $t_i = S_{i-1} \cdot t_{i-1}^r$ , then  $t_1 = S_0$ ,  $t_2 = S_1 S_0^r$  and therefore  $t_i = S_{i-1} (S_{i-2} \dots (S_2 (S_1 S_0^r)^r) \dots)^r = \prod_{k=0}^{i-1} S_{i-k-1}^{r^k}$ . Thus one can use the following Algorithm 6 to compute  $\xi$ . Computing  $\xi$  then requires at most  $2d$  exponentiations and  $2d$  multiplications.  $\square$

---

**Algorithm 6** Homomorphic linear prefix evaluation of the difference polynomial

---

**Input:**  $r, [S_0, \dots, S_{d-1}], [p_1, \dots, p_d]$ .

**Output:**  $\xi = \prod_{i=1}^d \left( \prod_{k=0}^{i-1} S_{i-k-1}^{r^k} \right)^{p_i}$ .

1:  $\xi = 1; t = 1;$

2: **for**  $i = 1$  **to**  $d$  **do**

3:    $t \leftarrow S_{i-1} \cdot t^r;$

4:    $\xi \leftarrow \xi \cdot t^{p_i}.$

5: **end for**

6: **return**  $\xi.$

---

$$\{t_i = \prod_{k=0}^{i-1} S_{i-k-1}^{r^k}\}$$

## C Paillier’s cryptosystem as the Linearly Homomorphic primitive

Paillier’s homomorphic system works modulo some RSA composite number  $N$ . Now it is possible to use it to compute evaluations modulo a different  $m$  (for instance a prime), provided that  $m$  is small enough: consider the modulo  $m$  operations to be over  $\mathbb{Z}$ , perform the homomorphic operations, and use  $m$  only to reduce *after* decryption. This is illustrated in [Algorithm 7](#).

---

**Algorithm 7** Homomorphic modular polynomial evaluation with a different Paillier modulus

---

**Input:** An integer  $r \in [0..m - 1]$ ;

**Input:** A Paillier cryptosystem  $(E, D)$  with modulus  $N > (m - 1)^2$ .

**Input:**  $(E(p_0), \dots, E(p_d)) \in \mathbb{Z}_N^{d+1}$ , such that  $\forall i, p_i \in [0..m - 1]$  and  $d < \frac{N}{(m-1)^2} - 1$ .

**Output:**  $c \in \mathbb{Z}_N$  such that  $D(c) \bmod m \equiv P(r) \bmod m \equiv \sum_{i=0}^d p_i r^i \bmod m$ .

1: let  $x_0 = 1$  and  $c = E(p_0)$ ;

2: **for**  $i = 1$  **to**  $d$  **do**

3:   let  $x_i \equiv x_{i-1} \cdot r \bmod m$ ;

{Now  $x_i \in [0..m - 1]$ }

4:   let  $c \leftarrow c \cdot E(p_i)^{x_i}$ ;

5: **end for**

6: **return**  $c$ .

---

**Lemma 15.** *Algorithm 7 is correct.*

*Proof.* If  $0 \leq p_i \leq (m - 1)$ , then as  $x_i$  is considered as an integer between 0 and  $m - 1$ , then  $0 \leq \sum_{i=0}^d p_i x_i \leq (d + 1)(m - 1)^2 < N$  by the constraints on  $d$  and  $N$ . Therefore  $\sum_{i=0}^d p_i x_i \bmod N = \sum_{i=0}^d p_i x_i \in \mathbb{Z}$  and now  $D(c) \bmod m = \sum_{i=0}^d p_i x_i \bmod m \equiv P(r)$ .  $\square$

## D Parallel prefix-like algorithm for the Server

We here provide the parallelization we used for the Server audits.

For the DPoR, the matrix-vector product part was already parallelized in [5, Table 6], a Server auditing the 1TB database in a few minutes.

For the polynomial part, as the dimensions become more rectangular, as we can see in [Table 9](#), the Server’s polynomial part is sometimes not negligible anymore, thus also benefits from some parallelization.

For this, we would need to parallelize both the homomorphic dot-product and the Horner-like pairings. On the one hand, the former operations, line 3 in [Algorithm 5](#), can be blocked in independent exponentiations and final multiplications in a binary tree. On the other hand, for the latter operations, a standard “baby steps / giant steps” approach can be employed for the iteration of lines 5-8 in [Algorithm 5](#):

- First, for steps of size  $k$ , compute  $t^{r^k}$ , then  $t^{r^{kj}}$  for  $j = 1..d/k$  as a parallel prefix; then iterates the multiplications by the coefficients of  $S$  in parallel for the  $d/k$  blocks.
- Second, then all the pairings could be computed in parallel and their final multiplications performed again with a binary tree.

This is exposed in [Algorithm 8](#).

**Lemma 16.** *Algorithm 8 is correct, work-optimal with work  $W_q(d) = O(d)$  and runs in time  $W_q/q + o(W_q)$  on  $q$  processors.*

*Proof.* Correctness of phases A, B and D stems directly from the correctness of [Algorithm 5](#). Phase C is correct since the new variables  $u_i$  satisfy  $\{u_0 = S_0, u_{i+1} = S_{i+1}u_i^r\}$ .

Then,  $p$  is the prime group order, and for any homomorphic system satisfying [Equation \(2\)](#) we have:

- Phase A: requires  $d$  multiplications modulo  $p$  with depth  $O(\log(d))$  and the parallelism is thus only bounded by Brent’s law [12, Lemma 2];

- Phase B: requires  $d+1$  homomorphic exponentiations and  $d$  homomorphic multiplications with a depth of  $b = d/q$  such operations and the parallelism is thus only bounded by Brent’s law;
- Phase C: requires  $d$  exponentiations and  $d-1$  multiplications in  $\mathbb{G}_2$ . But this is implemented in parallel with a depth of  $b = d/q$  such operations, only after precomputing  $q-1$  times  $b$  operations each with a depth of  $\log(b)$ ;
- Phase D: requires  $d$  pairings and  $d-1$  multiplications in  $\mathbb{G}_T$  with a depth of  $b = d/q$  such operations and the parallelism is thus only bounded by Brent’s law.

So only Phase C requires more operations in parallel than in sequence. And that number of operations is  $2d - 1/q$  if ran on  $q$  processors. This latter work is in fact optimal for prefix-like computations as shown in [41, Corollary 4] (see also [38]): indeed consider a family of binary gates  $\theta_{\rho_i}(a, b)$  that on inputs  $a$  and  $b$  compute  $a \cdot b^{\rho_i}$ . They satisfy the conditions of [41, Corollary 4] and thus computing all the  $u_i$  is lower bounded by  $2d - 1/q$ .  $\square$

**Remark 17.** *The accumulated independent exponentiations/pairings of lines 11, 16 and 26 of Algorithm 8 can in fact be gathered in small batches, where each batch can factorize some computations (e.g. using a generalized Shamir trick with multiple exponentiations in  $\mathbb{G}_2$ , or using NAF windows, etc.). Therefore, on the one hand, with respect to a purely sequential computation, the extra work required by Phase C (when used with more than 2 processors) is in fact batched. On the other hand, the other part of Phase C cannot benefit from these batches and is therefore dominant, but is more parallel. Therefore, as shown also in Table 11, this allows us to reach, on multiple cores, pretty good overall practical speed-ups.*

Table 11: Parallel Server-side VESPo

Degree	5816	18390	58154	186093	426519	4026778
1 core	4.6s	14.3s	46.0s	145.7s	335.7s	3 125.9s
4 cores	1.3s	4.1s	12.8s	40.6s	92.2s	867.9s
8 cores	0.7s	2.1s	6.7s	20.9s	48.1s	448.5s
16 cores	0.7s	1.8s	5.3s	16.4s	36.7s	340.0s

This parallelism can be used to further reduce the Server latency for large databases, to allow faster multi-user queries, and to make the scheme even more practically relevant.

---

**Algorithm 8** Parallel Server Eval

---

**Input:** Group order  $p$ , polynomial degree  $d$ , evaluation point  $r$  and vectors  $W, S, \bar{H}[j]$ , all as in [Algorithm 5](#).

**Input:** Cutting parameter  $q$  (e.g. can be the number of threads).

**Output:** SERVER  $\zeta, \bar{\xi}[j]$  for  $j = 1..2$ .

```
1: Let  $(b, r)$  s.t.  $d + 1 = bq + r$ , with  $0 \leq r < q$ ;  
2: Set  $b_k \leftarrow \begin{cases} k(b + 1) & k = 0..r - 1 \\ r(b + 1) + (k - r)b & k = r..q \end{cases}$   
   {PHASE A:  $r^i \pmod p$ , for  $i = 0..d$ }  
3:  $\rho_0 \leftarrow 1, \rho_1 \leftarrow r, i \leftarrow 1$ ;  
4: while  $i \leq d$  do { $\lceil \log_2(d) \rceil$  parallel steps}  
5:   parfor  $k = 1..max(i; d - i)$  do  
6:      $\rho_{i+k} \leftarrow \rho_i \cdot \rho_k \pmod p$ ;  
7:   end parfor  
8:    $i \leftarrow 2i$ ;  
9: end while  
   {PHASE B:  $\zeta = W^\top \square x = \prod_{i=0}^d w_i^{(r^i \pmod p)}$ }  
10: parfor  $k = 1..q$  do { $q$  Blocks of size  $b$  or  $b + 1$  in parallel}  
11:    $\zeta_k \leftarrow \prod_{i=b_{k-1}}^{b_k-1} w_i^{\rho_i}$   
12: end parfor  
13:  $\zeta \leftarrow \prod_{k=1}^q \zeta_k$  { $\lceil \log_2(q) \rceil$  parallel steps}  
   {PHASE C:  $u_i = \prod_{k=0}^i S_{i-k}^{r^k}$ , for  $i = 0..(d - 1)$ }  
14:  $u_0 \leftarrow S_0$ ;  
15: for  $k = 1$  to  $q - 1$  do  
16:    $u_{b_k} \leftarrow u_{b_{k-1}}^{\rho_{b_k - b_{k-1}}} \prod_{i=b_{k-1}+1}^{b_k} S_i^{\rho_{b_k - i}}$  { $\lceil \log_2(b + 1) \rceil$  parallel steps}  
17: end for  
18: parfor  $k = 0..q - 1$  do { $q$  Blocks of size  $b - 1$  or  $b$  in parallel}  
19:   for  $i = 0$  to  $b_{k+1} - b_k - 2$  do  
20:      $u_{b_k+i+1} \leftarrow S_{b_k+i+1} \cdot u_{b_k+i}^r$ ;  
21:   end for  
22: end parfor  
   {PHASE D:  $\bar{\xi} = \prod_{i=1}^d \prod_{k=0}^{i-1} e(\bar{H}_i; S_{i-k-1})^{r^k}$ }  
23:  $\bar{\xi} = [1_{\mathbb{G}_T}, 1_{\mathbb{G}_T}]^\top \in \mathbb{G}_T^2$ ;  
24: for  $j = 1$  to  $2$  do  
25:   parfor  $k = 1..q$  do { $q$  Blocks of size  $b$  or  $b + 1$  in parallel}  
26:      $\bar{\xi}_k[j] \leftarrow \prod_{i=b_{k-1}}^{b_k-1} e(\bar{H}_i[j]; u_i)$   
27:   end parfor  
28:    $\bar{\xi}[j] \leftarrow \prod_{k=1}^q \bar{\xi}_k[j]$  { $\lceil \log_2(q) \rceil$  parallel steps}  
29: end for
```

---