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LOCALLY MOVING GROUPS ACTING ON THE LINE AND \mathbb{R} -FOCAL ACTIONS

JOAQUÍN BRUM, NICOLÁS MATTE BON, CRISTÓBAL RIVAS, AND MICHELE TRIESTINO

ABSTRACT. We prove various results that, given a sufficiently rich subgroup G of the group of homeomorphisms on the real line, describe the structure of the other possible actions of G on the real line, and address under which conditions such actions must be semi-conjugate to the natural defining action of G . The main assumption is that G should be locally moving, meaning that for every open interval the subgroup of elements whose support is contained in such interval acts on it without fixed points. One example (among many others) is given by Thompson's group F . A first rigidity result implies that if G is a locally moving group, every faithful minimal action of G on the real line by C^1 diffeomorphisms is conjugate to its standard action. It turns out that the situation is much wilder when considering actions by homeomorphisms: for a large class of groups, including Thompson's group F , we describe uncountably many conjugacy classes of minimal faithful actions by homeomorphisms on the real line. To gain insight on such exotic actions, we introduce and develop the notion of \mathbb{R} -focal action, a class of actions on the real line that can be encoded by certain actions by homeomorphisms on planar real trees fixing an end. Under a suitable finite generation condition on a locally moving group G , we prove that every minimal faithful action of G on the line is either conjugate to the standard action, or it is \mathbb{R} -focal and the action on the associated real tree factors via a horofunction onto the standard action of G on the line. This establishes a tight relation between all minimal actions of G on the line and its standard action. Among the various applications of this result, we show that for a large class of locally moving groups, the standard action is locally rigid, in the sense that every sufficiently small perturbation in the compact-open topology gives a semi-conjugate action. This is based on an analysis of the space of harmonic actions on the line for such groups. Along the way we introduce and study several concrete examples.

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1. INTRODUCTION

1.1. Background and overview. The study of group actions on one-manifolds is a classical topic at the interface of dynamical systems, topology, and group theory, which is still under intense development and has been the object of several monographs in the recent years, such as Ghys [44], Navas [91], Clay and Rolfsen [26], Deroin, Navas, and the third named author [36], Kim, Koberda, and Mj [56]; see also the surveys by Mann [71] and Navas [92]. The ultimate goal would be, given a group G , to be able to describe all possible actions of G on a one-manifold M by homeomorphisms, and as the topology of the manifold is rather elementary one expects that this is more tractable than in higher dimension. In this framework, there is no much loss of generality of considering the following simplifying assumptions: the manifold M is connected (that is, either the real line or the circle), and the action of the group is *irreducible*, in the sense that it preserves the orientation and has no (global) fixed points. We then let $\text{Homeo}_0(M)$ be the group of orientation-preserving homeomorphisms of M , and write $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(M))$ for the space of irreducible actions of G on M . Moreover, to avoid a redundant description, we want to identify two actions if one is obtained from the other with a change of variables (what is called a *conjugacy*); more precisely, from a dynamical perspective, it is natural to study the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(M))$ up to *semi-conjugacy*. The definition of semi-conjugacy in this setting is recalled in Section 2; here let us simply remind that every action $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(M))$ of a *finitely generated* group is semi-conjugate either to a *minimal* action (i.e. an action all whose orbits are dense) or $\varphi(G)$ has a discrete orbit, in which case φ is semi-conjugate to *cyclic* action (i.e. an action by integer translations in the case of the real line or by rational rotations in the case of the circle); moreover this minimal or cyclic model is unique up to conjugacy (see e.g. Ghys [44]). Thus studying actions of finitely generated groups up to semi-conjugacy is essentially the same as to study its minimal actions up to conjugacy.

The situation is particularly nice when M is a circle, for which more tools are available. In particular the *bounded Euler class* is an complete algebraic invariant for classifying irreducible actions up to semi-conjugacy (see Ghys [42, 44]), which generalizes to arbitrary groups the rotation number for homeomorphisms. For instance, this has been successfully used to understand the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{S}^1))$ for various discrete subgroups of Lie groups (see for instance Burger and Monod [20], Matsumoto [79], Mann [70], or Mann and Wolff [74]), mapping class groups (Mann and Wolff [75]) or of Thompson’s group T [44, 45]. On the other

hand there is no known complete invariant to understand semi-conjugacy classes of actions on $M = \mathbb{R}$ (see the discussion in §1.6), and here there are fewer groups for which the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is well understood (beyond deciding whether it is empty). Most results concern rather small groups. For example, if G does not contain a free semigroup on two generators (e.g. if G has subexponential growth), then every action in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is semi-conjugate to an action taking values in the group of translations $(\mathbb{R}, +)$ (see Navas [90]), and actions up to semi-conjugacy can be completely classified for some classes of solvable groups such as polycyclic groups (Plante [97]) or the solvable Baumslag–Solitar groups $\text{BS}(1, n)$ (see [98]). Some further classification results for actions on the line can also be obtained by considering lifts of actions on the circle, but the scope of this method is rather limited: for instance one can show that the central lifts of the actions of the whole group $\text{Homeo}_0(\mathbb{S}^1)$ (Milton [85]), or of Thompson’s group T (see [81, Theorem 8.6]) are the unique actions of such groups on the line up to semi-conjugacy.

The main goal of this work is to prove structure theorems for actions on the real line of various classes of groups which arise as sufficiently rich subgroups of the group of homeomorphisms of an interval. Throughout the introduction we fix an open interval $X = (a, b) \subseteq \mathbb{R}$ with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$. We are interested in studying actions on the real line and intervals of groups $G \subset \text{Homeo}_0(X)$. The action of G on X will be referred to as *the standard action* of G . Of course there would be no loss of generality in assuming $X = \mathbb{R}$, but since we will be dealing with more general actions of G on the real line it is convenient to keep a separate notation for X . The precise assumptions we will make on G will depend on the situation, but the most important one is that G is a *locally moving* group. To recall this notion, given an open interval $I \subset X$, we denote by $G_I \subset G$ the subgroup consisting of elements that fix $X \setminus I$ pointwise (called the *rigid stabilizer* of I).

Definition 1.1. Let $X = (a, b)$ be an open interval. We say that a subgroup $G \subset \text{Homeo}_0(X)$ is *locally moving* if for every $I \subset X$ the subgroup G_I acts on I without fixed points.

The most basic example of a locally moving group is $\text{Homeo}_0(X)$ itself. However there are also many finitely generated (and even finitely presented) groups that admit a locally moving action on an interval: a relevant example is Thompson’s group F , as well as many other related groups studied in the literature.

The main question addressed in this work is the following.

Main Question. Let $G \subset \text{Homeo}_0(X)$ be a locally moving group of homeomorphisms of an open interval X . What are the possible actions of G on the real line, and in particular under which conditions an action $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ must be semi-conjugate to its standard action on X ?

Evidence towards rigidity of actions of locally moving groups on the real line comes from a general reconstruction theorem of Rubin [100, 101] holding for groups of homeomorphisms of locally compact spaces. In this (very special) case, Rubin’s theorem implies that any group isomorphism between two locally moving groups of homeomorphisms of intervals must be implemented by a topological conjugacy; equivalently for a locally moving group $G \subset \text{Homeo}_0(X)$, the standard action on X is the unique faithful *locally moving* action of G . This result does not allow to draw many conclusions on more general actions of G which may fail to be locally moving (see Lodha [64] for some conditions ensuring this). Some further evidence towards rigidity comes from the fact that the group $\text{Homeo}_0(\mathbb{R})$ is known to admit a unique action on the real line up to conjugacy by a result of Milton [85] (see also Mann [69]), and the same result was recently shown for the group of compactly supported diffeomorphisms

of \mathbb{R} by Chen and Mann [24]. However, for smaller (e.g. finitely generated) locally moving groups very little appears to be known, even for well-studied cases such as Thompson’s group F . In fact it turns out that smaller groups admit a much rich variety of actions on the line than one might guess based on the previous results, and satisfy a combination of rigidity and flexibility phenomena. In particular many of them (but not all!) admit a vast class of “exotic” actions on the real line.

Before proceeding to discuss our main results, let us clarify what we mean (or rather what we *don’t* mean) by “exotic” action in this context. When $G \subset \text{Homeo}_0(X)$ is a locally moving group acting on $X = (a, b)$, it is often possible to obtain new actions of G on the real line by considering actions induced from *proper quotients* of G . To explain this, we denote by $G_c \subset G$ the normal subgroup of G consisting of compactly supported elements (that is those elements that act trivially on a neighborhood of a and b), then a well-known simplicity argument shows that the commutator subgroup $[G_c, G_c]$ of G_c is simple and contained in every non-trivial normal subgroup of G (see Proposition 4.4). Thus, if $G = [G_c, G_c]$ then G is simple, and otherwise the group $G/[G_c, G_c]$ is the largest proper quotient of G . In particular whenever G is finitely generated the associated *groups of germs* $\text{Germ}(G, a)$ and $\text{Germ}(G, b)$ are non-trivial quotients of G which moreover can be faithfully represented inside $\text{Homeo}_0(\mathbb{R})$ (see [69]). Thus finitely generated locally moving groups admit non-trivial, yet non-faithful, actions on the real line factoring throughout its germs¹. Non-faithful actions of G correspond to actions of the largest quotient $G/[G_c, G_c]$ and can be studied separately. For example the largest quotient of Thompson’s group F coincides with its abelianization $F = F/[F, F] \cong \mathbb{Z}^2$, whose actions on the real line are always semi-conjugate to an action by translations arising from a homomorphism to $(\mathbb{R}, +)$. In view of this, when $G \subset \text{Homeo}_0(X)$ is locally moving, we will reserve the term *exotic action* for actions $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ which are *not semi-conjugate to the standard action of G on X , nor to any action induced from the quotient $G/[G_c, G_c]$.*

1.2. Actions by C^1 diffeomorphisms. We begin with a result for actions of locally moving groups on the line by diffeomorphisms of class C^1 , which states that such actions are never exotic in the sense clarified above.

Theorem 1.2 (C^1 rigidity of locally moving groups). *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a locally moving group. Then every irreducible action $\varphi: G \rightarrow \text{Diff}_0^1(\mathbb{R})$ is semi-conjugate either to the standard action of G on X , or to a non-faithful action (induced from an action of the largest quotient $G/[G_c, G_c]$).*

For actions on closed intervals one can rule out the non-faithful case under some mild additional assumption on G . Given an interval $X = (a, b)$, we say that a group $G \subset \text{Homeo}_0(X)$ has *independent groups of germs at the endpoints* if for every $g_1, g_2 \in G$, there is $g \in G$ which coincides with g_1 on a neighborhood of a and with g_2 on a neighborhood of b .

Corollary 1.3. *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a locally moving group with independent groups of germs at the endpoints. Let $\varphi: G \rightarrow \text{Diff}^1([0, 1])$ be a faithful action with no fixed point in $(0, 1)$. Then the φ -action of G on $(0, 1)$ is semi-conjugate to its standard action on X .*

¹Even if one restricts to study faithful actions, it should be kept in mind that a faithful action can be *semi-conjugate* to an action arising from a proper quotient. Indeed, one can start from a non-faithful action of a countable group and perform a “Denjoy’s blow up” by replacing each point of an orbit by an interval, and then extend the action to these intervals in a faithful way by identifying them equivariantly with X . This phenomenon can never happen for actions which are *minimal and faithful*, and we will often restrict to such.

The condition in the previous corollary is satisfied by many locally moving groups. In particular Corollary 1.3 applies to Thompson’s group F . However an interesting feature of the proof of Theorem 1.2 is that the group F plays a key role in an intermediate step. In fact this proof combines three ingredients. The first is a general trichotomy for C^0 actions of locally moving groups on the line (Theorem 5.3), which is the common building ground for all the results in this paper. The second is the fact that every locally moving group over an interval contains many copies of F (Proposition 4.8), as follows from an argument based on a presentation of F going back to Brin [18] and extensively exploited and popularized by Kim, Koberda, and Lodha [55] under the name “2-chain lemma”. These two results together imply that if a locally moving group G admits an exotic C^1 action on the line, then one can find an embedding of F in the C^1 centralizer of a diffeomorphism of a compact interval without fixed points in the interior. The third and last step consists in showing that the group F cannot admit such an embedding (this uses the C^1 Sacksteder’s theorem from [34] and an elementary version of Bonatti’s approximate linearization [12, 16], together with algebraic properties of F). We note that, while the abundance of copies of F inside rich groups of homeomorphisms of the line is a long-standing fact [18, 55], this seems to have been rarely exploited to prove general results about such groups.

1.3. On existence of exotic actions by homeomorphisms. The rigidity displayed in Theorem 1.2 fails in the C^0 setting. Perhaps the simplest way to give counterexamples is to consider countable groups of compactly supported homeomorphisms. For such groups, one can always obtain exotic actions via two general constructions presented in §5.3.1 and §5.3.2, which yield two proofs of the following fact.

Fact 1.4. Let X be an open interval, and let $G \subset \text{Homeo}_0(X)$ be a countable group of compactly supported homeomorphisms of X acting minimally on X . Then G admits irreducible actions on \mathbb{R} which are not semi-conjugate to its standard action on X , nor to any non-faithful action of G .

While this observation is formally sufficient to rule out the C^0 version of Theorem 1.2, it is not fully satisfactory, for instance because a group G as in Fact 1.4 cannot be finitely generated. In fact, the proofs from §5.3.1 and §5.3.2 yield actions which admit no non-empty closed invariant set on which the group acts minimally (in particular, the action is *not* semi-conjugate to any minimal action nor to any cyclic action); this phenomenon is somewhat degenerate, and cannot arise for a finitely generated group (see [91, Proposition 2.1.12]). Much more interesting is the fact that many (finitely generated) locally moving groups admit exotic actions which are *minimal and faithful*. Various constructions of such actions of different nature will be provided in Section 9, and more constructions can be found in Section 11 in the special case of Thompson’s group F . Here we only mention the following existence criteria, based on a construction in §9.2, which are satisfied by some well-studied groups.

Proposition 1.5 (Criteria for existence of minimal exotic actions). *For $X = (a, b)$, let $G \subset \text{Homeo}_0(X)$ be a finitely generated subgroup. Assume that G acts minimally on X and contains non-trivial elements of relatively compact support in X , and that at least one of the following holds.*

- (a) *The group G is a subgroup of the group of piecewise projective homeomorphisms of X .*
- (b) *The groups of germs $\text{Germ}(G, b)$ is abelian and its non-trivial elements have no fixed points in a neighborhood of b .*

Then there exists a faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ which is not topologically conjugate to the action of G on X (nor to any non-faithful action of G).

1.4. Actions on planar real trees and \mathbb{R} -focal actions. We now leave briefly aside locally moving groups to introduce a concept that plays a central role in this paper: the notion of \mathbb{R} -focal action. This will be the main tool to understand exotic actions on the line of a vast class of locally moving groups (see Theorem 1.13 below).

In order to define this notion, we say that a collection \mathcal{S} of open bounded real intervals is a *cross-free cover* if it covers \mathbb{R} and every two intervals in \mathcal{S} are either disjoint or one is contained in the other.

Definition 1.6. Let G be a group. An action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is \mathbb{R} -focal if there exists a bounded open interval $I \subset \mathbb{R}$ whose G -orbit is a cross-free cover.

The terminology comes from group actions on trees (and Gromov hyperbolic spaces). In this classical setting, an isometric group action on a tree \mathbb{T} is called *focal* if it fixes a unique end $\omega \in \partial\mathbb{T}$ and contains hyperbolic elements (which necessarily admit ω as an attracting or repelling fixed point). The dynamics of an \mathbb{R} -focal action on the line closely resembles the dynamics of the action on the boundary of a focal action on a tree. In fact this is more than an analogy: every \mathbb{R} -focal action on the line can be encoded by an action on a tree, except that we need to consider group actions on *real trees* (or \mathbb{R} -trees) by homeomorphisms (not necessarily isometric). Let us give an overview of this connection (for more precise definitions and details we refer to Section 8).

Recall that a real tree is a metrizable space \mathbb{T} where any two points can be joined by a unique path, and which admits a compatible metric which makes it geodesic. By a *directed tree* we mean a (separable) real tree \mathbb{T} together with a preferred end $\omega \in \partial\mathbb{T}$, called the *focus*². If \mathbb{T} is a directed tree with focus ω , we write $\partial^*\mathbb{T} := \partial\mathbb{T} \setminus \{\omega\}$. An action of a group G on a directed tree \mathbb{T} (by homeomorphisms) is always required to fix the focus. In this topological setting we will say that such an action is *focal* if for every $v \in \mathbb{T}$ there exists a sequence $(g_n) \subset G$ such that $(g_n.v)$ approaches ω along the ray $[v, \omega]$.

By a *planar directed tree* we mean a directed tree \mathbb{T} endowed with a *planar order*, which is the choice of a linear order on the set of directions below every branching point of \mathbb{T} (one can think of \mathbb{T} as embedded in the plane; see §8.2 for a formal definition). Note that in this case the set $\partial^*\mathbb{T}$ inherits a linear order \prec in a natural way. Assume that \mathbb{T} is a planar directed tree, and that $\Phi: G \rightarrow \text{Homeo}(\mathbb{T})$ is a focal action of a countable group which preserves the planar order. Then Φ induces an order-preserving action of G on the ordered space $(\partial^*\mathbb{T}, \prec)$. From this action one can obtain an action $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ on the real line (for instance by considering the Dedekind completion of any orbit in $\partial^*\mathbb{T}$, see §2.3.3 and §8.2 for details), which we call the *dynamical realization* of the action of Φ . It turns out that such an action is always minimal and \mathbb{R} -focal. In fact, we have the following equivalence, which can be taken as an alternative definition of \mathbb{R} -focal actions.

Proposition 1.7. *Let G be a countable group. An action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is minimal and \mathbb{R} -focal if and only if it is conjugate to the dynamical realization of a focal action by homeomorphisms of G on some planar directed tree.*

We will say that an \mathbb{R} -focal action φ can be represented by an action $\Phi: G \rightarrow \text{Homeo}(\mathbb{T})$ on a planar directed tree if it is conjugate to the dynamical realization of Φ . Note that in general such an action Φ representing φ is not unique.

Examples of \mathbb{R} -focal actions appear naturally in the context of solvable groups. In fact, the notion of \mathbb{R} -focal action was largely inspired by an action on the line of the group $\mathbb{Z} \wr \mathbb{Z}$

²While this is not exactly the definition of directed tree which we will work with (see Definition 8.1), it can be taken as equivalent definition for the purpose of this introduction (the connection is explained in §8.1.4).

constructed by Plante [97] to give an example of action of a solvable group on the line which is not semi-conjugate to any action by affine transformations (i.e. transformations of the form $x \mapsto ax + b$). See Example 7.10 for a generalization of Plante’s construction to arbitrary wreath products. For finitely generated solvable groups we obtain the following dichotomy.

Theorem 1.8. *Let G be a finitely generated solvable group. Then every irreducible action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is either semi-conjugate to an action by affine transformations, or to a minimal \mathbb{R} -focal action.*

Remark 1.9. This should be compared with the result by Plante, which motivated the construction of the action of $\mathbb{Z} \wr \mathbb{Z}$ mentioned above, that every irreducible action of a solvable group of *finite (Prüfer) rank* is semi-conjugate to an affine action (a weaker condition on the group is actually sufficient, see [97, Theorem 4.4] for details).

A distinctive feature of \mathbb{R} -focal actions is that the action of individual elements of the group satisfy a dynamical classification which resembles the classification of isometries of trees into elliptic and hyperbolic elements. Namely if $G \subset \text{Homeo}_0(\mathbb{R})$ is a subgroup whose action is \mathbb{R} -focal, then every element $g \in G$ satisfies one of the following (see Corollary 8.27).

- Either g is *totally bounded*: its set of fixed points accumulates on both $\pm\infty$.
- Or g is a *pseudohomothety*: it has a non-empty compact set of fixed points $K \subset \mathbb{R}$ and either every $x \notin [\min K, \max K]$ satisfies $|g^n(x)| \rightarrow \infty$ as $n \rightarrow +\infty$ (in which case we say that g is an *expanding* pseudohomothety), or the same holds as $n \rightarrow -\infty$ (in which case we say that g is *contracting*). If K is reduced to a single point, we further say that g is a *homothety*.

Moreover the dynamical type of each element can be explicitly determined from the G -action on a planar directed tree \mathbb{T} representing the \mathbb{R} -focal action by looking at the local dynamics near the focus $\omega \in \partial\mathbb{T}$.

We finally discuss another concept: the notion of *horograding* of an \mathbb{R} -focal action of a group G by another action of G . This will be crucial in the sequel, as it will allow us to establish a relation between exotic actions of various locally moving groups and their standard actions. Assume that \mathbb{T} is a directed tree with focus ω . An *increasing horograding* of \mathbb{T} by a real interval $X = (a, b)$ is a map $\pi: \mathbb{T} \rightarrow X$ such that for every $v \in \mathbb{T}$ the ray $[v, \omega[$ is mapped homeomorphically onto the interval $[\pi(v), b)$. This is a non-metric analogue of the classical metric notion of horofunction associated with ω . A *decreasing horograding* is defined analogously but maps $[v, \omega[$ to $(a, \pi(v)]$, and a *horograding* is an increasing or decreasing horograding. If G is a group acting both on \mathbb{T} and X we say that π is a *G -horograding* if it is G -equivariant. This leads to the following definition.

Definition 1.10. Assume that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a minimal \mathbb{R} -focal action, and that $j: G \rightarrow \text{Homeo}_0(X)$ is another action of G on some open interval X . We say that φ can be (increasingly or decreasingly) horograded by j if φ can be represented by an action on a planar directed tree $\Phi: G \rightarrow \text{Homeo}(\mathbb{T})$ which admits an (increasing or decreasing) G -horograding $\pi: \mathbb{T} \rightarrow X$.

The existence of such a horograding is a tight relation between φ and j , which is nevertheless quite different from the notion of semi-conjugacy: here the action of G on X plays the role of a hidden “extra-dimensional direction” with respect to the real line on which φ is defined. For instance, in the presence of an increasing G -horograding, the type of each element in φ can be determined from its germ on the rightmost point of X as follows.

Proposition 1.11. *Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action, and assume that φ can be increasingly horograded by an action $j: G \rightarrow \text{Homeo}_0(X)$ on an interval $X = (a, b)$. Then we have the following alternative.*

- (i) *If the fixed points of $j(g)$ accumulate on b then $\varphi(g)$ is totally bounded.*
- (ii) *Else $\varphi(g)$ is a pseudohomothety, which is expanding if $j(g)(x) > x$ for x in a neighborhood of b , and contracting otherwise. Moreover if $j(g)$ has no fixed points in X , then $\varphi(g)$ is a homothety.*

1.5. A structure theorem for C^0 actions. With the notion of \mathbb{R} -focal action in hand, we now go back to our original problem and state a structure theorem for actions on the line by homeomorphisms of a vast class of locally moving group.

Definition 1.12 (The classes \mathcal{F} and \mathcal{F}_0). For $X = (a, b)$, let $G \subset \text{Homeo}_0(X)$ be a subgroup, and recall that for an interval $I = (x, y) \subseteq X$ we denote by $G_{(x,y)}$ the rigid stabilizer of I . Write $G_+ := \bigcup_{x < b} G_{(a,x)}$ and $G_- := \bigcup_{x > a} G_{(x,b)}$ for the subgroups of elements with trivial germ at a and b respectively. Consider the following conditions.

- (i) G is locally moving.
- (ii) There exist two finitely generated subgroups $\Gamma_{\pm} \subset G_{\pm}$ and $x, y \in X$ such that $G_{(a,x)} \subset \Gamma_+$ and $G_{(y,b)} \subset \Gamma_-$.
- (iii) There exists an element of G without fixed points in X .

We say that G belongs to the class \mathcal{F} if it satisfies (i) and (ii), and that it belongs to the class \mathcal{F}_0 if it satisfies (i), (ii), and (iii).

Note that condition (ii) trivially holds true provided there exist $x, y \in X$ such that $G_{(a,x)}$ and $G_{(y,b)}$ are finitely generated. In practice this weaker condition will be satisfied in many examples, but (ii) is more flexible and more convenient to handle.

The class \mathcal{F}_0 contains many well-studied examples of finitely generated locally moving groups, including Thompson's group F , and all Thompson–Brown–Stein groups F_{n_1, \dots, n_k} [103], the groups of piecewise projective homeomorphisms of Lodha–Moore [65], and several other Bieri–Strebel groups [8]. It also contains various groups which are far from the setting of groups of piecewise linear or projective homeomorphisms: for instance every countable subgroup of $\text{Homeo}_0(X)$ is contained in a finitely generated group belonging to \mathcal{F}_0 (see Proposition 10.2).

Our main result is a qualitative description for the actions on the line of groups in the class \mathcal{F} , stating that such actions can be classified into three types of behavior.

Theorem 1.13 (Main structure theorem for actions of groups in \mathcal{F}). *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a group in the class \mathcal{F} . Then every irreducible action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is semi-conjugate to an action in one of the following families.*

- (1) (Non-faithful) *An action which factors through the largest quotient $G/[G_c, G_c]$.*
- (2) (Standard) *An action which is conjugate to the standard action of G on X .*
- (3) (Exotic) *A minimal faithful \mathbb{R} -focal action which can be horograded by the standard action of G on X .*

Note that many finitely generated groups which belong to the class \mathcal{F} , or even to the more restricted class \mathcal{F}_0 , do indeed admit exotic actions falling in (3), for instance as a consequence of Proposition 1.5 (but not all groups in \mathcal{F}_0 do, see Theorem 1.20 below). Moreover in some cases there are uncountably many non-semi-conjugate such actions (see for instance Theorem 1.18 below for the case of Thompson's group F), and the variety and flexibility of constructions suggest that in general it is too complicated to obtain a reasonably explicit description of

all semi-conjugacy classes of exotic actions (however the word *explicit* is crucial here, as we shall see in Theorem 1.14 that such a description exists in principle). Nevertheless, the main content of Theorem 1.13 is that every exotic action of a group in \mathcal{F} must be tightly related to the standard action of G , although not via a semi-conjugacy but at the level of a planar directed tree encoding the exotic (\mathbb{R} -focal) action. This relation can be effectively exploited to study such exotic actions (in particular the way in which every individual element of G acts on an exotic action is determined by its standard action, see Proposition 1.11). This leads to various applications that we describe now.

1.6. Space of semi-conjugacy classes and local rigidity for groups in \mathcal{F}_0 . The main structure theorem (Theorem 1.13) can be used to get an insight on the structure of semi-conjugacy classes inside the space of irreducible actions $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ whenever G is a group in \mathcal{F}_0 . Recall that the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ can be endowed with the natural *compact-open topology*, which means that a neighborhood basis of a given action $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is defined by considering for every $\varepsilon > 0$, finite subset $S \subset G$, and compact subset $K \subset \mathbb{R}$, the subset of actions

$$\left\{ \psi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) : \max_{g \in S} \max_{x \in K} |\varphi(g)(x) - \psi(g)(x)| < \varepsilon \right\}.$$

A natural way to understand actions of a group G up to semi-conjugacy would be to study the quotient of the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ by the semi-conjugacy equivalence relation. Unfortunately this quotient space is rather bad, for instance it is in general not Hausdorff and may even fail to have the structure of a standard measurable space (see Remark 3.23). In fact the semi-conjugacy equivalence relation need not be *smooth*, that is it need not be possible to classify the semi-conjugacy classes of actions in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ by a Borel complete invariant (see the discussion in §3.5).

However, for groups in the class \mathcal{F}_0 , Theorem 1.13 can be used to show that a large part of the set of semi-conjugacy classes in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(G))$ can in fact be parametrized by a well behaved space and it is possible to select a system of representatives of each semi-conjugacy class with nice properties. Under this identification, it turns out that the semi-conjugacy class of the standard action of G on X corresponds to an *isolated point* in this space. This is the content of the following result.

Theorem 1.14 (The space of semi-conjugacy classes for groups in \mathcal{F}_0). *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a finitely generated group in the class \mathcal{F}_0 . Denote by \sim the equivalence relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ given by positive semi-conjugacy, and let $\mathcal{U} \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ be the \sim -invariant set of irreducible actions that are not semi-conjugate to any action induced from the largest quotient $G/[G_c, G_c]$. Then \mathcal{U} is open and the following hold.*

- (i) *The quotient space \mathcal{U}/\sim , with the quotient topology, is Polish and locally compact.*
- (ii) *There exists a continuous section $\sigma: \mathcal{U}/\sim \rightarrow \mathcal{U}$ which is a homeomorphism onto its image, whose image is closed in \mathcal{U} and consists of faithful minimal actions.*
- (iii) *The semi-conjugacy class of the standard action of G on X is an isolated point in \mathcal{U}/\sim .*

Remark 1.15. In the previous theorem, ruling out the actions induced from the largest quotient $G/[G_c, G_c]$ is necessary, as in this generality it may happen that the semi-conjugacy equivalence relation on irreducible actions of $G/[G_c, G_c]$ might not be smooth, and *a fortiori* this may not be true for G . However the theorem implies that the semi-conjugacy equivalence relation on the set $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is smooth provided the same holds for irreducible actions of $G/[G_c, G_c]$ (see Corollary 10.24, or Theorem 11.6 in the case of Thompson's group F).

The main dynamical significance of the previous result is that it implies a local rigidity result. We say that an irreducible action $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ of a group G is *locally rigid* if there exists a neighborhood $\mathcal{U} \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ of φ such that every $\psi \in \mathcal{U}$ is semi-conjugate to φ . Otherwise, we say that the action of φ is *flexible*. Theorem 1.14 has the following direct consequence.

Corollary 1.16 (Local rigidity of the standard action for groups in \mathcal{F}_0). *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a finitely generated group in the class \mathcal{F}_0 . Then the natural action of G on X is locally rigid.*

Corollary 1.16 provides a vast class of locally rigid actions on the real line. For instance, by Proposition 10.2, it implies that every action on the real line of a countable group G can be extended to a locally rigid action of some finitely generated overgroup.

A well-developed approach to local rigidity of group actions on the line is through the space $\text{LO}(G)$ of left-invariant orders on G , which is a totally disconnected compact space. Every such order gives rise to an action in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$, and isolated points in $\text{LO}(G)$ produce locally rigid actions (see Mann and the third author [73, Theorem 3.11]). This can be used to show that some groups, for instance braid groups (see Dubrovin and Dubrovina [37], or the monograph by Dehornoy, Dynnikov, Rolfsen, and Wiest [31]), do have locally rigid actions. However the converse to this criterion does not hold, and this approach has been more fruitful in the opposite direction, namely for showing that a group has no isolated order from flexibility of the dynamical realization (see for instance the works by Navas [90], or by Alonso, and the first and third named authors [2, 3], as well as by Malicet, Mann, and the last two authors [66]). One difficulty underlying this approach is that it is usually not easy to determine when two orders in $\text{LO}(G)$ give rise to semi-conjugate actions.

To prove Theorem 1.14, we take a different approach to local rigidity, based on a compact space suggested by Deroin [32] as a dynamical substitute of the space of left-orders. One way to construct this space is based on work by Deroin, Kleptsyn, Navas, and Parwani [35] on symmetric random walks on $\text{Homeo}_0(\mathbb{R})$. Given a probability measure μ on G whose support is finite, symmetric, and generates G , one defines the *Deroin space* $\text{Der}_\mu(G)$ as the subspace of $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ of (normalized) *harmonic actions*, that is, actions of G for which the Lebesgue measure is μ -stationary (see §2.2 for details). The space $\text{Der}_\mu(G)$ is compact and Hausdorff, with a natural topological flow $\Phi : \mathbb{R} \times \text{Der}_\mu(G) \rightarrow \text{Der}_\mu(G)$ defined on it by the conjugation action of the group of translations $(\mathbb{R}, +)$, and has the property that two actions in $\text{Der}_\mu(G)$ are (positively semi-)conjugate if and only if they are on the same Φ -orbit. It was shown in [35] that every action in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ can be semi-conjugated to one inside $\text{Der}_\mu(G)$ by a probabilistic argument. In fact, we shall show in Section 3 that this can be done in such a way that the new action depends continuously on the original one (see Theorem 3.2). This continuous dependence implies a criterion for local rigidity of minimal actions within the space $\text{Der}_\mu(G)$ (see Corollary 3.4). The proof is based on an alternative description of $\text{Der}_\mu(G)$ as a quotient of the space of left-invariant *preorders* on G (Theorem 3.20), which also implies that $\text{Der}_\mu(G)$ does not depend on the choice of the probability measure μ up to homeomorphism.

With these tools in hand, to prove Theorem 1.14 we study the shape of orbits of the translation flow Φ on $\text{Der}_\mu(G)$ when G is a group in \mathcal{F}_0 . Faithful actions correspond to an open Φ -invariant subset of $\text{Der}_\mu(G)$. Using Theorem 1.13 and the properties of \mathbb{R} -focal actions we show that the restriction of the flow Φ to this open subset has the simplest possible type of orbit structure: there is a closed transversal \mathcal{S} intersecting every Φ -orbit, and Φ is conjugate to the vertical translation on $\mathcal{S} \times \mathbb{R}$, so that its space of orbits can be identified with \mathcal{S} . This

analysis yields Theorem 1.14. (See Figure 1 for a schematic representation for Thompson's group F .)

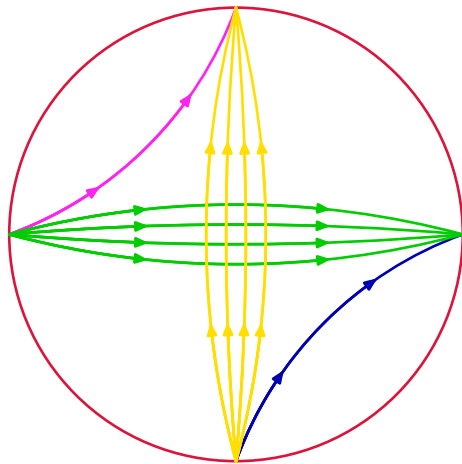


FIGURE 1. The Deroin space of Thompson's group F . The outer red circle parametrizes the non-faithful actions (of $F^{ab} \cong \mathbb{Z}^2$) and it is pointwise fixed by the flow Φ . The remaining Φ -orbits are the faithful actions. The purple and blue orbits correspond to the standard action and to its conjugate by the reflection, and are transversely isolated (this gives local rigidity). The pencils of green and yellow orbits are the \mathbb{R} -focal actions: both pencils contain uncountably many orbits, admit a compact transversal to the flow, and the shown convergence to limit points is uniform. See Figure 10 and §11.2 for details.

1.7. A criterion for non-orientation-preserving groups. We also observe that the situation is quite different if we leave the setting of orientation-preserving actions. Indeed, in this case we have the following result which follows from Corollary 1.16 (or can be more directly obtained in the course of its proof). The proof is given in §10.3. We denote by $\text{Homeo}(X)$ the group of all (not necessarily orientation-preserving) homeomorphisms of an interval X .

Corollary 1.17 (Global rigidity in the non-orientation-preserving case). *Let $X = (a, b)$ and let $G \subset \text{Homeo}(X)$ be a subgroup such that $G \subsetneq \text{Homeo}_0(X)$ and $G \cap \text{Homeo}_0(X)$ is in the class \mathcal{F} . Then every faithful minimal action $\varphi: G \rightarrow \text{Homeo}(\mathbb{R})$ is conjugate to the standard action on X .*

1.8. Some concrete groups in the class \mathcal{F}_0 . We have seen how Theorem 1.14 yields a global picture of actions on the line for groups in the class \mathcal{F}_0 . Nevertheless when G is a group in the class \mathcal{F} the rigidity or abundance of its exotic actions turns out to depend subtly on G . This difference is visible already among groups of piecewise linear homeomorphisms.

Given an open interval $X = (a, b)$ we denote by $\text{PL}(X)$ the group of orientation-preserving PL homeomorphisms of X , with finitely many discontinuity points for the derivatives. It turns out that this class of groups exhibits a surprising mixture of rigidity and flexibility properties and many illustrative examples of applications of our results arise as subgroups of $\text{PL}(X)$. Note that Proposition 1.5 implies that every finitely generated locally moving group $G \subset \text{PL}(X)$ admits a minimal faithful exotic action on the real line.

The most famous example of group of PL homeomorphisms is Thompson's group F . Recall that F is defined as the subgroup of $\text{PL}((0, 1))$ whose slopes are powers of 2 and whose constant

terms and breakpoints are in the ring $\mathbb{Z}[1/2]$ of dyadic rationals. The group F belongs to the class \mathcal{F}_0 and satisfies the assumptions of most of the results in this paper. The reader can find in Section 11 a detailed account of our results in this case. In particular every faithful action $\varphi: F \rightarrow \text{Diff}_0^1([0, 1])$ without fixed points in $(0, 1)$ is semi-conjugate to the standard action (Corollary 1.3), every irreducible exotic action $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ is \mathbb{R} -focal and horograded by the standard action of F on $(0, 1)$ (Theorem 1.13), and the standard action of F on $(0, 1)$ is locally rigid (Corollary 1.16).

Despite these rigidity results, it turns out that the group F admits a rich and complicated universe of minimal exotic actions, and our work leaves some interesting questions open: in particular we do not know whether F admits exotic actions that are locally rigid. In Section 11 we will discuss several different constructions of minimal \mathbb{R} -focal actions of F , which for instance leads to the following.

Theorem 1.18. *Thompson’s group F admits uncountably many actions on the line which are faithful, minimal, \mathbb{R} -focal and pairwise non-semi-conjugate. Moreover, there are uncountably many such actions whose restrictions to the commutator subgroup $[F, F]$ remain minimal and are pairwise non-semi-conjugate.*

This abundance of \mathbb{R} -focal actions can be explained in part by the fact that the group F admits many focal actions on *simplicial* trees by simplicial automorphisms (the Bass–Serre tree associated with any non-trivial splitting as an ascending HNN extension [41, Proposition 9.2.5]). Some of these actions preserve a planar order on the tree, and many \mathbb{R} -focal actions of F arise in this way (a characterization will be given in §11.3). However, we show in §11.6 that the group F also admits minimal \mathbb{R} -focal actions for which an associated planar directed tree cannot be chosen to be simplicial, or even to carry an invariant \mathbb{R} -tree metric. In fact the second claim in Theorem 1.18 is directly related to this phenomenon. We refer to Section 11 for further discussion on \mathbb{R} -focal actions of F .

In contrast, this abundance of exotic actions of the group F already fails for some tightly related groups of PL homeomorphisms. Given a real number $\lambda > 1$, we denote by $G(X; \lambda)$ the group of all PL homeomorphisms of X where all derivatives are integers powers of λ and all constant terms and breakpoints belong to $\mathbb{Z}[\lambda, \lambda^{-1}]$. Note that the group F is equal to $G((0, 1); 2)$. When $X = \mathbb{R}$, the group $G(\mathbb{R}; \lambda)$ belongs to \mathcal{F}_0 provided λ is algebraic, thus satisfies Theorem 1.13. However, in striking difference with the case of F , we have the following.

Theorem 1.19 (PL groups with finitely many exotic actions). *Let $\lambda > 1$ be an algebraic real number. Then the group $G = G(\mathbb{R}; \lambda)$ admits exactly three minimal faithful actions $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ on the real line up to (semi-)conjugacy, namely its standard action and two minimal \mathbb{R} -focal actions (which can be horograded by its standard action).*

Note that this shows that the two minimal exotic actions of $G(\mathbb{R}; \lambda)$ are locally rigid as well; in particular the standard action need not be the unique locally rigid action for a group in \mathcal{F}_0 .

Theorem 1.19 is in fact a special case of Theorem 12.3 which applies to a more general class of *Bieri–Strebel groups* of the form $G(\mathbb{R}; A, \Lambda)$, which are groups of PL homeomorphisms defined by restricting derivatives to belong to a multiplicative subgroup $\Lambda \subset \mathbb{R}_+^*$ and constant terms and breakpoints to belong to a $\mathbb{Z}[\Lambda]$ -submodule $A \subset \mathbb{R}$ (see §2.4). The two exotic actions of the group $G(\mathbb{R}; \lambda)$ arise as a special case of a construction of exotic actions of Bieri–Strebel groups explained in §9.3.

Building on the proof of Theorem 1.19, in §12.2 we also construct a finitely generated locally moving group G having no exotic actions at all. This group is obtained by perturbing the group $G(\mathbb{R}; 2)$ locally in a controlled manner.

Theorem 1.20 (A finitely generated locally moving group with no exotic actions). *There exists a finitely generated subgroup $G \subset \text{Homeo}_0(\mathbb{R})$ in the class \mathcal{F}_0 , such that every faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is conjugate to the standard action.*

1.9. Further consequences. We conclude this introduction by mentioning some additional results obtained along the way, which allow to recover and strenghten some previously known results in the unified setting developed here.

1.9.1. Very large locally moving groups. As mentioned at the beginning of the introduction, two rigidity results were known for some very large locally moving groups: namely a result of Militon shows that the group $\text{Homeo}_c(\mathbb{R})$ of compactly supported homeomorphisms of \mathbb{R} admits a unique irreducible action on \mathbb{R} up to conjugacy, and a recent result of Chen and Mann implies the same property for the group $\text{Diff}_c^r(\mathbb{R})$ of compactly supported diffeomorphisms in regularity $r \neq 2$. While the main focus of this paper is on countable groups, our method also allows to obtain a criterion for uniqueness of the action of a family of uncountable locally moving groups, which recovers and generalizes those results. In order to do this, we need the following relative version of a group property first considered by Schreier (see [83, Problem 111]).

Definition 1.21. Let G be a group and let H be a subgroup of G . We say that the pair (G, H) has *relative Schreier property* if every countable subset of H is contained in a finitely generated subgroup of G .³

The following result is a special case of what we show in Section 13.

Theorem 1.22 (Uniqueness of action for some uncountable locally moving groups). *Let $G \subset \text{Homeo}_c(\mathbb{R})$ be a perfect subgroup of compactly supported homeomorphisms. Suppose that for every bounded open interval $I \subset \mathbb{R}$ the following hold:*

- (1) *the G_I -orbit of every $x \in I$ is uncountable;*
- (2) *the pair (G, G_I) has relative Schreier property.*

Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is conjugate to the natural action of G .

1.9.2. Comparison with the case of groups acting on the circle. One may wonder whether a similar theory can be developed for locally moving groups acting on the *circle* rather than the real line. However this problem turns out to be considerably simpler and less rich, as in this setting locally moving groups essentially do not admit exotic actions at all. This phenomenon is already well-known in some cases. In particular a result of Matsumoto states that the group $\text{Homeo}_0(\mathbb{S}^1)$ admits only one action on \mathbb{S}^1 up to conjugacy, and an unpublished result of Ghys (announced in [44]) shows that every action of Thompson's group T on \mathbb{S}^1 is semi-conjugate to its standard action. The original proofs of these results are both based on the computation of the second bounded cohomology of these groups, and on its relation with actions on the circle established by Ghys [42] (a tool that does not have an analogue for actions on real line). For Thompson's group T another proof is provided in [60], which does not use the knowledge of bounded cohomology of T , but relies in part on the nature of T as a group of piecewise linear homeomorphisms.

³The choice of *relative Schreier property* is inspired by the work of Le Roux and Mann [63]. Apparently there is no standard name in the literature, and such choice may be ambiguous, as in other works *Schreier property* is used for different properties.

In fact, in Section 14 we observe that elementary considerations based on commutations suffice to obtain a more general criterion (Theorem 14.2), stating that if G is a group of homeomorphisms of an arbitrary compact space X satisfying suitable assumptions, then every minimal faithful action of G on the circle must factor onto its action on X . A special case of Theorem 14.2 (with $X = \mathbb{S}^1$) yields the following.

Theorem 1.23. *Let $G \subset \text{Homeo}_0(\mathbb{S}^1)$ be a locally moving subgroup. Then every faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ factors onto the standard action of G on \mathbb{S}^1 .*

In fact, under some mild additional assumption on G one obtains that the action φ is conjugate to (possibly a lift of) the standard action on \mathbb{S}^1 via a self-cover of \mathbb{S}^1 . We refer to Section 14 for more precise results and further discussion.

1.9.3. *An application to non-smoothability.* One possible direction of application of Theorem 1.2 is to use it as a tool to show that certain locally moving group acting on the line cannot be isomorphic to groups of diffeomorphisms (of suitable regularity). Results in this spirit were obtained by Bonatti, Lodha, and the fourth named author in [15] for various groups of piecewise linear and projective homeomorphisms. Here we consider the Thompson–Brown–Stein groups F_{n_1, \dots, n_k} , which are a natural generalization of Thompson’s group F associated with any choice of integers n_1, \dots, n_k whose logarithms are linearly independent over \mathbb{Q} (see Definition 2.45). It was shown in [15] that if $k \geq 2$, then the standard action of F_{n_1, \dots, n_k} on $(0, 1)$ cannot be conjugate to an action by C^2 diffeomorphisms. With our methods we obtain the following strengthening of that result.

Theorem 1.24. *When $k \geq 2$, any Thompson–Brown–Stein group of the form F_{n_1, \dots, n_k} admits no faithful action on the real line by diffeomorphisms of class $C^{1+\alpha}$ for any $\alpha > 0$.*

In this direction we also mention a recent result of Kim, Koberda, and third named author [57], who used our Theorem 1.2 to show that the free product $F * \mathbb{Z}$ of Thompson’s group and the integers admits no faithful action on $[0, 1]$ by C^1 diffeomorphisms.

1.10. **Outline of the paper.** This paper is structured as follows. Sections 2 and 3 contain the required general theory on group actions on the line. After recalling some elementary preliminaries and notations in Section 2, we focus in Section 3 on the Deroin space $\text{Der}_\mu(G)$ of μ -harmonic actions on the line of a finitely generated group G . The main purpose of that section is to prove that there exists a continuous retraction from $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ to the space $\text{Der}_\mu(G)$ and a related local rigidity criterion (Corollary 3.4), which will be used in the proofs of Theorem 1.14 and Corollary 1.16 (see the discussion in §1.6). These results appear to be new, although the arguments in the section are mostly based on elementary facts on left-invariant preorders.

The remaining sections can be roughly divided into four parts. Part I contains the central skeleton of the study locally moving groups on which the remaining parts are built, while Parts II–IV are essentially independent from each other and can be read after reading Part I (modulo some minor dependence between them, mostly limited to examples or secondary results).

Part I (Sections 4–5). **General theory on locally moving groups acting on the line.** After introducing notations and terminology on locally moving groups of homeomorphisms of intervals and proving some of their basic properties in Section 4, we begin in Section 5 the study of their actions on the real line. The main result of this part is Theorem 5.3, which establishes a dynamical trichotomy for actions on the line of a

locally moving group G of homeomorphisms, a crucial common ingredient for all the other results in this paper. This result imposes a restrictive technical condition on all exotic actions of G , which roughly speaking says that in every exotic action of G there is “domination” of certain subgroups of G over other. After proving this, we give in §5.2 some of its simpler applications, such as a rigidity result for actions with nice combinatorial properties (including actions by piecewise analytic transformations). We conclude the section by giving some first examples of exotic actions of locally moving groups in §5.3 (which allow to prove Fact 1.4).

Part II (Section 6). **Actions by C^1 diffeomorphisms.** This part consists of Section 6 alone, where we study C^1 actions of locally moving group and complete the proofs of Theorem 1.2 and Corollary 1.3.

Part III (Sections 7–12). **\mathbb{R} -focal actions and topological dynamics of exotic actions.** Here we develop the notion of \mathbb{R} -focal action and apply it to prove our main results for actions by homeomorphisms of locally moving groups in the class \mathcal{F} . In Section 7 we introduce \mathbb{R} -focal actions of groups on the line and establish their main properties. In Section 8 we illustrate the connection of \mathbb{R} -focal action with focal actions on trees. (Note that Sections 7–8 are not directly concerned with locally moving groups, and could also be read directly after reading Section 2.) In Section 9 we give various constructions and existence criteria for exotic minimal \mathbb{R} -focal actions of locally moving groups. Section 10 contains our main results on groups in the classes $\mathcal{F}_0 \subset \mathcal{F}$. We begin by proving our main structure theorem (Theorem 1.13) for actions of groups in \mathcal{F} , and then use it to study the Deroin space $\text{Der}_\mu(G)$ of a group in \mathcal{F}_0 . As a corollary of this analysis and of the results in Section 3 we obtain Theorem 1.14 and the local rigidity of the standard action for groups in \mathcal{F}_0 (Corollary 1.16). In Section 11 we focus on the example of Thompson’s group F : we illustrate our main rigidity results in this special case, and use them to analyze its Deroin space $\text{Der}_\mu(F)$ (see Figure 10); then we proceed to give various constructions of minimal exotic actions of the group F and discuss some of their properties. In Section 12 we provide examples of groups in the class \mathcal{F}_0 whose \mathbb{R} -focal actions are much more rigid: we classify the \mathbb{R} -focal actions of a class of Bieri–Strebel groups of PL homeomorphisms (in particular we prove Theorem 1.19) and construct a finitely generated locally moving group without exotic actions (Theorem 1.20).

Part IV (Sections 13–14) **Additional results.** In this part we prove some additional results which allow to recover and generalize some previously known results from the literature in the framework of this paper. In Section 13 we prove results on actions of some very large (uncountable) locally moving groups and prove Theorem 1.22. In Section 14 we prove a result for actions on the circle, which implies Theorem 1.23: this section is essentially self-contained and independent on the other results in this paper (it only relies on Proposition 5.4).

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2. NOTATION AND PRELIMINARIES

2.1. Actions on the line. In this work we are mainly concerned with orientation-preserving actions on the real line, that is, homomorphisms $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$. We will almost always be interested in actions without (global) fixed points, which will sometimes be called *irreducible* for short. Note that every action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ can be described in terms of irreducible actions, just considering all restrictions

$$\begin{array}{ccc} \varphi_J : G \rightarrow \text{Homeo}_0(J) \cong \text{Homeo}_0(\mathbb{R}) \\ g \mapsto \varphi(g) \downarrow_J \end{array},$$

of the action φ to minimal $\varphi(G)$ -invariant open intervals $J \subset \mathbb{R}$. We write $\text{Hom}(G, \text{Homeo}_0(\mathbb{R}))$ for the space of order-preserving actions of the group G , endowed with the compact open topology, and $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) \subseteq \text{Hom}(G, \text{Homeo}_0(\mathbb{R}))$ for the subspace of actions without fixed points.

2.1.1. Some notation on actions. Given $f \in \text{Homeo}(\mathbb{R})$, we write $\text{Fix}(f) = \{x \in \mathbb{R} : f(x) = x\}$ for the set of fixed points and $\text{Supp}(f) = \mathbb{R} \setminus \text{Fix}(f)$ for its *support*. Note that by definition $\text{Supp}(f)$ is an open set (we do *not* take its closure unless specified). For $G \subset \text{Homeo}_0(\mathbb{R})$ and $x \in \mathbb{R}$ we write $\text{Stab}_G(x)$ for the stabilizer of x . We denote by $\text{Homeo}_c(\mathbb{R})$ the group of homeomorphisms whose support is relatively compact. Given an action $\varphi : G \rightarrow \text{Homeo}(\mathbb{R})$ and $g \in G$ we set $\text{Fix}^\varphi(g) = \text{Fix}(\varphi(g))$ and $\text{Supp}^\varphi(g) = \text{Supp}(\varphi(g))$, and $\text{Stab}_G^\varphi(x)$ for the stabilizer. When there is no confusion we write $g.x$ instead of $\varphi(g)(x)$. The notation $g(x)$ will be reserved to the case when G is naturally given as a subgroup $G \subset \text{Homeo}_0(\mathbb{R})$ to refer to its standard action (but never to another action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$).

For $x \in \mathbb{R} \cup \{\pm\infty\}$ we denote by $\text{Germ}(x)$ the group of germs of homeomorphisms that fix x . Recall that this is defined as the group of equivalence classes of homeomorphisms $f \in \text{Homeo}_0(\mathbb{R})$ that fix x , where two such homeomorphisms are identified if they coincide on a neighborhood of x . By considering only one-sided neighborhoods, one gets two groups $\text{Germ}_-(x)$ and $\text{Germ}_+(x)$, the groups of right germs and the group of left germs respectively, such that $\text{Germ}(x) = \text{Germ}_-(x) \times \text{Germ}_+(x)$. If G is a group of homeomorphisms that fixes x , we denote by $\text{Germ}(G, x)$ the group of germs induced by elements of G , and by $\mathcal{G}_x : G \rightarrow \text{Germ}(G, x)$ the associated homomorphism. Similarly we have groups of one-sided germs $\text{Germ}_\pm(G, x)$. Note however that when G acts non-trivially only on one side of G (e.g. if x is an endpoint of its support, or if $x = \pm\infty$), we omit the sign \pm as this is clear from the context.

2.1.2. Commuting actions. A particular case of what said above is that every homeomorphism $g \in \text{Homeo}_0(\mathbb{R})$ is basically determined by its set of fixed points $\text{Fix}(g)$ and by how it acts on every connected component of its support $\text{Supp}(g) = \mathbb{R} \setminus \text{Fix}(g)$. Therefore, it is fundamental to understand the set of fixed points of a given element, or at least to be able to say whether it is empty or not. For this, a very useful relation is that for an element $g \in \text{Homeo}_0(\mathbb{R})$ and a subgroup $H \subseteq \text{Homeo}_0(\mathbb{R})$, one has $g(\text{Fix}(H)) = \text{Fix}(gHg^{-1})$. In particular, when $G \subseteq \text{Homeo}_0(\mathbb{R})$ and $H \triangleleft G$ is a normal subgroup, then for every $g \in G$ one has $g(\text{Fix}(H)) = \text{Fix}(H)$. This holds in particular for commuting subgroups, for which we have

the following observation, which is easily obtained using that the set of fixed point is a closed subset.

Lemma 2.1. *Consider two commuting subgroups H_1 and H_2 of $G \subseteq \text{Homeo}_0(\mathbb{R})$ (that is, $[h_1, h_2] = \text{id}$ for every $h_1 \in H_1$ and $h_2 \in H_2$). Suppose that both $\text{Fix}(H_1)$ and $\text{Fix}(H_2)$ are non-empty. Then H_1 and H_2 admit a common fixed point.*

This lemma will be used in the text without explicit reference.

2.1.3. Semi-conjugacy. It is customary in the field to consider actions up to *semi-conjugacy*. This means that not only we do not really take care of the choice of coordinate on \mathbb{R} (which corresponds to the classical notion of *conjugacy*), but we want to consider only the interesting part of the dynamics of the action. This was first formalized by Ghys in his work on bounded Euler class, but the definition has been unanimously fixed only recently. We follow here [56, Definition 2.1], although we allow order-reversing semi-conjugacies. For the statement we will say that a map $h : \mathbb{R} \rightarrow \mathbb{R}$ is *proper* if its image $h(\mathbb{R})$ is unbounded in both directions of the line.

Definition 2.2. Let $\varphi, \psi : G \rightarrow \text{Homeo}(\mathbb{R})$ be two actions of a group G on the real line. We say that φ and ψ are *semi-conjugate* if there exists a proper monotone map $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(2.1) \quad h\varphi(g) = \psi(g)h \quad \text{for every } g \in G.$$

Such a map h is called a *semi-conjugacy* between φ and ψ . Note that if φ and ψ are irreducible, the requirement that the map h be proper follows automatically from the equivariance (2.1) and thus can be dropped. When $h \in \text{Homeo}(\mathbb{R})$, we say that h is a *conjugacy* between φ and ψ , in which case we say that h is a *conjugacy*. When h is nondecreasing, we say that φ and ψ are *positively semi-conjugate*.

Remark 2.3. Both conjugacy and semi-conjugacy are equivalence relations (for conjugacies this is obvious, for semi-conjugacies the reader can check [56, Lemma 2.2] or [36, Proposition 1.1.16]). Notice that in the semi-conjugacy case, we do not require that h be continuous; indeed, being continuously semi-conjugate is not even a symmetric relation.

Rarely (essentially only in Section 14) we will need the analogous notion of semi-conjugacy for actions on the circle, which is defined as follows.

Definition 2.4. Let $\varphi, \psi : G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ be two actions on the circle. They are semi-conjugate if and only if there exists a group \tilde{G} which is a central extension of G of the form $1 \rightarrow C \rightarrow \tilde{G} \rightarrow G \rightarrow 1$, with $C \cong \mathbb{Z}$, and two semi-conjugate actions $\tilde{\varphi}, \tilde{\psi} : \tilde{G} \rightarrow \text{Homeo}_0(\mathbb{R})$ which both map C to the group \mathbb{Z} of integer translations, and which descend to the quotient respectively to the actions φ and ψ of $G = \tilde{G}/C$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Given an action $\varphi : G \rightarrow \text{Homeo}(\mathbb{R})$, one can consider the *reversed action* $\hat{\varphi} : G \rightarrow \text{Homeo}(\mathbb{R})$, defined by conjugating φ by the order-reversing isometry $x \mapsto -x$. After our definition, the actions φ and $\hat{\varphi}$ are conjugate.

Given a monotone map $h : \mathbb{R} \rightarrow \mathbb{R}$, we denote by $\text{Gap}(h) \subset \mathbb{R}$ the open subset of points at which h is locally constant. We also write $\text{Core}(h) = \mathbb{R} \setminus \text{Gap}(h)$. Note that when $h : \mathbb{R} \rightarrow \mathbb{R}$ is a semi-conjugacy between two actions $\varphi, \psi : G \rightarrow \text{Homeo}(\mathbb{R})$ without fixed points (in the sense that the equivariance (2.1) holds), then $\text{Core}(h)$ is a closed $\varphi(G)$ -invariant subset and $h(\text{Core}(h)) = h(\mathbb{R})$. The following folklore result is a sort of converse to this observation, and it describes the inverse operation of Denjoy's blow up of orbits (compare for instance [56, Theorem 2.2]).

Theorem 2.5. *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points and let $F \subset \mathbb{R}$ be a non-empty closed $\varphi(G)$ -invariant subset. Then there exists an action $\varphi_F : G \rightarrow \text{Homeo}_0(\mathbb{R})$ and a continuous positive semi-conjugacy $h : \mathbb{R} \rightarrow \mathbb{R}$ between φ and φ_F such that $\text{Core}(h) = F$.*

We follow with an easy application of Theorem 2.5 that will be repeatedly used in the article when discussing actions coming from quotients.

Corollary 2.6. *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points, and let $N \triangleleft G$ be a normal subgroup. Then φ is semi-conjugate to an action of the quotient G/N if and only if $\text{Fix}^\varphi(N) \neq \emptyset$.*

Proof. Notice that, since N is normal, the subset $F = \text{Fix}^\varphi(N)$ is closed and $\varphi(G)$ -invariant. Assume that $F \neq \emptyset$; we consider the action φ_F given by Theorem 2.5. Clearly we have $N \subseteq \ker \varphi_F$. The other implication is trivial. \square

The following lemma basically states that the semi-conjugacy class is determined by the action on one orbit.

Lemma 2.7. *Consider two actions $\varphi, \psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points. Let $\Omega \subset \mathbb{R}$ be a non-empty $\varphi(G)$ -invariant subset and $j_0 : \Omega \rightarrow \mathbb{R}$ a monotone map which is equivariant in the sense that it satisfies*

$$\psi(g)j_0 = j_0\varphi(g) \upharpoonright_\Omega \quad \text{for every } g \in G.$$

Then, j_0 can be extended to a semi-conjugacy $j : \mathbb{R} \rightarrow \mathbb{R}$ between φ and ψ .

Proof. Consider the map $j : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$j(x) = \sup\{j_0(y) : y \in \Omega \text{ and } y \leq x\}.$$

Since j_0 is monotone, we get that j is an extension of j_0 . Also note, it is direct from the definition of j that it is monotone and equivariant and therefore it defines a semi-conjugacy as desired. \square

2.1.4. *Canonical model.* We next introduce a family of canonical representatives for the semi-conjugacy relation when G is finitely generated. Such representatives are well defined up to conjugacy. Later, in §2.2, we will define a (much less redundant) family of representatives for the semi-conjugacy relation consisting on normalized μ -harmonic actions.

Definition 2.8. Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action, we say that a non-empty subset $\Lambda \subset \mathbb{R}$ is a *minimal invariant set* for $\varphi(G)$ if it is closed, $\varphi(G)$ -invariant and contains no proper, closed and $\varphi(G)$ -invariant subsets. When $\Lambda = \mathbb{R}$, we say that the action φ is *minimal*. When Λ is a proper perfect subset we say that Λ is an *exceptional minimal invariant set*. On the other hand, when the image of an action $\varphi(G)$ is generated by a homeomorphism without fixed points, we say that the action φ is *cyclic*.

Remark 2.9. Note that minimal invariant sets may not exist for general group actions. An archetypical example is given by an action of the group $\mathbb{Z}^\infty = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ in which each canonical generator has fixed points but there is no fixed point for the action (this happens, for instance, in the dynamical realization of the lexicographic ordering of \mathbb{Z}^∞). However, when the acting group is finitely generated there is always a minimal invariant set for the action; see e.g. [91, Proposition 2.1.12].

Remark 2.10. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a semi-conjugacy between two actions $\varphi, \psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$, in the sense that (2.1) holds. It is immediate to show that when ψ is minimal, the semi-conjugacy h is continuous, and that when φ is also minimal, then h is a conjugacy. Indeed, the subsets $\text{Core}(h)$ and $\overline{h(\mathbb{R})}$ are closed subsets which are respectively $\varphi(G)$ - and $\psi(G)$ -invariant.

The following notion corresponds to the *minimalization* considered in [56, Definition 2.3].

Definition 2.11. An action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a *canonical model* if either it is a minimal action or it is a cyclic action.

We have the following consequence of Theorem 2.5.

Corollary 2.12. *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points that admits a non-empty minimal invariant set (this is automatic if G is finitely generated). Then, φ is semi-conjugate to a canonical model. Moreover, the canonical model is unique up to conjugacy.*

A similar discussion holds for actions on the circle. In this case we say that an action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ is in canonical model if it is either minimal or conjugate to an action whose image is a finite cyclic group of rotations (in which case we say that it is *cyclic*). However a crucial simplifying difference is that in this case, by compactness, every group action on the circle admits a non-empty minimal invariant set (regardless on whether G is finitely generated or not). This yields the following well-known fact (see e.g. [44]).

Proposition 2.13. *Let G be a group, then every action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ without fixed points is semi-conjugate to a canonical model.*

After Corollary 2.12, semi-conjugacy classes of finitely generated group actions can be divided into two very different families: cyclic actions and minimal actions. However, for practical purposes, it is preferable to split further the case of minimal action. In this work the following classical notion will be important.

Definition 2.14. For $M \in \{\mathbb{R}, \mathbb{S}^1\}$, we say that a minimal action $\varphi : G \rightarrow \text{Homeo}_0(M)$ is *proximal* if for every non-empty open intervals $I, J \subsetneq M$ with I bounded, there exists an element $g \in G$ such that $g.I \subset J$.

Remark 2.15. Note that if a subgroup $G \subseteq \text{Homeo}_0(M)$ commutes with a non-trivial element $h \in \text{Homeo}_0(M)$, then the G -action cannot be proximal. In fact this is a well-known observation for group actions on arbitrary compact spaces.

A crucial feature of minimal actions on the circle and the line is that a converse also holds. In the case of the circle this goes back to Antonov [6] and was rediscovered by Margulis [77], and an analogous result for the real line is provided by Maljutin [67]. Given a subgroup $G \subseteq \text{Homeo}_0(M)$ with $M \in \{\mathbb{R}, \mathbb{S}^1\}$, its *centralizer* (in $\text{Homeo}_0(M)$) is the subgroup

$$C(G) := \{h \in \text{Homeo}_0(M) : gh = hg \text{ for every } g \in G\}.$$

We will write $C^\varphi := C(\varphi(G))$ when $\varphi : G \rightarrow \text{Homeo}_0(M)$ is an action. Then we have the following two results.

Theorem 2.16 (Antonov [6], see [44]). *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ be a minimal action. Then we have the following alternative:*

- either C^φ is trivial, in which case φ is proximal, or
- C^φ is a finite cyclic subgroup without fixed points, and the action on the topological circle \mathbb{S}^1/C^φ is proximal, or

- C^φ is conjugate to the group of rotations \mathbb{R}/\mathbb{Z} and G is conjugate to a subgroup of it.

Theorem 2.17 (Malyutin [67], see also [36]). *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal action. Then we have the following alternative:*

- either C^φ is trivial, in which case φ is proximal, or
- C^φ is a cyclic subgroup without fixed points, and the action on the topological circle \mathbb{R}/C^φ is proximal, or
- C^φ is conjugate to the group of translations $(\mathbb{R}, +)$ and G is conjugate to a subgroup of it.

2.2. Harmonic actions and Deroin spaces. In this subsection we recall the definition of the *Deroin space*. For that, we fix a finitely generated group G , a probability measure μ on G whose support is finite, generates G , and satisfies the symmetry property

$$\mu(g) = \mu(g^{-1}) \quad \text{for every } g \in G.$$

We will denote by $S \subset G$ the support of μ .

Definition 2.18. Given a nontrivial action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$, we say that a Radon measure ν on \mathbb{R} is *stationary* for the action φ if for every Borel subset $A \subset \mathbb{R}$ one has

$$\nu(A) = \sum_{g \in S} \nu(g^{-1} \cdot A) \mu(g).$$

When the Lebesgue measure Leb on \mathbb{R} is stationary, we say that the action φ is μ -*harmonic*.

Properties of μ -harmonic actions are discussed in [36, §4.4], and strongly rely on results of Deroin, Kleptsyn, Navas, and Parwani [35] on symmetric random walks in $\text{Homeo}_0(\mathbb{R})$. As remarked by Deroin, the space of μ -harmonic actions of G has several remarkable properties. To describe this, we need first to discuss two fundamental results.

Proposition 2.19 ([35]). *With notation as above, the following properties hold.*

- (1) *Every non-trivial μ -harmonic action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a canonical model.*
- (2) *Conversely, every canonical model $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is conjugate to a μ -harmonic action $\psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$.*
- (3) *Moreover, any two μ -harmonic actions ψ_1 and ψ_2 of a group G are conjugate by an affine homeomorphism.*

After a careful reading of the proof of [35, Proposition 8.1], there is a natural way to reduce the symmetries in part (3) of the statement above, to the group of translations. For this, given a homeomorphism $h \in \text{Homeo}_0(\mathbb{R})$ consider the following area function:

$$A^h(\xi) = \begin{cases} \int_{h^{-1}(\xi)}^{\xi} (h(\eta) - \xi) d\eta & \text{if } h(\xi) \geq \xi, \\ \int_{h(\xi)}^{\xi} (h^{-1}(\eta) - \xi) d\eta & \text{if } h(\xi) \leq \xi. \end{cases}$$

Note that $A^h(\xi) \geq 0$ for every $\xi \in \mathbb{R}$, and $A^h(\xi) = 0$ if and only if $\xi \in \text{Fix}(h)$. In the case $h(\xi) \geq \xi$, the quantity $A^h(\xi)$ represents indeed the area of the bounded planar region delimited by the segments $[h^{-1}(\xi), \xi] \times \{\xi\}$, $\{\xi\} \times [\xi, h(\xi)]$ and the graph of h (when $h(\xi) \leq \xi$, one has to switch the endpoints of the segments, and the same interpretation is valid). Being a two-dimensional area, it should be clear that when considering the map $\tilde{h} = aha^{-1}$, obtained by conjugating h by an affine map $a(\xi) = \lambda\xi + b$, one has the following relation:

$$(2.2) \quad \lambda^2 A^h(\xi) = A^{\tilde{h}}(a(\xi)) \quad \text{for every } \xi \in \mathbb{R}.$$

Lemma 2.20 ([35]). *With notation as above, let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a μ -harmonic action. Then the expected area function $A^\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$A^\varphi(\xi) = \sum_{g \in S} A^{\varphi(g)}(\xi) \mu(g),$$

is constant and positive.

Definition 2.21. With notation as above, we introduce the *Deroïn space* $\text{Der}_\mu(G)$ of normalized μ -harmonic actions as the subset of μ -harmonic actions $\varphi \in \text{Hom}(G, \text{Homeo}_0(\mathbb{R}))$ such that $A^\varphi = 1$, endowed with the induced compact-open topology.

Note that as the group G is finitely generated by the subset S , the compact-open topology of $\text{Hom}(G, \text{Homeo}_0(\mathbb{R}))$ is simply given by the product topology of $\text{Homeo}_0(\mathbb{R})^S$, where $\text{Homeo}_0(\mathbb{R})$ is considered with the topology of uniform convergence on compact subsets.

Theorem 2.22 ([35, 36]). *With notation as above, the following properties hold.*

- (1) *The Deroïn space $\text{Der}_\mu(G)$ is compact.*
- (2) *Every action $\varphi : G \rightarrow \text{Homeo}_0(G)$ without fixed points is positively semi-conjugate to a normalized μ -harmonic action $\psi \in \text{Der}_\mu(G)$, which is unique up to conjugacy by a translation.*
- (3) *Conjugacy by translations defines a topological flow $\Phi : \mathbb{R} \times \text{Der}_\mu(G) \rightarrow \text{Der}_\mu(G)$, called the translation flow. Explicitly, this is given by*

$$\Phi^t(\varphi)(g) = T_t \varphi(g) T_{-t},$$

where $T_t : \xi \rightarrow \xi + t$ denotes the translation by $t \in \mathbb{R}$.

Sketch of proof. After the identity (2.2) and Lemma 2.20, if φ is a normalized μ -harmonic action, then also $\Phi^t(\varphi)$ is normalized for every $t \in \mathbb{R}$. So the translation flow is well-defined.

After Corollary 2.12 and Proposition 2.19, we know that every action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points, is semi-conjugate to a μ -harmonic action, which is unique up to affine rescaling. Moreover, the identity (2.2) and Lemma 2.20 give that after an affine rescaling, we can assume that the μ -harmonic action is normalized. This gives (2).

Details for the fact that $\text{Der}_\mu(G)$ is compact can be found in [33, Proof of Theorem 5.4]. \square

Although this will not be used, note that the group G acts on the Deroïn space $\text{Der}_\mu(G)$ by the formula $g \cdot \varphi = \Phi^{\varphi(g)(0)}(\varphi)$ (see [36, (4.5)]). In other words, the action on the parameterized Φ -orbit of φ is basically the action φ .

Let us point out the following consequences of Theorem 2.22.

Corollary 2.23. *With notation as above, let φ_1 and φ_2 be two normalized μ -harmonic actions in $\text{Der}_\mu(G)$, which are positively conjugate by a homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ that fixes 0. Then $\varphi_1 = \varphi_2$.*

Corollary 2.24. *With notation as above, let φ_1 and φ_2 be two normalized μ -harmonic actions in $\text{Der}_\mu(G)$. Then φ_1 and φ_2 are positively conjugate if and only if they belong to the same Φ -orbit.*

2.3. Preorders and group actions.

2.3.1. *Preordered sets.* In this work, by a *preorder* on a set X we mean a transitive binary relation \leq which is *total*, in the sense that for every $x, y \in X$ we have $x \leq y$ or $y \leq x$ (possibly both relations can hold).

We write $x \lesssim y$ whenever $x \leq y$ but it does not hold that $y \leq x$. On the other hand, when $x \leq y$ and $y \leq x$ we say that x and y are *equivalent* and denote by $[x]_{\leq}$ the equivalence class of x (we will simply write $[x]$ when there is no risk of confusion).

Remark 2.25. A preorder \leq on X induces a *total* order on the set of equivalence classes $\{[x]_{\leq} : x \in X\}$, by declaring $[x] < [y]$ whenever $x \lesssim y$.

Definition 2.26. We say that a map between preordered sets $f : (X_1, \leq_1) \rightarrow (X_2, \leq_2)$ is *(pre)order-preserving* if $x \leq_1 y$ implies $f(x) \leq_2 f(y)$.

On the other hand, given a map $f : X_1 \rightarrow (X_2, \leq)$ we define the *pull-back* of \leq by f as the preorder $f^*(\leq)$ on X_1 , denoted by \preceq here, so that $x_1 \preceq x_2$ if and only if $f(x_1) \leq f(x_2)$.

Definition 2.27. Let (X, \leq) be a preordered set. An *automorphism* of (X, \leq) is an order-preserving bijection $f : (X, \leq) \rightarrow (X, \leq)$, whose inverse is also order-preserving. We denote by $\text{Aut}(X, \leq)$ the group of all automorphisms of (X, \leq) .

An *order-preserving action* of a group G on a preordered set (X, \leq) is a homomorphism $\psi : G \rightarrow \text{Aut}(X, \leq)$.

Remark 2.28. Let G be a group with actions $\psi_1 : G \rightarrow \text{Sym}(X_1)$ and $\psi_2 : G \rightarrow \text{Aut}(X_2, \leq)$. Let $f : X_1 \rightarrow (X_2, \leq)$ be an equivariant map. Then the preorder $f^*(\leq)$ is preserved by $\psi_1(G)$.

Definition 2.29. Let (X, \leq) be a preordered set and let $A \subset X$ be a subset. We say that A is *\leq -convex* if whenever $x \leq y \leq z$ and $x, z \in A$ it holds that $y \in A$.

Remark 2.30. It is a direct consequence of the definitions that when $f : (X_1, \leq_1) \rightarrow (X_2, \leq_2)$ is order-preserving and $A \subset (X_2, \leq_2)$ is \leq_2 -convex, then the preimage $f^{-1}(A)$ is \leq_1 -convex. This fact will be used several times without direct reference.

Definition 2.31. Given a partition \mathfrak{P} of X , denote by $\mathfrak{P}(x)$ the atom of \mathfrak{P} containing the point $x \in X$. We say that a partition \mathfrak{P} of a preordered set (X, \leq) is *\leq -convex* if every atom of \mathfrak{P} is a \leq -convex subset of (X, \leq) . Compare this with Remark 2.25.

Remark 2.32. When \mathfrak{P} is a \leq -convex partition of a preordered set (X, \leq) , there exists a total order $<_{\mathfrak{P}}$ on \mathfrak{P} defined by the condition that $\mathfrak{P}(x) <_{\mathfrak{P}} \mathfrak{P}(y)$ if and only if $\mathfrak{P}(x) \neq \mathfrak{P}(y)$ and $x \lesssim y$ (it is immediate to verify that this does not depend on the choice of points x and y).

2.3.2. *Preorders on groups.* A preorder on a group G is *left-invariant* if $h \leq k$ implies $gh \leq gk$ for all $g, h, k \in G$. In other words, the left-multiplication gives an action by automorphisms $G \rightarrow \text{Aut}(G, \leq)$. Recall also that a preorder on G is *bi-invariant* if it is preserved by left and right multiplications.

By invariance and Remark 2.25, given a left-invariant preorder \leq on G , the equivalence class $[1]_{\leq}$ is a subgroup of G and that \leq is a left-invariant order on G if and only if $[1]_{\leq} = \{1\}$. The subgroup $[1]_{\leq}$ is called the *residue*. We say that a left-invariant preorder \leq on G is *trivial* whenever $[1]_{\leq} = G$ and *non-trivial* otherwise. We denote by $\text{LPO}(G)$ the set of all non-trivial left-invariant preorders on G .

From now on, by a preorder on a group, we *always* mean a non-trivial left-invariant preorder. We endow $\text{LPO}(G)$ with the product topology induced by the realization of preorders as subsets of $\{\leq, \succ\}^{G \times G}$. It turns out that with this topology, $\text{LPO}(G)$ is a metrizable and totally disconnected topological space, which is moreover compact whenever G is finitely generated

(see Antolín and the third author [5], where preorders are called relative orders). For a modern treatment of left-orders and left-preorders see [36] and [4, 30].

Definition 2.33. Let G be a group and let $\leq \in \text{LPO}(G)$ be a preorder. The *positive cone* of \leq is the subset $P_{\leq} = \{g \in G : g \succeq 1\}$. Similarly, the subset $N_{\leq} = \{g \in G : g \preceq 1\}$ defines the *negative cone* of \leq .

Remark 2.34. A preorder $\leq \in \text{LPO}(G)$ induces a partition $G = P_{\leq} \sqcup [1]_{\leq} \sqcup N_{\leq}$. Also note that P_{\leq} and N_{\leq} are semigroups and satisfy $P_{\leq}^{-1} = N_{\leq}$ where $P_{\leq}^{-1} := \{g^{-1} : g \in P_{\leq}\}$. Reciprocally, given a partition $G = P \sqcup H \sqcup N$ such that

- (1) P is a semigroup,
- (2) $N = P^{-1}$,
- (3) H is a proper (possibly trivial) subgroup and
- (4) $HPH \subseteq P$,

there exists a preorder $\leq \in \text{LPO}(G)$ such that $P = P_{\leq}$, $H = [1]_{\leq}$ and $N = N_{\leq}$. See [30] for details.

When H is a \leq -convex subgroup, the left coset space $G/H = \{gH : g \in G\}$ defines a \leq -convex partition of G . The total order $<_{G/H}$ induced on G/H by \leq (see Remark 2.32) is preserved by left-multiplication of G on G/H . Given a \leq -convex subgroup $H \subseteq G$, we define the *quotient preorder* of \leq under H as the preorder $\leq_H \in \text{LPO}(G)$ given by the pull-back $\leq_H := p^*(<_{G/H})$, where $p : G \rightarrow G/H$ is the coset projection. Equivalently we can define \leq_H by setting $P_{\leq_H} = P_{\leq} \setminus H$.

Remark 2.35. After Remark 2.28, the identity map $\text{id} : (G, \leq) \rightarrow (G, \leq_H)$ is order-preserving and equivariant with respect to the actions of G by left-multiplication.

2.3.3. Dynamical realizations of actions on totally ordered sets. One general principle that we often use is that for building actions of a group G on the line by homeomorphisms, one may start by finding a totally ordered space $(X, <)$ and an action by order-preserving bijections $\psi : G \rightarrow \text{Aut}(X, <)$. Then if the order topology on X is nice enough, for instance when X is countable, then the action ψ can be “completed” to an action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ so that, there exists an order-preserving map $i : X \rightarrow \mathbb{R}$ satisfying that $\varphi(g)(i(x)) = i(\psi(g)(x))$ for all $g \in G$. Under some mild extra conditions, we call such φ a *dynamical realization* of ψ . See Definition 2.39.

Remark 2.36. It is a classical fact that a countable group is left-orderable if and only if it *embeds* into $\text{Homeo}_0(\mathbb{R})$ (a fact that we trace back to Conrad [27] in the abelian setting, see [36, 44] for a proof of the general case). In fact, one side of the proof consist in building a dynamical realization of the left-regular representation of a countable left-ordered group on itself. Analogously, one can show that a countable group admits a left-invariant preorder if and only if it admits a (possibly not faithful) non-trivial action by order-preserving homeomorphisms of the real line [5].

We now proceed to formally define what we mean by dynamical realization.

Definition 2.37. Let $(X, <)$ be a countable totally ordered set. We say that an injective order-preserving map $i : X \rightarrow \mathbb{R}$ is a *good embedding* if its image is unbounded in both directions and every connected component I of the complement of $i(X)$ satisfies $\partial I \subset i(X)$.

Remark 2.38. Following the classical construction of the dynamical realization of a countable left-ordered group (see [36, 44]), it follows that any countable and totally ordered set $(X, <)$

has a good embedding and that, given two different good embeddings $i_1, i_2 : X \rightarrow \mathbb{R}$, there exists $h \in \text{Homeo}_0(\mathbb{R})$ so that $i_2 = h \circ i_1$.

Definition 2.39. Assume that $\psi : G \rightarrow \text{Aut}(X, <)$ is a group action. An action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is said to be a *dynamical realization* of ψ if there exists a good embedding $i : X \rightarrow \mathbb{R}$ such that the following hold:

- (1) i is equivariant for φ and ψ ;
- (2) for every connected component I of the complement of $\overline{i(X)}$ the (setwise) stabilizer $\text{Stab}_G^\varphi(I)$ of I in G acts trivially on I .

Lemma 2.40. Consider an action $\psi : G \rightarrow \text{Aut}(X, <)$ where $(X, <)$ is a countable totally ordered set. Then a dynamical realization of ψ exists and is unique up to positive conjugacy.

Sketch of proof. Consider a good embedding $i : X \rightarrow \mathbb{R}$. Then, by transporting the action ψ through i on $i(X)$ and extending it by continuity to the closure we obtain an action $\varphi_0 : G \rightarrow \text{Homeo}_0(\overline{i(X)})$. For every $g \in G$ denote by $\varphi(g)$ the extension of $\varphi_0(g)$ which is affine on every connected component I of $\overline{i(X)}$. It is direct to check that $g \mapsto \varphi(g)$ is a dynamical realization of ψ .

Now, consider two actions $\varphi_1, \varphi_2 : G \rightarrow \text{Homeo}_0(\mathbb{R})$ which are dynamical realizations of ψ with associated good embeddings $i_1, i_2 : X \rightarrow \mathbb{R}$ respectively. Note that by Remark 2.38 there exists $h \in \text{Homeo}_0(\mathbb{R})$ with $i_2 = h \circ i_1$ and after conjugating φ_1 by h we can suppose that $i_1 = i_2 =: i$. By equivariance φ_1 and φ_2 must coincide on $\overline{i(X)}$. Let Ω be the set of connected components of $\mathbb{R} \setminus \overline{i(X)}$ and note that the G -actions on Ω induced by φ_1 and φ_2 coincide, so that the set of orbits Ω/G does not depend on the action. For $J \in \Omega$ we denote by $\alpha(J) \in \Omega/G$ its G -orbit. Pick a system of representatives $\{I_\alpha\}_{\alpha \in \Omega/G}$ of orbits. For $J \in \Omega$ choose $g_J \in G$ such that $g_J(J) = I_{\alpha(J)}$, and for $i \in \{1, 2\}$ set $f_{i,J} = \varphi_i(g_J) \upharpoonright_J$. Note that each $f_{i,J}$ is a homeomorphism from J to $I_{\alpha(J)}$ which does not depend on the choice of g_J by the assumption (2) in the definition of dynamical realization. Thus $f_{2,J}^{-1}f_{1,J}$ is a self-homeomorphism of J . Define a map $q : \mathbb{R} \rightarrow \mathbb{R}$ which is the identity on $\overline{i(X)}$ and satisfies $q \upharpoonright_J = f_{2,J}^{-1}f_{1,J}$ for every $J \in \Omega$. Then one readily checks that q conjugates φ_2 to φ_1 . \square

We proceed to give a sufficient condition for minimality of dynamical realizations of actions on totally ordered sets. For this, we need the following definition.

Definition 2.41. Let $(X, <)$ be a totally ordered set. We say that a subgroup $H \subseteq \text{Aut}(X, <)$ is of *homothetic type* if the following conditions are satisfied.

- (1) There exists $x_0 \in \text{Fix}(H)$.
- (2) For every $x \in X \setminus \{x_0\}$, there exists a sequence of elements $(h_n) \subset H$ such that $h_n(x) \rightarrow +\infty$ if $x > x_0$, and $h_n(x) \rightarrow -\infty$ if $x < x_0$.

When the cyclic subgroup $\langle g \rangle$ is of homothetic type we say that g is a *homothety*.

Lemma 2.42 (Minimality criterion). Let $(X, <)$ be a countable totally ordered set and consider an action $\psi : G \rightarrow \text{Aut}(X, <)$. Assume that for every four points x_1, x_2, y_1, y_2 of X with

$$x_1 < y_1 < y_2 < x_2,$$

there exists $g \in G$ such that

$$g.y_1 < x_1 < x_2 < g.y_2.$$

Then the dynamical realization of ψ is minimal.

Proof. Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a dynamical realization of ψ with its associated good embedding $i : X \rightarrow \mathbb{R}$ (Lemma 2.40). We first claim that i has dense image. Suppose by contradiction that it is not the case and take a connected component (η_1, η_2) of the complement of $\overline{i(X)}$. Since i is a good embedding we have that $\{\eta_1, \eta_2\} \subset i(X)$. Choose two points ξ_1 and ξ_2 in $i(X)$ such that $(\eta_1, \eta_2) \subsetneq (\xi_1, \xi_2)$; by our assumption, we can find an element $g \in G$ such that $(\eta_1, \eta_2) \supsetneq g \cdot (\xi_1, \xi_2)$, contradicting the $\varphi(g)$ -invariance of $i(X)$. This shows that i has dense image.

Suppose again by contradiction that there exists a proper closed $\varphi(G)$ -invariant subset $\Lambda \subset \mathbb{R}$. Take a connected component (η_1, η_2) of $\mathbb{R} \setminus \Lambda$. By density of $i(\mathbb{R})$, we can find four points $\xi_1, \xi_2, \zeta_1, \zeta_2$ in $i(X)$ such that

$$\zeta_1 < \eta_1 < \xi_1 < \xi_2 < \eta_2 < \zeta_2.$$

After our assumption, we can find an element $g \in G$ such that $(\eta_1, \eta_2) \supsetneq g \cdot (\zeta_1, \zeta_2)$. This contradicts the $\varphi(g)$ -invariance of Λ showing that φ is minimal, as desired. \square

The condition in Lemma 2.42 is met in the following situation.

Proposition 2.43. *Let $(X, <)$ be a countable totally ordered set and consider an action $\psi : G \rightarrow \text{Aut}(X, <)$. If for every $x \in X$ there exists a subgroup $H \subseteq G$ such that $\psi(H)$ is a subgroup of homothetic type fixing x , the dynamical realization of ψ is minimal.*

Proof. Consider four points x_1, x_2, y_1, y_2 of X with $x_1 < y_1 < y_2 < x_2$. By assumption, we can find an element $h_1 \in H_{y_1}$ such that $y_1 < h_1 \cdot y_2 < y_2$. Consider the point $x_* = h_1 \cdot y_2$, so that there exists $h_* \in H_{x_*}$ such that $h_* \cdot y_1 < x_1 < x_2 < h_* \cdot y_2$. Thus Lemma 2.42 applies. \square

2.4. Bieri–Strebel groups. To illustrate our results we will often consider examples of locally moving groups arising as groups of piecewise linear (PL) homeomorphisms of the line. Let us briefly fix the terminology and recall a large family of such groups, following Bieri and Strebel [8].

We say that a homeomorphism $f : X \rightarrow X$ of an interval $X \subset \mathbb{R}$ (bounded or unbounded) is *piecewise linear* (PL, for short) if there exists a discrete subset $\Sigma \subset X$ such that in restriction to $X \setminus \Sigma$, the map f is locally affine, that is, of the form $x \mapsto \lambda x + a$, with $\lambda > 0$ and $a \in \mathbb{R}$. We denote by $\text{BP}(f)$ the minimal subset Σ satisfying such condition, and points of $\text{BP}(f)$ will be the *breakpoints* of f . When $\text{BP}(f)$ is finite, we say that f is *finitary* PL. Note that with this definition, a PL homeomorphism always preserves the orientation.

Definition 2.44. Given a non-trivial multiplicative subgroup $\Lambda \subseteq \mathbb{R}_+^*$, and a non-trivial $\mathbb{Z}[\Lambda]$ -submodule $A \subset \mathbb{R}$, the *Bieri–Strebel group* $G(X; A, \Lambda)$ is the group of finitary PL homeomorphisms $f : X \rightarrow X$ with the following properties:

- (1) breakpoints of f are in A ,
- (2) in restriction to $X \setminus \text{BP}(f)$, the map f is locally of the form $\lambda x + a$, with $\lambda \in \Lambda$ and $a \in A$.

For example, Thompson’s F is $G([0, 1]; \mathbb{Z}[1/2], \langle 2 \rangle_*)$. Other interesting examples are provided by the so-called Thompson–Brown–Stein groups.

Definition 2.45. Let $1 < n_1 < \dots < n_k$ be natural numbers such that the multiplicative subgroup $\Lambda = \langle n_i \rangle \subseteq \mathbb{R}_+^*$ is an abelian group of rank k . Denote by A the ring $\mathbb{Z}[1/m]$, where m is the least common multiple of the n_i . The group

$$F_{n_1, \dots, n_k} := G([0, 1]; A, \Lambda)$$

is the corresponding *Thompson–Brown–Stein group*.

The group F_2 is simply Thompson's group F . For every $n \geq 2$, the group F_n is isomorphic to a subgroup of F , and these groups were first considered by Brown in [19], inspired by the so-called Higman–Thompson groups. Later Stein [103] started the investigation of the more general class of groups F_{n_1, \dots, n_k} . She proved that every Thompson–Brown–Stein group is finitely generated and even finitely presented. Moreover, given any m -adic interval $I \subset [0, 1]$ (that is, an interval with endpoints in $\mathbb{Z}[1/m]$), the subgroup $(F_{n_1, \dots, n_k})_I$ is isomorphic to F_{n_1, \dots, n_k} .

We refer to the classical monograph by Bieri and Strebel [8] for an extensive investigation of the groups $G(X; A, \Lambda)$.

Remark 2.46. It would be interesting to see how the results of this text apply to other groups of piecewise projective homeomorphisms, such as Monod's groups $H(A)$ [86] (we will not pursue this task).

3. DEROIN SPACES AND PREORDERS

Recall that we denote by $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ the space of representations of G without fixed points. The following notion is classical in dynamical systems.

Definition 3.1. An action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is *locally rigid* if there exists a neighborhood \mathcal{U} of φ in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ consisting of representations all positively semi-conjugate to φ .

In other terms, every sufficiently small perturbation of the action φ gives a positively semi-conjugate action. The main goal of this section is to characterize local rigidity in terms of the topology of the orbits along the translation flow in the Deroïn space (see §2.2 for a definition of this space). This will be a consequence of the following result.

Theorem 3.2. *Let G be a finitely generated group, and consider a symmetric probability measure μ whose support is finite and generates G . There exists a continuous retraction*

$$r : \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) \rightarrow \text{Der}_\mu(G)$$

which preserves positive semi-conjugacy classes.

Definition 3.3. The map $r : \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) \rightarrow \text{Der}_\mu(G)$ from Theorem 3.2 is called the *harmonic retraction* of the representation space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$.

Corollary 3.4 (Local rigidity criterion). *Let G be a finitely generated group, and consider a symmetric probability measure μ whose support is finite and generates G . For $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$, let $\psi \in \text{Der}_\mu(G)$ be a normalized μ -harmonic action which is positively semi-conjugate to φ . If the orbit along the translation flow of ψ is open in $\text{Der}_\mu(G)$, then φ is locally rigid. The converse hold provided φ is minimal.*

Proof. Let $\mathcal{I} = \{\Phi^t(\psi) : t \in \mathbb{R}\}$ be the orbit of ψ in $\text{Der}_\mu(G)$ and let $r : \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) \rightarrow \text{Der}_\mu(G)$ be the harmonic retraction (Theorem 3.2). Then $r(\varphi)$ is an element of $\text{Der}_\mu(G)$ which is positively semi-conjugate to ψ and thus belongs to \mathcal{I} , i.e. $\varphi \in r^{-1}(\mathcal{I})$. If \mathcal{I} is open, then $r^{-1}(\mathcal{I})$ is an open neighborhood of φ consisting only of actions positively semi-conjugate to ψ , so the claim follows. Conversely, assume that φ is minimal. Since local rigidity is preserved under conjugacy we can assume that $\varphi = \psi$. Thus if \mathcal{I} is not open, using the translation flow we find that φ can be approximated by μ -harmonic actions in different Φ -orbits. Since actions in $\text{Der}_\mu(G)$ belonging to different Φ -orbits are not positively semi-conjugate (Corollary 2.24), this implies that φ is not locally rigid. \square

Remark 3.5. The requirement that φ be minimal in the second part of Corollary 3.4 is essential. To see this consider the action $\varphi : \mathbb{F}_2 \times \mathbb{Z} \rightarrow \text{Homeo}_0(\mathbb{R})$ obtained as the lift of the ping-pong action $\varphi_0 : \mathbb{F}_2 \rightarrow \text{PSL}(2, \mathbb{R}) \subset \text{Homeo}_0(\mathbb{S}^1)$, given by the action of the fundamental group of a hyperbolic *one-holed* torus on the boundary at infinity of the hyperbolic plane; to make this more concrete, one can consider the subgroup of $\text{PSL}(2, \mathbb{R})$ generated by the two homographies $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. The action φ_0 is then non-minimal and locally rigid so the same holds for φ . However, the canonical model of φ_0 corresponds to the action of the fundamental group of a hyperbolic *one-punctured* torus on the boundary at infinity of the hyperbolic plane, and so this canonical model has a parabolic element (the commutator of the generators). In particular, this minimal action and its corresponding lift to the line are not locally rigid. Thus the representative of φ in $\text{Der}_\mu(\mathbb{F}_2 \times \mathbb{Z})$ has non-isolated orbit along the translation flow.

In order to prove Theorem 3.2, we obtain an order-theoretic description of the Deroin space. Note that not every group action is the dynamical realization of some order, so we are forced to consider the much more flexible notion of preorder. However, different preorders may correspond to the same conjugacy class of a minimal action, and we have to translate this into an equivalence relation on preorders. The appropriate equivalence relation on preorders is introduced in §3.1. In §3.2, we make explicit the relations between properties of preorders and their dynamical realizations. In §3.3 we investigate the topology of the corresponding quotient space. Finally, in §3.4 we identify such quotient space with the Deroin space and provide the proof of Theorem 3.2. As a by-product of this, we will get that the topology of $\text{Der}_\mu(G)$ does not depend on the choice of the probability measure μ .

3.1. Equivalence of preorders. We start with the following definition which gives an order-theoretic analogue of continuous positive semi-conjugacies between representations in $\text{Homeo}_0(\mathbb{R})$. Part of our discussion is close to the exposition of Decaup and Rond [30].

Definition 3.6. Let G be a group and let $\leq, \preceq \in \text{LPO}(G)$ be two preorders. We say that \leq *dominates* \preceq if the identity map $\text{id} : (G, \preceq) \rightarrow (G, \leq)$ is order-preserving.

Remark 3.7. The direction in the definition of domination could appear counterintuitive, but in fact it is justified when thinking in terms of preorders (it is even clearer in terms of dynamical realization). Indeed, take an element $g \in G$, and suppose that we want to know whether $g \succeq 1$. Then we first check whether $g \geq 1$ (or $g \not\geq 1$), and only in the case $g \in [1]_{\leq}$, we take the preorder \preceq into consideration.

Note that when \leq dominates \preceq , the subgroup $H = [1]_{\preceq}$ is \preceq -convex, thus the quotient preorder \preceq_H is well-defined. The following lemma characterizes domination in terms of positive cones.

Lemma 3.8. *Let G be a group and let $\leq, \preceq \in \text{LPO}(G)$ be two preorders. Then, the following are equivalent.*

- (1) \leq dominates \preceq .
- (2) $P_{\leq} \subseteq P_{\preceq}$.
- (3) The subgroup $H = [1]_{\preceq}$ is \preceq -convex, and $P_{\leq} = P_{\preceq} \setminus H$ (so that \leq coincides with the quotient preorder \preceq_H).

Proof. We prove that (1) implies (3). As $\text{id} : (G, \preceq) \rightarrow (G, \leq)$ is order-preserving, the subgroup $H = [1]_{\preceq}$, which coincides with its preimage, is \preceq -convex. Moreover, we have the following inclusions:

$$P_{\preceq} \subseteq P_{\leq} \sqcup [1]_{\preceq}, \quad [1]_{\preceq} \subseteq [1]_{\leq}, \quad N_{\preceq} \subseteq [1]_{\preceq} \sqcup N_{\leq}.$$

As $G = P_{\preceq} \sqcup [1]_{\preceq} \sqcup N_{\preceq}$, we deduce $P_{\preceq} = P_{\leq} \setminus [1]_{\leq}$. Clearly (3) implies (2). Finally, by left-invariance, we have $g \preceq h$ if and only if $h^{-1}g \in G \setminus P_{\preceq}$, and clearly the same holds for the preorder \leq . This gives that (1) and (2) are equivalent. \square

Let G be a group. For a given a preorder $\leq \in \text{LPO}(G)$, the collection of proper \leq -convex subgroups is totally ordered by inclusion. The subgroup $[1]_{\leq}$ is always the least such subgroup, however a maximal proper \leq -convex subgroup may not exist. This issue is the order-theoretic analogue of the problem of existence of a minimal invariant set (Remark 2.9). To simplify the discussion, we will systematically assume finite generation in the rest of the section, in which case there exists a maximal proper \leq -convex subset, that we will denote by H_{\leq} . This leads to the notion of minimal model of a preorder which is an order-theoretic analogue of canonical models for actions on the line.

Definition 3.9. Let G be a finitely generated group. The *minimal model* of a preorder $\leq \in \text{LPO}(G)$ is the quotient preorder \leq_H , where $H = H_{\leq}$ is the maximal proper \leq -convex subgroup. The minimal model will be denoted by \leq^* . When the preorder \leq coincides with \leq^* , we say that $\leq \in \text{LPO}(G)$ is a minimal model.

Remark 3.10. Minimal preorders are exactly those whose structure of convex subgroups is the simplest possible. Indeed, after Lemma 3.8, the minimal model dominates the original preorder. Therefore, H_{\leq} is the unique proper \leq^* -subgroup of (G, \leq^*) , and thus $H_{\leq} = [1]_{\leq^*}$. Reversely, if $[1]_{\leq} = H_{\leq}$ is the unique proper \leq -convex subgroup, then \leq is a minimal model.

It should be not surprising that the minimal model is unique, in the following sense.

Lemma 3.11. *Let G be a finitely generated group and consider preorders $\leq, \preceq \in \text{LPO}(G)$ such that \leq is a minimal model and dominates \preceq . Then, \leq is the minimal model of \preceq .*

Proof. As \leq dominates \preceq , the subgroup $H = [1]_{\preceq}$ is \preceq -convex, so $H \subseteq H_{\preceq}$. This gives $P_{\preceq} \supseteq P_{\leq^*}$, so by Lemma 3.8, the map

$$\text{id} : (G, \preceq) \rightarrow (G, \leq^*)$$

is order-preserving. Using Remark 3.10, as $H_{\preceq} = [1]_{\preceq^*}$, we deduce that H_{\preceq} is \leq -convex, and thus that $H_{\preceq} = H$. \square

We next introduce the following equivalence relation on preorders.

Definition 3.12. Let G be a finitely generated group. We say two preorders are *equivalent* if they have the same minimal model. We denote by $[\leq]$ the equivalence class of $\leq \in \text{LPO}(G)$ and we write $[\text{LPO}](G) = \{[\leq] : \leq \in \text{LPO}(G)\}$ for the corresponding quotient.

It is immediate to verify that this is indeed an equivalence relation. We have the following result.

Proposition 3.13. *Let G be a finitely generated group and consider preorders $\leq_1, \leq_2 \in \text{LPO}(G)$. Then, there exists a preorder \leq that dominates \leq_1 and \leq_2 , if and only if the equivalence classes $[\leq_1]$ and $[\leq_2]$ are the same.*

Proof. We have that the minimal order \leq^* dominates \leq and thus both \leq_1 and \leq_2 . From Lemma 3.11, we conclude that \leq^* is the minimal model of both \leq_1 and \leq_2 . The converse statement follows directly from Remark 3.10. \square

3.2. Relations between preorders and actions on the real line. We can now make explicit the relation between minimal models and their dynamical counterpart. For this, note that if G is a countable group and $\leq \in \text{LPO}(G)$ is a preorder, we can consider a *dynamical realization* φ_{\leq} for the action of G on the totally ordered set $(G/H, <_{G/H})$, which exists by Lemma 2.40. Conversely, given an action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points, we denote by $\leq_{\varphi} \in \text{LPO}(G)$ the preorder induced by the $\varphi(G)$ -orbit of 0.

Proposition 3.14. *Let G be a finitely generated group and let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a canonical model. Then the preorder \leq_{φ} is a minimal model.*

Proof. Suppose by way of contradiction that there exists a proper \leq_{φ} -convex subgroup $H \neq [1]_{\leq_{\varphi}}$. Then, for each $gH \in G/H$, denote by $I_{gH} \subset \mathbb{R}$ the interior of the convex hull of $\varphi(gH)(0)$. Since H is \leq_{φ} -convex, we get that $\mathbb{R} \setminus \bigcup_{g \in G} I_{gH}$ is a proper closed $\varphi(G)$ -invariant subset. On the other hand, since H strictly contains $[1]_{\leq_{\varphi}}$ the $\varphi(G)$ -stabilizer of each I_g acts non-trivially on I_g , and this implies that the action φ is not cyclic, a contradiction. \square

Proposition 3.15. *Let G be a finitely generated group and consider a preorder $\leq \in \text{LPO}(G)$. Then the dynamical realization φ_{\leq} is a canonical model if and only if \leq is a minimal model.*

Moreover, the dynamical realization φ_{\leq^} is a canonical model for φ_{\leq} .*

The rest of the subsection is devoted to the proof of Proposition 3.15. We will need a preliminary result.

Lemma 3.16. *Let G be a group. Consider preorders $\leq_1, \leq_2 \in \text{LPO}(G)$ so that \leq_2 dominates \leq_1 . Then the dynamical realizations φ_{\leq_1} and φ_{\leq_2} are positively semi-conjugate.*

Proof. For $i \in \{1, 2\}$ write $H_i := [1]_{\leq_i}$ and $\iota_i : G/H_i \rightarrow \mathbb{R}$ for the corresponding good embeddings. As $\text{id} : (G, \leq_1) \rightarrow (G, \leq_2)$ is order-preserving, we have $H_1 \subseteq H_2$ and the quotient map

$$i_0 : (G/H_1, <_1) \rightarrow (G/H_2, <_2)$$

is order-preserving and equivariant (here $<_i$ denotes the total ordered induced by \leq_i for $i \in \{1, 2\}$). Write $X = \iota_1(G/H_1)$ and define $j_0 : X \rightarrow \mathbb{R}$ as

$$j_0(\iota_1(gH_1)) = \iota_2(i_0(gH_1)) = \iota_2(gH_2).$$

It follows directly from the definitions that X is $\varphi_{\leq_1}(G)$ -invariant and that j_0 is order-preserving and equivariant. We conclude using Lemma 2.7. \square

Let us immediately point out the following conclusion.

Corollary 3.17. *For finitely generated groups, equivalent preorders have positively semi-conjugate dynamical realizations.*

Proof. After Lemma 3.16, the dynamical realizations of equivalent preorders are positively semi-conjugate to the dynamical realization of their minimal model. As positive semi-conjugacy is an equivalence relation, the conclusion follows. \square

Proof of Proposition 3.15. Suppose first that φ_{\leq} is not a canonical model. Then φ_{\leq} is neither minimal nor cyclic and so φ_{\leq} has an exceptional minimal invariant set $\Lambda \subset \mathbb{R}$ or φ_{\leq} has a closed discrete orbit but its image is not cyclic. We write $X = \iota(G/[1]_{\leq})$ for the image of the good embedding $\iota : G/[1]_{\leq} \rightarrow \mathbb{R}$ associated with φ_{\leq} , and remark that X consists of a single $\varphi_{\leq}(G)$ -orbit.

Case I. φ_{\leq} has an exceptional minimal invariant set.

We first show the following.

Claim. X is contained in $\mathbb{R} \setminus \Lambda$.

Proof of claim. We assume by contradiction that $X \cap \Lambda \neq \emptyset$. As X consists of a single orbit, by invariance of Λ we get that $X \subset \Lambda$ and thus $\Lambda = \overline{X}$ by minimality of Λ . Consider a connected component $I = (\xi, \eta)$ of $\mathbb{R} \setminus \Lambda$ and note that, by Definition 2.37 of good embedding, we have $\{\xi, \eta\} \subseteq X$ and thus the points ξ and η are in the same orbit. This shows that Λ is discrete, which is a contradiction. \square

Consider now the connected component U of $\mathbb{R} \setminus \Lambda$ that contains $\xi_0 = \iota([1]_{\leq})$ and write $H = \text{Stab}_G^{\varphi_{\leq}}(U)$. Note that $H = \iota^{-1}(U)$ is a \leq -convex subgroup. We will show that H strictly contains $[1]_{\leq}$, which by Remark 3.10, implies that \leq is not a minimal model. Equivalently, we need to show that $U \cap X$ strictly contains ξ_0 . Assume that ξ_0 is an isolated point of X , otherwise there is nothing to prove. Let $\lambda \in \Lambda$ be the rightmost point of U . Note that \overline{X} is $\varphi_{\leq}(G)$ -invariant, so we must have $\overline{X} \supseteq \Lambda$. Thus $\lambda \in \overline{X}$. If $X \cap (\xi_0, \lambda) = \emptyset$, then (ξ_0, λ) is a connected component of $\mathbb{R} \setminus \overline{X}$. As ι is a good embedding, this gives $\lambda \in X$, contradicting the claim.

Case II. φ_{\leq} has a closed discrete orbit but its image is not cyclic.

Take a point $\xi \in \mathbb{R}$ with closed discrete orbit and write $H = \text{Stab}_G^{\varphi_{\leq}}(\xi)$ and $F = \text{Fix}(H)$. Note that H is the kernel of the morphism $G \rightarrow \mathbb{Z}$, induced from the orbit of ξ . Since φ_{\leq} is assumed to be non-cyclic we have that F is a proper closed subset, and it is $\varphi_{\leq}(G)$ -invariant, for H is normal. If $X \cap F \neq \emptyset$, then the fact that X is a single $\varphi_{\leq}(G)$ -orbit gives $X \subseteq F = \text{Fix}(H)$, and as φ_{\leq} is a dynamical realization of \leq , this implies $H = \varphi_{\leq}([1]_{\leq})$, so the image $\varphi_{\leq}(G)$ is cyclic. A contradiction. Thus $X \subseteq \mathbb{R} \setminus F$. More precisely, we have the following.

Claim. $F = \text{Fix}(H)$ is contained in $\overline{X} \setminus X$.

Proof of claim. Assume for contradiction that there is a point of F in the complement of \overline{X} , and let I be corresponding connected component of I . As the action φ_{\leq} is a dynamical realization (Definition 2.39) and $F = \text{Fix}(H)$, this gives that the closure \overline{I} is fixed by H , so that $\overline{I} \subseteq F$. However, ι is a good embedding, so $\partial I \in X$, which contradicts the fact that $X \cap F = \emptyset$. \square

We can now argue analogously as in Case I to find a proper \leq -convex subgroup strictly containing $[1]_{\leq}$.

For the converse, assume that \leq is not a minimal model and take a proper \leq -convex subgroup $H \neq [1]_{\leq}$. Denote by U the interior of the convex hull of $\iota(H) \subset \mathbb{R}$ and notice that the orbit $\varphi_{\leq}(G)(U)$ is a proper open $\varphi_{\leq}(G)$ -invariant subset. Also notice that the stabilizer $\text{Stab}_G^{\varphi_{\leq}}(U)$ acts non-trivially on U which implies that φ_{\leq} is not a cyclic action. The last two facts together imply that φ_{\leq} is not a canonical model.

It remains to prove that φ_{\leq^*} is a canonical model for φ_{\leq} . After Remark 3.10 and Lemma 3.16, the actions φ_{\leq^*} and φ_{\leq} are positively semi-conjugate. On the other hand, by the first part of this proposition, the action φ_{\leq^*} is a canonical model. \square

3.3. Topology of the space of equivalence classes of preorders. Recall that we consider $\text{LPO}(G)$ as a subspace of $\{\leq, \geq\}^{G \times G}$ endowed with the product topology, and that this makes $\text{LPO}(G)$ a metrizable and totally disconnected topological space, which is compact when G is finitely generated.

Lemma 3.18. *Let G be a finitely generated group. The subset*

$$\mathcal{E} = \{(\leq_1, \leq_2) \in \text{LPO}(G) \times \text{LPO}(G) : [\leq_1] = [\leq_2]\}$$

is closed in $\text{LPO}(G) \times \text{LPO}(G)$. Therefore, $[\text{LPO}](G)$ considered with the quotient topology is a Hausdorff topological space.

Proof. Consider a convergent sequence $(\preceq_n, \preceq'_n) \rightarrow (\preceq_\infty, \preceq'_\infty)$ so that $(\preceq_n, \preceq'_n) \in \mathcal{E}$ for every $n \in \mathbb{N}$. We want to show that $[\preceq_\infty] = [\preceq'_\infty]$ and this amounts to show, after Proposition 3.13, that there exists a preorder that dominates both \preceq_∞ and \preceq'_∞ .

For every $n \in \mathbb{N}$, using Proposition 3.13 we find a preorder \leq_n that dominates both \preceq_n and \preceq'_n . As $\text{LPO}(G)$ is compact, upon passing to a subsequence, we can suppose that \leq_n has a limit \leq_∞ . We shall prove that \leq_∞ dominates both \preceq_∞ and \preceq'_∞ , and by Lemma 3.8 this amounts to show that $P_{\leq_\infty} \subseteq (P_{\preceq_\infty} \cap P_{\preceq'_\infty})$. For this, note that by definition of product topology, if $g \in P_{\leq_\infty}$ then $g \in P_{\leq_n}$ for n large enough, which by Lemma 3.8, implies that $g \in P_{\preceq_n} \cap P_{\preceq'_n}$ for n large enough. This implies that $g \in P_{\preceq_\infty} \cap P_{\preceq'_\infty}$ as desired. \square

Lemma 3.19. *Let G be a finitely generated group. Consider preorders $(\leq_n)_{n \in \mathbb{N}}$ and \leq in $\text{LPO}(G)$. Assume that for every finite subset $F \subseteq P_\leq$ there exists $k_0 \in \mathbb{N}$ such that $F \subseteq P_{\leq_k}$ for every $k \geq k_0$. Then $[\leq_k] \rightarrow [\leq]$ in $[\text{LPO}](G)$.*

In particular, if $\varphi_n, \varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ are representations such that $\varphi_n \rightarrow \varphi$ in the compact open topology we get $[\leq_{\varphi_n}] \rightarrow [\leq_\varphi]$ in $[\text{LPO}](G)$.

Proof. By hypothesis, we have that the limit of every convergent subsequence $\leq_{k_n} \rightarrow \leq_\infty$ satisfies $P_\leq \subseteq P_{\leq_\infty}$. This implies that \leq dominates \leq_∞ and therefore, by Proposition 3.13, we get $[\leq] = [\leq_\infty]$. Since every convergent subsequence of $(\leq_n)_{n \in \mathbb{N}}$ converges to a preorder equivalent to \leq , we conclude that $[\leq_n] \rightarrow [\leq]$ in $[\text{LPO}](G)$ as desired. \square

3.4. Retraction to the Deroin space. As in §2.2, we fix a finitely generated group G , a symmetric probability measure μ on G whose support is finite, and generates G . We will show that $[\text{LPO}](G)$ with the quotient topology is homeomorphic to $\text{Der}_\mu(G)$. In particular, this will show that the topology of the Deroin space does not depend on the choice of the probability measure μ .

Theorem 3.20. *With notation as above, the Deroin space $\text{Der}_\mu(G)$ is homeomorphic to $[\text{LPO}](G)$. In particular, the homeomorphism type of $\text{Der}_\mu(G)$ does not depend on μ .*

Proof. Consider the map

$$I : \text{Der}_\mu(G) \rightarrow [\text{LPO}](G) \\ \varphi \mapsto [\leq_\varphi].$$

We will show that I is a homeomorphism. By Proposition 3.15 we know that if $\preceq \in \text{LPO}(G)$ is a minimal model, then φ_\preceq is a canonical model. On the other hand, by Proposition 2.19, we can choose φ_\preceq to be μ -harmonic. We introduce the map

$$(3.1) \quad J : [\text{LPO}](G) \rightarrow \text{Der}_\mu(G),$$

where J assigns to $[\preceq]$ the μ -harmonic dynamical realization of \preceq^* , whose associated good embedding satisfies $\iota_{\preceq^*}([1]_{\preceq^*}) = 0$. We will show that J is the inverse of I . The equality $I \circ J = \text{id}$ is given by the fact that \leq_{φ_\preceq} coincides with \leq for every preorder $\preceq \in \text{LPO}(G)$, which is a direct consequence of the definitions.

To verify that $J \circ I = \text{id}$, fix $\varphi \in \text{Der}_\mu(G)$. Since φ is a canonical model (Proposition 2.19), Proposition 3.14 implies that \leq_φ is a minimal model. Therefore, $J([\leq_\varphi])$ is a dynamical realization of \leq_φ . Set $X = \varphi(G)(0)$ and notice that there exists an order-preserving map

$j_0 : X \rightarrow \mathbb{R}$ which is equivariant for the actions φ and $J([\leq_\varphi])$. Since both actions are canonical models, it follows from Lemma 2.7 and Corollary 2.12 that j_0 can be extended to a positive conjugacy. Moreover, since $j_0(0) = 0$, this conjugacy fixes 0 and therefore, by Corollary 2.23, we conclude that $\varphi = J([\leq_\varphi])$ as desired.

The continuity of I follows directly from Lemma 3.19. Finally, since $\text{Der}_\mu(G)$ is compact (Theorem 2.22) and $[\text{LPO}](G)$ is a Hausdorff topological space (Lemma 3.18), we conclude that I is a homeomorphism. \square

We can now prove the main result of this section.

Proof of Theorem 3.2. We consider the map

$$\begin{array}{ccc} r : \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) & \rightarrow & \text{Der}_\mu(G) \\ \varphi & \mapsto & J([\leq_\varphi]) \end{array},$$

where J is the map introduced in (3.1) and \leq_φ is the preorder induced by 0. It follows directly from the definitions that r is the identity in restriction to $\text{Der}_\mu(G)$. Also, by Lemma 3.19 and Theorem 3.20 we conclude that r is continuous. Finally, to check that r preserves positive semi-conjugacy classes first note that φ is positively semi-conjugate to the dynamical realization of \leq_φ and that, by Corollary 3.17, the dynamical realizations of \leq_φ and \leq_φ^* are also positively semi-conjugate. \square

3.5. Application to the Borel reducibility of the semi-conjugacy relation. For a given group G it is natural to try to determine how difficult it is to distinguish actions in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ up to semi-conjugacy, and whether it is possible to classify them. To conclude this section, we highlight that the Deroin space and of Theorem 3.2 shed light on this question, and suggest a line of research for group actions on the line. An appropriate framework to formalize the question is the theory of Borel reducibility of equivalence relations. Most notions of isomorphisms can be naturally interpreted as equivalence relations on some standard Borel space, so that this theory offers tools to study various classification problems and compare the difficulty of one with respect to another. Let us recall some basic notions from this setting following the exposition of Kechris [54] (to which we refer for more details). Recall that a standard Borel space \mathcal{Z} is a measurable space isomorphic to a complete separable metric space endowed with its Borel σ -algebra. A subset of such a space is *analytic* if it is the image of a Borel set under a Borel map from a standard Borel space. An analytic equivalence relation on \mathcal{Z} (henceforth just *equivalence relation*) is an equivalence relation \mathcal{R} which is an analytic subset $\mathcal{R} \subset \mathcal{Z} \times \mathcal{Z}$. This standing assumption is convenient to develop the theory, and general enough to model most natural isomorphism problems. It includes in particular the class of Borel equivalence relations (those which are Borel subsets of $\mathcal{Z} \times \mathcal{Z}$), which is much better behaved but too restricted for some purposes (see below). We write $x\mathcal{R}y$ if $(x, y) \in \mathcal{R}$.

For $i \in \{1, 2\}$ let \mathcal{R}_i be an equivalence relation on a standard Borel space \mathcal{Z}_i . We say that \mathcal{R}_1 is *reducible* to \mathcal{R}_2 , and write $\mathcal{R}_1 \leq_B \mathcal{R}_2$, if there exists a Borel map $q : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$ such that $x\mathcal{R}_1y$ occurs if and only if $q(x)\mathcal{R}_2q(y)$. This means that distinguishing classes of \mathcal{R}_1 is “easier” than for \mathcal{R}_2 . If $\mathcal{R}_1 \leq_B \mathcal{R}_2$ and $\mathcal{R}_2 \leq_B \mathcal{R}_1$, we say that \mathcal{R}_1 and \mathcal{R}_2 are *bireducible*.

The simplest type of equivalence relations from the perspective of reducibility are the *smooth* ones. An equivalence relation \mathcal{R} on \mathcal{Z} is said smooth if there exist a standard Borel space \mathcal{W} and a Borel map $q : \mathcal{Z} \rightarrow \mathcal{W}$ which is a complete invariant for the equivalence classes of \mathcal{R} , in the sense that $x\mathcal{R}y$ if and only if $q(x) = q(y)$.

The next class of equivalence relations are the *hyperfinite* ones. An equivalence relation \mathcal{R} is called *hyperfinite* if it is a countable union $\mathcal{R} = \bigcup \mathcal{R}_n$ of equivalence relations whose classes

are finite, and essentially hyperfinite if it is bireducible to a hyperfinite one. An example of hyperfinite equivalence relation which is not smooth, denoted \mathcal{E}_0 , is the equivalence relation on the set of one-sided binary sequence $\{0, 1\}^{\mathbb{N}}$, where (x_n) and (y_n) are equivalent if $x_n = y_n$ for large enough n . In fact, up to bireducibility the relation \mathcal{E}_0 is the unique essentially hyperfinite equivalent relation which is not smooth, and moreover every Borel equivalence relation \mathcal{R} is either smooth or satisfies $\mathcal{E}_0 \leq_B \mathcal{R}$ [47], so that \mathcal{E}_0 can be thought of as the simplest non-smooth Borel equivalence relation.

The essentially hyperfinite equivalence relations form a strict subset of the essentially countable ones (those that are bireducible to a relation whose classes are countable). These are precisely those induced by orbits of actions of countable groups, and the poset of bireducibility types of such relations is quite complicated (see [1]). Essentially countable relations are themselves a strict subset of Borel equivalence relations, after which we find general (analytic) equivalence relations.

The bireducibility type of the conjugacy relation of various classes of dynamical systems and group actions has been extensively studied. For many classes of topological or measurable dynamical systems, the conjugacy relations is known or conjectured to be quite complicated from the perspective of bireducibility [39, 40, 62, 102]. As an example, the conjugacy relation of elements of $\text{Homeo}_0(\mathbb{R})$ is bireducible to the isomorphism relation of all countable (but not necessarily locally finite) graphs, which is analytic but not Borel (see [48, Theorem 4.9]), and the conjugacy of homeomorphisms of the plane is strictly more complicated by a result of Hjorth [48, Theorem 4.17]. As a consequence, one may expect that the semi-conjugacy relation on the space of irreducible action of a given group G should also be complicated and might not even be Borel. In contrast, the existence of Deroin space and Theorem 3.2 imply the following.

Corollary 3.21. *Let G be a finitely generated group. Then the semi-conjugacy relation on the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is essentially hyperfinite (in particular, it is Borel).*

Proof. Theorem 3.2 shows that this relation is bireducible to the orbit equivalence relation on the Deroin space $\text{Der}_\mu(G)$ induced by the translation flow Φ . On the other hand, the orbit equivalence relation of any Borel flow on a standard Borel space is essentially hyperfinite [54, Theorem 8.32]. \square

After Corollary 3.21 and the previous discussion we may further distinguish two cases: either the semi-conjugacy relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is smooth, or it is not, in which case it is bireducible to \mathcal{E}_0 . As mentioned above, the first case corresponds to groups for which actions up to semi-conjugacy can be completely classified by a Borel invariant, so that it is natural to ask which groups have this property. This can be rephrased using the Deroin space as follows.

Corollary 3.22. *Let G be a finitely generated group endowed with a symmetric finitely supported probability measure μ . Then the semi-conjugacy relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is smooth if and only if the translation flow Φ on the Deroin space $\text{Der}_\mu(G)$ admits a Borel cross section (that is a Borel subset which intersects every Φ -orbit in exactly one point).*

Proof. Given a flow Φ on a Borel space \mathcal{Z} , the orbit equivalence relation induced by Φ is smooth if and only if Φ admits a Borel cross section [54, Proposition 3.12]. \square

We propose to keep Corollary 3.22 in mind as a goal when studying Deroin space of groups. We will present some applications in §10.4.3.

Remark 3.23. There exist finitely generated groups G for which the semi-conjugacy relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is not smooth (and in particular, the space of semi-conjugacy classes is

not a standard measurable space). For example it is not difficult to show this for the free group \mathbb{F}_2 as follows. Fix a ping-pong pair of homeomorphisms g, h of \mathbb{R}/\mathbb{Z} , with $\text{Fix}(g) = \{0, 1/2\}$ and $\text{Fix}(h) = \{1/4, 3/4\}$, where both g, h have one attracting and one repelling fixed point and such that $\langle g, h \rangle$ acts minimally on \mathbb{R}/\mathbb{Z} . Let $\tilde{g}, \tilde{h} \in \text{Homeo}_0(\mathbb{R})$ be two lifts, with $\text{Fix}(\tilde{g}) = \frac{1}{2}\mathbb{Z}$ and $\text{Fix}(\tilde{h}) = \frac{1}{2}\mathbb{Z} + \frac{1}{4}$. Given a sequence $\omega = (\omega_n) \in \{+1, -1\}^{\mathbb{Z}}$, define an element $\tilde{g}_\omega \in \text{Homeo}_0(\mathbb{R})$ by $\tilde{g}_\omega(x) = \tilde{g}^{\omega_n}(x)$ if $x \in [\frac{1}{2}n, \frac{1}{2}n + 1]$, and if \mathbb{F}_2 is the free group with free generators a, b , define a representation $\varphi_\omega: \mathbb{F}_2 \rightarrow \text{Homeo}_0(\mathbb{R})$ by $\varphi_\omega(a) = \tilde{g}_\omega$ and $\varphi_\omega(b) = \tilde{h}$. It is not difficult to check that the map $\omega \mapsto \varphi_\omega$ is Borel (actually continuous). since $\varphi_\omega(\mathbb{F}_2) = \langle \tilde{g}_\omega, \tilde{h} \rangle$ acts on \mathbb{R} with the same orbits as $\langle \tilde{g}, \tilde{h} \rangle$, every action φ_ω is minimal, and this implies for $\omega, \omega' \in \{\pm 1\}^{\mathbb{Z}}$ the actions φ_ω and $\varphi_{\omega'}$ are positively semi-conjugate if and only if they are positively conjugate, and this happens if and only if ω and ω' belong to the same orbit of the \mathbb{Z} -shift. Since the orbit equivalence relation of the shift is not smooth, the conclusion follows.

Remark 3.24. In contrast, for every countable group G the semi-conjugacy relation on the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{S}^1))$ of irreducible actions on the circle is always smooth. One way to prove this is to repeat the above argument by fixing a generating probability measure μ on G and recall that every action is semi-conjugate to an action for which the Lebesgue measure is stationary (unlike for the case of the real line, this is a straightforward consequence of compactness). This can be used to construct an analogue of the Deroin space, where the translation flow should be replaced by an action of the group \mathbb{S}^1 , and since this is a compact group the action must always admit a Borel cross section. Another route would be to prove that the bounded Euler class is a Borel complete invariant under semi-conjugacy. This difference can be seen as a formalization of the observation, mentioned in the introduction, that studying actions on the circle up to semi-conjugacy is easier than for the real line thanks to compactness.

4. MICRO-SUPPORTED AND LOCALLY MOVING GROUPS

4.1. Definitions. Throughout this section (and mostly in the rest of the paper), we let $X = (a, b)$ be an open interval, with endpoints $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Recall that for a subgroup $G \subseteq \text{Homeo}_0(X)$ and a subinterval $I \subset X$, we denote by G_I the subgroup of G consisting of elements that fix pointwise $X \setminus I$.

Definition 4.1. A subgroup $G \subseteq \text{Homeo}_0(X)$ is *micro-supported* if for every non-empty subinterval $I \subset X$ the subgroup G_I is non-trivial. We also say that G is *locally moving* if for every open subinterval $I \subset X$ the subgroup G_I acts on I without fixed points.

Given open subintervals I and J we write $I \Subset J$ if I is relatively compact in J . For $G \subseteq \text{Homeo}_0(X)$, we denote by G_c the normal subgroup of elements with relatively compact support in X , that is, $G_c = \bigcup_{I \Subset X} G_I$. We also let $\text{Germ}(G, a)$ and $\text{Germ}(G, b)$ be the *groups of germs* of elements of G at the endpoints of X . Recall that the germ of an element $g \in G$ at a is the equivalence class of g under the equivalence relation that identifies two elements $g_1, g_2 \in G$ if they coincide on some interval of the form (a, x) , with $x \in X$. The germ of g at b is defined similarly. We denote by $\mathcal{G}_a: G \rightarrow \text{Germ}(G, a)$ and $\mathcal{G}_b: G \rightarrow \text{Germ}(G, b)$ the two natural germ homomorphisms and their kernels by G_- and G_+ , respectively. Note that

$$G_- = \bigcup_{x \in X} G_{(x, b)} \quad \text{and} \quad G_+ = \bigcup_{x \in X} G_{(a, x)}.$$

When G acts minimally, the micro-supported condition is equivalent to the non-triviality of the subgroup G_c :

Proposition 4.2. *For $X = (a, b)$, assume that the subgroup $G \subseteq \text{Homeo}_0(X)$ acts minimally on X . Then G is micro-supported if and only if it contains a non-trivial element with relatively compact support.*

Proof. The forward implication is obvious. Conversely, assume that there exists a relatively compact subinterval $I \Subset X$ for which $G_I \neq \{\text{id}\}$. It follows that the centralizer of G_I in $\text{Homeo}_0(X)$ must fix the infimum of the support of every non-trivial element of G_I . Then by Theorem 2.17, the action of G on X is proximal. Therefore for every non-empty open subinterval $J \subset X$ there exists $g \in G$ such that $g(I) \subset J$, which implies that the group G_J is non-trivial. \square

Let us summarize some basic observations on the locally moving condition in the following lemma.

Lemma 4.3. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then the following hold for every non-empty open subinterval $I \subset X$.*

- (1) *The subgroup G_I acts minimally on I . In particular G acts minimally on X .*
- (2) *The derived subgroup $[G_I, G_I]$ also acts without fixed points on I .*

Proof. Write $I = (c, d)$ for a non-empty open subinterval. Fix $x, y \in I$ and assume, say, that $x < y$. Since the subgroup $G_{(c,y)} \subseteq G_I$ has no fixed point in (c, y) , there exist elements $g \in G_{(c,y)}$ such that $g(x)$ is arbitrarily close to y . Thus the G_I -orbit of x accumulates at y . By a symmetric argument, the same holds if $y < x$. Since x and y are arbitrary, this shows that every G_I -orbit in I is dense in I . Finally if $[G_I, G_I]$ admits fixed points, its set of fixed points is closed and G_I -invariant and thus by minimality of the action of G_I we deduce that $[G_I, G_I]$ is trivial, and therefore that G_I is abelian and conjugate to a group of translations. This is not possible since the action of G_I on I is micro-supported. \square

4.2. Structure of normal subgroups. The following proposition shows that locally moving groups are close to be simple. This follows from well-known arguments, that we repeat here for completeness.

Proposition 4.4 (Structure of normal subgroups). *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a micro-supported subgroup whose action is minimal. Then every non-trivial normal subgroup of G contains $[G_c, G_c]$. Moreover, if $[G_c, G_c]$ acts minimally, then it is simple.*

In particular, when G is locally moving, then $[G_c, G_c]$ is simple and contained in every non-trivial normal subgroup of G .

The proof uses the following classical observation on normal subgroups of homeomorphisms, sometimes known as the “double-commutator lemma”. With this formulation it is [95, Lemma 4.1]

Lemma 4.5. *Let H be a group of homeomorphisms of a Hausdorff space Z , and N be a non-trivial group of homeomorphisms of Z normalized by H . Then there exists a non-empty open subset $U \subset Z$ such that N contains $[H_U, H_U]$, where H_U is the pointwise fixator of $Z \setminus U$.*

Proof of Proposition 4.4. Suppose that N is a non-trivial normal subgroup of G . Then by Lemma 4.5, N contains $[G_I, G_I]$ for some non-empty open subinterval $I \subset X$. Take now $g \in [G_c, G_c]$. Then $g \in [G_J, G_J]$ for some non-empty open subinterval $J \Subset X$. Note that the centralizer of G in $\text{Homeo}_0(X)$ is trivial, for the action is micro-supported (see the proof of Proposition 4.2). Then by Theorem 2.17, the action is proximal, so that we can find $h \in G$ such that $h(J) \subset I$, so that $hgh^{-1} \in [G_I, G_I] \subseteq N$ and since N is normal we have $g \in N$.

Since g is arbitrary, we have $[G_c, G_c] \subseteq N$. Note that this implies in particular that $[G_c, G_c]$ is perfect, since its commutator subgroup is normal in G and thus must coincide with $[G_c, G_c]$.

Assume now that $[G_c, G_c]$ acts minimally on X . Then it is micro-supported (Proposition 4.2), and by the previous part every non-trivial normal subgroup of $[G_c, G_c]$ must contain the derived subgroup of $[G_c, G_c]$. Since $[G_c, G_c]$ is perfect, this implies that it is simple. \square

It follows that when $G \subseteq \text{Homeo}_0(X)$ is micro-supported and acts minimally (in particular, when G is locally moving), the quotient group $\overline{G} := G/[G_c, G_c]$ is the largest proper quotient of G , and thus plays an important role. Note that the (*a priori* smaller) quotient G/G_c has a natural dynamical interpretation, namely it is naturally a subgroup of the product of groups of germs $\text{Germ}(G, a) \times \text{Germ}(G, b)$. The largest quotient \overline{G} is an extension of G/G_c with abelian kernel:

$$1 \rightarrow G_c/[G_c, G_c] \rightarrow \overline{G} \rightarrow G/G_c \rightarrow 1.$$

The abelian group $G_c/[G_c, G_c]$ can be difficult to identify in general. However for some relevant examples of locally moving groups it is known that the group G_c is perfect, so that $\overline{G} = G/G_c$.

Example 4.6. Consider the Brin–Navas group, which was introduced independently by Brin [18] and Navas [88], and further studied by Bleak [9] who showed that B is contained in any non-solvable subgroup of $\text{PL}([0, 1])$. The group B has the following presentation (see [11] and Proposition 9.33):

$$B = \left\langle f, w_n (n \in \mathbb{Z}) \mid fw_n f^{-1} = w_{n+1} \forall n \in \mathbb{Z}, [w_i, w_n^m w_j w_n^{-m}] = 1 \forall n > i, j, \forall m \in \mathbb{Z} \setminus \{0\} \right\rangle.$$

That is, the group B is defined as an HNN extension of the group generated by the w_n , $n \in \mathbb{Z}$, and this group is a bi-infinitely iterated wreath product of \mathbb{Z} . Following the notation in [9], we write $(\wr \mathbb{Z})^\infty$ for the subgroup generated by the w_n in B , so that $B = (\wr \mathbb{Z})^\infty \rtimes \mathbb{Z}$. A minimal micro-supported action on $(0, 1)$ of B is realized in the group $\text{PL}([0, 1])$ of piecewise linear homeomorphisms, choosing generators (see [9] and Figure 2)

$$f(x) = \begin{cases} \frac{1}{4}x & x \in [0, \frac{1}{4}], \\ x - \frac{3}{16} & x \in [\frac{1}{4}, \frac{7}{16}], \\ 4x - \frac{3}{2} & x \in [\frac{7}{16}, \frac{9}{16}], \\ x + \frac{3}{16} & x \in [\frac{9}{16}, \frac{3}{4}], \\ \frac{1}{4}x + \frac{3}{4} & x \in [\frac{3}{4}, 1], \end{cases} \quad w_0(x) = \begin{cases} x & x \in [0, \frac{7}{16}], \\ 2x - \frac{7}{16} & x \in [\frac{7}{16}, \frac{15}{32}], \\ x + \frac{1}{32} & x \in [\frac{15}{32}, \frac{1}{2}], \\ \frac{1}{2}x + \frac{9}{32} & x \in [\frac{1}{2}, \frac{9}{16}], \\ x & x \in [\frac{9}{16}, 1]. \end{cases}$$

In this case, the subgroup B_c is the normal subgroup $(\wr \mathbb{Z})^\infty$, and $B_c/[B_c, B_c] \cong \mathbb{Z}^\infty$, so that the largest proper quotient $\overline{B} \cong \mathbb{Z}^\infty \rtimes \mathbb{Z} = \mathbb{Z} \wr \mathbb{Z}$ is the lamplighter group. Observe that the bi-infinite wreath product B_c does not act minimally on $(0, 1)$.

Example 4.7. A rich source of examples of micro-supported, and actually locally moving groups, are the Bieri–Strebel groups introduced with the Definition 2.44. Recall that these are defined as the groups of piecewise linear homeomorphisms of an interval with prescribed breakpoints and slopes. For a quite simple class of examples, fix $n \geq 2$ and consider the group $G = F_n$ of all piecewise linear homeomorphisms of $[0, 1]$ such that all derivatives are powers of n and the breakpoints are in the ring $A = \mathbb{Z}[1/n]$. When $n = 2$, this is Thompson’s group F . The subgroup G_c of compactly supported elements consists exactly of elements which have derivative 1 at the endpoints of $[0, 1]$, so that $G/G_c \cong \mathbb{Z}^2$. However, the abelianization of G is larger for $n \geq 3$, as $G^{ab} \cong \mathbb{Z}^n$. This is determined by the “homomorphism into the slope group” ν discussed in [8, §C11]. The homomorphism ν is obtained by gathering together the

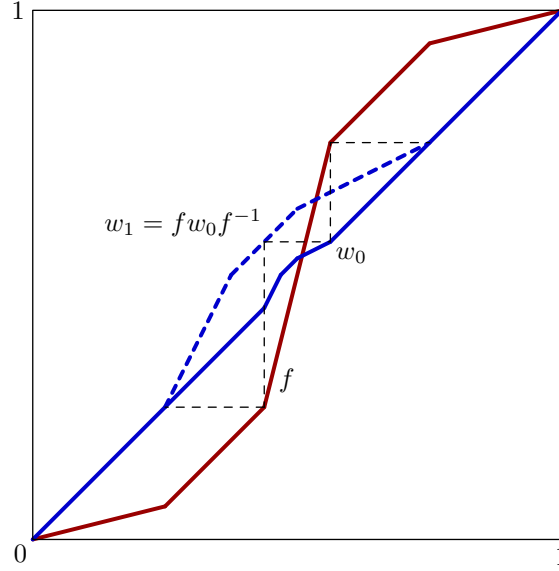


FIGURE 2. PL realization of the Brin–Navas group with minimal micro-supported action.

homomorphisms $\nu_\Omega : G \rightarrow \mathbb{Z}$, where Ω is a G -orbit of breakpoints (there are exactly $n - 1$ many of them), which are defined by considering the total jump of derivatives at points in the orbit:

$$\nu_\Omega(f) = \log_n \prod_{a \in \Omega} \frac{D^+ f(a)}{D^- f(a)}, \quad f \in G.$$

4.3. Subgroups isomorphic to Thompson’s group. Thompson’s group F plays a special role among locally moving groups, due to the following proposition, which will be crucial in our proof of the C^1 rigidity of locally moving groups (Theorem 6.9).

Proposition 4.8. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then G contains a subgroup isomorphic to Thompson’s group F .*

Its proof is based on the following variant of the “2-chain lemma” of Kim, Koberda, and Lodha [55]. The key idea can be traced to Brin [18], and has been also largely developed in [10]. It is based on the following two properties: every non-trivial quotient of F is abelian, and the finite presentation of F ,

$$(4.1) \quad F = \langle a, b \mid [a, (ba)b(ba)^{-1}] = [a, (ba)^2 b(ba)^{-2}] = 1 \rangle,$$

where the two relations have in fact an interesting dynamical interpretation.

Lemma 4.9 (Noisy 2-chain lemma). *Take two homeomorphisms $f, g \in \text{Homeo}_0(\mathbb{R})$, write $d = \sup \text{Supp}(f)$ and $c = \inf \text{Supp}(g)$, and assume the following:*

- (1) $c < d$;
- (2) $c \notin \text{Fix}(f)$ and $d \notin \text{Fix}(g)$;
- (3) d and $f(c)$ are in the same connected component of $\text{Supp}(g)$.

Then $\langle f, g \rangle$ contains a subgroup isomorphic to Thompson’s F

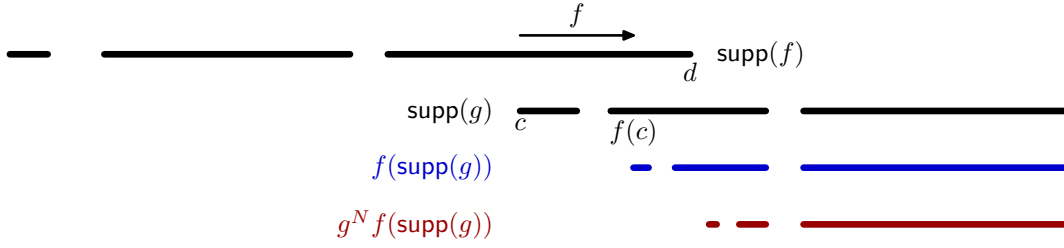


FIGURE 3. Proof on the noisy 2-chain lemma (Lemma 4.9).

Proof. After the assumptions, there exists $N \in \mathbb{Z}$ such that $g^N f(c) > d$. Thus we also have $(g^N f)^2(c) = g^{2N} f(c) > d$. Hence, for $i \in \{1, 2\}$, we have that the subset

$$(g^N f)^i(\text{Supp}(g)) = \text{Supp}\left((g^N f)^i g (g^N f)^{-i}\right) = \text{Supp}\left((g^N f)^i g^N (g^N f)^{-i}\right)$$

is disjoint from $\text{Supp}(f)$ (see Figure 3). We deduce that the elements $a = f$ and $b = g^N$ satisfy the two relations in the presentation (4.1), and thus $\langle f, g^N \rangle$ is isomorphic to a quotient of F . As the supports $\text{Supp}(f)$ and $\text{Supp}(g^N) = \text{Supp}(g)$ overlap, we deduce that the subgroup $\langle f, g^N \rangle$ is non-abelian and thus isomorphic to F . \square

Proof of Proposition 4.8. Take $f \in G_c$. Using Lemma 4.3, it is not difficult to find an element g , conjugate to f , such that conditions (1–2) in Lemma 4.9 are satisfied by f and g . Write $c = \inf \text{Supp}(g)$ and $d = \sup \text{Supp}(f)$. Up to replace f by its inverse, we can assume $f(c) > c$. If condition (3) in Lemma 4.9 is not satisfied, we use Lemma 4.3 again to find an element $h \in G_{(c,d)}$ such that $h(f(c))$ belongs to the same connected component of $\text{Supp}(g)$ as d . Replace f by the conjugate hfh^{-1} and now property (3) is also satisfied. \square

5. DYNAMICAL TRICHOTOMY FOR ACTIONS OF LOCALLY MOVING GROUPS ON THE LINE

5.1. The trichotomy. Let $X = (a, b)$ be an open interval. The goal of this section is to establish a first result on the possible actions $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$, when $G \subseteq \text{Homeo}_0(X)$ is locally moving, showing that such actions fall into one of three kinds. This trichotomy will be the starting point of most of our results.

In the sequel we will often be dealing with two different actions of the same group G , namely its standard action on X and different, given action of G on \mathbb{R} . Recall that to avoid confusion we fix the following notation throughout the paper.

Notation 5.1. Let $G \subseteq \text{Homeo}_0(X)$ be a subgroup and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action on the real line. For $g \in G$ and $x \in X$ we use the notation $g(x)$ to refer to the standard action on X , while for $\xi \in \mathbb{R}$ we will write $g.\xi := \varphi(g)(\xi)$. We also write $\text{Fix}^\varphi(H)$ for the set of fixed points of a subgroup $H \subseteq G$ with respect to the action φ , and $\text{Supp}^\varphi(H) = \mathbb{R} \setminus \text{Fix}^\varphi(H)$.

Also, we will often write $K = G_c$ and $N = [G_c, G_c]$, although this will be systematically recalled.

Let us also introduce the following terminology, which will simplify many statements.

Definition 5.2. Let G be a group and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action on the real line. Let $A, B \subseteq G$ be subgroups. We say that A is *totally bounded* if $\text{Fix}^\varphi(A)$ accumulates at both $\pm\infty$ (equivalently, if the $\varphi(A)$ -orbit of every $\xi \in \mathbb{R}$ is bounded).

We say that A is *dominated by commuting elements* if its φ -image has fixed points, while the image of its centralizer $\varphi(C_G(A))$ has none. If moreover $B \subseteq G$ is a subgroup such

that the φ -image of $C_B(A)$ has no fixed point, we say that A is dominated by commuting elements *within* B . Note that since $C_B(A)$ preserves the subset $\text{Fix}^\varphi(A)$ and acts without fixed points, this implies that A is totally bounded. Finally we say that A is *locally* dominated by commuting elements (within B) if all finitely generated subgroups of A are dominated by commuting elements (within B).

Recall that given a subgroup $G \subseteq \text{Homeo}_0(X)$, with $X = (a, b)$, we denote by G_- and G_+ the kernels of the two germs homomorphisms $\mathcal{G}_a: G \rightarrow \text{Germ}(G, a)$ and $\mathcal{G}_b: G \rightarrow \text{Germ}(G, b)$, respectively. Recall also that we let G_c be the subgroup of compactly supported elements in G , and that $[G_c, G_c]$ is simple and is the smallest normal subgroup of G (Proposition 4.4).

The following theorem says that all “exotic” actions of G have an abundance of subgroups dominated by commuting elements.

Theorem 5.3 (Dynamical trichotomy for locally moving groups). *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points satisfies one of the following statements.*

- (i) (Induced from a quotient) *It is semi-conjugate to an action that factors through the largest quotient $G/[G_c, G_c]$.*
- (ii) (Standard) *It is semi-conjugate to the standard action of G on X .*
- (iii) (Exotic) *One of the subgroups G_+ and G_- (possibly both) is locally dominated by commuting elements within $[G_c, G_c]$.*

Theorem 5.3 leverages the abundance of commutation in a locally moving group. This is quite natural to exploit, and can be compared with some rigidity results for actions of certain large groups on one-manifolds, which take advantage of commutation for actions with some regularity, for instance by diffeomorphisms in class C^2 (see for example Ghys and Sergiescu [45, Theorem K] for Thompson’s group T , or the results of Mann [68] and the alternative proofs by Matsumoto [80]). Indeed recall that Kopell’s lemma states that if f, g are two C^2 diffeomorphisms of a compact interval and if f has no fixed point in the interior, then neither does g . This gives a powerful tool to reconstruct the standard action by keeping track of supports of elements. In contrast, for C^0 homeomorphisms it is much easier to commute, so that mere commutation is not sufficient to reconstruct support of elements and get rigidity (and indeed exotic actions in the sense of Theorem 5.3 exist in abundance). The conclusion of Theorem 5.3 describes precisely this failure of rigidity.

We now prove Theorem 5.3. We begin with a useful observation based on Theorem 2.17.

Proposition 5.4 (Actions of direct products). *Let $M \in \{\mathbb{R}, \mathbb{S}^1\}$. Let Γ_1 and Γ_2 be two groups, and if $M = \mathbb{R}$ assume that Γ_1 is finitely generated. Then for every action $\varphi: \Gamma_1 \times \Gamma_2 \rightarrow \text{Homeo}_0(M)$, there exists $i \in \{1, 2\}$ such that the image of $[\Gamma_i, \Gamma_i]$ has fixed points.*

Proof. Note that we identify here Γ_1 with the subgroup $\Gamma_1 \times \{1\}$, and similarly for Γ_2 . Suppose that $\varphi(\Gamma_1)$ admits no fixed point (otherwise, the conclusion is true). If the action of Γ_1 admits a closed discrete orbit, then the action on this discrete orbit factors through a cyclic quotient of Γ_1 and so $[\Gamma_1, \Gamma_1]$ fixes it pointwise, and the conclusion holds true. If not, let $\Lambda \subset M$ be the unique minimal invariant set for $\varphi(\Gamma_1)$, which exists when $M = \mathbb{R}$ since we assume Γ_1 finitely generated. Then Λ is preserved by $\varphi(\Gamma_2)$ and the action of Γ_1 is semi-conjugate to a minimal action, obtained by collapsing the connected components of $M \setminus \Lambda$. In the case $M = \mathbb{R}$, we apply Theorem 2.17 to this minimal action. If this action is by translations then again $[\Gamma_1, \Gamma_1]$ acts trivially on Λ . Otherwise its centralizer is trivial or cyclic, so that $[\Gamma_2, \Gamma_2]$ fixes Λ pointwise. When $M = \mathbb{S}^1$, we argue similarly using Theorem 2.16, which gives that

either the action is conjugate to an action by rotations (in which case $[\Gamma_1, \Gamma_1]$ acts trivially), or the centralizer of $\varphi(\Gamma_1)$ is finite cyclic (in which case $[\Gamma_2, \Gamma_2]$ fixes every point of Λ). \square

Remark 5.5. Note that here we will only use this for $M = \mathbb{R}$ (but an application of the case $M = \mathbb{S}^1$ will be given in Section 14). In this case, the assumption that Γ_1 be finitely generated cannot be dropped, as shown by the following example. Let $(H, <)$ be any left-ordered non-abelian countable group. Let $G = \bigoplus_{n \in \mathbb{N}} H$ be the infinite restricted product, i.e. the group of all sequences (h_n) in H such that $h_n = 1$ for all but finitely many n , with pointwise multiplication. Consider on G the lexicographic order given by $(h_n) < (h'_n)$ if $h_m < h'_m$ for $m = \max\{n \in \mathbb{N} : h_n \neq h'_n\}$, and let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be the dynamical realization of this order. Let $\Gamma_1 \subseteq G$ be the subgroup consisting of all sequences (h_n) such that $h_n = 1$ for n even, and Γ_2 be the subgroup of sequences such that $h_n = 1$ for n odd, so that $G = \Gamma_1 \times \Gamma_2$. Then it is easy to see that neither the image of $[\Gamma_1, \Gamma_1]$ nor of $[\Gamma_2, \Gamma_2]$ have fixed points.

Remark 5.6. A special case of Proposition 5.4 appears in [80, Proposition 3.1] (for $M = \mathbb{S}^1$ and assuming Γ_i are simple).

Proposition 5.4 implies the following in the setting of micro-supported groups.

Corollary 5.7. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup acting without fixed points on X . Let $\Gamma \subseteq G_c$ be a finitely generated subgroup. Then for every action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ the image $\varphi([\Gamma, \Gamma])$ has fixed points.*

Proof. Let $I \Subset X$ be a relatively compact subinterval such that $\Gamma \subseteq G_I$. Since the action of G has no fixed points, we can find $g \in G$ such that $g(I) \cap I = \emptyset$. Then $g\Gamma g^{-1} \subseteq G_{g(I)}$ commutes with Γ , so that Proposition 5.4 applied to the subgroups $\Gamma_1 := \Gamma$ and $\Gamma_2 := g\Gamma g^{-1}$ implies in either case that $\varphi([\Gamma, \Gamma])$ has fixed points. \square

Remark 5.8. Again, finite generation is essential: it is not true that for every interval $I \subset X$, the φ -image of $[G_I, G_I]$ must have fixed points. Counterexamples can be found in §5.3.3.

The next lemma is an improvement of the previous corollary, and will be used repeatedly.

Lemma 5.9. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup acting without fixed points, and set $K = G_c$. Let I and J be non-empty disjoint open subintervals of X . Let $\Gamma \subseteq G_I$ be a finitely generated subgroup. Then for every action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$, at least one of the following holds.*

- (i) Γ is totally bounded (with respect to the action φ).
- (ii) The image $\varphi([K_J, K_J])$ has fixed points.

Proof. We assume that (i) does not hold and prove that (ii) must hold. First of all we note that if $\text{Fix}^\varphi(\Gamma)$ is non-empty, but does not accumulate at both $\pm\infty$, then its supremum (or infimum) must be fixed by $\varphi(G_J) \supseteq \varphi([K_J, K_J])$, since the two subgroups commute. Thus we can suppose that $\text{Fix}^\varphi(\Gamma) = \emptyset$. In this case, since Γ is finitely generated, there exists a compact interval L which intersects all $\varphi(\Gamma)$ -orbits. Let \mathcal{S} be the collection of finitely generated subgroups of K_J . For $\Delta \in \mathcal{S}$, the subset $T_\Delta := \text{Fix}^\varphi([\Delta, \Delta])$ is non-empty by Corollary 5.7, and it is $\varphi(\Gamma)$ -invariant. Thus, we have $T_\Delta \cap L \neq \emptyset$. Moreover for $\Delta_1, \dots, \Delta_n \in \mathcal{S}$ we have $T_{\Delta_1} \cap \dots \cap T_{\Delta_n} \supset T_{\langle \Delta_1, \dots, \Delta_n \rangle} \neq \emptyset$, so that by compactness of L we have that the subset $T := \bigcap_{\Delta \in \mathcal{S}} T_\Delta$ is non-empty. Since $[K_J, K_J] = \bigcup_{\Delta \in \mathcal{S}} [\Delta, \Delta]$, every point of T is fixed by $\varphi([K_J, K_J])$. This proves the lemma. \square

We now need to address a minor technical point. Let $G \subseteq \text{Homeo}_0(X)$ be locally moving, and set $K = G_c$ and $N = [K, K]$. Let $I \subset X$ be a subinterval. We can then consider the two subgroups N_I and $[K_I, K_I]$. A little thinking gives that $[K_I, K_I] \subseteq N_I$, but in general this inclusion is strict. However the following lemma allows to think of it as an equality for some practical purposes.

Lemma 5.10. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Let $I \subset J \subset X$ be open nested subintervals with no common endpoint in $X \setminus \{a, b\}$. With the notation as above, we have $[K_I, K_I] \subseteq N_I \subseteq [K_J, K_J]$.*

Proof. Only the second inclusion needs to be justified. Let us give the proof in the case where $I = (a, x)$ and $J = (a, y)$ for some $y > x$ (the case where both endpoints of I are in the interior of X is analogous). Take $g \in N_I$ and let us show that $g \in [K_J, K_J]$. Since $g \in N$, it can be written as a product of commutators involving finitely many elements of G_c , which all belong to $K_{(a,z)}$ for some $z < b$ large enough. Thus we have $g \in [K_{(a,z)}, K_{(a,z)}]$ for some $z < b$ large enough, and we can suppose that $z > y$ since otherwise $g \in [K_J, K_J]$. Since the group $G_{(x,b)}$ acts without fixed points on (x, b) , there exists $h \in G_{(x,b)}$ such that $h(z) \in (z, y)$. But h commutes with g , so we have $g = hgh^{-1} \in [K_{(a,h(z))}, K_{(a,h(z))}] \subseteq [K_J, K_J]$. \square

We now give a criterion for an action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ to be semi-conjugate either to the standard action, or to an action induced from a proper quotient.

Proposition 5.11. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving, and write $N = [G_c, G_c]$. Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points. Suppose that there exist $x, y \in X$ such that both images $\varphi(N_{(a,x)})$ and $\varphi(N_{(y,b)})$ admit fixed points. Then φ is semi-conjugate either to the standard action on X , or to an action which factors through the quotient G/N .*

Proof. First of all, observe that if $\varphi(N)$ admits fixed points, then $\text{Fix}^\varphi(N)$ is a closed $\varphi(G)$ -invariant subset of \mathbb{R} which accumulates at both $\pm\infty$, and the action on it factors through G/N . In this case we deduce that φ is semi-conjugate to an action of G/N . Thus we will assume that $\varphi(N)$ has no fixed point, and show that in this case the action must be semi-conjugate to the standard action of G on X .

Note that since there exists x such that $N_{(a,x)}$ has fixed points for φ then this is true for every $x \in X$, since a subgroup $N_{(a,x)}$ is always conjugate into every other $N_{(a,y)}$, for $y \in X$. The same holds true for the subgroups of the form $N_{(x,b)}$. In particular, for every $x \in X$, the images $\varphi(N_{(a,x)})$ and $\varphi(N_{(x,b)})$ both admit fixed points, and since they commute they admit common fixed points. Thus for every $x \in X$ the φ -image of the subgroup $H_x := \langle N_{(a,x)}, N_{(x,b)} \rangle$ admits fixed points. The idea is to construct a semi-conjugacy $q: X \rightarrow \mathbb{R}$ by setting $q(x) = \inf \text{Fix}^\varphi(H_x)$. We need to check that such map q is well-defined (i.e. that the subsets $\text{Fix}^\varphi(H_x)$ are bounded) and monotone. Let us first prove the following claim.

Claim 1. *Let x and y be two distinct points of X . Then we have either $\sup \text{Fix}^\varphi(H_x) < \inf \text{Fix}^\varphi(H_y)$ or viceversa.*

Proof of claim. If the conclusion does not hold, then upon exchanging the roles of x and y if needed, we can find two distinct points $\xi, \eta \in \text{Fix}^\varphi(H_x)$ and $\delta \in \text{Fix}^\varphi(H_y)$ such that $\xi \leq \delta \leq \eta$. Assume first that $x < y$. Recall that we are supposing that $\varphi(N)$ has no fixed point, so we can choose $g \in N$ such that $g.\xi > \eta$. Let $z > x$ be a point of X such that $g \in N_{(a,z)}$. By Lemma 4.3, the action of $N_{(x,b)}$ on (x, b) has no fixed point, so we can find $h \in N_{(x,b)}$ such that $h(z) \in (x, y)$. Note that $h \in H_x$, so that $\varphi(h)$ fixes both ξ and η . On the other hand the

element $k = hgh^{-1}$ belongs to $N_{(a,y)}$, thus to H_y , and therefore $\varphi(k)$ fixes δ . Thus, writing $g = h^{-1}kh$, we have

$$g.\xi = h^{-1}k.\xi \leq h^{-1}k.\delta = h^{-1}.\delta \leq h^{-1}.\eta = \eta,$$

contradicting that $g.\xi > \eta$ by the choice of g . The case $y < x$ is treated analogously. \square

After Claim 1, the map $q: X \rightarrow \mathbb{R}$, $x \mapsto \inf \text{Fix}^\varphi(H_x)$ is well-defined and injective. We next have to verify that it is monotone.

Claim 2. *Let $x_1 < x_2 < x_3$ be point of X . Then either $q(x_1) < q(x_2) < q(x_3)$ or $q(x_1) > q(x_2) > q(x_3)$.*

Proof of claim. The arguments are similar to the proof of the previous claim. For $i \in \{1, 2, 3\}$, set $\xi_i = q(x_i)$ and note that by the previous claim the points ξ_1, ξ_2, ξ_3 are pairwise distinct. We divide the proof into cases according to their relative position. We will detail the case $\xi_1 < \xi_2$ (the case $\xi_1 > \xi_2$ being totally analogous); for this, we will assume for contradiction that $\xi_3 < \xi_2$ and we further split into two subcases depending on the position of ξ_3 .

Case I. *We have $\xi_1 < \xi_3 < \xi_2$.*

In this case, we choose an element $g \in N$ such that $g.\xi_1 > \xi_2$. Let $y > x_2$ be a point of X such that $g \in N_{(a,y)}$. Let $h \in N_{(x_2,b)}$ be such that $h(y) \in (x_2, x_3)$. Note that $h \in H_{x_2}$ and since $(x_2, b) \subset (x_1, b)$ we also have $h \in H_{x_1}$, so that $\varphi(h)$ fixes ξ_2 and ξ_1 . On the other hand the element $k = hgh^{-1}$ belongs to $N_{(a,h(y))}$, and since $(a, h(y)) \subset (a, x_3)$ we have $k \in H_{x_3}$ so that $\varphi(k)$ fixes ξ_3 . Writing $g = h^{-1}kh$, we have

$$g.\xi_1 = h^{-1}k.\xi_1 < h^{-1}k.\xi_3 = h^{-1}.\xi_3 < h^{-1}.\xi_2 = \xi_2,$$

contradicting that $g.\xi_1 > \xi_2$.

Case II. *We have $\xi_3 < \xi_1 < \xi_2$.*

In this case, choose an element $g \in N$ such that $g.\xi_2 < \xi_3$. Let $y < x_2$ be a point of X such that $g \in N_{(y,b)}$. Let $h \in N_{(a,x_2)}$ be such that $h(y) \in (x_1, x_2)$. Note that $h \in H_{x_2}$ and since $(a, x_2) \subset (a, x_3)$ we also have $h \in H_{x_3}$, so that $\varphi(h)$ fixes ξ_2 and ξ_3 . On the other hand the element $k = hgh^{-1}$ belongs to $N_{(h(y),b)}$, and since $(h(y), b) \subset (x_1, b)$ we have $k \in H_{x_1}$ so that $\varphi(k)$ fixes ξ_1 . Writing $g = h^{-1}kh$, we have

$$g.\xi_2 = h^{-1}k.\xi_2 > h^{-1}k.\xi_1 = h^{-1}.\xi_1 > h^{-1}.\xi_3 = \xi_3,$$

contradicting that $g.\xi_2 < \xi_3$.

Thus the unique possibility is that $\xi_1 < \xi_2 < \xi_3$, as desired. \square

The claim implies that the map $q: X \rightarrow \mathbb{R}$ is monotone (increasing or decreasing). Moreover, it is clearly equivariant by construction:

$$g.q(x) = g.\inf \text{Fix}^\varphi(H_x) = \inf g.\text{Fix}^\varphi(H_x) = \inf \text{Fix}^\varphi(H_{g(x)}) = q(g(x)).$$

Therefore the map q establishes a semi-conjugacy between φ and the standard action of G on X . \square

We get to the following statement, which is a more explicit, but slightly more technical, version of Theorem 5.3. Here we keep the notation $N = [G_c, G_c]$.

Proposition 5.12. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points satisfies one of the following statements.*

- (i) (Induced from a quotient) φ is semi-conjugate to an action that factors through the largest quotient G/N .
- (ii) (Standard) φ is semi-conjugate to the standard action of G on X .
- (iii) (Exotic) One of the following holds for every point $x \in X$:
 - (iii.a) (Right-hand side domination) every finitely generated subgroup of G_+ is totally bounded, while $N_{(x,b)}$ acts without fixed points;
 - (iii.b) (Left-hand side domination) every finitely generated subgroup of G_- is totally bounded, while $N_{(a,x)}$ acts without fixed points.

Proof of Theorem 5.3 from Proposition 5.12. Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ and assume that it corresponds to the exotic case in Proposition 5.12, say in case (iii.a). Since every finitely generated subgroup of $\Gamma \subseteq G_+$ is contained in $G_{(a,x)}$ for some x , and it commutes with $N_{(x,b)} \subseteq N$, it follows that G_+ is locally dominated by commuting elements within N . Case (iii.b) is analogous. \square

Proof of Proposition 5.12. Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points, which is not semi-conjugate to the standard action, nor to any action of G/N . Then, by Proposition 5.11, there exists $y \in X$ such that either $\varphi(N_{(y,b)})$ or $\varphi(N_{(a,y)})$ has no fixed point. Assume that the first case holds. Note that this implies that $\varphi(N_{(x,b)})$ has no fixed point for every $x \in X$, since $N_{(x,b)}$ is conjugate into $N_{(y,b)}$. Moreover, by Lemma 5.10 this also implies that the image $\varphi\left([K_{(x,b)}, K_{(x,b)}]\right)$ has no fixed point for every $x \in X$, where $K = G_c$. Then Lemma 5.9 implies that every finitely generated subgroup $\Gamma \subseteq G_{(a,x)}$ is totally bounded. This shows that φ falls in case (iii.a) of Proposition 5.12. The case where $\varphi(N_{(a,y)})$ has no fixed point is analogous and leads to case (iii.b). \square

5.2. First consequences of the trichotomy. Let us describe some elementary consequences of Theorem 5.3. First we record for later use the following observation.

Lemma 5.13. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving, and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal faithful action. Then the action φ is proximal.*

Proof. The standard action is proximal (cf. the proof of Proposition 4.2). So we assume to be in case (iii.a) of Proposition 5.12 (case (iii.b) being analogous). After Theorem 2.17, we assume by contradiction that the action has a non-trivial centralizer, which is necessarily an infinite cyclic group generated by an element without fixed points τ , which we can assume satisfies $\tau(x) > x$ for every $x \in X$. Fix $x \in X$ and write $I = [x, \tau(x)]$, which is a fundamental domain for τ . Let $\Gamma \subset G_+$ be a finitely generated subgroup, and note that as it admits fixed points, it must admit fixed points in the fundamental domain I . As I is compact, the intersection of the subsets $\text{Fix}^\varphi(\Gamma) \cap I$, when Γ runs over the finitely generated subgroups of G_+ , is non-empty, and actually coincides with $\text{Fix}^\varphi(G_+) \cap I$. In particular, this implies that G_+ , and thus G_c , admits fixed points. As $N_{(x,b)} = [G_c, G_c]_{(x,b)} \subset G_c$, we reach a contradiction. \square

We next describe sets of fixed points for exotic actions.

Corollary 5.14. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving, and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action which is not semi-conjugate to the standard action of G on X , nor to any action of a proper quotient. Then the following hold.*

- (i) For every $g \in G$ the support $\text{Supp}^\varphi(g)$ accumulates at both $\pm\infty$.

(ii) *Every finitely generated subgroup $\Gamma \subseteq G_c$ is totally bounded with respect to φ and its support accumulates at both $\pm\infty$, moreover the boundary $\partial \text{Fix}^\varphi(\Gamma)$ of its set of fixed points is non-discrete.*

Proof. Set again $N = [G_c, G_c]$. We first prove (i), which only requires Proposition 5.11. Assume by contradiction that $g \in G$ is such that $\text{Supp}^\varphi(g)$ is upper-bounded (the case where it is lower-bounded is analogous). Then the germ homomorphism $\mathcal{G}_{+\infty} \circ \varphi: G \rightarrow \text{Germ}(\varphi(G), +\infty)$ is not injective, and by Proposition 4.4 its kernel contains N . Thus for $h \in N$ we have that $\text{Supp}^\varphi(h)$ is also upper-bounded, and the point $\xi = \sup \text{Supp}^\varphi(h)$ must be fixed by the centralizer of h . Since the centralizer of h contains $N_{(a,x)}$ and $N_{(y,b)}$ for suitable $x, y \in X$, Proposition 5.11 gives that φ must be semi-conjugate to the standard action on X or to an action of G/N , which is a contradiction.

Let us now prove (ii). We apply Theorem 5.3, and φ must be in the exotic case. Since $G_c = G_+ \cap G_-$, in both subdivisions of such case, we have that every finitely generated subgroup $\Gamma \subseteq G_c$ is dominated by commuting elements (within N), and thus totally bounded. The fact that its support accumulates at both $\pm\infty$ is a consequence of part (i), applied to any element in a finite generating subset of Γ . It remains to prove that $\partial \text{Fix}^\varphi(\Gamma)$ is non-discrete. For this, let $I = (\xi_1, \xi_2)$ be a connected component of $\text{Supp}^\varphi(\Gamma)$. Since the image of $C_N(\Gamma)$ has no fixed point, we can choose $h \in C_N(\Gamma)$ such that $h.\xi_1 > \xi_2$. But since $h \in N \subseteq G_c$, the element h is also totally bounded. Thus $h^n.\xi_1$ must converge as $n \rightarrow +\infty$ towards a fixed point $\zeta \in \text{Fix}^\varphi(h)$. Since $\partial \text{Fix}^\varphi(\Gamma)$ is a closed $\varphi(h)$ -invariant set, we have that $h^n.\xi_1$ is a sequence in $\partial \text{Fix}^\varphi(\Gamma)$ which converges non-trivially to $\zeta \in \partial \text{Fix}^\varphi(\Gamma)$, proving the statement. \square

A relevant special case of the second part of Corollary 5.14 arises when $\Gamma = \langle g \rangle$ is generated by a single element $g \in G_c$. Indeed it implies that if $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is an exotic action, the set of fixed points of every element $g \in G_c$ is unbounded, with unbounded complement, and has non-discrete boundary. This can be seen as a first rigidity result of combinatorial nature, saying that actions with particularly “nice” structure of fixed points are semi-conjugate to the standard action or to an action of the largest quotient. For instance, recall from the introduction that two locally moving actions on the line of the same group G must be conjugate (this is customarily deduced from the much more general results of Rubin [100, 101]). The first statement in Corollary 5.14 recovers this result in the following slightly more general form.

Corollary 5.15. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(\mathbb{R})$ be locally moving. Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed point whose image contain elements of relatively compact support. Then φ is semi-conjugate to the standard action of G on X .*

As another application, let us consider actions by *piecewise real-analytic homeomorphisms*, in the following sense.

Definition 5.16. Let $\text{PDiff}_0^\omega([0, 1])$ be the group of all homeomorphisms f of $[0, 1]$ such that there exist finitely many points

$$0 = a_0 < a_1 < \dots < a_n < a_{n+1} = 1$$

such that for every $i \in \{0, \dots, n\}$, the restriction $f \upharpoonright_{[a_i, a_{i+1}]}$ coincides with an analytic diffeomorphism defined on some open neighborhood of $[a_i, a_{i+1}]$ in the complex plane (in particular f should be analytic on a neighborhood of the endpoints 0 and 1).

Given an open interval (α, β) (possibly unbounded) we also define the larger group $\text{PDiff}_{loc}^\omega((\alpha, \beta))$ of homeomorphisms f such that there exists an increasing sequence $(a_n)_{n \in \mathbb{Z}}$, with $\lim_{n \rightarrow -\infty} a_n = \alpha$ and $\lim_{n \rightarrow +\infty} a_n = \beta$, such that for every $i \in \mathbb{Z}$, the restriction $f \upharpoonright_{[a_i, a_{i+1}]}$

coincides with an orientation-preserving diffeomorphism which is analytic on a neighborhood of $[a_i, a_{i+1}]$ in the complex plane (with no condition at α and β). Note that this includes the more classical setting of piecewise projective homeomorphisms with a discrete set of breakpoints.

Since the set of fixed points of any analytic diffeomorphism is isolated, for every element $g \in \text{PDiff}_{loc}^\omega((\alpha, \beta))$ the boundary $\partial \text{Fix}(g)$ is a discrete subset of (α, β) . From Corollary 5.14 we deduce the following.

Corollary 5.17 (Rigidity of actions by piecewise analytic homeomorphisms). *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. The following hold.*

- (a) *For every faithful action $\varphi: G \rightarrow \text{PDiff}_0^\omega([0, 1])$ without fixed points in the interior $(0, 1)$, the φ -action of G on $(0, 1)$ is semi-conjugate to its standard action on X .*
- (b) *For every interval (α, β) and faithful action $\varphi: G \rightarrow \text{PDiff}_{loc}^\omega((\alpha, \beta))$ without fixed points, one of the following holds:*
 - (b.1) *either φ is semi-conjugate to the standard action of G on X ,*
 - (b.2) *or $\varphi(G)$ admits a closed discrete orbit $O \subset (\alpha, \beta)$, so that φ is semi-conjugate to a cyclic action. Moreover, in this case the φ -image of $N := [G_c, G_c]$ fixes O pointwise, and its action on each connected component of $\text{Supp}^\varphi(N)$ is semi-conjugate to its standard action on X .*

Proof. Let $\varphi: G \rightarrow \text{PDiff}_0^\omega([0, 1])$ be a faithful action, without fixed points in $(0, 1)$. Since for every element $g \in \text{PDiff}_0^\omega([0, 1])$ the set $\partial \text{Fix}(g)$ is discrete, Corollary 5.14 implies that the φ -action on $(0, 1)$ must be semi-conjugate either to the standard action of G on X or to an action of G/N , with $N = [G_c, G_c]$. Assume by contradiction that the second case holds. Then $\varphi(N)$ admits fixed points in $(0, 1)$ and since $\text{Fix}^\varphi(N)$ is $\varphi(G)$ -invariant it must accumulate at both 0 and 1; moreover its complement $\text{Supp}^\varphi(N) = (0, 1) \setminus \text{Fix}^\varphi(N)$ is non-empty and also accumulates at both 0 and 1 by G -invariance, so that every neighborhood of 0, respectively 1, contains connected components of $\text{Supp}^\varphi(N)$. Since N is a simple group (Proposition 4.4), its action on each connected component of $\text{Supp}^\varphi(N)$ must be faithful. Thus for every $g \in N$ the set $\partial \text{Fix}^\varphi(g)$ must accumulate at both 0 and 1, and thus cannot be discrete. This is the desired contradiction, and it proves part (a).

Let now $\varphi: G \rightarrow \text{PDiff}_{loc}^\omega((\alpha, \beta))$ be a faithful action. Again, by Corollary 5.14, φ is semi-conjugate to the standard action or to an action of G/N , and we only need to analyze the second case. A similar reasoning as in the compact case, relying on the simplicity of N , shows that $\partial \text{Fix}^\varphi(N)$ must be discrete. Since the latter is a $\varphi(G)$ -invariant set, the G -action on it must factor through an epimorphism $\tau: G \rightarrow \mathbb{Z}$, and φ is semi-conjugate to a cyclic action. Finally, the action of N on each connected component of its support must be faithful and semi-conjugate to its standard action on X , by part (a) for actions on compact intervals. \square

Remark 5.18. Corollary 5.17 can be compared with the work of Lodha [64], who develops a way to study embeddings between certain groups of PL transformations, such as Thompson–Brown–Stein groups F_{n_1, \dots, n_k} , and proves non-embedding results between them. His approach is based on Rubin’s theorem; namely he identifies a class of group actions that he calls *coherent* and shows that such actions are automatically locally moving. The coherent condition is simpler to check than the locally moving condition in some situations. In particular he shows that certain embeddings between groups of PL homeomorphisms must give rise to coherent actions (up to semi-conjugacy), and thus, by Rubin’s theorem, to a semi-conjugation to the standard action.

5.3. First examples of exotic actions. In this subsection we give some simple examples of actions of locally moving groups which are exotic according to the classification in Proposition 5.12. The first two examples show two general mechanisms giving order-preserving actions of the group $\text{Homeo}_c(\mathbb{R})$ of compactly supported homeomorphisms of \mathbb{R} on ordered spaces. Thus for countable subgroups $G \subseteq \text{Homeo}_c(\mathbb{R})$, we can consider the dynamical realization of these actions and obtain actions on \mathbb{R} , which turn to be exotic. The role of our third example is to explain that it is crucial to consider finitely generated subgroups in the conclusion of Proposition 5.12. Indeed we will construct a locally moving subgroup $G \subseteq \text{Homeo}_0(X)$ and an exotic action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ such that, for every interval $I \Subset X$, the group $[G_I, G_I]$ acts without fixed points (recall that all its finitely generated subgroups must admit fixed points, by Corollary 5.7).

The reader should be warned that all examples of exotic actions discussed in this subsection admit no minimal invariant set. This is a somewhat degenerate phenomenon; for instance it can never arise for finitely generated groups (indeed all groups that we consider here are not finitely generated, and this is essential to the constructions). It is more interesting and less clear how to construct examples of exotic *minimal* actions of locally moving groups. Various such examples will be given later (see Section 9 and §§11.4–11.6).

5.3.1. Orders of germ type. In this paragraph we build exotic actions on the line that are obtained as dynamical realizations of some left-invariant preorder on compactly supported locally moving groups. Our preorders here are inspired by the well known construction of bi-invariant orders on the group of orientation-preserving piecewise linear homeomorphisms of an interval (see [23, 94]).

Let $G \subset \text{Homeo}_c(\mathbb{R})$ be a countable locally moving group. Note that $G = G_c$, so that $N = [G, G]$ is the minimal proper normal subgroup of G . For $x \in \mathbb{R}$, we denote by $\text{Germ}(G, x)$ the group of germs of $\text{Stab}_G(x)$ at x , and for $g \in G$ we let $\mathcal{G}_x(g)$ be its germ. With abuse of notation, we will denote by id the trivial germ, without reference to the base point.

Recall that groups of germs of interval homeomorphisms are left-orderable (see [36]) and therefore, for each $x \in \mathbb{R}$ there exists a left-invariant order $<^{(x)}$ on $\text{Germ}(G_{(-\infty, x)}, x)$. Choose a collection of such orders $\{<^{(x)}: x \in \mathbb{R}\}$. Therefore, we can define

$$P = \left\{ g \in G : \mathcal{G}_{p_g}(g) >^{(p_g)} \text{id} \right\}$$

where $p_g := \sup\{x \in X : g(x) \neq x\}$. It is straightforward to check that P is a semigroup disjoint from P^{-1} , and that it defines a partition $G = P \sqcup \{1\} \sqcup P^{-1}$. Thus, P is the positive cone of a left-invariant order $< \in \text{LO}(G)$ (see Remark 2.34). Denote by $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ its dynamical realization. We want to show that φ is an exotic action (according to the classification in Proposition 5.12).

First we will show that φ is not semi-conjugate to any action induced from a quotient. For this purpose, we claim that the group $\varphi(N)$ acts on \mathbb{R} without fixed points. Indeed to show this it is enough to show that the orbit of id under the action of N is unbounded above and below in the ordered space $(G, <)$. For this, fix an element $h \in G$ with $h \succ \text{id}$. Since N is normal in G , the subset $\{p_g : g \in N\} \subseteq X$ is G -invariant, so that by minimality there exists $g \in N$ with $p_g > p_h$. Upon replacing g with g^{-1} suppose that $\mathcal{G}_{p_g}(g) >^{(p_g)} \text{id}$. Then, since $p_{gh^{-1}} = p_g$ and thus

$$\mathcal{G}_{p_{gh^{-1}}}(gh^{-1}) = \mathcal{G}_{p_g}(g) >^{(p_g)} \text{id},$$

we have $g \cdot \text{id} = g \succ h$. Similarly, for every $h \prec \text{id}$, we can find an element $g' \in N$ such that $g' \cdot \text{id} \prec h$. This shows that the N -orbit of id is unbounded in both directions, as desired. In

particular, φ is not semi-conjugate to any action that factors through G/N . Finally, in order to show that φ is not semi-conjugate to the action of G , note that the previous argument can be improved to show that for every $x \in \mathbb{R}$, the image $\varphi(N_{(x,+\infty)})$ acts without fixed points: it is enough to take $g \in N$ as above with $p_g > \max\{x, p_h\}$, and as G is locally moving, we can also assume that $\text{Supp}(g) \subseteq (x, p_g)$. From this construction we get the following.

Proposition 5.19. *Every countable locally moving group $G \subseteq \text{Homeo}_c(\mathbb{R})$ of compactly supported homeomorphisms admits a faithful exotic action on the line.* \square

By varying on the choice of orders on the groups of germs, one in fact gets an uncountable family of non-semi-conjugate actions. In fact it is not difficult to show the following.

Lemma 5.20. *Let $G \subset \text{Homeo}_c(\mathbb{R})$ be countable and locally moving. For $i \in \{1, 2\}$, let $\mathcal{F}_i = \{\prec_i^{(x)} : x \in \mathbb{R}\}$ be a collection of left-orders on $\text{Germ}(G_{(-\infty, x)})$. Let \prec_i be the associated orders on G , and let φ_i be their dynamical realizations. Then φ_1 and φ_2 are positively semi-conjugate if and only if there exists $y \in \mathbb{R}$ such that $\prec_1^{(x)} = \prec_2^{(x)}$ for all $x \geq y$.*

Proof. First assume that there exists $y \in \mathbb{R}$ as in the statement. Note that the group $G_{(-\infty, y)}$ is convex with respect to both orders \prec_1, \prec_2 , and that both induce the same order on the coset space $G/G_{(-\infty, y)}$. In other words, the quotient preorders of \prec_1 and \prec_2 with respect to this convex subgroup coincide, and thus \prec_1 and \prec_2 admit a common dominating preorder. It follows that \prec_1 and \prec_2 are equivalent preorders, so that their dynamical realizations are positively semi-conjugate (see Lemma 3.16).

Conversely assume that φ_1 and φ_2 are positively semi-conjugate, and let $h: \mathbb{R} \rightarrow \mathbb{R}$ be a monotone increasing semi-conjugacy from φ_1 to φ_2 . For $i \in \{1, 2\}$, let $\iota_i: G \rightarrow \mathbb{R}$ be the good embedding associated with \prec_i (Definition 2.37), which we assume both to satisfy $\iota_i(\text{id}) = 0$. Also, for $x \in \mathbb{R}$, denote by $I_{i,x} \subset \mathbb{R}$ the interior of the convex hull of $\iota_i(G_{(-\infty, x)})$. Note that for fixed i the intervals $I_{i,x}$ define an increasing exhaustion of \mathbb{R} as $x \rightarrow +\infty$. Since the subgroup $G_{(-\infty, x)}$ is \prec_i -convex, we have that for every $g \notin G_{(-\infty, x)}$, the element $\varphi_i(g)$ must map $I_{i,x}$ disjointly from itself, and either to the right or to the left according to the sign of g with respect to the order \prec_i . Now choose $y \in \mathbb{R}$ large enough so that $I_{2,y}$ contains $h(0)$. Then if $g \in G$ is such that $p_g > y$ and $g \succ_2 \text{id}$, we have $\varphi_2(g)(h(0)) > h(0)$, so by equivariance of h also $\varphi_1(g)(0) > 0$, i.e. $g \succ_1 \text{id}$. This implies that $\prec_1^{(x)} = \prec_2^{(x)}$ for $x > y$. \square

Recall from the discussion in §3.5 that one way to get insight on the variety of actions on the line of a given group is to study the semi-conjugacy equivalence relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ up to Borel reducibility. The best case scenario would be when this equivalence relation is smooth. However the previous statement implies that this is never the case for locally moving groups of compactly supported homeomorphisms.

Corollary 5.21. *Let $G \subset \text{Homeo}_c(\mathbb{R})$ be countable and locally moving. Then the semi-conjugacy equivalence relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is not smooth.*

Proof. Recall from §3.5 that we denote by \mathcal{E}_0 the equivalence relation on one-sided binary sequences $\{0, 1\}^{\mathbb{N}}$, where (x_n) and (y_n) are equivalent if $x_n = y_n$ for all but finitely many n , and that \mathcal{E}_0 is not smooth. From Lemma 5.20 it is easy to construct a family of actions of G indexed by $\{0, 1\}^{\mathbb{N}}$ (in a Borel way), which are positively semi-conjugate if and only if the two sequences are \mathcal{E}_0 equivalent. For instance start from any collection of orders $\mathcal{F} = \{\prec^{(x)}, x \in \mathbb{R}\}$ on $\text{Germ}(G_{(-\infty, x)})$. For every $\omega \in \{0, 1\}^{\mathbb{N}}$, define a new family \mathcal{F}_ω , where for $x \in [n, n+1)$ we leave the order $\prec^{(x)}$ unchanged if $\omega(n) = 0$ and consider the opposite order if $\omega(n) = 1$. This together with Lemma 5.20 shows the desired claim. \square

This applies for instance to the commutator $[F, F]$ of Thompson's group F , and should be compared with Theorem 1.14 for locally moving groups in the class \mathcal{F}_0 (see also Theorem 11.6 for Thompson's group F).

5.3.2. Escaping sequences. Let $G \subseteq \text{Homeo}_c(\mathbb{R})$ be a locally moving group of compactly supported homeomorphisms. The action of G on \mathbb{R} defines a diagonal action of G on the space of sequences $\mathbb{R}^{\mathbb{N}}$. For the second construction, we fix a sequence $\omega_0 \in \mathbb{R}^{\mathbb{N}}$ of distinct points in \mathbb{R} , where we write $\omega_0(n)$ for the n -th term, and we suppose that ω_0 is *escaping*, in the sense that every compact subset of \mathbb{R} contains only finitely many terms of the sequence (in other terms, it has no accumulation point). Then we write $S = G.\omega_0$ for the G -orbit of the sequence ω_0 . Since elements of G have compact support, and the sequence ω_0 is escaping, it holds that every $\omega \in S$ agrees with ω_0 on all but finitely many entries. Thus, for $\omega, \omega' \in S$, we can declare that $\omega \prec \omega'$ if for the largest entry on which they disagree (say n), we have that $\omega(n) < \omega'(n)$. Note as G is locally moving, the diagonal action of G on S is faithful. Moreover, it preserves the total order \prec .

Now suppose that S is countable (this holds, for instance, when G is a countable group) and consider the dynamical realization $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ of the diagonal action of G on (S, \prec) . We claim that φ is an exotic action. To see this, consider two sequences $\omega \prec \omega'$ in S and let $n \in \mathbb{N}$ be such that the k -th entries of both ω and ω' agree with the k -th entry of ω_0 for all $k \geq n$. So, if $g \in G$ satisfies $g(\omega_0(n)) \neq \omega_0(n)$, then we have

$$\min\{g.\omega_0, g^{-1}.\omega_0\} \prec \omega \prec \omega' \prec \max\{g.\omega_0, g^{-1}.\omega_0\}.$$

This implies, on the one hand, that the action of $N = [G, G] = [G_c, G_c]$ (which is a subgroup without fixed points) on S is cofinal (i.e. every N -orbit is unbounded in both directions), and hence $\varphi(N)$ acts without fixed points. In particular $\varphi(G)$ is not semi-conjugate to any action coming from a factor of G (see Proposition 4.4). On the other hand, since ω_0 is escaping, for any compact interval $I \subset \mathbb{R}$, the action of $G_{\mathbb{R} \setminus I} = \{g \in G : I \subseteq \text{Fix}(g)\}$ on S is cofinal as well, so $\varphi(G_{\mathbb{R} \setminus I})$ has no fixed points. In particular, $\varphi(G)$ is not semi-conjugate to the standard action.

We also remark that this construction can be used to show the following (to be compared with Corollary 1.16).

Proposition 5.22. *Let $G \subset \text{Homeo}_c(\mathbb{R})$ be a locally moving group of compactly supported homeomorphisms. Then the standard action of G on \mathbb{R} is not locally rigid.*

Proof. Given an escaping sequence ω , denote by \preceq_ω the preorder induced on G by the action on the orbit S_ω , with $g \preceq_\omega h$ if $g.\omega \preceq h.\omega$. Denote also by \prec_0 the preorder on G induced by its standard action and the point $0 \in \mathbb{R}$, by $g \prec_0 h$ if $g(0) \leq h(0)$. Choose a sequence $\omega_n \in \mathbb{R}^{\mathbb{N}}$ of escaping sequences such that $\omega_n(0) = 0$ and $|\omega_n(j)| > n$ if $j > 0$. Then for every finite subset $S \subset G$ and n large enough we have that every $s \in S$ satisfies $s \prec_{\omega_n} \text{id}$ if and only if $s \prec_0 \text{id}$. Thus the sequence of preorders (\prec_{ω_n}) converges to \prec_0 in $\text{LPO}(G)$. Thus by an easy adaptation to preorders of the argument in [73, Proposition 3.3], this implies that the sequence of actions associated to ω_n as above can be positively conjugated to a sequence (φ_n) which converges to the standard action of G in the compact open topology. \square

5.3.3. An example where the rigid stabilizers have no fixed points. We write $\text{PL}_{\mathbb{Q}}((0, 1))$ for the Bieri–Strebel group $G((0, 1); \mathbb{Q}, \mathbb{Q}_{>0})$ of piecewise linear homeomorphisms that preserve $\mathbb{Q} \cap (0, 1)$. For a prime number $q \in \mathbb{N}$, consider the corresponding q -adic order $\nu_q : \mathbb{Q}_{>0} \rightarrow \mathbb{Z}$ (these are the functions such that $r = \prod_{q \text{ prime}} q^{\nu_q(r)}$ for every $r \in \mathbb{Q}_{>0}$). We then write

$\pi_q(r) = q^{\nu_q(r)}$, which corresponds to the power of the prime q appearing in the factorization of $r \in \mathbb{Q}_{>0}$. Given an element $f \in \text{PL}_{\mathbb{Q}}((0, 1))$, we define $D_q^- f := \pi_q \circ D^- f$. Observe that the chain rule works for D_q^- , namely

$$(5.1) \quad D_q^-(fg)(x) = D_q^- f(g(x)) D_q^- f(x) \quad \text{for every } x \in (0, 1) \text{ and } f, g \in \text{PL}_{\mathbb{Q}}((0, 1)).$$

Every prime number $q \in \mathbb{N}$ defines a preorder on $\text{PL}_{\mathbb{Q}}((0, 1))$ in the following way. We consider the subset $H_q \subset \text{PL}_{\mathbb{Q}}((0, 1))$ defined by

$$H_q = \left\{ f \in \text{PL}_{\mathbb{Q}}((0, 1)) : D_q^- f(x) = 1 \quad \text{for every } x \in (0, 1) \right\},$$

and then the subset

$$P_q = \left\{ f \in \text{PL}_{\mathbb{Q}}((0, 1)) \setminus H_q : D_q^- f(x_f) > 1 \right\},$$

where for $f \in \text{PL}_{\mathbb{Q}}((0, 1))$, we set $x_f = \max\{x \in (0, 1) : D_q^- f(x) \neq 1\}$. We have the following.

Proposition 5.23. *For every prime $q \in \mathbb{N}$, the subset P_q is the positive cone of a preorder \preceq_q of $\text{PL}_{\mathbb{Q}}((0, 1))$.*

Proof. The chain rule (5.1) easily gives that P_q is a semigroup, $\text{PL}_{\mathbb{Q}}((0, 1)) = P_q \sqcup H_q \sqcup P_q^{-1}$, H_q is a subgroup and $H_q P_q H_q \subseteq P_q$. The result then follows from Remark 2.34. \square

We are ready to construct our action. For this, given $f \in \text{PL}_{\mathbb{Q}}((0, 1))$ consider the subset $E_f = \{q : q \notin H_q\}$, which is the collection of primes appearing in the factorization of the left derivative $D^- f(x)$, for some $x \in (0, 1)$. Note that E_f is a finite set, so we can consider the prime $p_f := \max E_f$ (where we are considering the standard ordering of primes). Then, define

$$P := \{f \in \text{PL}_{\mathbb{Q}}((0, 1)) : \text{id} \prec_{p_f} f\},$$

where \preceq_{p_f} is the preorder on $\text{PL}_{\mathbb{Q}}((0, 1))$ from Proposition 5.23.

Proposition 5.24. *The subset P is the positive cone of an order \prec on $\text{PL}_{\mathbb{Q}}((0, 1))$. Furthermore, if $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is the dynamical realization of \prec , then for every non-empty subinterval $I \subseteq (0, 1)$ the φ -image of $[G_I, G_I]$ acts without fixed points.*

Proof. Write $G = \text{PL}_{\mathbb{Q}}((0, 1))$. For a non-trivial element $f \in G$, the chain rule (5.1) applied to the inverse function gives that $E_f = E_{f^{-1}}$ and that $f \in P$ if and only if $f^{-1} \notin P$. Thus, we have $\text{PL}_{\mathbb{Q}}((0, 1)) = P \sqcup \{1\} \sqcup P^{-1}$. Also note that if $f, g \in P$, it holds that $p_{fg} = \max\{p_f, p_g\}$ and that $\text{id} \preceq_{p_{fg}} f$ and $\text{id} \preceq_{p_{fg}} g$, one inequality being strict. Thus, $fg \in P$ showing that P is a positive cone. Now let $I \subseteq (0, 1)$ be a non-empty subinterval, let G_I be its rigid stabilizer inside G and take $f \in G$. Since E_f is finite, there is $g \in [G_I, G_I]$ such that $p_g > \max E_f$, so in particular $\min\{g, g^{-1}\} \preceq f \preceq \max\{g, g^{-1}\}$. This shows that the $[G_I, G_I]$ -orbit of id in (G, \prec) is unbounded in both directions, and thus $\text{Fix}^\varphi([G_I, G_I]) = \emptyset$, as desired. \square

Analogous constructions will be presented in §9.3 for general Bieri–Strebel groups.

6. DIFFERENTIABLE ACTIONS OF LOCALLY MOVING GROUPS

In this section we are interested in actions on a closed interval, or on the real line, which are by diffeomorphisms of class C^1 . First, let us observe that a rather direct consequence of Theorem 5.3 is a rigidity result for actions on the real line by diffeomorphisms of class C^2 , namely every action $\varphi : G \rightarrow \text{Diff}_0^2(\mathbb{R})$ of a locally moving group (without fixed points) is semi-conjugate to the standard action or to an action of $G/[G_c, G_c]$. Indeed, recall that Kopell’s lemma states that whenever f and g are non-trivial commuting C^2 diffeomorphisms

of a compact interval, and if f has no fixed point in its interior, then neither does g . If φ falls into the exotic case of Theorem 5.3, it is easy to check (as in the proof of Corollary 5.14) that the image of G must contain an abundance of commuting pairs of elements which preserve a compact interval and do not satisfy the conclusion of Kopell's lemma. However we do not elaborate on this, because we will show in this section that this rigidity actually holds for C^1 actions (for which Kopell's lemma fails, see e.g. Bonatti and Farinelli [13]).

We start by recalling some classical results in this setting.

6.1. Conradian actions and C^1 actions. Let $G \subseteq \text{Homeo}_0(\mathbb{R})$ be a subgroup. A *pair of successive fixed points* for G is a pair $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ such that there is an element $g \in G$ for which (a, b) is a connected component of $\text{Supp}(g)$. A *linked pair of fixed points* for G consists of pairs a, b and c, d in $\mathbb{R} \cup \{\pm\infty\}$ such that:

- (1) there are elements $f, g \in G$ such that a, b is a pair of successive fixed points of f and c, d is a pair of successive fixed points of g ;
- (2) either $\{a, b\} \cap (c, d)$ or $(a, b) \cap \{c, d\}$ is a point.

As pointed out by Navas [90], the previous notion is the dynamical counterpart of Conradian orderings on groups. Following the terminology of [93], we will say that an action of a group G on an interval is *Conradian* if it has no pair of linked fixed points and there is no (global) fixed point. We have the following fundamental fact (see for instance [93, Theorem 2.14]).

Theorem 6.1. *Any Conradian action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ of a finitely generated group G on the real line is semi-conjugate to an action by translations. In particular, there exists a non-trivial morphism $\tau : G \rightarrow \mathbb{R}$ (the Conrad homomorphism), unique up to positive rescaling, such that if $g \in G$ is such that $\tau(g) > 0$ and $f \in \ker \tau$, then for every $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}$ one has $f^n \cdot \xi \leq g \cdot \xi$.*

Moreover, assume that $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is Conradian, let $H \subseteq G$ be a finitely generated subgroup, and let $I \subset \mathbb{R}$ be a connected component of $\text{Supp}^\varphi(H)$. Then the action of H induced on I by restriction is still Conradian, therefore it is also semi-conjugate to an action by translations.

The following result is the version of Sacksteder's theorem for C^1 pseudogroups, as established by Deroin, Kleptsyn, and Navas in [34] (see also Bonatti and Farinelli [13] for a simplified proof).

Theorem 6.2. *Let $G \subseteq \text{Diff}_0^1([0, 1])$ be a subgroup acting with a linked pair of fixed points. Then there exists a point $x \in (0, 1)$, and an element $h \in G$ for which x is a hyperbolic fixed point: $h(x) = x$ and $h'(x) < 1$.*

Remark 6.3. In fact, the proof of Theorem 6.2 gives a more precise statement, which we point out as it will be useful for the sequel.

Write $I = [0, 1]$. It is not difficult to see that the existence of a linked pair of fixed points for G , as in Theorem 6.2, gives a subinterval $J \subset I$ and two elements $f, g \in G$ such that the images $f(J)$ and $g(J)$ are both contained in J , and are disjoint: $f(J) \cap g(J) = \emptyset$. (This situation is the analogue of a Smale's horseshoe for one-dimensional actions.) It follows that every element $h \in \langle f, g \rangle_+$ in the (free) semigroup generated by f and g satisfies $h(J) \subset J$, and moreover the images $h(J)$, where h runs through the 2^n elements of length n in the semigroup $\langle f, g \rangle_+$ (with respect to the generating system $\{f, g\}$), are pairwise disjoint. Clearly the inclusion $h(J) \subset J$ gives that every h admits a fixed point in $h(J)$. Using a probabilistic argument, and uniform continuity of f' and g' on J , one proves that as n goes to infinity, most of the elements h of

length n are uniform contractions on J . This implies that most elements $h \in \langle f, g \rangle_+$ of length n , when n is large enough, have a unique fixed point in J , which is hyperbolic.

We deduce that if $\Lambda \subset J$ is an invariant Cantor set for f and g (and thus for $\langle f, g \rangle_+$), then the hyperbolic fixed point for a typical long element $h \in \langle f, g \rangle_+$ will never belong to the closure of a gap J_0 of Λ : otherwise, h would fix the whole gap J_0 , and therefore there would be a point $y \in J_0$ for which $h'(y) = 1$, contradicting the fact that h is a uniform contraction.

We point out a straightforward consequence of Theorem 6.2 (largely investigated in [13]).

Corollary 6.4. *Let $G \subseteq \text{Diff}_0^1([0, 1])$ be a subgroup acting with a linked pair of fixed points, then there is no non-trivial element $f \in \text{Diff}_0^1([0, 1])$ without fixed points in $(0, 1)$, centralizing G .*

6.2. Conradian C^1 actions of Thompson's group F . Before discussing C^1 actions of general locally moving groups, we first prove a preliminary result in the case of Thompson's group F , namely we rule out the existence of Conradian C^1 faithful actions of F . In fact, this will be used when studying general locally moving groups.

The first step is to analyze actions which are sufficiently close to the trivial action, in a spirit similar to the works of Bonatti [12] and McCarthy [84] (see also the related works [14, 16]). For the statement, we recall that the C^1 topology on $\text{Diff}_0^1([0, 1])$ is defined by the C^1 distance

$$d_{C^1}(f, g) = \sup_{\xi \in [0, 1]} |f(\xi) - g(\xi)| + \sup_{\xi \in [0, 1]} |f'(\xi) - g'(\xi)|.$$

When G is a finitely generated group endowed with a finite symmetric generating set S , we consider the induced topology on $\text{Hom}(G, \text{Diff}_0^1([0, 1]))$, saying that two representations φ and ψ are δ -close if $d_{C^1}(\varphi(g), \psi(g)) \leq \delta$ for every $g \in S$. Moreover, given $\varphi \in \text{Hom}(G, \text{Diff}_0^1([0, 1]))$, an element $g \in G$, and a point $\xi \in [0, 1]$, we write $\Delta_\xi^\varphi(g) = g \cdot \xi - \xi$ for the displacement of the point ξ . Clearly $\Delta_\xi^\varphi(g) = 0$ for every $\xi \in [0, 1]$ if and only if $g \in \ker \varphi$.

Lemma 6.5. *There exists a neighborhood \mathcal{V} of the trivial representation in $\text{Hom}(F, \text{Diff}_0^1([0, 1]))$ such that if $\varphi \in \mathcal{V}$ has no linked pair of fixed points, then $[F, F] \subseteq \ker \varphi$.*

Proof. Let $\varphi \in \text{Hom}(F, \text{Diff}_0^1([0, 1]))$ be an action with no linked pair of fixed points. Consider two dyadic open subintervals $I \Subset J \Subset X = (0, 1)$ (i.e. with dyadic rational endpoints). Let $h \in F$ be such that $h(J) = I$. We choose an element $f \in F_I \cong F$ without fixed points in I , and set $g = h^{-1}fh \in F_J$ for the conjugate element (which acts without fixed points on J). Note that both f and g belong to the subgroup $H := \langle g, [F_J, F_J] \rangle$. The group H is finitely generated (it is generated by g and by the group F_L for any dyadic subinterval $L \Subset J$ such that $g(L) \cap L \neq \emptyset$), and since $[F_J, F_J]$ is simple and normal in H , the abelianization of H is infinite cyclic, generated by the image of g . By Theorem 6.1 (applied to every action obtained by taking restriction of φ to a connected component of $\text{Supp}^\varphi(H)$), this implies that

$$(6.1) \quad \left| \Delta_\xi^\varphi(f^n) \right| \leq \max \left\{ \left| \Delta_\xi^\varphi(g) \right|, \left| \Delta_\xi^\varphi(g^{-1}) \right| \right\} \quad \text{for every } \xi \in [0, 1] \text{ and } n \in \mathbb{Z}.$$

Claim. *Fix $\gamma > 0$. If $\varphi(f)$ and $\varphi(g)$ are sufficiently close to the identity in the C^1 topology, for all $\xi \in [0, 1]$ we have*

$$(2 - \gamma) \left| \Delta_\xi^\varphi(f) \right| \leq (1 + \gamma) \left| \Delta_\xi^\varphi(g) \right|.$$

Proof of claim. Fix $\gamma > 0$. Using the mean value theorem (in more general form, this is Bonatti's approximate linearization estimate [12]; see also [14]), we get that if $\varphi(f)$ and $\varphi(g)$ are sufficiently C^1 close to the identity, for all $\xi \in [0, 1]$ we have

$$\left| \Delta_\xi^\varphi(f^2) - 2\Delta_\xi^\varphi(f) \right| \leq \gamma \left| \Delta_\xi^\varphi(f) \right|$$

and

$$\left| \left| \Delta_\xi^\varphi(g^{-1}) \right| - \left| \Delta_\xi^\varphi(g) \right| \right| = \left| \Delta_\xi^\varphi(g^{-1}) + \Delta_\xi^\varphi(g) \right| \leq \gamma \left| \Delta_\xi^\varphi(g) \right|.$$

From (6.1) and the two estimates above, we obtain the inequalities

$$(2 - \gamma) \left| \Delta_\xi^\varphi(f) \right| \leq \left| \Delta_\xi^\varphi(f^2) \right| \leq (1 + \gamma) \left| \Delta_\xi^\varphi(g) \right|. \quad \square$$

Claim. Fix $\gamma > 0$. If $\varphi(g)$ and $\varphi(h)$ are sufficiently close to the identity in the C^1 topology, for all $\xi \in [0, 1]$ we have

$$(1 - \gamma) \left| \Delta_\xi^\varphi(g) \right| \leq \left| \Delta_{h,\xi}^\varphi(f) \right|.$$

Proof of claim. We have

$$\begin{aligned} \Delta_{h,\xi}^\varphi(f) &= fh.\xi - h.\xi = hg.\xi - h.\xi \\ &= \varphi(h)'(\xi) \cdot \Delta_\xi^\varphi(g) + o\left(\Delta_\xi^\varphi(g)\right). \end{aligned}$$

Assume that $o\left(\Delta_\xi^\varphi(g)\right) \leq \frac{\gamma}{2} \left| \Delta_\xi^\varphi(g) \right|$ and $|\varphi(h)'(\xi) - 1| \leq \frac{\gamma}{2}$. Then by the triangle inequality we get

$$\left| \Delta_{h,\xi}^\varphi(f) - \Delta_\xi^\varphi(g) \right| \leq |\varphi(h)'(\xi) - 1| \cdot \left| \Delta_\xi^\varphi(g) \right| + o\left(\Delta_\xi^\varphi(g)\right) \leq \gamma \left| \Delta_\xi^\varphi(g) \right|. \quad \square$$

Choose now $\gamma > 0$ small enough, so that $\frac{(2-\gamma)(1-\gamma)}{1+\gamma} = \lambda > 1$, and consider the appropriate neighborhood \mathcal{V} of the trivial representation in $\text{Hom}\left(F, \text{Diff}_0^1([0, 1])\right)$, so that the conditions in the claims are satisfied for every Conradian action $\varphi \in \mathcal{V}$. Fix $\varphi \in \mathcal{V}$ Conradian. Combining the claims, we conclude that for every $\xi \in [0, 1]$, one has $\lambda \left| \Delta_\xi^\varphi(f) \right| \leq \left| \Delta_{h,\xi}^\varphi(f) \right|$. As this holds for any $\xi \in [0, 1]$, we get that for any $n \in \mathbb{N}$ and $\xi \in [0, 1]$ one has $\lambda^n \left| \Delta_\xi^\varphi(f) \right| \leq \left| \Delta_{h^n,\xi}^\varphi(f) \right|$. As $\left| \Delta_{h^n,\xi}^\varphi(f) \right|$ is bounded by 1 (the length of $[0, 1]$), we deduce that the element f belongs to the kernel of φ , as desired. In particular $\ker \varphi$ is a non-trivial normal subgroup of F , and thus it contains $[F, F]$. \square

The previous statement, which is of local nature (perturbations of the trivial actions), is used to obtain a global result. For this, we recall a trick attributed to Muller [87] and Tsuboi [106] (see also [16]) after which, given any C^1 action $\varphi : G \rightarrow \text{Diff}_0^1([0, 1])$ of a group G on $[0, 1]$ and a finite subset $S \subset G$, for any $\varepsilon > 0$ there exists $\delta > 0$ and a C^1 action $\psi : G \rightarrow \text{Diff}_0^1([0, 1])$, which is C^0 conjugate to the original one, such that

$$\left| \psi(g)(\xi) - \xi \right| + \left| \psi(g)'(\xi) - 1 \right| \leq \varepsilon \quad \text{for every } \xi \in [0, \delta] \text{ and } g \in S.$$

Lemma 6.6. *Every Conradian C^1 action of F on the closed interval $[0, 1]$ has abelian image.*

Proof. Let $\varphi \in \text{Hom}\left(F, \text{Diff}_0^1([0, 1])\right)$ be a Conradian action. We want to prove that $\ker \varphi$ contains $[F, F]$, and for this it is enough to show that $\ker \varphi$ is non-trivial.

Let us fix a dyadic subinterval $I \Subset X = (0, 1)$. After Theorem 6.1, φ is semi-conjugate to an action by translations, and this is given by the Conrad homomorphism $\tau : F \rightarrow \mathbb{R}$. We can

find a minimal non-empty closed $\varphi(F)$ -invariant subset $\Lambda \subset (0, 1)$ such that $\ker \tau$ pointwise fixes Λ . We write \mathcal{J} for the collection of connected components of $(0, 1) \setminus \Lambda$.

Note that $F_I \subseteq \ker \tau$ and therefore, given an interval $J \in \mathcal{J}$, we have an induced action $\varphi_J \in \text{Hom}\left(F_I, \text{Diff}_0^1(\bar{J})\right)$ obtained by restriction of φ . Note that as φ is Conradian, then φ_J has no linked pair of fixed points. We also remark that if J and J' are in the same $\varphi(F)$ -orbit, then φ_J and $\varphi_{J'}$ are conjugate. Let \mathcal{V} be the neighborhood of the trivial representation provided by Lemma 6.5, and denote by $\mathcal{V}_J \subset \text{Hom}\left(F_I, \text{Diff}_0^1(\bar{J})\right)$ the corresponding neighborhood obtained after considering an identification $\text{Hom}\left(F_I, \text{Diff}_0^1(\bar{J})\right) \cong \text{Hom}\left(F, \text{Diff}_0^1([0, 1])\right)$. Using the trick of Muller and Tsuboi to the action φ , we can assume that there exists $\delta > 0$ such that if $J \subset (0, \delta)$, then $\varphi_J \in \mathcal{V}_J$. In particular for such intervals $J \in \mathcal{J}$, we have $[F_I, F_I] \subseteq \ker \varphi_J$. As the $\varphi(F)$ -orbit of every $J \in \mathcal{J}$ has an element contained in $(0, \delta)$, and two intervals in the same $\varphi(F)$ -orbit lead to conjugate actions, we deduce that $[F_I, F_I] \subseteq \ker \varphi_J$ for every $J \in \mathcal{J}$. As $\varphi(F_I)$ pointwise fixes Λ , which is the complement of $\bigcup_{J \in \mathcal{J}} J$ in $(0, 1)$, we deduce that $[F_I, F_I] \subseteq \ker \varphi$, as desired. \square

With similar proof, we can extend the previous result to C^1 actions on the real line.

Proposition 6.7. *Every Conradian C^1 action of F on the real line has abelian image.*

Proof. We proceed as in the proof of Lemma 6.6, and for this reason we skip some detail. We start with a Conradian action $\varphi \in \text{Hom}\left(F, \text{Diff}_0^1(\mathbb{R})\right)$, and consider the Conradian homomorphism $\tau : F \rightarrow \mathbb{R}$, and a minimal invariant subset $\Lambda \subset \mathbb{R}$, pointwise fixed by $\ker \tau$. We denote by \mathcal{J} the collection of connected components of $\mathbb{R} \setminus \Lambda$. For a given dyadic subinterval $I \in \mathcal{J}$, this gives rise to actions $\varphi_J \in \text{Hom}\left(F_I, \text{Diff}_0^1(J)\right)$ without linked pairs of fixed points ($J \in \mathcal{J}$). After Lemma 6.6 (applied to the restriction of φ_J to the closure of every connected component of $\text{Supp}^{\varphi^J}(F_I)$), we deduce that $[F_I, F_I] \subseteq \ker \varphi_J$ and therefore $[F_I, F_I] \subseteq \ker \varphi$. \square

6.3. C^1 actions of general locally moving groups. Using that any locally moving group contains a copy of F (Proposition 4.8), we get the following consequence of Proposition 6.7.

Proposition 6.8. *Locally moving groups admit no faithful Conradian C^1 actions on the real line.*

We actually have the following alternative, which is a more precise formulation of Theorem 1.2 from the introduction.

Theorem 6.9. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then every action $\varphi : G \rightarrow \text{Diff}_0^1(\mathbb{R})$ without fixed points, satisfies one of the following:*

- i) either φ is semi-conjugate to the standard action of G on X ;*
- ii) or φ is semi-conjugate to an action that factors through the quotient $G/[G_c, G_c]$.*

Moreover the second case occurs if and only if $\varphi([G_c, G_c])$ has fixed points, in which case the action of $[G_c, G_c]$ on each connected component of its support is semi-conjugate to its standard action on X .

Proof of Theorem 6.9. Assume by contradiction that $\varphi : G \rightarrow \text{Diff}_0^1(\mathbb{R})$ is an action which is not semi-conjugate to the standard action on X , nor to any action of the quotient $G/[G_c, G_c]$. In particular φ must be faithful.

Note that G_c is locally moving, so that by Proposition 4.8 we can find a subgroup $\Gamma \subseteq G_c$ isomorphic to F . In particular, Γ is finitely generated, and we can apply Corollary 5.14. Let

$I \subset \text{Supp}^\varphi(\Gamma)$ be a connected component, and take an element $f \in [G_c, G_c]$ centralizing Γ , such that $f.I \cap I = \emptyset$ (this always exists, as pointed out in the proof of Corollary 5.14.(ii)). Let J be the connected component of $\text{Supp}^\varphi(f)$ containing I , which is bounded after Corollary 5.14.(i). Applying Corollary 6.4 to the induced action of Γ on \bar{J} obtained by restriction of φ , we deduce from Lemma 6.6 that the restriction of $\varphi(\Gamma)$ to I is abelian. As $I \subset \text{Supp}^\varphi(\Gamma)$ was arbitrary, we get that $\varphi(\Gamma)$ is abelian, and therefore $\varphi : G \rightarrow \text{Diff}_0^1(\mathbb{R})$ is not faithful, which is an absurd.

The last statement is a consequence of the fact that the group $[G_c, G_c]$ is still locally moving (by Lemma 4.3), thus we can apply the first part of the theorem to its action on the connected components of the support, and since it is simple (Proposition 4.4) only the first case can occur. \square

The last result of this section corresponds to Corollary 1.3, with a more precise statement.

Definition 6.10. For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. We say that G has *independent groups of germs at the endpoints* if for every pair of elements $\gamma_1 \in \text{Germ}(G, a)$ and $\gamma_2 \in \text{Germ}(G, b)$, there exists $g \in G$ such that $\mathcal{G}_a(g) = \gamma_1$ and $\mathcal{G}_b(g) = \gamma_2$.

Note that this is equivalent to the condition that the natural injective homomorphism $G/G_c \hookrightarrow \text{Germ}(G, a) \times \text{Germ}(G, b)$ be onto.

Corollary 6.11. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving with independent groups of germs at the endpoints. Then the following hold.*

- i) *Every faithful action $\varphi : G \rightarrow \text{Diff}_0^1([0, 1])$ without fixed points is semi-conjugate to the standard action on X .*
- ii) *Every faithful action $\varphi : G \rightarrow \text{Diff}_0^1(\mathbb{R})$ without fixed points is either semi-conjugate to the standard action on X , or to a cyclic action.*

In the latter case, if $\tau : G \rightarrow \mathbb{Z}$ is the homomorphism giving the semi-conjugate action, then the action of $\ker \tau$ on each connected component of its support is semi-conjugate to its standard action on X .

To prove Corollary 6.11 we need the following lemma.

Lemma 6.12. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving with independent groups of germs at the endpoints. Then for every pair of points $c < d$ in X , the subgroup $\langle G_{(a,c)}, G_{(d,b)} \rangle$ projects onto the largest quotient $G/[G_c, G_c]$.*

Proof. Given $\gamma \in \text{Germ}(G, a)$ the assumption implies that there exists $g \in G$ with $\mathcal{G}_a(g) = \gamma$ and $\mathcal{G}_b(g) = \text{id}$. Thus, we have $g \in G_{(a,x)}$ for some $x \in X$. If we choose $h \in G_c$ such that $h(x) < c$ the element $g' = hgh^{-1}$ belongs to $G_{(a,c)}$ and satisfies $\mathcal{G}_a(g') = \gamma$. We conclude that the group $G_{(a,c)}$ projects onto $\text{Germ}(G, a)$ and similarly $G_{(d,b)}$ projects onto $\text{Germ}(G, b)$. Therefore the subgroup $\langle G_{(a,c)}, G_{(d,b)} \rangle$ projects onto $\text{Germ}(G, a) \times \text{Germ}(G, b) \simeq G/G_c$. It is therefore enough to show that its image in $G/[G_c, G_c]$ contains $G_c/[G_c, G_c]$. This is the case because every $g \in G_c$ is conjugate inside G_c to an element of $\langle G_{(a,c)}, G_{(d,b)} \rangle$, which has the same image in the abelianisation $G_c/[G_c, G_c]$. \square

Proof of Corollary 6.11. In the proof we set $N = [G_c, G_c]$. Let $Y = [0, 1]$ or $Y = \mathbb{R}$ according to the case in the statement. By Theorem 6.9 the φ -action of G on Y is either semi-conjugate to the standard action on X or to an action that factors through G/N . Assume that the second condition holds: after Theorem 6.9, we know more precisely that if I is a connected component of $\text{Supp}^\varphi(N)$, then the induced action of N on I is semi-conjugate to the standard

action on X . In particular, it admits linked pairs of fixed points and by Theorem 6.2 we can find $h \in N$ with a hyperbolic fixed point $\xi \in I$. We make a slightly more elaborate argument than the one that is needed for Corollary 6.4. Let $(c, d) \in X$ be such that $h \in N_{(c,d)}$. Then the subgroup $H = \langle G_{(a,c)}, G_{(d,b)} \rangle$ centralizes h and thus every point in the orbit $H.\xi$ is fixed by h , with derivative always equal to $\varphi(h)'(\xi) \neq 1$. As the derivative $\varphi(h)'$ is continuous, it must be that the orbit $H.\xi$ is discrete in Y . In the case $Y = [0, 1]$, the only possibility is that ξ is fixed by $\varphi(H)$, and thus after Lemma 6.12, the quotient action of G/N would have a fixed point as well. Similarly, in the case $Y = \mathbb{R}$, $\varphi(H)$ cannot fix ξ , so that the orbit $H.\xi$ is infinite and discrete. Using Lemma 6.12 again, we deduce that the quotient action of G/N has an infinite discrete orbit, which means that it is semi-conjugate to a cyclic action.

For the last statement, apply the case $Y = [0, 1]$ to the action of $\ker \tau$. \square

6.4. An application to non-smoothability. In Theorem 6.9 it may happen that the standard action of G is not semi-conjugate to an action of a given regularity, so that the first possibility is not realizable for actions in that regularity. Here we discuss two applications of this to certain groups of piecewise linear homeomorphisms, which improve results on non-smoothability of such groups of Bonatti, Lodha and the fourth author [15].

6.4.1. Thompson–Brown–Stein groups. Here we discuss an application to differentiable actions of Thompson–Brown–Stein groups F_{n_1, \dots, n_k} introduced in Definition 2.45, which are natural generalizations of Thompson’s group F . Such groups are clearly locally moving. It was shown in [15] that when $k \geq 2$ the standard action of F_{n_1, \dots, n_k} cannot be conjugate to any C^2 action. Here we show the following.

Theorem 6.13. *Let $r > 1$. For any $k \geq 2$ and choice of n_1, \dots, n_k as in Definition 2.45, the Thompson–Brown–Stein group F_{n_1, \dots, n_k} admits no faithful C^r action on the real line.*

We first need a lemma for the usual Higman–Thompson groups F_n . Corollary 6.11 applies to them, so every faithful action of F_n of class C^1 is semi-conjugate to its standard action on the interval $[0, 1]$. However, as pointed out by Ghys and Sergiescu [45], the group F (and every F_n) actually admits C^1 (even C^∞) actions which are not conjugate to the standard action (see Remark 11.2). In these examples, the action is obtained by blowing up the orbit of dyadic rationals. The next lemma, which is a consequence of Sacksteder’s theorem (Theorem 6.2) and the structure of the group, gives a restriction on possible semi-conjugate but not conjugate C^1 actions.

Lemma 6.14. *For $n \geq 2$, let $\varphi : F_n \rightarrow \text{Diff}_0^1(\mathbb{R})$ be a faithful action which is semi-conjugate to the standard action $\varphi_0 : F_n \rightarrow \text{PL}([0, 1])$, but not conjugate. Let $h : \mathbb{R} \rightarrow (0, 1)$ be the corresponding continuous monotone map such that $h\varphi = \varphi_0 h$. Then there exists a rational point $p \in [0, 1]$ which is not n -adic (i.e. $p \in (\mathbb{Q} \setminus \mathbb{Z}[1/n]) \cap [0, 1]$), such that the preimage $\xi = h^{-1}(p)$ is a singleton, and an element $g \in F$ for which ξ is a hyperbolic fixed point.*

Proof. The proof is a tricky refinement of arguments in [15, §5.1]. In the following, we write $G = F_n$. Given an action φ as in the statement, denote by $\Lambda \subset \mathbb{R}$ the corresponding minimal invariant Cantor set. Fix an open n -adic subinterval $I \Subset (0, 1)$, then $\varphi(G_I)$ preserves the interval $h^{-1}(I)$ and $\Lambda_I := \Lambda \cap h^{-1}(I)$ is the minimal invariant subset for the restriction $\varphi_I : G_I \rightarrow \text{Diff}_0^1(h^{-1}(\bar{I}))$ induced by φ .

After Proposition 6.2 and subsequent Remark 6.3 applied to φ_I , there exists an element $g \in G_I$ and a point $\xi \in \Lambda_I$ such that $g.\xi = \xi$ and $\varphi(g)'(\xi) < 1$. Moreover, such a point ξ cannot belong to the closure of a gap of Λ_I (see Remark 6.3), or in other terms the semi-conjugacy h

must be injective at ξ . It is well known that if a point $p \in I$ is an isolated fixed point for some element of G_I in the standard action, then p is rational (the point p must satisfy a rational equation $n^k p + \frac{a}{n^b} = p$; see Lemma 6.19 below). From this we deduce that the point $p = h(\xi)$ is rational. Moreover, the point p cannot be n -adic: take any element $f \in G_I$ such that $\varphi_0(f)$ coincides with $\varphi_0(g)$ in restriction to $[p, 1]$ and is the identity in restriction to $[0, p]$. Then the right derivative of $\varphi(f)$ at ξ must be equal to $\varphi(g)'(\xi) < 1$ and the left derivative of $\varphi(f)$ at ξ must be equal to 1, contradicting the fact that φ is a C^1 action. This concludes the proof. \square

As a consequence of Lemma 6.14, we get a strong improvement of [15, Theorem 3.4], on regularity of actions of Thompson–Brown–Stein groups.

The idea is to replace the use of the Szekeres vector field (which requires C^2 regularity), with Sternberg’s linearization theorem, which works in C^r regularity ($r > 1$), but requires hyperbolicity (granted from Lemma 6.14). In this form, these results can be found in [108, Appendice 4] or [91, Theorems 3.6.2 and 4.1.11] (a detailed proof when $r < 2$ appears in [105, §6.2.1]). A similar approach, although less technical, appears in [76] to exhibit examples of groups at “critical regularity”.

Theorem 6.15 (Sternberg). *Fix $r > 1$ and let f be a C^r diffeomorphism of the half-open interval $[0, 1)$ with no fixed point in $(0, 1)$ and such that $f'(0) \neq 1$. Then there exists a diffeomorphism $h : [0, 1) \rightarrow [0, +\infty)$ of class C^r such that*

- (1) $h'(0) = 1$,
- (2) the conjugate map hfh^{-1} is the scalar multiplication by $f'(0)$.

Theorem 6.16 (Szekeres). *Fix $r > 1$ and let f be a C^r diffeomorphism of the half-open interval $[0, 1)$ with no fixed point in $(0, 1)$ and such that $f'(0) \neq 1$. Then there exists a unique C^{r-1} vector field \mathcal{X} on $[0, 1)$ with no singularities on $(0, 1)$ such that*

- (1) f is the time-1 map of the flow $\{\phi_{\mathcal{X}}^s\}$ generated by \mathcal{X} ,
- (2) the flow $\{\phi_{\mathcal{X}}^s\}$ coincides with the C^1 centralizer of f in $\text{Diff}_0^1([0, 1))$.

The following statement is the analogue of [15, Proposition 7.2].

Proposition 6.17. *Take $a \in (0, 1)$ and $r > 1$. Assume that two homeomorphisms $f, g \in \text{Homeo}_0([0, 1])$ satisfy the following properties.*

- (1) *The restrictions of f and g to $[0, a]$ are C^2 contractions, namely the restrictions are C^2 diffeomorphisms onto their images such that*

$$f(x) < x \quad \text{and} \quad g(x) < x \quad \text{for every } x \in (0, a].$$

- (2) *f and g commute in restriction to $[0, a]$, that is,*

$$fg(x) = gf(x) \quad \text{for every } x \in [0, a].$$

- (3) *The C^2 germs of f and g at 0 generate an abelian free group of rank 2.*

Then, for every homeomorphism $\psi \in \text{Homeo}_0([0, 1])$ such that:

- (1) $\psi f \psi^{-1}$ and $\psi g \psi^{-1}$ are C^r in restriction to $[0, \psi(a)]$,
- (2) $(\psi f \psi^{-1})'(0) < 1$,

one has that the restriction of ψ to $(0, a]$ is C^r .

Sketch of proof. The proof is basically the same as in [15], and we only give a sketch. As f and g are C^2 contractions near 0 and commute, they can be simultaneously linearized by considering the Szekeres vector field \mathcal{X} for f . As their germs generate a rank 2 abelian group,

we can find a dense subset of times $A \subset \mathbb{R}$, such that for every $\alpha \in A$, there exists an element $h_\alpha \in \langle f, g \rangle$ such that the restriction of h_α to $[0, a]$ coincides with the time- α map of the flow $\phi_{\mathcal{X}}^\alpha$. Given a map ψ as in the statement, we can also simultaneously linearize $\psi f \psi^{-1}$ and $\psi g \psi^{-1}$, using Sternberg's linearization theorem (Theorem 6.15). If \mathcal{Y} is the corresponding vector field from Theorem 6.16, we deduce that the restriction of ψ to $(0, a]$ is C^1 and sends one vector field to the other: $(\psi|_{[0, a]})_* \mathcal{X} = \mathcal{Y}$. Writing this relation more explicitly, we get

$$\psi'(x) = \frac{\mathcal{Y}(\psi(x))}{\mathcal{X}(x)} \quad \text{for every } x \in (0, a],$$

whence we deduce that ψ is C^r in restriction to $(0, a]$. \square

The next technical result is an adaptation of classical arguments in one-dimensional dynamics, which can be traced back to Hector and Ilyashenko (see specifically [89, Proposition 3.5] and [52, Lemma 3]).

Proposition 6.18. *For $r \in (1, 2]$, let $f, g \in \text{Diff}_0^r(\mathbb{R})$ be two diffeomorphisms fixing 0 with the following properties:*

- (1) $f'(0) = \lambda < 1$ and $g'(0) = \mu \geq 1$,
- (2) for every $(l, m) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, there exists $\varepsilon > 0$ such that $g^m f^l(x) \neq x$ for every $x \in (0, \varepsilon)$.

Then there exists $\delta > 0$ such that the $\langle f, g \rangle$ -orbit of every point $x \in (0, \delta)$ is dense in $(0, \delta)$.

Proof. By Sternberg's linearization theorem (Theorem 6.15), we can take a C^r coordinate $h : U \rightarrow \mathbb{R}$ on a neighborhood U of 0 so that the map f becomes the scalar multiplication by λ on \mathbb{R} (more precisely, we take as U the maximal open interval containing no other fixed points for f). Write $V = h(U \cap g^{-1}(U))$, $\bar{f} = h f h^{-1}$ and $\bar{g} = h g h^{-1}|_V$. Note that $\bar{g}'(0) = g'(0) = \mu$.

We first rule out the case where $\log \lambda$ and $\log \mu$ are rationally dependent. So take $(l, m) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ such that $\lambda^l \mu^m = 1$ and consider the composition $\gamma := g^m \circ f^l$, which satisfies $\gamma'(0) = 1$, and write $\bar{\gamma} = \bar{g}^m \circ \bar{f}^l$ for the corresponding map defined on an appropriate open subinterval $V' \subset V$ containing 0. Then, for every $x \in V'$ and $n \in \mathbb{N}$, we have $\bar{f}^{-n} \bar{\gamma} \bar{f}^n(x) = \frac{\bar{\gamma}(\lambda^n x)}{\lambda^n}$, from which we deduce that $\bar{f}^{-n} \bar{\gamma} \bar{f}^n \rightarrow \text{id}$ uniformly on compact subsets of V' , as $n \rightarrow \infty$. Going back to the original coordinates, we get that there exists $\delta > 0$ such that $f^{-n} \gamma f^n \rightarrow \text{id}$ uniformly on $[0, \delta]$ as $n \rightarrow \infty$. Take now $x \in (0, \delta)$ and let K be the closure of the $\langle f, g \rangle$ -orbit of x , which clearly contains the point 0. Assume by contradiction that $K \cap [0, \delta] \neq [0, \delta]$, and let I be a connected component of $[0, \delta] \setminus K$. By $\langle f, g \rangle$ -invariance of K and the established uniform convergence to the identity, there exists $n_0 \in \mathbb{N}$ such that $f^{-n} \gamma f^n(I) = I$ for every $n \geq n_0$. This gives that the element $\gamma = g^m \circ f^l$ preserves all the intervals of the form $f^n(I)$, for $n \geq n_0$, and in particular it admits infinitely many fixed points accumulating on 0, which contradicts the assumption (2).

We assume next that $\log \lambda$ and $\log \mu$ are rationally independent, and in particular that $\mu > 1$. Applying Sternberg's linearization theorem again, take a C^r coordinate $k : W \rightarrow \mathbb{R}$ on a neighborhood W of 0 so that the map \bar{g} becomes the scalar multiplication by μ on \mathbb{R} ; more precisely, upon exchanging the roles of f and g , we take as W the maximal open interval containing no other fixed points for \bar{g} , and assume that $W \subset U$. Note that after Theorem 6.15, we can take k such that $k'(0) = 1$, so that $k(x) = x + O(x^r)$ as $x \rightarrow 0$. Given $\nu > 0$, there exist two increasing sequences $(l_n)_{n \in \mathbb{N}}, (m_n)_{n \in \mathbb{N}} \subset \mathbb{N}$, such that $\lambda^{l_n} \mu^{m_n} \rightarrow \nu$ as $n \rightarrow \infty$. For $n \in \mathbb{N}$, the composition $g_n = \bar{g}^{m_n} \circ \bar{f}^{l_n}$ is defined on W , as \bar{f} contracts W and \bar{g} preserves it.

Fix $x \in W$, so that

$$\begin{aligned} g_n(x) &= k^{-1} \left(\mu^{m_n} k(\lambda^{l_n} x) \right) = k^{-1} \left(\mu^{m_n} (\lambda^{l_n} x + O(\lambda^{r l_n} x^r)) \right) \\ &= k^{-1} \left(\mu^{m_n} \lambda^{l_n} x + O(\lambda^{(r-1)l_n} x^r) \right) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We deduce the convergence $g_n(x) \rightarrow k^{-1}(\nu x)$ as $n \rightarrow \infty$. As $\nu > 0$ was arbitrary, this gives that the orbit of every $x \in W \cap (0, +\infty)$ is dense in $W \cap (0, +\infty)$, as desired. \square

Finally, we also need a basic fact.

Lemma 6.19. *Let $\ell \in \mathbb{N}$ be an integer, and $p \in \mathbb{Q}$ any rational. Then there exists a non-trivial ℓ -adic affine map $g \in \text{Aff}(\mathbb{Z}[1/\ell], \langle \ell \rangle_*) \subseteq \text{Aff}(\mathbb{R})$ such that $g(p) = p$.*

Proof. Write $g(x) = \ell^k x + \frac{a}{\ell^b}$, with $a \in \mathbb{Z}$ and $b, k \in \mathbb{N}$, for a generic ℓ -adic affine map. Note that the condition $g(p) = \ell^k p + \frac{a}{\ell^b} = p$ gives $p = \frac{a}{\ell^b(\ell^k - 1)}$, which can be any rational number (choosing appropriate $a \in \mathbb{Z}$ and $b, k \in \mathbb{N}$). \square

We can now prove the main result of this section.

Proof of Theorem 6.13. We argue by way of contradiction. Write $G = F_{n_1, \dots, n_k}$ and let $\varphi : G \rightarrow \text{Diff}_0^r(\mathbb{R})$ be a faithful action ($r > 1$) without fixed points. After Corollary 6.11, φ is either semi-conjugate to the standard action on X , or to a cyclic action. Assume first that the former occurs and write $h : \mathbb{R} \rightarrow X = (0, 1)$ for the semi-conjugacy. Using Lemma 6.14 applied to the action of $F_{n_1} \subseteq G$, we find a rational point $p \in X$ and an element $f \in F_{n_1}$, such that $\xi = h^{-1}(p)$ is a hyperbolic fixed point for $\varphi(f)$. Using Lemma 6.19, we can find an element $g \in F_{n_2}$ for which $p \in X$ is an isolated fixed point. In particular, f and g commute in restriction to a right neighborhood $[p, q]$ of p , and their right germs at p generate an abelian free group of rank 2 (they are scalar multiplications by powers of n_1 and n_2 respectively). Thus, up to considering inverse powers, the assumptions of Proposition 6.18 are satisfied by the maps $\varphi(f)$ and $\varphi(g)$, from which we deduce that the action of $\langle \varphi(f), \varphi(g) \rangle$ is minimal in restriction to an interval of the form $(\xi, \xi + \delta)$, with $\delta > 0$. Hence, the semi-conjugacy $h : \mathbb{R} \rightarrow X$ considered above is a conjugacy (that is, h is a homeomorphism). On the other hand, up to considering inverse powers, the elements f and g satisfy the assumptions of Proposition 6.17. We deduce that h is C^r in restriction to $[p, q]$.

We conclude as in [15, Proof of Theorem 7.3]. Take an element $\gamma \in G$ with a discontinuity point $r \in [p, q]$ for its derivative; then also the derivative $\varphi(\gamma)'$ has a discontinuity point at $h^{-1}(r)$. This gives the desired contradiction.

In the case of cyclic action, considering the corresponding homomorphism $\tau : G \rightarrow \mathbb{Z}$, we have after Corollary 6.11 that $\ker \tau$ acts on every connected component of its support semi-conjugate to the standard action. Thus one can reproduce the previous argument, adapted to $\ker \tau$. This is a little tricky, as the abelianization $F_{n_i}/[F_{n_i}, F_{n_i}] \cong \mathbb{Z}^{n_i}$ is larger than the quotient $F_{n_i}/(F_{n_i})_c \cong \mathbb{Z}^2$. Start with an element $f \in F_{n_1} \cap \ker \tau$ with a hyperbolic fixed point ξ , as in the previous case, and then choose $g_1 \in (F_{n_2})_c$ fixing $p = h^{-1}(\xi)$ playing the role of g in the previous case. However, it could be that $g_1 \notin \ker \tau$, so for this, take an element $g_2 \in G$ such that $g_2(\text{Supp}(g_1)) \cap \text{Supp}(g_1) = \emptyset$. Then the commutator $g = [g_1, g_2]$ coincides with g_1 on $\text{Supp}(g_1)$, and belongs to $\ker \tau$. \square

The method presented here cannot be improved further to go down to exclude C^1 smoothability. We believe however this can be achieved with a different approach. Let us point out that all known examples of C^1 smoothable groups of PL homeomorphisms embed in Thompson's

F . It would be tempting to conjecture that this is also a necessary condition. However, we estimate that little is known about other groups of PL homeomorphisms, such as those defined by irrational slopes (see however the very recent work [51], where it is proved that several such groups do not embed in F). So, let us highlight the following concrete problem.

Question 6.20. *Fix an irrational $\tau \in \mathbb{R} \setminus \mathbb{Q}$, and write $\Lambda = \langle \tau \rangle_*$ and $A = \mathbb{Z}[\Lambda]$ (as an explicit case, one can take the golden ratio $\tau = \frac{\sqrt{5}-1}{2}$). Consider the irrational slope Thompson's group $F_\tau = G([0, 1]; A, \Lambda)$. Is the action of F_τ on the interval C^1 smoothable?*

6.4.2. *An application to Bieri–Strebel groups on the line.* Given a real number $\lambda > 1$, we consider the Bieri–Strebel groups acting on the line $G(\lambda) = G(\mathbb{R}; \mathbb{Z}[\lambda, \lambda^{-1}], \langle \lambda \rangle_*)$. It was remarked in [15] that the standard action of $G(\lambda)$ cannot be conjugate to any C^1 action. One of the main results of [15] states that for certain choices of λ the group $G(\lambda)$ does not admit any faithful C^1 action on the line. Here we generalize this result by removing all restrictions on λ .

Corollary 6.21. *For $\lambda > 1$, there is no faithful C^1 action of the Bieri–Strebel group $G(\lambda)$ on the closed interval.*

Proof. Indeed, it is proved in [15, Theorem 6.10] that the standard action of $G(\lambda)$ on the line cannot be conjugate to any C^1 action on the closed interval, but a closer look at the proof (notably using the results from [16, §4.2]) shows that even a semi-conjugacy is impossible, as the action of the affine subgroup of $G(\lambda)$ must be minimal. Hence the result follows from Theorem 6.9. \square

In Section 12 we will classify C^0 actions of $G(\lambda)$ up to semi-conjugacy whenever λ is algebraic.

7. \mathbb{R} -FOCAL ACTIONS

In this section and the next we leave temporary aside the study of locally moving groups to introduce and study the notion of \mathbb{R} -focal action. In this section we give the definition and some first properties of \mathbb{R} -focal actions and, as initial motivation, we show that they arise naturally in some situations, for instance for actions of solvable groups. The meaning of this notion will be further clarified in Section 8, where we will reinterpret \mathbb{R} -focal actions in terms of actions on planar directed trees, and study their dynamical properties more in details.

7.1. Cross-free covers and \mathbb{R} -focal actions.

Definition 7.1. Let Ω be a set and let $I, J \subset \Omega$ be two subsets. We say that I and J do not cross if either $I \subset J$ or $J \subset I$, or $I \cap J = \emptyset$. A collection of subsets \mathcal{S} is *cross-free* if I and J do not cross for every $I, J \in \mathcal{S}$.

When (Ω, \prec) is a totally ordered space, we say that a cross-free collection \mathcal{S} is a *cross-free cover* (CF-cover for short) if the following conditions are satisfied:

- (C1) every element of \mathcal{S} is \prec -convex, open and bounded (with respect to the order topology),
- (C2) there exists a subcollection of \mathcal{S} which is totally ordered by inclusion and covers Ω .

Remark 7.2. When (Ω, \prec) is the real line \mathbb{R} with its standard order, condition (C1) for a CF-cover \mathcal{S} means that every element of \mathcal{S} is a bounded open interval, whilst condition (C2) amounts to requiring that \mathcal{S} be a cover of \mathbb{R} , justifying the terminology. Indeed if this holds, then if \mathcal{C}_1 and \mathcal{C}_2 are maximal totally ordered subcollections of \mathcal{S} (with respect to inclusion), then the cross-free property implies that the unions $\bigcup_{I \in \mathcal{C}_1} I$ and $\bigcup_{J \in \mathcal{C}_2} J$ are either equal or

disjoint, and hence \mathbb{R} can be written as a disjoint union of open subsets of this form. By connectedness of \mathbb{R} , there can be only one such maximal subcollection (and there is at least one by Zorn's lemma).

Definition 7.3. An action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is said to be \mathbb{R} -focal if it has no fixed points and there exists a bounded open interval $I \subset \mathbb{R}$ whose $\varphi(G)$ -images form a CF-cover of \mathbb{R} .

Remark 7.4. Note that equivalently an action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is \mathbb{R} -focal if and only if it admits an invariant CF-cover \mathcal{S} and an interval $J \in \mathcal{S}$ whose $\varphi(G)$ -orbit is *cofinal* in \mathcal{S} with respect to inclusion, meaning that for every $J' \in \mathcal{S}$ there exists $g \in G$ such that $J' \subset g.J$. Indeed the latter condition clearly implies that the orbit of J is a CF-cover. Conversely if I is a bounded open interval whose orbit is a CF-cover, then condition (C2) implies that the orbit of I is cofinal.

We now study some basic properties of \mathbb{R} -focal actions.

Lemma 7.5. *Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action admitting an invariant CF-cover. Then for every element $g \in G$, the image $\varphi(g)$ has fixed points.*

A more precise analysis of the dynamics of individual elements in an \mathbb{R} -focal action will be given in §8.3.

Proof of Lemma 7.5. Let \mathcal{S} be an invariant CF-cover and fix $g \in G$. For every $\xi \in \mathbb{R}$, from condition (C2) we can find an interval $J \in \mathcal{S}$ which contains ξ and $g.\xi$, so that $g.J \cap J \neq \emptyset$. Then the cross-free property implies that either g or g^{-1} must map J into itself. By the intermediate value theorem, $\varphi(g)$ has fixed points inside J . \square

Proposition 7.6. *Let G be a group, and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an \mathbb{R} -focal action. Then the following holds.*

- (i) *If $\psi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is an action semi-conjugate to φ then ψ is \mathbb{R} -focal.*
- (ii) *$\varphi(G)$ has a unique minimal invariant set $\Lambda \subseteq \mathbb{R}$, which is not discrete.*

In particular, every \mathbb{R} -focal action is semi-conjugate to a minimal \mathbb{R} -focal action.

Proof. Let $I \subset \mathbb{R}$ be a bounded open interval such that $\mathcal{S} = \{\varphi(g)(I) : g \in G\}$ is a CF-cover. To prove (i), let $h: \mathbb{R} \rightarrow \mathbb{R}$ be a semi-conjugacy from φ to ψ (in the sense that (2.1) holds). After condition (C2), \mathcal{S} contains an increasing sequence of intervals that exhaust \mathbb{R} , thus, as h is proper, we can find $g \in G$ such that $h(\varphi(g)(I)) = \psi(g)(h(I))$ is not a singleton, so that actually $J = h(I)$ is a bounded open interval. The equivariance of h implies that the $\psi(G)$ -orbit of J is a CF-cover.

To prove (ii), observe first that $\varphi(G)$ has no closed discrete orbit. If this was the case, then the number of points of such an orbit inside intervals in \mathcal{S} must be finite and constant, which is clearly in contradiction with condition (C2). Thus $\varphi(G)$ has at most one non-empty closed minimal subset, and it is enough to show that such a set exists. By condition (C2), every non-empty closed G -invariant subset of \mathbb{R} intersects an element of \mathcal{S} , so by $\varphi(G)$ -invariance it intersects I . By compactness of \bar{I} , a standard application of Zorn's lemma gives a non-empty minimal invariant set. Finally note that since φ admits a non-discrete minimal invariant set, it is semi-conjugate to a minimal action (Corollary 2.12), which must be \mathbb{R} -focal by (i). \square

After the previous proposition there is little loss of generality if we restrict to the study of \mathbb{R} -focal *minimal* actions, and we will systematically do so. Note that in this case the notion of \mathbb{R} -focal action can be reformulated by simply requiring the existence of a CF-cover. More precisely, we have the following result.

Proposition 7.7. *Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal action. Then φ is \mathbb{R} -focal if and only if it preserves a non-empty cross-free family of intervals. Moreover if this is the case, then φ is proximal.*

Proof. The forward implication is obvious from the definition of \mathbb{R} -focal action, by taking the family consisting of $\varphi(G)$ -images of I . Conversely assume φ is minimal and that \mathcal{S} is an invariant cross-free family of intervals. Then, for every $I \in \mathcal{S}$, the family $\mathcal{S}_0 = \{\varphi(g)(I) : g \in G\}$ is also cross-free. The union of elements of \mathcal{S}_0 is an invariant open subset which is non-trivial, so that by minimality of the action, it has to coincide with the whole real line. In particular we have that \mathcal{S}_0 is a CF-cover, so φ is an \mathbb{R} -focal action.

Let us now show that φ must be proximal. By Theorem 2.17 if this is not the case then there exists an element $\tau \in \text{Homeo}_0(\mathbb{R})$ without fixed points which centralizes $\varphi(G)$. Upon conjugating the action, we can assume that $\tau(x) = x + 1$. Now, given a bounded interval $I \subset \mathbb{R}$ denote by n_I the largest $n \geq 0$ such that $\tau^n(I) \cap I \neq \emptyset$. Since τ commutes with $\varphi(G)$, we must have that $n_{\varphi(g)(I)} = n_I$ holds for every g . We deduce that the orbit of every bounded interval I must consist of intervals whose length is bounded by $n_I + 1$, and thus φ cannot be \mathbb{R} -focal. \square

We conclude with a simple lemma, which says that minimal \mathbb{R} -focal actions are determined, uniquely up to conjugacy, by the combinatorics of the CF-cover. This will be used in the next section.

Lemma 7.8. *Let G be a group and for $i \in \{1, 2\}$, let $\varphi_i: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action with invariant CF-cover \mathcal{S}_i . Assume there exists a G -equivariant map $f: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ such that for every $I, J \in \mathcal{S}_1$ with $\sup I \leq \sup J$, one has $\sup f(I) \leq \sup f(J)$. Then the actions φ_1 and φ_2 are positively conjugate.*

Remark 7.9. After Proposition 7.7, the actions in the statement of Lemma 7.8 are proximal, so by Remark 2.15 the positive conjugacy $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ between φ_1 and φ_2 is unique. One can actually prove that the conjugacy induces the map f , in the sense that $h(I) = f(I)$ for every $I \in \mathcal{S}_1$.

Proof of Lemma 7.8. Given a bounded interval I , we write $I_+ = \sup I$ for simplicity. Consider the subset $\mathcal{S}_1^+ = \{I_+ : I \in \mathcal{S}_1\} \subseteq \mathbb{R}$ of rightmost points of elements of \mathcal{S}_1 , and introduce the function $j: \mathcal{S}_1^+ \rightarrow \mathbb{R}$ given by

$$j(\xi) = \sup\{f(I)_+ : I \in \mathcal{S}_1, I_+ = \xi\}.$$

Note that j is a monotone non-decreasing equivariant map: monotonicity follows from the assumption that $I_+ \leq J_+$ implies $f(I)_+ \leq f(J)_+$, and equivariance follows from equivariance of f . By Lemma 2.7, the map j extends to a positive semi-conjugacy h between φ_1 and φ_2 , which is actually a conjugacy because φ_1 and φ_2 are minimal. \square

Probably the first example of \mathbb{R} -focal action appearing in the literature is an action of the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$ studied by J. F. Plante [97]. The following example generalizes this construction to arbitrary wreath products of countable groups. We give will use it repeatedly along this section and the next to illustrate the notions discussed.

Example 7.10 (Plante actions of wreath products). Recall that for general groups G and H , the wreath product $H \wr G$ is defined by the semidirect product $(\bigoplus_G H) \rtimes G$, where G acts on the direct sum by shift of indices. More explicitly, considering the direct sum $\bigoplus_G H$ as the set of functions $s: G \rightarrow H$ which are trivial at all but finitely elements of G , the action of $h \in G$ is given by $\sigma(g)(s)(x) = s(g^{-1}x)$.

Given left-invariant orders $\prec_G \in \text{LO}(G)$ and $\prec_H \in \text{LO}(H)$, we can consider an order \prec of lexicographic type on $\bigoplus_G H$, as follows. We denote by \mathbf{e} the trivial element of $\bigoplus_G H$, that is the function satisfying $\mathbf{e}(x) = 1_H$ for every $x \in G$, and we define

$$P = \left\{ \mathbf{s} \in \bigoplus_G H : \mathbf{s} \neq \mathbf{e}, \mathbf{s}(x_s) >_H 1_H \right\},$$

where for $\mathbf{s} \neq \mathbf{e}$ we set $x_s = \max_{\prec_G} \{x \in G : \mathbf{s}(x) \neq 1_H\}$. It is not difficult to check that P defines a positive cone, and thus a left-invariant order \prec on the direct sum $\bigoplus_G H$, which is also invariant under the shift action σ of H . This gives an order-preserving action $\Psi : H \wr G \rightarrow \text{Aut}(\bigoplus_G H, \prec)$, that we call the *Plante product* of \prec_G and \prec_H . When G and H are countable so is $\bigoplus_G H$, and thus we may consider the dynamical realization of Ψ , which we call the *Plante action* associated with \prec_G and \prec_H . In this situation we let $\iota : (\bigoplus_G H, \prec) \rightarrow \mathbb{R}$ be the associated good embedding and let $\varphi : H \wr G \rightarrow \text{Homeo}_0(\mathbb{R})$ be the dynamical realization. When $G = H = \mathbb{Z}$ and \prec_G and \prec_H are the standard left orders of \mathbb{Z} , this construction yields the action of $\mathbb{Z} \wr \mathbb{Z}$ considered by Plante (see [36, §3.2]), an illustration of which appears in Figure 4.

We claim that the Plante action φ is minimal and \mathbb{R} -focal. To prove minimality, note that the stabilizer of the trivial element $\mathbf{e} \in \bigoplus_G H$ coincides with G , and its orbit is the whole subgroup $\bigoplus_G H$. Thus, after Proposition 2.43, it is enough to check that $\Psi(G)$ is of homothetic type. For this, consider four elements $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_2 \in \bigoplus_G H$ such that $\mathbf{t}_1 \prec \mathbf{s}_1 \prec \mathbf{e} \prec \mathbf{s}_2 \prec \mathbf{t}_2$. Set $y_* = \max_{\prec_G} \{x_{\mathbf{t}_1}, x_{\mathbf{t}_2}\}$ and $x_* = \min_{\prec_G} \{x_{\mathbf{s}_1}, x_{\mathbf{s}_2}\}$, and consider an element $g \in G$ such that $g^{-1}x_* >_G y_*$. Then it is immediate to check that

$$\sigma(g)(\mathbf{s}_1) \prec \mathbf{t}_1 \prec \mathbf{e} \prec \mathbf{t}_2 \prec \sigma(g)(\mathbf{s}_2),$$

which gives the desired conclusion. To check that φ is \mathbb{R} -focal it is enough to check that $(\bigoplus_G H, \prec)$ admits a CF-cover \mathcal{S} which is invariant under the Ψ -action of $H \wr G$. Indeed, considering the collection of interiors of closures of images of elements of \mathcal{S} by the good embedding $\iota : \bigoplus_G H \rightarrow \mathbb{R}$ associated with φ , we obtain a CF-family which is invariant for $\varphi(H \wr G)$ (an argument analogous to that in Proposition 7.6 shows that this is non-trivial), so that we can conclude by Proposition 7.7. For this, for $\mathbf{s} \in \bigoplus_G H$ and $g \in G$, we set

$$C_{\mathbf{s},g} = \left\{ \mathbf{t} \in \bigoplus_G H : \mathbf{t}(x) = \mathbf{s}(g) \text{ for every } x >_G g \right\}.$$

Clearly every $C_{\mathbf{s},g}$ is a convex and bounded subset of $(\bigoplus_G H, \prec)$ and

$$\mathcal{S}_0 = \left\{ C_{\mathbf{s},g} : \mathbf{s} \in \bigoplus_G H, g \in G \right\}$$

defines a cover of $\bigoplus_G H$, which is $\Psi(H \wr G)$ -invariant. It only remains to check that \mathcal{S}_0 is a cross-free family. For this, take two elements $C_{\mathbf{s},g}$ and $C_{\mathbf{s}',g'}$ in \mathcal{S}_0 with $g \leq_G g'$, and assume there is some element \mathbf{t} in their intersection. It follows that \mathbf{s} , \mathbf{t} , and \mathbf{s}' all agree on $\{x \in G : x >_G g'\}$ and so $C_{\mathbf{s},g} \subseteq C_{\mathbf{s}',g'}$. This shows that \mathcal{S}_0 is a CF-cover and thus the Plante action is \mathbb{R} -focal.

7.2. A condition for \mathbb{R} -focality. The following criterion implies the \mathbb{R} -focality of a vast class of actions.

Proposition 7.11. *Consider a minimal faithful action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ and let $N \triangleleft G$ be a normal subgroup which is not a cyclic subgroup of the center of G . Assume that $\varphi(N)$ does not act minimally on \mathbb{R} . Then φ is an \mathbb{R} -focal action.*

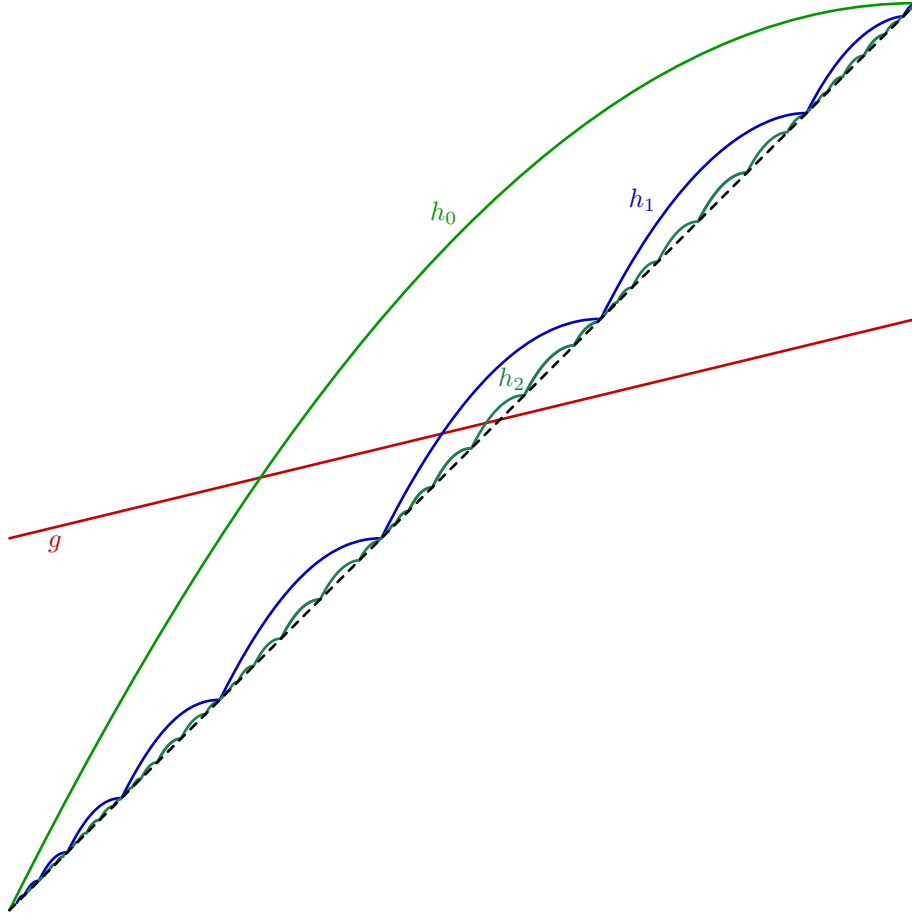


FIGURE 4. Plante action of $\mathbb{Z} \wr \mathbb{Z}$ on the line. One factor is generated by g which acts as a homothety. The generator of the other factor is h_0 , and we have $h_n = g^n h_0 g^{-n}$ for every $n \in \mathbb{Z}$, where the h_n commute and are a basis of the lamp group $\bigoplus_{\mathbb{Z}} \mathbb{Z}$.

First we recall the following lemma. Its proof is well-known but we include it for completeness.

Lemma 7.12. *Let G be a group, and let N be a non-trivial normal subgroup which is not a cyclic subgroup of the center of G . Assume that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a minimal faithful action of G . Then either the image of N acts minimally, or it admits no minimal invariant set.*

Proof. Assume that there exists a non-empty minimal $\varphi(N)$ -invariant set $\Lambda \subset \mathbb{R}$, and let us show that $\Lambda = \mathbb{R}$. Note that Λ cannot be a fixed point otherwise we get a contradiction from Corollary 2.6. Then either Λ is the unique minimal $\varphi(N)$ -invariant set, or a closed orbit. In the first case, we have that Λ is preserved by the whole group G and thus $\Lambda = \mathbb{R}$ by minimality. In the second case we have that the action of N must be semi-conjugate to a cyclic action coming from a homomorphism $\tau: N \rightarrow \mathbb{Z}$, and that $\ker \tau$ acts trivially on Λ . Since $\ker \tau$ is precisely the subset of N acting with fixed points, the subgroup $\ker \tau$ is necessarily normal in the whole group G , and so, as before, we must have $\ker \tau = \{1\}$. Thus N is infinite cyclic and acts as a group of translations. Since N is normal in G , this implies that N is central in G , contradicting the assumption. \square

Proof of Proposition 7.11. We first observe the following.

Claim. *For every open interval $I \subset \mathbb{R}$, there exists an element $h \in N$ such that $h.I \cap I \neq \emptyset$ and $h \upharpoonright_I \neq \text{id}$.*

Proof of claim. Consider the subset of points where the condition in the statement fails, namely

$$A := \{\xi \in \mathbb{R} : \exists \text{ neighborhood } V \ni \xi \text{ s.t. } \forall h \in N \text{ either } h \upharpoonright_V = \text{id} \text{ or } h.V \cap V = \emptyset\}.$$

It is clear from the definition that A is open, and the fact that N is normal in G gives that it is G -invariant. As we are assuming that the action is minimal, we conclude that A is either empty or the whole real line. Assume for contradiction that $A = \mathbb{R}$, and note that this implies that every element $h \in N$ either acts trivially or has no fixed point at all. The first possibility is ruled out by the fact that the action φ is minimal, so that by Hölder theorem (see [44, Theorem 6.10]), we have that $\varphi(N)$ is semi-conjugate to a group of translations and in particular $\varphi(N)$ admits a minimal invariant set. From Lemma 7.12 we get that this minimal set must be the whole real line which contradicts our hypothesis that $\varphi(N)$ does not act minimally. \square

Now, for each interval $I \subset \mathbb{R}$ denote by $\text{Stab}_N^\varphi(I)$ the stabilizer of I in N , and by $\text{Fix}_I^\varphi(N)$ the subset of fixed points of $\text{Stab}_N^\varphi(I)$ in I . Since $\varphi(N)$ does not act minimally, there exists a proper, closed and $\varphi(N)$ -invariant subset of the line that we denote by Λ . Let U be a connected component of $\mathbb{R} \setminus \Lambda$. The claim implies that the stabilizer $\text{Stab}_N^\varphi(U)$ is non-trivial, and moreover the subset $\text{Fix}_U^\varphi(N)$ has empty interior, so that it is strictly contained in U . Hence the subset $\Lambda_1 := \Lambda \cup (\bigcup_U \text{Fix}_U^\varphi(N))$ (where the union runs over the connected components of $\mathbb{R} \setminus \Lambda$) is a proper closed subset of \mathbb{R} , which is $\varphi(N)$ -invariant (indeed, for $g \in N$ we have $g.\text{Fix}_U^\varphi(N) = \text{Fix}_{g.U}^\varphi(N)$) and moreover $\text{Fix}_V^\varphi(N) = \emptyset$ for every connected component V of $\mathbb{R} \setminus \Lambda_1$.

Consider now the collection \mathcal{S} of all non-empty open bounded intervals $I \subset \mathbb{R}$ such that

- for every $h \in N$, either $h.I = I$ or $h.I \cap I = \emptyset$, and
- $\text{Fix}_I^\varphi(N) = \emptyset$.

The collection \mathcal{S} is non-empty since it contains each connected component of $\mathbb{R} \setminus \Lambda_1$. Also note that, since N is normal, the family \mathcal{S} is $\varphi(G)$ -invariant. We will show that \mathcal{S} is cross-free. To see this, suppose that $I, J \in \mathcal{S}$ are crossed, and let ξ and η be the points in the intersections $I \cap \partial J$ and $\partial I \cap J$, respectively. Since $\text{Stab}_N^\varphi(J)$ acts on J without fixed points and fixes ξ , we can find an element $h \in \text{Stab}_N^\varphi(J)$ which moves η and thus satisfies $h.I \cap I \neq \emptyset$, but $h.I \neq I$ (see Figure 5). This gives the desired contradiction, and we conclude that \mathcal{S} is a $\varphi(G)$ -invariant cross-free family of intervals. As φ is minimal, Proposition 7.7 implies that it is \mathbb{R} -focal. \square

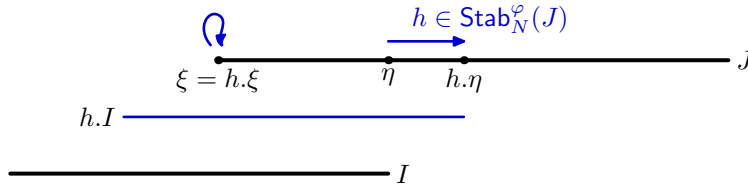


FIGURE 5. End of the proof of Proposition 7.11.

Remark 7.13. The converse to Proposition 7.11 is not true: there exist \mathbb{R} -focal actions of groups such that all non-trivial normal subgroups act minimally (see §9.3 for counterexamples), and even simple groups which admit minimal \mathbb{R} -focal actions (such as the commutator subgroup

$[F, F]$ of Thompson's group F , see §11.6). However a partial converse to Proposition 7.11 will be given in §8.5

7.3. \mathbb{R} -focal actions and alternatives. Here we discuss two situations where \mathbb{R} -focal actions appear naturally.

7.3.1. \mathbb{R} -focal actions and micro-supported groups. In the next proposition we show that if a micro-supported group acting minimally is not locally moving (see §4.1), then its action is necessarily \mathbb{R} -focal.

Proposition 7.14 (Alternative for micro-supported groups). *A micro-supported subgroup $G \subseteq \text{Homeo}_0(\mathbb{R})$ whose action is minimal, is either locally moving or \mathbb{R} -focal.*

For the proof, given a micro-supported group $G \subseteq \text{Homeo}(\mathbb{R})$ and an interval $I \subset \mathbb{R}$ (not necessarily bounded) we write $\text{Fix}_I := \text{Fix}(G_I) \cap I$ for the set of fixed points inside I of the rigid stabilizer of I . We first need the following observation.

Lemma 7.15. *Let $G \subseteq \text{Homeo}_0(\mathbb{R})$ be a micro-supported subgroup whose action is minimal. Suppose there exist $\xi, \eta \in \mathbb{R}$ such that $\text{Fix}_{(-\infty, \xi)} = \text{Fix}_{(\eta, +\infty)} = \emptyset$. Then G is locally moving.*

Proof. Take a bounded open interval $I = (\alpha, \beta)$ and $x \in I$. We need to show that $g(x) \neq x$ for some $g \in G_I$. From the minimality of the action of G , after possibly conjugating $G_{(\eta, +\infty)}$ and $G_{(-\infty, \xi)}$, we can assume that ξ and η satisfy that

$$\alpha < \xi < x < \eta < \beta.$$

Since G is micro-supported, the subgroup G_c of compactly supported elements is non-trivial. Since the action of G is minimal and G_c is normal, we have that G_c has no fixed points, and thus there exists $h \in G_c$ such that $h(x) \neq x$. Consider the smallest interval (a, b) containing $\text{Supp}(h)$. From the assumption, we can find elements $k_1 \in G_{(-\infty, \xi)}$ and $k_2 \in G_{(\eta, +\infty)}$ such that $k_1(a) \in (\alpha, \xi)$ and $k_2(b) \in (\eta, b)$. Write $k = k_1 k_2$. Then $g = khk^{-1} \in G_I$ and $g(x) \neq x$, as desired. \square

Proof of Proposition 7.14. Suppose that G is not locally moving. After Lemma 7.15, we can assume that there exists $\xi \in \mathbb{R}$ such that $\text{Fix}_{(\xi, +\infty)} \neq \emptyset$ (the symmetric case $\text{Fix}_{(-\infty, \xi)} \neq \emptyset$ can be treated similarly). Therefore, the support of the subgroup $G_{(\xi, +\infty)}$ has a bounded connected component, let us call it $U = (\alpha, \beta) \subseteq (\xi, +\infty)$. We claim that the G -orbit of U is cross-free.

Looking for a contradiction, suppose that there is $g \in G$ such that U and $g(U)$ are crossed. Clearly $g(U)$ is a connected component of the support of $gG_{(\xi, +\infty)}g^{-1} = G_{(g(\xi), +\infty)}$. But, up to changing g by its inverse, we can assume that $\xi \leq g(\xi)$, and hence that $G_{(g(\xi), +\infty)}$ is a subgroup of $G_{(\xi, +\infty)}$. In particular we get that U is fixed by every element in $G_{(g(\xi), +\infty)}$ and therefore it is not possible that U is crossed with a component of support of $G_{(g(\xi), +\infty)}$. This contradicts our assumption that U and $g(U)$ are crossed, and so the claim follows. From Proposition 7.7, we get that the action of G is \mathbb{R} -focal.

To conclude, we need to show that the action of G cannot be simultaneously locally moving and \mathbb{R} -focal. For this, note that when G is locally moving the diagonal action of G on the set $X_2 := \{(x, y) \in \mathbb{R}^2 : x < y\}$ is minimal, while in the case when the action is \mathbb{R} -focal this is not the case. For instance, if $U = (\alpha, \beta)$ is an interval whose G -orbit has no crossed elements, then (α, β) cannot accumulate on (α', β') for any $\alpha' < \alpha < \beta' < \beta$. \square

We have the following straightforward application.

Corollary 7.16. *If $G \subseteq \text{Homeo}_0(\mathbb{R})$ is a group acting minimally on \mathbb{R} and containing an element of relatively compact support and an element without fixed points, then G is locally moving.*

Proof. The fact that G acts minimally on \mathbb{R} and contains an element of compact support implies that G is micro-supported by Proposition 4.2. The assumption that G contains an element without fixed points excludes the case of \mathbb{R} -focal action, as every element should have with fixed points (Lemma 7.5). By Proposition 7.14, the action is locally moving. \square

7.3.2. Actions of solvable groups. We conclude this section with the following application of Proposition 7.11, which shows that \mathbb{R} -focal actions appear naturally in the context of solvable groups. Its derivation from Proposition 7.11 uses ideas of Rivas and Tessera in [99]. By an affine action of G on \mathbb{R} we mean an action by affine transformations, i.e. transformations of the form $x \mapsto ax + b$ with $a > 0$ and $b \in \mathbb{R}$.

Theorem 7.17 (Alternative for solvable groups). *Let G be a finitely generated solvable group and let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points. Then either φ is semi-conjugate to an affine action, or it is \mathbb{R} -focal.*

Proof. If φ is semi-conjugate to a cyclic action then the first case holds. So up to semi-conjugacy we can assume that φ is minimal; moreover upon replacing G by a quotient we can suppose that it is faithful. Let $G^{(n)}$ denote the derived series of G , and consider $k \in \mathbb{N}$ so that $G^{(k)} \neq \{1\}$ and $G^{(k+1)} = \{1\}$. Then $H := G^{(k)}$ is an abelian and normal subgroup of G . As H is normal, we deduce that $\varphi(H)$ cannot have fixed points (otherwise we would get a fixed point for the action of G).

Assume first that H is cyclic contained in the center of G . Then $\varphi(H)$ is conjugate to the cyclic group generated by a translation and the action φ induces a minimal action on the circle $\mathbb{R}/\varphi(H)$. As G is solvable and thus amenable, it preserves a Borel probability measure on the circle, which must be of total support for the action is minimal; after [91, Proposition 1.1.1], we get that the action is conjugate to a minimal action by rotations (basically, this is the action defined by the rotation number homomorphism $\text{rot} : G \rightarrow \mathbb{S}^1$). Therefore, $\varphi(G)$ is the lift of a group of rotations of the circle and thus conjugate to a group of translations, so that we get that the action is affine in this case.

Assume next that $\varphi(H)$ acts minimally. Take a non-trivial $h \in H$ and note as before that $\varphi(h)$ has no fixed points (otherwise we would get a fixed point for $\varphi(H)$). Thus we obtain a minimal action of H on the circle $\mathbb{R}/\langle \varphi(h) \rangle$, and the argument for the previous case leads to the conclusion that $\varphi(H)$ is conjugate to a minimal group of translations. Then the set of invariant Radon measures for $\varphi(H)$ corresponds to the one-parameter family of positive multiples of the Lebesgue measure, and this family must be preserved by $\varphi(G)$, for H is normal in G . By a standard argument, we deduce that $\varphi(G)$ is conjugate to a group of affine transformations (see for instance [97]).

If neither of the previous cases hold, by Proposition 7.11 we get that φ is \mathbb{R} -focal as desired. \square

Since in a minimal \mathbb{R} -focal action every element has fixed points (Lemma 7.5), we deduce the following result first obtained Guelman and Rivas in [46].

Corollary 7.18. *Let G be a finitely generated solvable group, and let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action such that some element acts without fixed points. Then, the action φ is semi-conjugate to an affine action.*

8. PLANAR TREES AND HOROGRADED \mathbb{R} -FOCAL ACTIONS

In this section we study in more detail \mathbb{R} -focal actions and reinterpret them in terms of actions on planar directed trees, which provides a visual formalism to study them. This will allow to define a natural invariant associated to an \mathbb{R} -focal action, called the *focal germ representation*, which bears a lot of information about the action. It is also the natural setting to introduce the tightly related notion of *horograding* of an \mathbb{R} -focal action of a group G by another action of G , which allows to compute its focal germ representation. This notion will be important later to establish a relation between exotic actions of a class of a locally moving groups to their standard action.

8.1. Preliminaries on trees. We first make a digression to set some terminology about (real) *trees*.

8.1.1. Directed trees. Roughly speaking, a real tree is a space \mathbb{T} obtained by gluing copies of the real line in such a way that no closed loops appear. We are specifically interested in *directed* real trees, which are trees together with a preferred direction to infinity (equivalently a preferred end $\omega \in \partial\mathbb{T}$). Usually real trees are defined as metric spaces, but we adopt the point of view of Favre and Johnson [38], and introduce the following definition which does not refer to any metric or topology. (The equivalence with the more familiar metric notion will be discussed in §8.1.4 below.)

Definition 8.1. A (non-metric) countably branching directed tree (hereinafter just *directed tree*) is a partially ordered set (*poset* for short) $(\mathbb{T}, \triangleleft)$ with the following properties.

- (T1) For every $v \in \mathbb{T}$ the subset $\{u \in \mathbb{T} : v \trianglelefteq u\}$ is totally ordered and order-isomorphic to a half-line $[0, +\infty)$.
- (T2) Every pair of points $v, u \in \mathbb{T}$ has a smallest common upper bound, denoted $v \wedge u$.
- (T3) There exist a countable subset $\Sigma \subset \mathbb{T}$ such that for every distinct $u, v \in \mathbb{T}$ with $u \trianglelefteq v$ there exists $z \in \Sigma$ such that $u \trianglelefteq z \trianglelefteq v$.

We say that a point v is *below* u (or that u is *above* v) if $v \neq u$ and $v \trianglelefteq u$, and write $v \triangleleft u$.

Condition (T3) is a technical separability assumption. A first consequence of it is that all totally ordered subsets of \mathbb{T} are isomorphic to subsets of the real line. More precisely we have the following lemma. (Recall that every totally ordered subset of a poset is contained in a maximal one, by Zorn's lemma.)

Lemma 8.2. *Let $(\mathbb{T}, \triangleleft)$ be a directed tree. If $\ell \subset \mathbb{T}$ is a maximal totally ordered subset, then it is order-isomorphic to either $[0, +\infty)$ or to \mathbb{R} . If ℓ_1, ℓ_2 are two maximal totally ordered subsets, then either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2$ is order-isomorphic to $[0, +\infty)$.*

Proof. First of all, observe that for every $v \in \ell$ we have $\{u : v \trianglelefteq u\} \subset \ell$, or otherwise $\ell \cup \{u : v \trianglelefteq u\}$ would be a strictly larger totally ordered set contradicting maximality of ℓ . Thus if ℓ has a minimum v , then we must have $\ell = \{v : v \trianglelefteq u\} \cong [0, +\infty)$ by (T1). If not, then (T3) implies that for every maximal totally ordered subset $\ell \subset \mathbb{T}$ contains a strictly decreasing sequence (v_n) such that $\ell = \bigcup_n \{u : v_n \trianglelefteq u\}$. Since by (T1), every element in the union is isomorphic to $[0, +\infty)$ it follows that ℓ is isomorphic to \mathbb{R} . Let ℓ_1 and ℓ_2 be two maximal totally ordered subsets. If $\ell_1 \neq \ell_2$ then there exist $v_1 \in \ell_1$ and $v_2 \in \ell_2$ which are not comparable. Then $v_1 \wedge v_2$ (which exists by (T2)) is the smallest element in $\ell_1 \cap \ell_2$ and it follows from (T1) that $\ell_1 \cap \ell_2 = \{w : v_1 \wedge v_2 \trianglelefteq w\} \cong [0, +\infty)$. \square

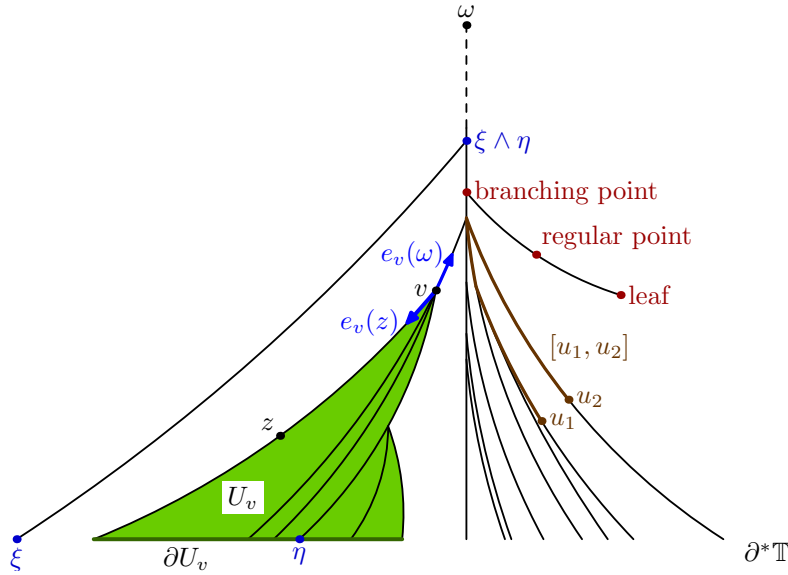


FIGURE 6. A directed tree $(\mathbb{T}, \triangleleft)$ with focus ω , and the corresponding defined objects: arc between two points (brown), directions (blue), smallest upper bound (blue), shadow (green).

8.1.2. *End-completion and boundary.* From Lemma 8.2 we can introduce a notion of boundary for a directed tree.

Definition 8.3. Given a directed tree $(\mathbb{T}, \triangleleft)$, we define its *end-completion* $(\overline{\mathbb{T}}, \triangleleft)$ as the poset obtained by adding points to \mathbb{T} as follows. First we add a point ω , called the *focus*, which is the unique maximal point of $(\overline{\mathbb{T}}, \triangleleft)$. In addition, for each maximal totally ordered subset $\ell \subset \mathbb{T}$ without minimum, we add a point $\xi \in \overline{\mathbb{T}}$ which satisfies $\xi \triangleleft v$ for every $v \in \ell \cup \{\omega\}$. The subset $\partial\mathbb{T} = \overline{\mathbb{T}} \setminus \mathbb{T}$ is called the *boundary* of \mathbb{T} . We will also write $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$.

We next introduce further terminology associated to the end-completion. Note that any pair of points $x, y \in \overline{\mathbb{T}}$ still admits a unique smallest upper bound, which we continue to denote by $x \wedge y$, and we have $x \wedge y \in \mathbb{T}$ unless $x = \omega$ or $y = \omega$. Given distinct points $u, v \in \overline{\mathbb{T}}$ we define the *arc* between them as the subset

$$[u, v] = \left\{ z \in \overline{\mathbb{T}} : u \trianglelefteq z \trianglelefteq (v \wedge u) \text{ or } v \trianglelefteq z \trianglelefteq (v \wedge u) \right\}.$$

and set $]u, v[= [u, v] \setminus \{u, v\}$. The subsets $[u, v[$ and $]u, v]$ are defined similarly. A subset $Y \subseteq \overline{\mathbb{T}}$ is *path-connected* if $[u, v] \subseteq Y$ for every $u, v \in Y$. Every subset of $\overline{\mathbb{T}}$ is a disjoint union of maximal path-connected subsets, called its *path-components*. Given $v \in \mathbb{T}$, we define the set E_v of *directions* at v as the set of path-components of $\overline{\mathbb{T}} \setminus \{v\}$. The cardinal $|E_v|$ is called the *degree* of v . Points of degree 1 are called *leaves*; these are precisely the minimal elements in $(\mathbb{T}, \triangleleft)$. We say that v is *regular* if $|E_v| = 2$ and a *branching point* if $|E_v| \geq 3$. The set of branching points of \mathbb{T} will be denoted by $\text{Br}(\mathbb{T})$. For $z \in \overline{\mathbb{T}} \setminus \{v\}$, we let $e_v(z) \in E_v$ denote the direction containing z . If ω is the focus, we let $E_v^- = E_v \setminus \{e_v(\omega)\}$ denote the set of *directions below* v . Finally, we denote by $U_v \subset \mathbb{T}$ the subset of points below v . The corresponding subset of the boundary $\partial U_v \subset \partial^*\mathbb{T}$ is called the *shadow* of v . See Figure 6 for an illustration of these definitions.

Remark 8.4. Note that condition (T3) implies that the subset $\text{Br}(\mathbb{T})$ is at most countable, and for every $v \in \text{Br}(\mathbb{T})$ the set of directions E_v is also countable. Indeed if $\Sigma \subset \mathbb{T}$ is a countable subset as in (T3) then every $v \in \text{Br}(\mathbb{T})$ has a lower bound in Σ . It then follows that every direction in E_v^- contains a point in Σ , and hence E_v is countable. In particular every branching point v can be written as $v = z_1 \wedge z_2$ for some $z_1, z_2 \in \Sigma$, so that the subset $\text{Br}(\mathbb{T})$ is at most countable.

Remark 8.5. There is little loss of generality in assuming that \mathbb{T} has no leaves. Indeed by removing all leaves we obtain a directed tree \mathbb{T}' and the removed leaves now correspond to points in $\partial^*\mathbb{T}'$. For this reason we shall often restrict to directed trees without leaves (see also Remark 8.15).

8.1.3. Horogradings.

Definition 8.6. Let $(\mathbb{T}, \triangleleft)$ be a directed tree. An increasing (respectively, decreasing) *horograding* of \mathbb{T} is an increasing (respectively, decreasing) map $\pi: \mathbb{T} \rightarrow \mathbb{R}$ such that for every $u, v \in \mathbb{T}$ with $u \triangleleft v$ the restriction of π to the arc $[u, v] \subset \mathbb{T}$ is an order-preserving bijection onto the interval $[\pi(u), \pi(v)]$ (respectively $[\pi(v), \pi(u)]$).

Remark 8.7. Assume that π is an increasing horograding. If we let $\omega \in \partial\mathbb{T}$ denote the focus of \mathbb{T} , and set $b = \sup \pi(\mathbb{T}) \in \mathbb{R} \cup \{+\infty\}$, then the definition implies that for every $v \in \mathbb{T}$ the restriction of π to $[v, \omega]$ is an increasing bijection onto $[\pi(v), b)$. This is in fact an equivalent formulation of the definition. There is an analogous characterization for decreasing horogradings.

Definition 8.8. Assume that $\pi: \mathbb{T} \rightarrow \mathbb{R}$ is an increasing horograding and write $X = \pi(\mathbb{T})$, with $a = \inf X$ and $b = \sup X$. Notice that π naturally extends to the end-completion $(\overline{\mathbb{T}}, \triangleleft)$, taking values in $\overline{X} = [a, b]$. We denote by $\overline{\pi}$ this extension and define the π -complete boundary as the subset

$$\partial_{\overline{\pi}}^*\mathbb{T} := \{\xi \in \partial^*\mathbb{T} : \overline{\pi}(\xi) = a\}.$$

The analogous definition can be given in the case of decreasing horograding.

The following proposition is yet another consequence of condition (T3).

Proposition 8.9. *Every directed tree admits a horograding.*

Proof. Assume first that \mathbb{T} has no leaves (that is, the poset $(\mathbb{T}, \triangleleft)$ has no minimal elements), and let $\omega \in \partial\mathbb{T}$ be its focus. Let $\Sigma = \{v_n : n \in \mathbb{N}\} \subset \mathbb{T}$ be a countable collection as in (T3), and note that every element of \mathbb{T} has a lower bound in Σ , so that $\mathbb{T} = \bigcup_{n \in \mathbb{N}} [v_n, \omega[$. We define π inductively on each ray $[v_n, \omega[$. We begin by choosing arbitrarily a bijection $\pi: [v_0, \omega[\rightarrow [0, b)$ for some $b \in \mathbb{R} \cup \{+\infty\}$. Assume that π has already been defined on $\bigcup_{j=0}^{n-1} [v_j, \omega[$. If $v_j \triangleleft v_n$ for some $j < n$, then $[v_n, \omega[\subset [v_j, \omega[$ so that π has already been defined on $[v_n, \omega[$. Otherwise set $w_n = \min_{\triangleleft} \{v_n \wedge v_j : j < n\}$. Note that the points $v_n \wedge v_j$ are all contained in the totally ordered subset $[v_n, \omega[$ and thus the minimum is well defined; moreover the point w_n satisfies $v_j \triangleleft w_n$ for some $j < n$ and thus π has already been defined on the ray $[w_n, \omega[$. We define π on $[v_n, w_n[$ by choosing arbitrarily an order-preserving bijection of $[v_n, w_n[$ onto an interval of the form $[x_n, \pi(w_n))$ for some $x_n < \pi(w_n)$. Proceeding in this way for every n we obtain the desired horograding $\pi: \mathbb{T} \rightarrow \mathbb{R}$.

Now if \mathbb{T} has leaves, let \mathbb{T}' be the directed tree obtained by removing all leaves. By the previous construction we can find a horograding $\pi: \mathbb{T}' \rightarrow \mathbb{R}$, and we can do the construction so that its image is bounded below. Then we can extend π to \mathbb{T} by setting $\pi(v) = \inf \pi(u)$ where the infimum is taken over all $u \in \mathbb{T}'$ such that $v \triangleleft u$. \square

8.1.4. *Metric and topologies on directed trees.* A *metric \mathbb{R} -tree* is a metric space (\mathbb{T}, d) such that every pair of points $u, v \in \mathbb{T}$ can be connected by a unique injective continuous arc $\gamma: [0, 1] \rightarrow \mathbb{T}$, and this arc can be chosen to be geodesic (i.e. an isometric embedding). This notion is well-studied in geometric group theory (though not exclusively); see for instance the survey of Bestvina [7]. Let us clarify the connection between \mathbb{R} -trees and directed trees in the sense of Definition 8.1.

A directed tree as in Definition 8.1 can always be endowed with a metric that makes it an \mathbb{R} -tree. Indeed assume that $(\mathbb{T}, \triangleleft)$ is a directed tree, and choose a horograding $\pi: \mathbb{T} \rightarrow X$ (see Proposition 8.9). We can then define a distance on \mathbb{T} by the formula

$$(8.1) \quad d_\pi(u, v) = |\pi(u \wedge v) - \pi(u)| + |\pi(u \wedge v) - \pi(v)|.$$

This distance turns \mathbb{T} into an \mathbb{R} -tree; see [38]. We call a distance of this form a *compatible \mathbb{R} -tree metric* on $(\mathbb{T}, \triangleleft)$, associated with the horograding π .

Conversely assume that (\mathbb{T}, d) is a separable metric \mathbb{R} -tree with no leaves. Let $(\hat{\mathbb{T}}, d)$ be the metric completion of \mathbb{T} , and set $\partial_f \mathbb{T} := \hat{\mathbb{T}} \setminus \mathbb{T}$ (the set $\partial_f \mathbb{T}$ consists of leaves of $\hat{\mathbb{T}}$). Further let $\partial_\infty \mathbb{T}$ be the *Gromov* (or *visual*) *boundary* of $\hat{\mathbb{T}}$, namely $\partial_\infty \mathbb{T}$ is the set of equivalence classes of geodesic rays in $\hat{\mathbb{T}}$, where a geodesic ray is a subset $\rho \subset \mathbb{T}$ isometric to $[0, +\infty)$, and two rays ρ_1, ρ_2 are equivalent if $\rho_1 \cap \rho_2$ is a geodesic ray. We define the end-boundary of \mathbb{T} as the set $\partial \mathbb{T} = \partial_f \mathbb{T} \sqcup \partial_\infty \mathbb{T}$. The choice of a point $\omega \in \partial \mathbb{T}$ defines naturally a partial order \triangleleft on \mathbb{T} , by the condition that $v \triangleleft u$ when $u \in [v, \omega[$, where $[v, \omega[$ is the ray from v to ω . Moreover the choice of a point $z_0 \in \mathbb{T}$ allows to define a *horofunction* $\pi_{\omega, z_0}: \mathbb{T} \rightarrow \mathbb{R}$ centered at ω (normalized to vanish on z_0), by the formula

$$\pi_{\omega, z_0}(v) = d(z_0, v \wedge z_0) - d(v, v \wedge z_0).$$

This function is a horograding in the sense of Definition 8.6, and commonly the subsets $\{v \in \mathbb{T}: \pi_{\omega, z_0}(v) = n\}$ are called *horospheres*, see for instance [17, Chapter II.8].

One can readily check that these two constructions are inverse to each other. Note however that the partition of the boundary $\partial \mathbb{T} = \partial_f \mathbb{T} \sqcup \partial_\infty \mathbb{T}$ appearing above depends on the choice of a compatible \mathbb{R} -tree metric and cannot be reconstructed from the poset structure of $(\mathbb{T}, \triangleleft)$. Notice also that, in the case that $\partial_\infty \mathbb{T}$ is non-empty, it holds that $\partial_\infty \mathbb{T} \cap \partial^* \mathbb{T} = \partial_\pi^* \mathbb{T}$. That is, up to possibly the focus, the Gromov boundary of \mathbb{T} and the π -complete boundary coincide.

Remark 8.10. We note that every directed tree $(\mathbb{T}, \triangleleft)$ the set \mathbb{T} can also be endowed with an intrinsic topology (which does not depend on the choice of a horograding), called the *weak topology* in [38] (or *observers' topology* in Coulbois, Hilion, and Lustig [28]). A subbasis of open subsets for the weak topology is given by the set of directions $e_v(w)$ for $v \in \mathbb{T}$ and $w \in \mathbb{T}$. This topology turns the end-completion $\bar{\mathbb{T}} = \mathbb{T} \sqcup \partial \mathbb{T}$ into a compact space [28].

8.1.5. *Case of simplicial trees.* A simple special case of directed trees are directed *simplicial trees*. A simplicial tree is a connected graph \mathbb{T} without cycles. Its boundary $\partial \mathbb{T}$ is defined as its set of ends (or equivalently as its Gromov boundary with respect to the simplicial metric). Assume that \mathbb{T} is a simplicial tree of countable degree (that is, the degree at every point is at most countable) and infinite diameter. Then the choice of a point $\omega \in \partial \mathbb{T}$ turns it into a directed tree, with respect to the partial order $u \triangleleft v$ if v lies on the ray from u to ω in \mathbb{T} . We will say that a directed tree $(\mathbb{T}, \triangleleft)$ is *simplicial* if it arises in this way from a simplicial tree. Note that in this case the set of directions E_v^- below a vertex v can be identified with the set of edges at v which are below v .

Remark 8.11. Note that simplicial trees, admit natural horogradings mapping vertices to integers. Moreover, for such horogradings, the boundaries $\partial^* \mathbb{T}$ and $\partial_\pi^* \mathbb{T}$ coincide.

8.1.6. Planar directed trees.

Definition 8.12. Let $(\mathbb{T}, \triangleleft)$ be a directed tree without leaves, and denote by $\omega \in \partial\mathbb{T}$ its focus. A *planar order* on $(\mathbb{T}, \triangleleft)$ is a collection $\prec = \{\prec^v : v \in \text{Br}(\mathbb{T})\}$ of total orders defined on the set E_v^- for each branching point $v \in \text{Br}(\mathbb{T})$. The triple $(\mathbb{T}, \triangleleft, \prec)$ will be called a *planar directed tree*.

A planar order induces a total order on $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$, that we will still denote by \prec , as follows. Given two distinct ends $\xi_1, \xi_2 \in \partial^*\mathbb{T}$, write $v = \xi_1 \wedge \xi_2$, and note that this gives $e_v(\xi_1) \neq e_v(\xi_2)$ in E_v^- . We set $\xi_1 \prec \xi_2$ if $e_v(\xi_1) <^v e_v(\xi_2)$. It is straightforward to check that this defines an order on $\partial^*\mathbb{T}$. Moreover, we say that the planar order \prec is *proper* if for every $v \in \mathbb{T}$ the shadow ∂U_v is bounded above and below in $(\partial^*\mathbb{T}, \prec)$.

Remark 8.13. Note that for every planar order \prec on $(\mathbb{T}, \triangleleft)$ and $v \in \mathbb{T}$, the shadow ∂U_v is a convex subset of $(\partial^*\mathbb{T}, \prec)$.

8.1.7. *Automorphisms and focal actions.* Given a directed tree $(\mathbb{T}, \triangleleft)$ we denote by $\text{Aut}(\mathbb{T}, \triangleleft)$ the group of its order-preserving bijections. We will be particularly interested in the following class of subgroups of automorphisms.

Definition 8.14. A subgroup $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$ is *focal* if for every $u, v \in \mathbb{T}$ there exists an element $g \in G$ such that $v \triangleleft g(u)$.

Remark 8.15. Note that if a directed tree $(\mathbb{T}, \triangleleft)$ admits a focal action, then it has no leaves. As an equivalent definition, a subgroup $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$ is focal if for every $v \in \mathbb{T}$ there exists a sequence $(g_n) \subset G$ such that $g_n(v)$ tends to ω along the ray $[v, \omega[$.

Given a planar order \prec on $(\mathbb{T}, \triangleleft)$, we denote by $\text{Aut}(\mathbb{T}, \triangleleft, \prec)$ the subgroup of $\text{Aut}(\mathbb{T}, \triangleleft)$ which preserves the planar order \prec , meaning that for every element $g \in G$ the corresponding bijections between E_v^- and $E_{g(v)}^-$ induce isomorphisms between \prec_v and $\prec_{g(v)}$. Note that the induced action of $\text{Aut}(\mathbb{T}, \triangleleft, \prec)$ on $\partial^*\mathbb{T}$ preserves the total order \prec .

Remark 8.16. A subgroup $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$ of automorphisms of a directed tree admits an invariant planar order if and only if for every $v \in \text{Br}(\mathbb{T})$ there is an order on E_v^- which is invariant under $\text{Stab}_G(v)$. Indeed it is enough to choose such an order for v in a system of representatives of G -orbits $B \subseteq \text{Br}(\mathbb{T})$, and then extend this collection to all $v \in \text{Br}(\mathbb{T})$ uniquely in a G -invariant way.

The terminology extends to group actions $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$. For instance, we say that the action Φ is focal if the image $\Phi(G)$ is a focal subgroup. In the particular case when $(\mathbb{T}, \triangleleft)$ is simplicial (see §8.1.5), we say that an action $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ is *simplicial* if it arises from a simplicial action of G on \mathbb{T} preserving the end $\omega \in \partial\mathbb{T}$.

Assume now that G is a countable group and $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ is an action on a planar directed tree. For every end $\xi \in \partial^*\mathbb{T}$ let us denote by $\mathcal{O}_\xi \subset \partial^*\mathbb{T}$ its G -orbit, which is thus an at most countable totally ordered set on which G acts by order-preserving bijections. We can therefore consider the dynamical realization $\varphi_\xi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ of the action of G on (\mathcal{O}_ξ, \prec) (see §2.3.3 for details). We will see in the next subsection (Proposition 8.18) the dynamical realization φ_ξ does not depend on ξ , up to positive conjugacy. For this, we will need the following result.

Lemma 8.17. *Let $(\mathbb{T}, \triangleleft, \prec)$ be a planar directed tree with $|\partial^*\mathbb{T}| \geq 2$, and $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ be a focal action of a group G . Then the following hold.*

- (1) The planar order \prec is proper.
- (2) For every $\xi \in \partial^*\mathbb{T}$, the orbit \mathcal{O}_ξ is cofinal (that is, unbounded in both directions) and densely ordered (that is, for every $\eta_1, \eta_2 \in \mathcal{O}_\xi$ with $\eta_1 \prec \eta_2$, there exists $\xi_0 \in \mathcal{O}_\xi$ with $\eta_1 \prec \xi_0 \prec \eta_2$).

Proof. We first prove (1). Since $|\partial^*\mathbb{T}| \geq 2$, we can find two points $u, v \in \mathbb{T}$ such that neither is below the other, so that the shadows ∂U_u and ∂U_v are disjoint and \prec -convex (Remark 8.13). Therefore at least one of them is bounded below and at least one of them is bounded above. As the action is focal, for every $z \in \mathbb{T}$, there exist elements $g, h \in G$ such that $g.\partial U_u = \partial U_{g.u} \supset \partial U_z$ and $h.\partial U_v = \partial U_{h.v} \supset \partial U_z$. It follows that the shadow ∂U_z is bounded above and below, so that \prec is proper.

We next prove (2). First of all, observe that by focality, we have $\mathcal{O}_\xi \cap \partial U_v \neq \emptyset$ for every $v \in \mathbb{T}$. Indeed it is enough to choose u above ξ and $g \in G$ such that $u \triangleleft g.v$, so that $\xi \in \partial U_{g.v}$, or equivalently $g^{-1}.\xi \in \partial U_v$. This immediately gives that the orbit \mathcal{O}_ξ is cofinal in $(\partial^*\mathbb{T}, \prec)$. Let us show that it is densely ordered. Assume by contradiction that there is a pair $\xi_1 \prec \xi_2$ with no elements of \mathcal{O}_ξ between them, and let $g \in G$ be such that $\xi_2 = g.\xi_1$. Since the action is order-preserving, applying g^{-1} we deduce that the point $\xi_0 = g^{-1}.\xi_1$ satisfies $\xi_0 \prec \xi_1 \prec \xi_2$, and there is no $\eta \in \mathcal{O}_\xi$ satisfying $\xi_0 \prec \eta \prec \xi_1$ nor $\xi_1 \prec \eta \prec \xi_2$. Now choose $v \in \mathbb{T}$ such that $\xi_1 \in \partial U_v$ and $\xi_0, \xi_2 \notin \partial U_v$. Since ∂U_v is a \prec -convex subset, we deduce that $\partial U_v \cap \mathcal{O}_\xi = \{\xi_1\}$. But clearly we can find $u \in \mathbb{T}$ such that $|\partial U_u \cap \mathcal{O}_\xi| \geq 2$ (just take $u = \xi_1 \wedge \xi_2$). This gives a contradiction since, by focality, there exists $h \in G$ so that $h.\partial U_u \subset \partial U_v$, and therefore $\{h.\xi_1, h.\xi_2\} \subset \partial U_v$. See Figure 7. \square

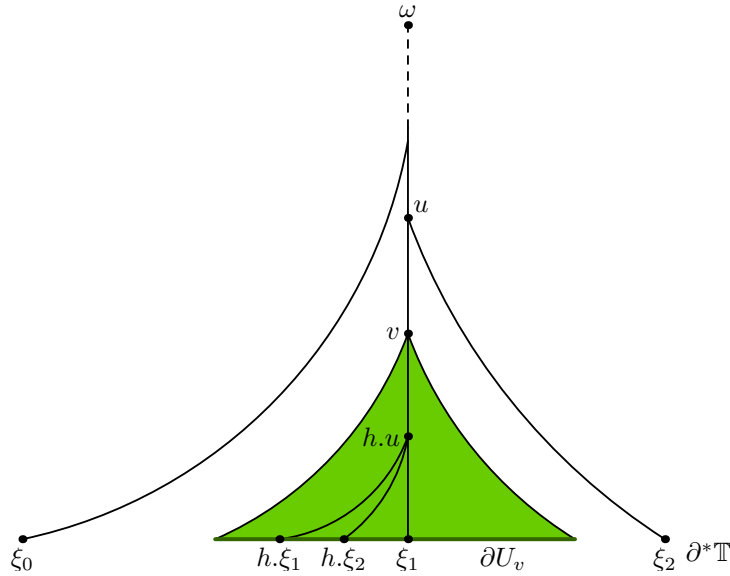


FIGURE 7. Proof of (2) in Lemma 8.17.

A more detailed description of the dynamics of a focal action will be given in §8.3.

8.2. Planar directed trees and \mathbb{R} -focal actions. The connection between focal and \mathbb{R} -focal actions is clarified by Propositions 8.18 and 8.19 below, which allow to make a transition from one to the other.

Proposition 8.18 (From planar directed trees to \mathbb{R} -focal actions). *Let $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ be a focal action of a countable group G on a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ with $|\partial^*\mathbb{T}| \geq 2$. Then, for $\xi \in \partial^*\mathbb{T}$, the dynamical realization $\varphi_\xi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a minimal \mathbb{R} -focal action, which does not depend on the choice of ξ up to positive conjugacy.*

Proof. Consider points $\zeta_1 \prec \xi_1 \prec \xi_2 \prec \zeta_2$ of \mathcal{O}_ξ . By Lemma 8.17 we can choose $\eta \in \mathcal{O}_\xi$ such that $\xi_1 \prec \eta \prec \xi_2$. Let $v, w \in \mathbb{T}$ be such that ∂U_v contains $\{\zeta_1, \xi_1, \eta, \zeta_2, \xi_2\}$ and ∂U_w separates η from $\{\zeta_1, \xi_1, \zeta_2, \xi_2\}$ (one can take for instance $v = \zeta_1 \wedge \zeta_2$ and $w \in]\eta, \xi_1 \wedge \xi_2[$). By focality, there exists $g \in G$ such that $g \cdot \partial U_v \subset \partial U_w$. Since ∂U_w is convex, this implies that $\xi_1 \prec g(\zeta_1) \prec g(\zeta_2) \prec \xi_2$. It follows that φ_ξ is minimal from Lemma 2.42. To prove that the action φ_ξ is \mathbb{R} -focal, we observe that the collection $\mathcal{S} = \{\partial U_v \cap \mathcal{O}_\xi : v \in \mathbb{T}\}$ is a CF-cover of (\mathcal{O}_ξ, \prec) which is $\Psi(G)$ -invariant. As in Example 7.10, we get a CF-cover \mathcal{S}_ξ of \mathbb{R} which is $\varphi_\xi(G)$ -invariant, by taking the interior of the closures of elements in \mathcal{S} under a good embedding $\iota_\xi : \mathcal{O}_\xi \rightarrow \mathbb{R}$. Now, given $\xi, \xi' \in \partial^*\mathbb{T}$, the map $f : \mathcal{S}_\xi \rightarrow \mathcal{S}_{\xi'}$ given by

$$\text{Int } \overline{\iota_\xi(\partial U_v \cap \mathcal{O}_\xi)} \mapsto \text{Int } \overline{\iota_{\xi'}(\partial U_v \cap \mathcal{O}_{\xi'})}$$

satisfies all conditions in Lemma 7.8, whence we deduce that φ_ξ and $\varphi_{\xi'}$ are positively conjugate. \square

After Proposition 8.18, there is no much ambiguity when saying that the action $\varphi_\xi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is the *dynamical realization* of the focal action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$. Given an action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$, we will say that φ is *represented* by a focal action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ if φ is positively conjugate to the dynamical realization of Φ . Conversely to Proposition 8.18, we have the following.

Proposition 8.19 (From \mathbb{R} -focal actions to planar directed trees). *Let G be a countable group and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action. Then there exists a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ and a focal action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ representing φ .*

Proof. Let \mathcal{S} be an invariant CF-cover for the \mathbb{R} -focal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$. We want to define a planar directed tree starting from the poset (\mathcal{S}, \subset) . However, for general \mathcal{S} , none of the properties (T1)–(T3) for a directed tree is satisfied, and we will need to consider a completion of (\mathcal{S}, \subset) .

Let us start with some preliminary considerations on the structure of (\mathcal{S}, \subset) . Firstly, from the cross-free property we have that for every $I \in \mathcal{S}$, the subset $\{J \in \mathcal{S} : J \supseteq I\}$ is totally ordered, while from condition (C2), we see that the collection \mathcal{S} cannot have a maximal element. This is in the direction of condition (T1), although we cannot guarantee that $\{J \in \mathcal{S} : J \supseteq I\}$ is order-isomorphic to $[0, +\infty)$.

To ensure a property analogue to (T2), we will assume that \mathcal{S} is closed, in the sense that the set of pairs of points defined by intervals in \mathcal{S} is a closed subset of $\mathbb{R}^2 \setminus \text{diag}$. There is no loss of generality in assuming this, as the closure of \mathcal{S} in the above sense is still an invariant CF-cover. In particular, this implies that any $I, J \in \mathcal{S}$ have a smallest common upper bound in \mathcal{S} .

We next observe that minimality of the action ensures that (\mathcal{S}, \subset) basically has no leaves (and this is a necessary condition to define a planar order). Namely, we have that every maximal totally ordered subset $\mathcal{C} \subset \mathcal{S}$ must contain intervals of arbitrarily small length. Indeed, if the length of every interval in \mathcal{C} is uniformly lower-bounded, the intersection $L = \bigcap_{I \in \mathcal{C}} I$ is an interval of non-empty interior. However, as we are assuming that φ is minimal, then it is proximal (Proposition 7.7), so any interval of \mathcal{S} can be mapped by some element $g \in G$ into

L , and this contradicts maximality of \mathcal{C} , because of the invariance of \mathcal{S} . In particular, every maximal totally ordered set \mathcal{C} defines a point $x_{\mathcal{C}} \in \mathbb{R}$, as the unique point in the intersection:

$$(8.2) \quad \{x_{\mathcal{C}}\} := \bigcap_{I \in \mathcal{C}} \bar{I}.$$

Note that an element $I \in \mathcal{S}$ belongs to \mathcal{C} if and only if $x_{\mathcal{C}} \in I$. In particular if we choose a countable collection (\mathcal{C}_n) of maximal totally ordered subsets such that the sequence $(x_{\mathcal{C}_n})$ is dense in \mathbb{R} then $\mathcal{S} = \bigcup_n \mathcal{C}_n$ (by minimality of the G -action on \mathbb{R} , we may for instance choose the G -orbit of some given \mathcal{C}). This observation will guarantee condition (T3).

Let us now proceed to introduce a good completion. Consider the map $\pi: \mathcal{S} \rightarrow (0, +\infty)$, that associates to an interval I its length $|I|$. Then π is clearly increasing, and maps every maximal totally ordered subset bijectively onto a closed subset of $(0, +\infty)$ (the fact that the image is closed follows from the fact that we are assuming the collection \mathcal{S} closed). Call a pair of distinct intervals $(I, J) \in \mathcal{S}^2 \setminus \text{diag}$ a *gap* if $I \subset J$ and there exists no $L \in \mathcal{S} \setminus \{I, J\}$ such that $I \subset L \subset J$. Let $\mathcal{G} \subset \mathcal{S}^2 \setminus \text{diag}$ be the set of gaps. For each $(I, J) \in \mathcal{G}$ let $L_{(I,J)}$ be a homeomorphic copy of the interval $(\pi(I), \pi(J))$. As a set, we define a directed tree \mathbb{T} as the disjoint union

$$\mathbb{T} = \mathcal{S} \sqcup \left(\bigsqcup_{(I,J) \in \mathcal{G}} L_{(I,J)} \right).$$

The partial order \subset on \mathcal{S} extends in a natural way to a partial order \triangleleft on \mathbb{T} , by declaring each $L_{I,J}$ to be an order-isomorphic copy of $(\pi(I), \pi(J))$ lying between I and J , and extend the map $\pi: \mathcal{S} \rightarrow (0, +\infty)$ to \mathbb{T} by in the obvious way. After our preliminary discussion, we are sure that the poset $(\mathbb{T}, \triangleleft)$ is a directed tree.

The action of G on \mathcal{S} extends to \mathbb{T} by letting each $g \in G$ map $L_{I,J}$ affinely onto $L_{g(I),g(J)}$, and this action is order-preserving. The fact that it is focal follows from the assumption that the action of G on \mathbb{R} is minimal and thus proximal (Proposition 7.7).

We next introduce a planar order \prec on $(\mathbb{T}, \triangleleft)$ which is preserved by the action of G . Note first that by construction all $v \in \mathbb{T} \setminus \mathcal{S}$ have degree 2. Thus every branching point $v \in \mathbb{T}$ is of the form $v = J \in \mathcal{S}$, and the set E_v^- of components below v is in one-to-one correspondence with gaps of the form (I, J) . Let $e_1, e_2 \in E_v^-$ be two distinct components below $v = J$ and let $I_1, I_2 \in \mathcal{S}$ be the intervals defining the corresponding gaps with J . Note that I_1 and I_2 are necessarily disjoint, so we can declare $e_1 \prec^v e_2$ if $\sup I_1 \leq \inf I_2$. The family $\{\prec^v: v \in \text{Br}(\mathbb{T})\}$ defines a G -invariant planar order \prec on $(\mathbb{T}, \triangleleft)$.

Finally note that every maximal totally ordered subset $\mathcal{C} \subset \mathbb{T}$ defines a point $\xi_{\mathcal{C}} \in \partial^* \mathbb{T}$, represented by any decreasing proper ray taking values in \mathcal{C} . By construction, the G -orbit of $\xi_{\mathcal{C}}$ can be identified with the orbit of the point $x_{\mathcal{C}} \in \mathbb{R}$ defined by (8.2), in an order-preserving way. Thus φ is conjugate to the dynamical realization of the action on $(\mathcal{O}_{\xi_{\mathcal{C}}}, \prec)$. This concludes the proof. \square

Remark 8.20. The proof of Proposition 8.19 provides a simple canonical way to associate a planar directed tree with every invariant CF-cover \mathcal{S} of \mathbb{R} . However we point out that the planar directed tree satisfying the conclusion of the proposition is not unique, and this general construction yields a tree which is often “too large”. In practice it is useful to find smaller tree encoding the action having some additional structure (such as a G -equivariant horograding, or a G -invariant compatible \mathbb{R} -tree metric, etc.) In most cases of our interest, a more direct construction having such additional structure will be available.

Let us just observe here that the action of G on the planar directed tree \mathbb{T} constructed in the proof is continuous with respect to a compatible \mathbb{R} -tree metric on \mathbb{T} , which can be chosen

to be complete. Indeed, the map $\pi: \mathbb{T} \rightarrow (0, +\infty)$ appearing in the proof is a horograding of $(\mathbb{T}, \triangleleft)$. In particular it defines a compatible \mathbb{R} -tree metric d_π on \mathbb{T} by the expression (8.1), and one can check the action of G on \mathbb{T} is continuous with respect to the topology defined by this metric. Moreover if we reparametrize π by post-composing it by an orientation-preserving homeomorphism $H: (0, +\infty) \rightarrow \mathbb{R}$, then \mathbb{R} -tree metric associated with the new horograding is complete and induces the same topology (so that the G -action remains continuous).

Example 8.21 (Planar directed trees for Plante actions). Let us illustrate how to get a focal action on a planar directed tree in the case of a Plante action of a wreath product. As in Example 7.10, we consider two countable groups G and H and fix left-invariant orders $\triangleleft_G \in \text{LO}(G)$ and $\triangleleft_H \in \text{LO}(H)$. We also take a good embedding $\iota: G \rightarrow \mathbb{R}$. For every $x \in \mathbb{R}$, consider the collection of functions

$$\mathbb{S}_x = \{s: (x, +\infty) \rightarrow H: s(y) = 1_H \text{ for all but finitely many } y, \text{ in } \iota(G)\}.$$

As a set, we define \mathbb{T} as the disjoint union $\bigsqcup_{x \in \mathbb{R}} \mathbb{S}_x$. We declare that $s \trianglelefteq t$ if $s \in \mathbb{S}_x, t \in \mathbb{S}_y$, with $x \leq y$ and $s \upharpoonright_{(y, +\infty)} = t$. With this definition it is immediate to see that for every $s \in \mathbb{S}_x$, the subset $\{u: s \trianglelefteq u\} = \{s \upharpoonright_{(y, +\infty)}: y \geq x\}$ is order-isomorphic to $[x, +\infty)$, giving (T1). To verify (T2), given $s \in \mathbb{S}_x$ and $t \in \mathbb{S}_y$, the restriction $s \upharpoonright_{(-\infty, x_*)}$ with $x_* = \max\{z: s(z) \neq t(z)\}$ gives the desired smallest common upper bound. Finally, we have that the collection $\Sigma = \bigsqcup_{x \in \iota(G)} \mathbb{S}_x$ is countable, as G and H are countable, and this gives (T3). Note that Σ coincides with the collection of branching points $\text{Br}(\mathbb{T})$. For $x \in \iota(G)$ and $s \in \mathbb{S}_x \subset \text{Br}(\mathbb{T})$, we note that E_s^- is in one-to-one correspondence with H , the identification being given by the value of s at x ; this allows to put on E_s^- the total order coming from \triangleleft_H . This defines a planar order \triangleleft on $(\mathbb{T}, \triangleleft)$. A horograding $\pi: \mathbb{T} \rightarrow \mathbb{R}$ is simply given by $\pi(s) = x$, where $x \in \mathbb{R}$ is such that $s \in \mathbb{S}_x$.

In order to describe the boundaries $\partial^* \mathbb{T}$ and $\partial_\pi^* \mathbb{T}$, set

$$\mathbb{S} = \{s: (x, +\infty) \rightarrow H: s \upharpoonright_{(y, +\infty)} \in \mathbb{S}_y \text{ for every } y > x\}$$

and notice that the partial order \triangleleft naturally extends to \mathbb{S} . Denote by $\mathbb{S}^* \subseteq \mathbb{S}$ the subset of the functions which are minimal for the relation \triangleleft . That is, an element $s \in \mathbb{S}$ is in \mathbb{S}^* if and only if, either s is defined over \mathbb{R} or the support of s is infinite. There is a correspondence between \mathbb{S}^* and $\partial^* \mathbb{T}$ so that, the function $s: (x, +\infty) \rightarrow H$ corresponds to the equivalence class of the ray $y \mapsto s \upharpoonright_{(y, +\infty)}$. Under this correspondence, the π -complete boundary $\partial_\pi^* \mathbb{T}$ is identified with those functions in \mathbb{S}^* whose domain is \mathbb{R} . In particular, we can see $\bigoplus_G H$ as a subset of $\partial_\pi^* \mathbb{T}$ (a function $s: G \rightarrow H$ can be seen as a function $s: \mathbb{R} \rightarrow H$ with support in $\iota(G)$). Notice that the order induced on $\bigoplus_G H$ by \triangleleft under this identification coincides with that introduced in Example 7.10.

The wreath product $H \wr G$ acts on $(\mathbb{T}, \triangleleft, \triangleleft)$. Indeed, every $g \in G$ sends \mathbb{S}_x to $\mathbb{S}_{g.x}$ by precomposition with g^{-1} , where the action on \mathbb{R} is the dynamical realization of \triangleleft_G corresponding to the good embedding ι . If $s \in \bigoplus_G H$, then we can see it as a function defined on \mathbb{R} as before, and thus s acts on \mathbb{T} by pointwise multiplication of functions. It is direct to check that this action preserves the poset structure \triangleleft and the planar order \triangleleft .

Note that this action naturally extends to \mathbb{S}^* , giving the action of $H \wr G$ on $\partial^* \mathbb{T}$ through the correspondence between \mathbb{S}^* and $\partial^* \mathbb{T}$. Moreover, the π -complete boundary $\partial_\pi^* \mathbb{T}$ and (the copy of) $\bigoplus_G H$ are invariant subsets for this action. Indeed, the restriction to $\bigoplus_G H$ coincides with the action Ψ (discussed in Example 7.10). In other terms, we constructed an order-preserving and equivariant “embedding” of the Plante product $\Psi: H \wr G \rightarrow \text{Aut}(\bigoplus_G H, \triangleleft)$ defined in Example 7.10, into the restriction to the boundary of an action on a planar directed tree.

Finally, note that the horograding π satisfies $\pi(g.s) = g.\pi(s)$ for every $g \in G$ and $s \in \mathbb{T}$, while it is constant on $\bigoplus_G H$ -orbits. In the terminology that will be introduced later in §8.4,

the action of $H \wr G$ on $(\mathbb{T}, \triangleleft)$ is increasingly horograded by the action $j : H \wr G \rightarrow \text{Homeo}_0(\mathbb{R})$ defined as $j = j_0 \circ pr_G$, where j_0 is the dynamical realization of $\langle G$ and $pr_G : H \wr G \rightarrow G$ is the projection to the factor G . See Figure 8.

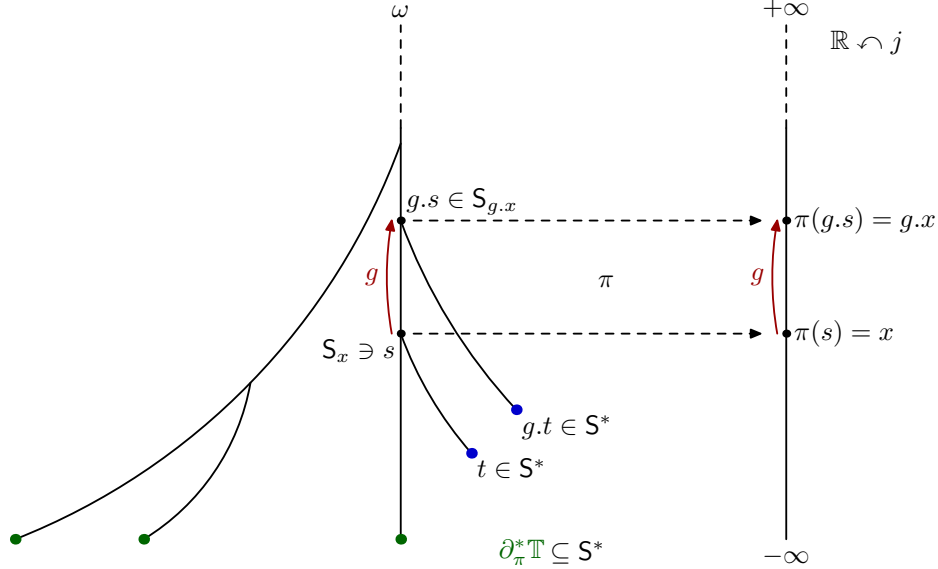


FIGURE 8. The directed tree for the Plante action (Example 8.21).

8.3. Focal germ representation and dynamical classification of elements. In this section we will show that \mathbb{R} -focal actions admit a dynamical classification of elements which closely resembles that of isometries of trees into hyperbolic and elliptic elements. Moreover the type of every element can be determined by an invariant called the *focal germ representation*. Let us begin by defining this invariant.

Definition 8.22. Let $(\mathbb{T}, \triangleleft)$ be a directed tree with focus $\omega \in \partial\mathbb{T}$. Choose also an increasing horograding $\pi : \mathbb{T} \rightarrow \mathbb{R}$, and set $b = \sup \pi(\mathbb{T})$. Then for every $v \in \mathbb{T}$ the ray $[v, \omega[$ can be identified with the interval $[\pi(v), b)$. Let $g \in \text{Aut}(\mathbb{T}, \triangleleft)$ be an automorphism. Since g maps $[v, \omega[$ to $[g(v), \omega[$, it induces a homeomorphism between the intervals $[\pi(v), b)$ and $[\pi(g.v), b)$. Since for any other v' the element $v \wedge v'$ belongs to $[v, \omega[$, we have that the left germ at b of the homeomorphism induced by g does not depend on the choice of $v \in \mathbb{T}$. In this way we obtain a homomorphism from $\text{Aut}(\mathbb{T}, \triangleleft)$ to the group of (left) germs at b :

$$\tau : \text{Aut}(\mathbb{T}, \triangleleft) \rightarrow \text{Germ}(b).$$

We call this homomorphism the *focal germ representation* of $\text{Aut}(\mathbb{T}, \triangleleft)$. When $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ is a group action, we can consider the composition of Φ with the focal germ representation. We will still denote by $\tau : G \rightarrow \text{Germ}(b)$ such composition.

Remark 8.23. Note that the focal germ representation $\tau : \text{Aut}(\mathbb{T}, \triangleleft) \rightarrow \text{Germ}(b)$ does not depend on the choice of the horograding π , up to conjugacy of germs.

When $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a minimal \mathbb{R} -focal action, we say that $\tau : G \rightarrow \text{Germ}(b)$ is a focal germ representation *associated to* φ if it is conjugate to the focal germ representation of a focal action $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ which represents φ . As discussed in Remark 8.20, a given \mathbb{R} -focal action can be represented φ by a focal action on a planar directed tree in several ways,

and this may lead to different (i.e. non-conjugate) focal germ representations associated to φ . However, we can introduce the following notion of semi-conjugacy of germs so that the focal germ representation of an \mathbb{R} -focal actions is well defined up to it.

Definition 8.24. For $i \in \{1, 2\}$, let $\tau_i : G \rightarrow \text{Germ}(b_i)$ be two germ representations of a group G . We say that τ_1 and τ_2 are *semi-conjugate* if there exist $\varepsilon_1, \varepsilon_2 > 0$ and a non-decreasing map $h : (b_1 - \varepsilon_1, b_1) \rightarrow (b_2 - \varepsilon_2, b_2)$ with $\lim_{t \rightarrow b_1} h(t) = b_2$ such that for every $g \in G$ one has $h\tau_1(g) = \tau_2(g)h$ at the level of germs.

Lemma 8.25. *Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal actions, and for $i \in \{1, 2\}$, let $\tau_i : G \rightarrow \text{Germ}(b_i)$ be a focal germ representation associated to φ . Then τ_1 and τ_2 are semi-conjugate.*

Proof. Suppose for simplicity that $b_1 = b_2 =: b$. Consider focal actions on planar directed trees $\Phi_1 : G \rightarrow \text{Aut}(\mathbb{T}_1, \triangleleft_1, \prec_1)$ and $\Phi_2 : G \rightarrow \text{Aut}(\mathbb{T}_2, \triangleleft_2, \prec_2)$ representing φ . For $i \in \{1, 2\}$, choose an increasing horograding $\pi_i : \mathbb{T}_i \rightarrow (a_i, b)$ so that the focal germ representation associated to π_i is τ_i . Let us see how to construct the desired semi-conjugacy. For $i \in \{1, 2\}$, fix $\xi_i \in \partial^*\mathbb{T}_i$ and let $\iota_i : \mathcal{O}_{\xi_i} \rightarrow \mathbb{R}$ be an order-preserving embedding such that the corresponding dynamical realization is exactly the action φ . We first introduce a G -equivariant map $H : \mathbb{T}_1 \rightarrow \mathbb{T}_2$, as follows: for $v \in \mathbb{T}_1$, the image $H(v)$ is the smallest $w \in \mathbb{T}_2$ such that $\overline{\iota_2(\partial U_w \cap \mathcal{O}_{\xi_2})}$ contains $\iota_1(\partial U_v \cap \mathcal{O}_{\xi_1})$. The map H is clearly G -equivariant and monotone on maximal totally ordered subsets of $(\mathbb{T}_1, \triangleleft_1)$. When restricted to a ray converging to the focus ω_1 of \mathbb{T}_1 , it gives a semi-conjugacy h between τ_1 and τ_2 . \square

The focal germ representation carries a huge amount of information about the dynamics of \mathbb{R} -focal actions and actions of planar trees. We now proceed to give a dictionary between the two.

We say that a germ $\gamma \in \text{Germ}(b)$ has *fixed points near b* if there is an increasing sequence (z_n) converging to b such that every representative of γ fixes z_n for n large enough. A subgroup Γ of $\text{Germ}(b)$ has *fixed points near b* if there is such a sequence which works simultaneously for every element of Γ . Note that if $\gamma \in \text{Germ}(b)$ has no fixed points near b , then every representative h of γ satisfies that either $h(x) > x$ or $h(x) < x$ for all x close enough to b . In that case we say that γ is *positive* or *negative* respectively. Note that if $\tau : G \rightarrow \text{Germ}(b)$ is a germ representation, then for $g \in G$ the type of $\tau(g)$ according to this classification is invariant under semi-conjugacy of τ .

By extension to the case of the real line, we say that a group Γ of automorphisms of an ordered set (Ω, \prec) is totally bounded if all of its orbits are bounded above and below with respect to the order \prec . Similarly an automorphism g is totally bounded if $\langle g \rangle$ is. We say that g is an expanding (respectively contracting) *pseudohomothety* of (Ω, \prec) if there exists $\alpha \prec \beta$ such that $g^n(\alpha) \rightarrow -\infty$ and $g^n(\beta) \rightarrow +\infty$ in (Ω, \prec) as $n \rightarrow +\infty$ (respectively as $n \rightarrow -\infty$). Finally we say that g is an (expanding or contracting) *homothety* if it has a unique fixed point $\xi \in \Omega$ and the previous conclusion holds for every $\alpha \prec \xi \prec \beta$.

Proposition 8.26 (Dynamical classification of automorphisms of planar directed trees). *Let $(\mathbb{T}, \triangleleft, \prec)$ be a planar directed tree with $|\partial^*\mathbb{T}| \geq 2$. Let $\tau : \text{Aut}(\mathbb{T}, \triangleleft, \prec) \rightarrow \text{Germ}(b)$ be the associated focal germ representation. Then the following hold.*

- (1) *if $\Gamma \subseteq \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ is a finitely generated subgroup such that $\tau(\Gamma)$ has fixed points near b , then the image of Γ in $\text{Aut}(\partial^*\mathbb{T}, \prec)$ is totally bounded. In particular every element $g \in \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ such that $\tau(g)$ has fixed points near b , gives a totally bounded automorphism of $(\partial^*\mathbb{T}, \prec)$.*

- (2) If $g \in \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ is such that $\tau(b)$ has no fixed points near b , then g defines a pseudohomothety of $(\partial^*\mathbb{T}, \prec)$, which is expanding if $\tau(b)$ is positive, and contracting otherwise. If moreover g has no fixed points inside \mathbb{T} , then it is a homothety of $(\partial^*\mathbb{T}, \prec)$.

Proof. Note that by Lemma 8.17, the planar order \prec is proper. Let $\pi: \mathbb{T} \rightarrow \mathbb{R}$ be the increasing horograding used to define the focal germ representation. Assume that (z_n) is an increasing sequence converging to b such that every germ in $\tau(\Gamma)$ eventually fixes z_n . Fix $v_0 \in \mathbb{T}$ such that $\pi(v_0) = z_0$. Then we can choose an increasing sequence $(v_n) \subset [v_0, \omega[$ such that $\pi(v_n) = z_n$. Since Γ is finitely generated, the assumption implies that it fixes v_n for all n large enough. Thus it preserves the sequence of shadows ∂U_{v_n} which form an increasing exhaustion of $\partial^*\mathbb{T}$ by \prec -bounded subsets (because \prec is proper). It follows that every Γ -orbit in $\partial^*\mathbb{T}$ is contained in of these subsets, and thus Γ is totally bounded. This proves (1).

Assume now that g is such that $\tau(g)$ has no fixed points near b . Without loss of generality we can suppose that $\tau(g)$ is positive. This implies that we can find a point $v \in \mathbb{T}$ such that $g^n.v$ approaches ω along the ray $[v, \omega[$ as $n \rightarrow \infty$. Thus the sequence of shadows $\partial U_{g^n(v)} = g_n.\partial U_v$ is an increasing exhaustion of $(\partial^*\mathbb{T}, \prec)$ by \prec -convex subsets, which are bounded because \prec is proper. This implies that g is an expanding pseudohomothety. Assume now that g has no fixed point inside \mathbb{T} . Consider the sequence $\{g^n.v: n \in \mathbb{Z}\}$, where v is as above. This is a totally ordered subset of \mathbb{T} , and thus it has an infimum $\xi \in \mathbb{T} \cup \partial^*\mathbb{T}$, which must be fixed by g and so $\xi \in \partial^*\mathbb{T}$. Since $g^n.v$ must approach ξ along the ray $] \xi, \omega[$ as $n \rightarrow -\infty$, the intersection of the subsets $g^n.\partial U_v$ for $n \in \mathbb{Z}$ is reduced to $\{\xi\}$. This implies that g is a homothety with fixed point ξ . \square

From Proposition 8.26, we get the analogous result for \mathbb{R} -focal actions.

Corollary 8.27 (Dynamical classification of elements for \mathbb{R} -focal actions). *Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action, with associated focal germ representation $\tau: G \rightarrow \text{Germ}(b)$. Then the following hold.*

- (1) If $\Gamma \subseteq G$ is a finitely generated subgroup such that $\tau(\Gamma)$ has fixed points near b , then $\varphi(\Gamma)$ is totally bounded. In particular if $g \in G$ is an element such that $\tau(g)$ has fixed points near b , then $\varphi(g)$ is totally bounded.
- (2) If $g \in G$ is an element such that $\tau(g)$ has no fixed points near b , then $\varphi(g)$ is a pseudohomothety, which is expanding if $\tau(g)$ is positive, and contracting otherwise.

Proof. Assume that φ is represented by a focal action $\Phi: G \rightarrow \text{Homeo}(\mathbb{T}, \triangleleft, \prec)$. Note that if Φ represents φ , we necessarily have $|\partial^*\mathbb{T}| \geq 2$. Take $\xi \in \partial^*\mathbb{T}$, and up to conjugating φ , assume that φ is the dynamical realization of the action induced by Φ on (\mathcal{O}_ξ, \prec) , and let $\iota: \mathcal{O}_\xi \rightarrow \mathbb{R}$ be the associated good embedding. Observe that as φ is minimal, the image $\iota(\mathcal{O}_\xi)$ is dense. Moreover, Lemma 8.17 guarantees that \mathcal{O}_ξ is cofinal in $(\partial^*\mathbb{T}, \prec)$, whence (1) and the first part of (2) readily follow from Proposition 8.26. \square

8.4. Horograded \mathbb{R} -focal actions. After Proposition 8.9, every directed tree $(\mathbb{T}, \triangleleft)$ admits a horograding $\pi: \mathbb{T} \rightarrow \mathbb{R}$ (which, moreover, can be taken to be continuous with respect to some metric, see §8.1.4). When in addition we have a group action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$, then the compatibility between the horograding and the action becomes relevant. For instance, if $\Phi(G)$ preserves the horospheres defined by π (in the sense that $\pi(u) = \pi(v)$ implies that $\pi(\Phi(g)(u)) = \pi(\Phi(g)(v))$ for all $g \in G$), then G naturally acts on the image of π simply by $g.\pi(v) = \pi(\Phi(g)(v))$. This motivates the following definition, which will play a crucial role in the sequel (see for instance Theorem 10.3).

Definition 8.28. Let $X = (a, b)$ be an interval, and let $j: G \rightarrow \text{Homeo}_0(X)$ be an action of a group G . We say that an action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ on a directed tree is increasingly (respectively, decreasingly) *horograded* by j if there exists an increasing (respectively, decreasing) G -equivariant surjective horograding $\pi: \mathbb{T} \rightarrow X$.

In addition, we say that a minimal \mathbb{R} -focal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is horograded by j if φ can be represented by a focal action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ on a planar directed tree which is horograded by j .

The following remark serves as an easy example of the above definition.

Remark 8.29. One can consider the more restrictive assumption that a group action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ on a directed tree preserves a compatible \mathbb{R} -tree metric on \mathbb{T} , in the sense of §8.1.4. From the discussion in §8.1.4, the reader may realize that this occurs if and only if the Φ -action is horograded by an action by translations of G on the line (the horograding is in this case a metric horofunction). Thus the existence of an equivariant horograding to a more general action can be seen as a relaxation of the existence of such an invariant metric.

Remark 8.30. A first class of examples of horograded \mathbb{R} -focal actions are the Plante actions of wreath products $H \wr G$ (Example 8.21) where the horograding was given by the dynamical realization of the order $<_G$ chosen on G . We will also see later with Theorem 10.3, that for a large class of locally moving groups, every exotic action can be horograded by its locally moving action.

Remark 8.31. Notice that every action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ can be extended to an action on the end-completion $(\overline{\mathbb{T}}, \triangleleft)$. If in addition $\pi: \mathbb{T} \rightarrow X$ is a G -equivariant horograding, the π -complete boundary $\partial_\pi^* \overline{\mathbb{T}}$ is always G -invariant.

If a minimal \mathbb{R} -focal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is increasingly horograded by an action $j: G \rightarrow \text{Homeo}_0((a, b))$, then its associated focal germ representation can be identified by the representation $\tau: G \rightarrow \text{Germ}(b)$ induced by the action j . In particular, in this situation we have the following.

Proposition 8.32 (Dynamical classification of elements in the horograded case). *For $X = (a, b)$, let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action which is increasingly horograded by an action $j: G \rightarrow \text{Homeo}_0(X)$. Then the following hold.*

- (i) *If $\Gamma \subseteq G$ is a finitely generated subgroup such that $j(\Gamma)$ has fixed points arbitrarily close to b , then $\varphi(\Gamma)$ is totally bounded.*
- (ii) *If $g \in G$ is an element without fixed points in a neighborhood of b then $\varphi(g)$ is a pseudohomothety which is expanding if $g(x) > x$ for every x sufficiently close to b , and contracting otherwise. Moreover if g has no fixed points in X then $\varphi(g)$ is a homothety.*

Proof. Choose an action on a planar directed tree $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ which represents φ and is increasingly horograded by j , with horograding $\pi: \mathbb{T} \rightarrow X$. Then the focal germ representation associated to Φ is conjugate to the germ representation $\mathcal{G}_b: G \rightarrow \text{Germ}(G, b)$. So (i) and the first sentence in (ii) are direct consequences of Corollary 8.27. Assume now that g has no fixed point in X . Then g has no fixed point in \mathbb{T} (as the image of a fixed point under π would be fixed in X). Thus by Proposition 8.26 $\Phi(g)$ is a homothety of $(\partial^* \mathbb{T}, \prec)$; let $\eta \in \partial^* \mathbb{T}$ be its unique fixed point. Since φ is conjugate to the dynamical realisation of the G -action on the orbit $(\mathcal{O}_\eta, \prec)$ (Proposition 8.18), there is an order-preserving equivariant map $\iota: \mathcal{O}_\eta \rightarrow \mathbb{R}$ with dense image. Since $\phi(g)$ is a homothety on $(\mathcal{O}_\eta, \prec)$, this implies that $\varphi(g)$ is a homothety with unique fixed point $\iota(\eta)$. \square

8.5. \mathbb{R} -focal actions arising from simplicial trees. In view of the correspondence between \mathbb{R} -focal action and actions on planar directed trees, perhaps the simplest type of \mathbb{R} -focal actions are those that can be represented by an action on a *simplicial* planar directed tree (of countable degree) by simplicial automorphisms (see §8.1.5). This is characterised by the following result, which is a partial converse to Proposition 7.11. This result will be invoked in §11.3 when studying \mathbb{R} -focal actions of Thompson's group F .

Proposition 8.33. *Let G be a group not isomorphic to \mathbb{Z}^2 and let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a faithful minimal action. Then the following are equivalent.*

- (i) φ is \mathbb{R} -focal and can be represented by a focal action on a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$, such that \mathbb{T} is a simplicial tree of countable degree and the action of G on \mathbb{T} is by simplicial automorphisms.
- (ii) There exists a non-trivial normal subgroup $N \trianglelefteq G$ such that $G/N \cong \mathbb{Z}$ and $\varphi(N)$ does not act minimally.

Proof. Let us prove that (i) implies (ii). Assume that φ is the dynamical realization of an action on a planar directed simplicial tree $(\mathbb{T}, \triangleleft, \prec)$ by simplicial automorphisms. Let $\mathbb{T}_0 \subset \mathbb{T}$ be the vertex set of \mathbb{T} , as simplicial complex. The focal germ representation associated with this action (see §8.3) provides a homomorphism $\tau: G \rightarrow \mathbb{Z}$, which does not vanish precisely on the elements which act as hyperbolic isometries on \mathbb{T} and it is given by the translation length along their axes. Let N be its kernel. Fix also $v_0 \in \mathbb{T}_0$ and consider the horofunction $\pi: \mathbb{T}_0 \rightarrow \mathbb{Z}$ given by $\pi(v) = d_{\mathbb{T}}(v \wedge v_0, v_0) - d_{\mathbb{T}}(v \wedge v_0, v)$ where $d_{\mathbb{T}}$ is the simplicial distance. This function is G -equivariant (with respect to the action on \mathbb{Z} given by τ), so that the action of N on \mathbb{T} must preserve each horosphere $S_n := \pi^{-1}(n)$, for $n \in \mathbb{Z}$. For $u, v \in S_n$, the shadows $\partial U_u, \partial U_v \subset \partial^* \mathbb{T}$ span two disjoint intervals $I_u, I_v \subset \mathbb{R}$. It follows that for every $n \in \mathbb{Z}$, $\varphi(N)$ preserves the open subset $\bigcup_{v \in S_n} I_v$ and thus it does not act minimally.

Let us now proceed to prove the converse statement. The assumption that $G \neq \mathbb{Z}^2$ rules out that N is a cyclic subgroup of the center of G . Hence by Proposition 7.11 the action is \mathbb{R} -focal. Let $\Lambda \subset \mathbb{R}$ be a proper closed $\varphi(N)$ -invariant subset, and as in the proof of Proposition 7.11 we assume that for every connected component U of $\mathbb{R} \setminus \Lambda$ we have $\text{Fix}_U^{\varphi(N)}(H) = \emptyset$. Fix such a component U . The argument in the proof of Proposition 7.11 shows that the G -orbit of U under G defines a CF-cover \mathcal{S} invariant under $\varphi(G)$.

Choose $f \in G$ which projects to a generator of $G/N \cong \mathbb{Z}$, so that $g = N \times \langle f \rangle$. Since φ is \mathbb{R} -focal the image of f must have a fixed point ξ (Lemma 7.5). Since moreover the action is minimal, we can choose $g \in G$ such that $g.\xi \in U$, so that upon replacing f by gfg^{-1} we can assume that f has a fixed point in U , so that $f.U \cap U \neq \emptyset$. Note also that f fixes no endpoint η of U ; indeed since $\eta \in \Lambda$ this would imply that for every $g \in G$, writing $g = hf^n$ with $h \in N$, we would have $g.\eta = h.\eta \in \Lambda$, contradicting that the G -orbit of η is dense. Thus since the intervals U and $f.U$ intersect non-trivially and cannot cross, we must either have $U \subseteq f.U$ or $f.U \subseteq U$, and upon replacing f by its inverse we assume that $U \subseteq f.U$.

For $n \in \mathbb{Z}$ write $\mathcal{S}_n = \{hf^n.U : h \in N\}$, so that $\mathcal{S} = \bigcup_{n \in \mathbb{Z}} \mathcal{S}_n$. Note that every $V \in \mathcal{S}_n$ is a connected component of $\mathbb{R} \setminus \Lambda_n$, with $\Lambda_n := f^n.\Lambda$, so that in particular every $V_1, V_2 \in \mathcal{S}_n$ are either equal or disjoint. Moreover, every $V \in \mathcal{S}_n$ is contained in a unique $W \in \mathcal{S}_{n+1}$, indeed $V = hf^n.U \subseteq hf^{n+1}.U$. It follows that the set \mathcal{S} is naturally the vertex set of a directed simplicial tree $(\mathbb{T}, \triangleleft)$. The group G acts on \mathbb{T} by simplicial automorphisms. Moreover the G -action preserves the natural planar order $\prec = \{\prec^W : W \in \mathcal{S}\}$ on \mathbb{T} , where two edges (V_1, W) and (V_2, W) with $V_1, V_2 \in W$ are ordered according to the order in which V_1 and V_2 appear in W . By construction φ is conjugate to the dynamical realization of the action on $(\mathbb{T}, \triangleleft, \prec)$. \square

8.6. Constructing \mathbb{R} -focal actions from actions on ultrametric spaces. We conclude this section by explaining a way to construct horograded \mathbb{R} -focal actions based on the a well-known correspondence between trees and *ultrametric spaces* (see for instance [25, 49]), which will be usefull for certain examples. Recall that an ultrametric space is a metric space (\mathcal{Z}, δ) which satisfies the *ultrametric inequality*, that is, for every $\xi_1, \xi_2, \xi_3 \in \mathcal{Z}$ it holds that

$$\delta(\xi_1, \xi_2) \leq \max\{\delta(\xi_1, \xi_3), \delta(\xi_3, \xi_2)\}.$$

The next definition is slightly more general:

Definition 8.34. Let $X = (a, b)$ be an interval, and let \mathcal{Z} be a set. A function $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X \cup \{a\}$ is an *ultrametric kernel* if it is symmetric, it satisfies the ultrametric inequality and one has $\delta(\xi_1, \xi_2) = a$ if and only if $\xi_1 = \xi_2$. We say that it is unbounded if $\sup_{\xi_1, \xi_2 \in \mathcal{Z}} \delta(\xi_1, \xi_2) = b$.

Given an ultrametric kernel, for $\xi \in \mathcal{Z}$ and $t \in X$, we write $B_\delta(\xi, t) = \{\xi' \in \mathcal{Z} : \delta(\xi, \xi') \leq t\}$. We refer to the sets of this form as δ -balls.

Equivalently, δ is an ultrametric kernel if and only if $H \circ \delta$ is an ultrametric distance on \mathcal{Z} for whenever H is a homeomorphism $H : [a, b) \rightarrow [0, +\infty)$.

Note that the ultrametric inequality implies that if (\mathcal{Z}, δ) is a set with a ultrametric kernel, then the δ -balls form a cross-free family of subsets of \mathcal{Z} in the sense of Definition 7.1.

Remark 8.35. Consider a directed tree $(\mathbb{T}, \triangleleft)$ endowed with a horograding map $\pi : \mathbb{T} \rightarrow X$, where $X = (a, b)$. Recall that we write $\partial_\pi^* \mathbb{T}$ for the set of π -complete ends, i.e. the subset of ends $\xi \in \partial^* \mathbb{T}$ such that $\pi(\downarrow \xi, \omega] = X$. Then the set $\partial_\pi^* \mathbb{T}$ admits an ultrametric kernel $\delta : \partial_\pi^* \mathbb{T} \times \partial_\pi^* \mathbb{T} \rightarrow [a, b)$ defined as $\delta(\xi_1, \xi_2) = \pi(\xi_1 \wedge \xi_2)$ if $\xi_1 \neq \xi_2$, and $\delta(\xi_1, \xi_2) = a$ if $\xi_1 = \xi_2$. It is straightforward to check that δ is an ultrametric kernel. Moreover if G is a group endowed with actions $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ and $j : G \rightarrow \text{Homeo}_0(X)$ such that π is equivariant, then the map δ is equivariant in the sense that $\delta(\Phi(g)(\xi_1), \Phi(g)(\xi_2)) = j(g)(\delta(\xi_1, \xi_2))$. This remark motivates our previous and next definitions.

The following construction is a special case of the well-known correspondence between real trees and ultrametric spaces.

Definition 8.36. Assume that (\mathcal{Z}, δ) be a space with an unbounded ultrametric kernel $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X$. We define a directed tree $(\mathbb{T}, \triangleleft)$, called the directed tree *associated with* (\mathcal{Z}, δ) , endowed with a horograding $\pi : \mathbb{T} \rightarrow X$ and an injective map $\iota : \mathcal{Z} \rightarrow \partial^* \mathbb{T}$ identifying \mathcal{Z} with a subset of $\partial_\pi^* \mathbb{T}$, as follows.

As a set, the tree \mathbb{T} is the quotient $\mathbb{T} = \mathcal{Z} \times X / \sim$ with respect to the equivalence relation

$$(\xi_1, t) \sim (\xi_2, s) \quad \text{if and only if} \quad t = s \text{ and } t \geq \delta(\xi_1, \xi_2).$$

Note that by the ultrametric inequality, the condition $t \geq \delta(\xi_1, \xi_2)$ is equivalent to $B_\delta(\xi_1, t) = B_\delta(\xi_2, t)$, and from this one immediately sees that the above relation is an equivalence relation. Denote by $p : \mathcal{Z} \times X \rightarrow \mathbb{T}$ the quotient map. The partial order \triangleleft on \mathbb{T} is defined by declaring $u \triangleleft v$ if for some (equivalently, for every) representatives of u and v , say $u = p(\xi_1, s)$ and $v = p(\xi_2, t)$, it holds that $s \leq t$ and $B_\delta(\xi_1, s) \subset B_\delta(\xi_2, t)$ (equivalently that $s \leq t$ and $\delta(\xi_1, \xi_2) \leq t$). With these definitions $(\mathbb{T}, \triangleleft)$ is a directed tree [38, §3.1.6]. The map $\pi : \mathbb{T} \rightarrow X$, defined by $\pi(p(\xi_1, t)) = t$ is an increasing horograding of $(\mathbb{T}, \triangleleft)$. Finally the set \mathcal{Z} can be naturally identified with a subset of $\partial_\pi^* \mathbb{T}$, by the map $i : \mathcal{Z} \rightarrow \partial^* \mathbb{T}$, mapping each $\xi \in \mathcal{Z}$ to the unique infimum in $\partial^* \mathbb{T}$ of the ray $\{p(\xi, t) : t \in X\}$.

We now consider the previous situation in the presence of a group G acting.

When G is a group acting on both \mathcal{Z} and X , by actions $\varphi : G \rightarrow \text{Sym}(\mathcal{Z})$ and $j : G \rightarrow \text{Homeo}_0(X)$ respectively, we say that an ultrametric kernel $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X \cup \{a\}$ is G -equivariant if for every $\xi_1, \xi_2 \in \mathcal{Z}$ and $g \in G$ one has

$$(8.3) \quad \delta(\varphi(g)(\xi_1), \varphi(g)(\xi_2)) = j(g)(\delta(\xi_1, \xi_2)).$$

In what follows, to simplify notation, we will write as usual $g.\xi$ and $g.t$ instead of $\varphi(g)(\xi)$ and $j(g)(t)$, respectively. We place ourselves in the following setting.

Assumption 8.37. Let G be a group, and consider actions $\varphi : G \rightarrow \text{Sym}(\mathcal{Z})$ on a countable set, and $j : G \rightarrow \text{Homeo}_0(X)$ on an open interval $X = (a, b)$. We let $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X \cup \{a\}$ be an unbounded G -equivariant ultrametric kernel.

The equivariance relation (8.3) implies that for every $g \in G$, $\xi \in \mathcal{Z}$, and $t \in X$ one has

$$(8.4) \quad g.B_\delta(\xi, t) = B_\delta(g.\xi, g.t).$$

In particular, the set of δ -balls is G -invariant.

Under Assumption 8.37, we will say that the action of G on \mathcal{Z} *expands δ -balls* if for every ball $B_\delta(\xi, t)$ there exists a sequence of elements $(g_n) \subset G$ such that the sequence of balls $g_n.B_\delta(\xi, t)$ is an increasing exhaustion of \mathcal{Z} .

Proposition 8.38. *Under Assumption 8.37, let $(\mathbb{T}, \triangleleft)$ be the directed tree associated with (\mathcal{Z}, δ) . Then there exists an action $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ increasingly horograded by j , and such that the map $i : \mathcal{Z} \rightarrow \partial^*\mathbb{T}$ is G -equivariant. Moreover, the action Φ is focal if and only if the action of G on \mathcal{Z} expands δ -balls.*

Proof. The diagonal action (induced by φ and j) of G on $\mathcal{Z} \times X$ descends to the quotient to an action on the directed tree $(\mathbb{T}, \triangleleft)$. Thus, we get an action $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$, given by $\Phi(g)(p(\xi, t)) = p(g.\xi, g.t)$. It is evident from the definition of the tree \mathbb{T} and from (8.4) that the horograding $\pi : \mathbb{T} \rightarrow X$ from Definition 8.36 is G -equivariant and thus, Φ is increasingly horograded by j . Moreover, notice that the natural map $i : \mathcal{Z} \rightarrow \partial^*\mathbb{T}$ defined previously is also G -equivariant with respect to the actions φ and Φ . Finally if the action of G on \mathcal{Z} expands δ -balls then it follows from the construction of the tree \mathbb{T} that for every $v \in \mathbb{T}$ there exists a sequence of elements (g_n) such that $g_n.v$ tends to the focus ω along the ray $[v, \omega[$, which implies that Φ is a focal action. \square

The next step is to introduce an invariant planar order on the tree \mathbb{T} , whenever the set \mathcal{Z} is endowed with an appropriate total order.

Definition 8.39. Given an ultrametric kernel $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X \cup \{a\}$, we say that a total order $<$ on \mathcal{Z} is δ -convex if all δ -balls are $<$ -convex subsets.

Proposition 8.40. *Under Assumption 8.37, consider the action $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ on the directed tree associated with (\mathcal{Z}, δ) . Consider also a φ -invariant total order $<$ on \mathcal{Z} .*

Then, the order $<$ is induced by a Φ -invariant planar order \prec (via the map $i : \mathcal{Z} \rightarrow \partial^\mathbb{T}$ defined in Proposition 8.38) if and only if $<$ is δ -convex.*

Proof. Recall that we denote by $p : \mathcal{Z} \times X \rightarrow \mathbb{T}$ the quotient projection. Also, we denote by ∂U_v the shadow in $\partial^*\mathbb{T}$ of a point $v \in \mathbb{T}$. Then, for every $\xi \in \mathcal{Z}$ and $t \in X$, we have

$$(8.5) \quad i(B_\delta(\xi, t)) = \partial U_{p(\xi, t)} \cap i(\mathcal{Z}).$$

Assume first that $<$ is induced by a Φ -invariant planar order \prec on $(\mathbb{T}, \triangleleft)$ and take a δ -ball $B_\delta(\xi, t)$ in \mathcal{Z} . Since shadows of vertices are \prec -convex and $<$ is induced by \prec , the equality (8.5) ensures that $B_\delta(\xi, t)$ is $<$ -convex, as desired.

Conversely, assume that $<$ is δ -convex. Let $v \in \text{Br}(\mathbb{T})$, and let us define an order \prec^v on E_v^- . For this, let $e_1, e_2 \in E_v^-$ be distinct directions in E_v^- . By construction of $(\mathbb{T}, \triangleleft)$ there exist $\xi_1, \xi_2 \in i(\mathcal{Z}) \subseteq \partial^*\mathbb{T}$ with $e_v(\xi_i) = e_k$ for $k \in \{1, 2\}$ (according to the notation in §8.2). Notice that in this case $\xi_1 \wedge \xi_2 = v$. We define \prec^v on E_v^- by setting $e_1 \prec^v e_2$ if and only if $\xi_1 < \xi_2$. To show that \prec^v is well defined, consider different choices of representatives ξ'_1 and ξ'_2 for the corresponding directions. By definition, for $k \in \{1, 2\}$ we have that $\{\xi_k, \xi'_k\}$ is contained in the δ -ball $B_\delta(\xi_k, \delta(\xi_k, \xi'_k))$, which is $<$ -convex by hypothesis. Thus, in order to show that \prec^v is well defined, it is enough to show that $B_\delta(\xi_1, \delta(\xi_1, \xi'_1)) \cap B_\delta(\xi_2, \delta(\xi_2, \xi'_2)) = \emptyset$. For this, for $k \in \{1, 2\}$ write $v_k = \xi_k \wedge \xi'_k$ and notice that, since $v_k \in e_k$ and $e_1 \neq e_2$, we have $\partial U_{v_1} \cap \partial U_{v_2} = \emptyset$. Finally, since for $k \in \{1, 2\}$ one has $p(\xi_k, \delta(\xi_k, \xi'_k)) = v_k$, the equality (8.5) implies that $i(B_\delta(\xi_k, \delta(\xi_k, \xi'_k))) = \partial U_{v_k} \cap i(\mathcal{Z})$. This shows that \prec^v is well defined. The fact that $<$ is induced by \prec and that \prec is Φ -invariant is clear from the construction and the φ -invariance of $<$. \square

We record the following consequence of the previous discussion.

Corollary 8.41. *Let $X = (a, b)$ be an interval, $(\mathcal{Z}, <)$ a totally ordered countable set, and G be a group together with two actions $\varphi : G \rightarrow \text{Aut}(\mathcal{Z}, <)$ and $j : G \rightarrow \text{Homeo}_0(X)$. Assume further that $\delta : \mathcal{Z} \times \mathcal{Z} \rightarrow X \cup \{a\}$ is a G -equivariant ultrametric kernel such that the order $<$ is δ -convex and the action of G on \mathcal{Z} expands δ -balls. Let $\psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be the dynamical realization of φ . Then ψ is a minimal \mathbb{R} -focal action, increasingly horograded by j .*

Proof. Applying Proposition 8.38 we get an action of G on a directed tree $(\mathbb{T}, \triangleleft)$ and a G -equivariant map $i : \mathcal{Z} \rightarrow \partial^*\mathbb{T}$. On the other hand, since $<$ is φ -invariant and δ -convex, Proposition 8.40 implies the existence of a Φ -invariant planar order \prec inducing $<$ on $i(\mathcal{Z})$. Thus, by the definition of dynamical realization of a focal action (notice that Φ is focal by hypothesis) we get that the dynamical realization of φ and that of Φ coincide. Then, Proposition 8.18 implies that the dynamical realization of φ (and also of Φ) is a minimal \mathbb{R} -focal action. Finally, since Φ is increasingly horograded by j we conclude that the dynamical realization of φ is increasingly horograded by j . \square

Example 8.42 (Ultrametric kernels for Plante actions). We revisit the example of the Plante product of two left-orders, already treated in Examples 7.10 and 8.21, but now with the perspective of ultrametric kernels. Recall that for given countable left-ordered groups $(G, <_G)$ and $(H, <_H)$, we defined their Plante product as an action on the ordered space $\varphi : H \wr G \rightarrow \text{Aut}(\bigoplus_G H, \prec)$ of the wreath product $H \wr G = \bigoplus_G H \rtimes G$, where $\bigoplus_G H$ is the set of functions $s : G \rightarrow H$ which are trivial at all but finitely many elements of G . In the action φ , the subgroup G acts by pre-composing with left translations, whilst $\bigoplus_G H$ acts by pointwise left-multiplication. We already explained in Example 8.21 how to get a focal action on a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ representing the dynamical realization of the Plante product, and we observed at the end that the focal action is increasingly horograded by the dynamical realization of $<_G$. Here we will get the same conclusion, by simply introducing an ultrametric kernel (which actually corresponds to the ultrametric kernel for a horograded focal action defined in Remark 8.35).

Consider a good embedding $\iota : G \rightarrow \mathbb{R}$ associated with the total order $<_G$. We define an ultrametric kernel

$$\delta : \bigoplus_G H \times \bigoplus_G H \rightarrow \mathbb{R} \cup \{-\infty\}$$

as follows: when $\mathbf{s} \neq \mathbf{t}$, we define $\delta(\mathbf{s}, \mathbf{t}) = \iota(x_*)$ where $x_* = \max_{<_G} \{x \in G : \mathbf{s}(x) \neq \mathbf{t}(x)\}$, while in the remaining case, we set $\delta(\mathbf{s}, \mathbf{s}) = -\infty$ for every $\mathbf{s} \in \bigoplus_G H$. It is direct to check that δ is an

ultrametric kernel. On the other hand, with arguments similar to those in Example 7.10 we get that $<$ is a δ -convex total order. This ultrametric kernel δ is $H \wr G$ -equivariant if we consider the action φ and $j : H \wr G \rightarrow \text{Homeo}_0(\mathbb{R})$ defined as $j = j_0 \circ pr_G$ where $pr_G : H \wr G \rightarrow G$ is the projection and j_0 is the dynamical realization of $<_G$ corresponding to the good embedding ι .

We next verify that the action of $H \wr G$ on $\bigoplus_G H$ expands δ -balls. By transitivity of the action, it is enough to consider δ -balls centered at the trivial element $\mathbf{e} \in \bigoplus_G H$, and we can simply consider the action of G , which is exactly the stabilizer of \mathbf{e} . Thus for every $x \in \mathbb{R}$ and $g \in G$, we have that $g.B_\delta(\mathbf{e}, x) = B_\delta(\mathbf{e}, g.x)$. As the action of G on \mathbb{R} is given by the dynamical realization of $<_G$, for every $x \in \mathbb{R}$ we can find a sequence of elements $(g_n) \subset G$ such that $g_n.x \rightarrow \infty$ as $n \rightarrow \infty$. This proves that the action expands δ -balls.

Finally, applying Corollary 8.41 we get that the Plante action of $H \wr G$ associated with $<_G$ and $<_H$ is \mathbb{R} -focal, increasingly horograded by j .

9. SOME MINIMAL \mathbb{R} -FOCAL ACTIONS

The goal of this section is to present some general constructions of exotic actions on \mathbb{R} . The first three examples describe \mathbb{R} -focal actions of locally moving groups. As in Section 4, given $X = (a, b)$ and a locally moving subgroup $G \subseteq \text{Homeo}_0(X)$, we say that an action $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ is exotic if it is not semi-conjugate to the standard action on X , nor to any action of the largest quotient $G/[G_c, G_c]$. Some basic examples of exotic actions have already been provided in §5.3; those examples are not satisfactory because they yield actions which are not minimal (and that do not admit any non-empty closed minimal invariant subset), and they essentially rely on the fact that the groups considered there are not finitely generated. On the contrary, here we will construct examples of exotic *minimal* actions for some classes of locally moving groups, including finitely generated ones. Along the way we will observe that these constructions share the property to produce \mathbb{R} -focal actions.

Finally, the last example is a generalization of the Brin–Navas group defined in Example 4.6. The new examples, called *generalized Brin–Navas groups*, serve to show that some results in this article concerning locally moving groups do not generalize to the general context of micro-supported groups (notably Corollary 5.15 and Theorem 6.9 fail).

9.1. Locally moving groups with cyclic group of germs. Throughout the subsection we let $X = (a, b)$, and we let $G \subseteq \text{Homeo}_0(X)$ be a locally moving group such that $\text{Germ}(G, b)$ is infinite cyclic and acts freely near b (that is, for every $g \in G$ whose projection to $\text{Germ}(G, b)$ is non-trivial, there is an interval (x, b) on which g has no fixed points). We will say for short that G has *cyclic germs* at b .

Example 9.1. One example of group with cyclic germs is Thompson’s group $F \subseteq \text{Homeo}_0((0, 1))$, and more generally any Higman–Thompson’s group F_n (see §2.4). A much wider class of examples with this property is given by chain groups in the sense of Kim, Koberda, and Lodha [55].

9.1.1. Actions from escaping orbits. For $G \subseteq \text{Homeo}_0(X)$ with cyclic germs, we present a mechanism build a continuum of pairwise non-conjugate minimal exotic actions, which is a modification of the escaping sequence construction from §5.3. For this, we identify $\text{Germ}(G, b)$ with \mathbb{Z} in such a way that a germ for which b is an attractive fixed point is sent to the positive generator of \mathbb{Z} . We denote by $\tau : G \rightarrow \mathbb{Z}$, the homomorphism obtained via this identification. Then we fix an element $f_0 \in G$ such that $\tau(f_0) = 1$ (that is, the germ of f_0 generates $\text{Germ}(G, b)$) and we have $f_0(x) > x$ near b). Choose next a bi-infinite sequence $\mathbf{s} = (s_n)_{n \in \mathbb{Z}} \subset X$ with

$$(9.1) \quad s_{n+1} = f_0(s_n) \text{ for } n \in \mathbb{Z}, \quad \text{and.}$$

Consider the action of the group G on the set of sequences $(0, 1)^{\mathbb{Z}}$, where the action of $g \in G$ on a sequence $\mathbf{t} = (t_n)_{n \in \mathbb{Z}}$ is given by

$$(9.2) \quad g \cdot \mathbf{t} = \left(g(t_{n-\tau(g)}) \right)_{n \in \mathbb{Z}}.$$

It is straightforward to check that this defines an action of G on $X^{\mathbb{Z}}$, using that τ is a homomorphism. We let $S \subset X^{\mathbb{Z}}$ be the orbit of \mathbf{s} under this action. Note that \mathbf{s} is fixed by f_0 .

Lemma 9.2. *With notation as above, for every sequence $\mathbf{t} = (t_n)_{n \in \mathbb{Z}} \in S$, there exists $n_0 \in \mathbb{Z}$ such that $t_n = s_n$ for every $n \geq n_0$.*

Proof. Let $g \in G$ be such that $\mathbf{t} = g \cdot \mathbf{s}$. Then $t_n = g(s_{n-\tau(g)})$. As we required $\lim_{n \rightarrow +\infty} s_n = b$ in (9.1), and g coincides with $f_0^{\tau(g)}$ on a neighborhood of b , the conclusion follows. \square

It follows from the lemma that for every two distinct sequences $\mathbf{t} = (t_n)$ and $\mathbf{t}' = (t'_n)$ in S , the integer

$$(9.3) \quad m(\mathbf{t}, \mathbf{t}') = \max \{ n \in \mathbb{Z} : t_n \neq t'_n \}$$

is well-defined and finite. Thus we can introduce the total order relation \prec on S , given by $\mathbf{t} \prec \mathbf{t}'$ if and only if $t_m < t'_m$, with $m = m(\mathbf{t}, \mathbf{t}')$

Lemma 9.3. *With notation as above, the total order \prec on S is preserved by the action of G on S defined by (9.2). Moreover, the element f_0 acts as a homothety on (S, \prec) (in the sense of Definition 2.41) with fixed point \mathbf{s} .*

Proof. It is routine verification that the order \prec is G -invariant. Let us check that f_0 is a homothety. We have already noticed that the sequence \mathbf{s} is a fixed point for f_0 . Fix sequences $\mathbf{t}, \mathbf{t}' \in S$ such that

$$(9.4) \quad \mathbf{s} \prec \mathbf{t} \prec \mathbf{t}'.$$

We need to show that there exists $n \in \mathbb{Z}$ such that $f_0^n \cdot \mathbf{t} \succ \mathbf{t}'$ (in fact, we will find some $n \geq 1$, showing that f_0 acts as an *expanding* homothety). Write $m_0 = m(\mathbf{t}, \mathbf{s})$ and $m_1 = m(\mathbf{t}', \mathbf{s})$ and note that the condition (9.4) gives $m_0 \leq m_1$ and $t_{m_0} > s_{m_0}$. We claim that $n = m_1 - m_0 + 1$ is fine for our purposes. For this, we compute directly:

$$\begin{aligned} (f_0^n \cdot \mathbf{t})_{m_1+1} &= f_0^n(t_{m_1+1-n}) = f_0^n(t_{m_0}) \\ &> f_0^n(s_{m_0}) = s_{m_0+n} = s_{m_1+1} = t'_{m_1+1}, \end{aligned}$$

while for every $m > m_1 + 1 = n + m_0$ we have

$$(f_0^n \cdot \mathbf{t})_m = f_0^n(t_{m-n}) = f_0^n(s_{m-n}) = s_m = t'_m.$$

Thus $m(\mathbf{t}', \mathbf{t}) = m_0 + 1$ and $f_0^n \cdot \mathbf{t} \succ \mathbf{t}'$, as desired. Similarly one argues for $\mathbf{t}' \prec \mathbf{t} \prec \mathbf{s}$. \square

Assume now that G is countable, so that the set S is countable as well. Then we can consider the dynamical realization $\varphi_{\mathbf{s}}: G \rightarrow \text{Homeo}_0(\mathbb{R})$ of the action of G on (S, \prec) .

Proposition 9.4. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a countable locally moving group with cyclic germs at b . For every sequence $\mathbf{s} = (s_n)_{n \in \mathbb{Z}}$ as in (9.1), the action $\varphi_{\mathbf{s}}: G \rightarrow \text{Homeo}_0(\mathbb{R})$ constructed above is minimal and faithful. Moreover if \mathbf{s}' is another such sequence, whose image is different from that of \mathbf{s} (that is, if they are not the same after a shift of indices), then $\varphi_{\mathbf{s}}$ and $\varphi_{\mathbf{s}'}$ are not conjugate. In particular G has uncountably many minimal faithful non-conjugate actions on the real line.*

Proof. The fact that $\varphi_{\mathbf{s}}$ is minimal follows from Lemma 9.3 and Proposition 2.43 (the action on \mathbf{S} is transitive, so it is enough to describe what happens at \mathbf{s}). Let $\iota: (\mathbf{S}, \prec) \rightarrow \mathbb{R}$ be an equivariant good embedding associated with $\varphi_{\mathbf{s}}$. Since f_0 is a homothety on (\mathbf{S}, \prec) , it follows that its image of $\varphi_{\mathbf{s}}$ is a homothety of \mathbb{R} whose unique fixed point is $\iota(\mathbf{s})$. In particular the stabilizer of this point inside $G_+ = \ker \tau$, which after (9.2) coincides with the stabilizer of \mathbf{s} for the natural diagonal action of G_+ on $X^{\mathbb{Z}}$, is a well-defined invariant of the conjugacy class of $\varphi_{\mathbf{s}}$. Now note that if \mathbf{s} and \mathbf{s}' are sequences with distinct images, using that G is locally moving it is not difficult to construct $g \in G_+$ such that $g \cdot \mathbf{s} = \mathbf{s}$ and $g \cdot \mathbf{s}' \neq \mathbf{s}'$, showing that $\varphi_{\mathbf{s}}$ and $\varphi_{\mathbf{s}'}$ are not conjugate. \square

9.1.2. *A simplicial tree and \mathbb{R} -focality.* We keep the same standing assumption on G and the same setting as above, with a fixed sequence $\mathbf{s} = (s_n)_{n \in \mathbb{Z}}$ as in (9.1). We now want to observe that the action $\varphi_{\mathbf{s}}$ constructed above is \mathbb{R} -focal. What is more, we will interpret the action $\varphi_{\mathbf{s}}$ as the dynamical realization of a G -action on a planar directed tree which is *simplicial* (of infinite degree), by simplicial automorphisms. For $n \in \mathbb{Z}$ denote by $\mathbb{Z}_{\geq n}$ the set of integers $j \geq n$. We let $\mathbf{S}_{\geq n} \subset X^{\mathbb{Z}_{\geq n}}$ be the subset of sequences indexed by $\mathbb{Z}_{\geq n}$ obtained by restricting sequences in \mathbf{S} to $\mathbb{Z}_{\geq n}$:

$$\mathbf{S}_n = \{(t_j)_{j \geq n} : (t_j)_{j \in \mathbb{Z}} \in \mathbf{S}\}.$$

We will call *truncation* this operation. Given a sequence $(t_j)_{j \geq n} \in \mathbf{S}_n$ we say that $(t_j)_{j \geq n+1} \in \mathbf{S}_{n+1}$ is its *successor*. The disjoint union $\bigsqcup_{n \in \mathbb{Z}} \mathbf{S}_n$ is naturally the vertex set of a simplicial tree \mathbb{T} , obtained by connecting each element to its successor. Indeed it is clear to see that the graph obtained in this way has no cycles; moreover it is connected, because of the fact that all elements of \mathbf{S} eventually coincide with the sequence \mathbf{s} (Lemma 9.2). Thus we obtain a simplicial tree.⁴

If we endow all edges of \mathbb{T} with the orientation from a point to its successor, then all edges point to a common natural end $\omega \in \partial \mathbb{T}$, we get a directed tree $(\mathbb{T}, \triangleleft)$ with focus ω . Note that every $\mathbf{t} = (t_j)_{j \in \mathbb{Z}} \in \mathbf{S}$ defines a bi-infinite ray of \mathbb{T} , whose vertices are the successive truncations of \mathbf{t} . For every $\mathbf{t} \in \mathbf{S}$, this sequence converges to ω as $n \rightarrow +\infty$. As $n \rightarrow -\infty$, it converges to some end $\alpha_{\mathbf{t}} \in \partial^* \mathbb{T} = \partial \mathbb{T} \setminus \{\omega\}$. The map $\mathbf{t} \mapsto \alpha_{\mathbf{t}}$ is clearly injective, and thus allows to identify \mathbf{S} with a subset of $\partial^* \mathbb{T}$.

The group G has a natural action on $(\mathbb{T}, \triangleleft)$, namely for every vertex $v = (t_j)_{j \geq n}$ of \mathbb{T} and $g \in G$ we set

$$g \cdot (t_j)_{j \geq n} = \left(g(t_{j-\tau(g)}) \right)_{j \geq n+\tau(g)}.$$

Note in particular that if $v \in \mathbf{S}_n$ then $g \cdot v \in \mathbf{S}_{n+\tau(g)}$. This action is by simplicial automorphism and fixes the end ω . Moreover if $g \in G$ is such that $\tau(g) \neq 0$, then g has no fixed point on \mathbb{T} , and thus acts as a hyperbolic isometry. If $\tau(g) = 0$, then g preserves each of the sets \mathbf{S}_n and acts as an elliptic isometry (indeed since g acts trivially on some neighborhood of b in x , it must fix all vertices $(s_j)_{j \geq n}$ for n large enough).

Let us now define a planar order on $(\mathbb{T}, \triangleleft)$. In this case, this just means an order $<^v$ for every $v = (t_j)_{j \geq n} \in \mathbb{T}$ on the set of edges E_v^- which lie below v (i.e. opposite to ω). Fix $v = (t_j)_{j \geq n}$, and consider two distinct edges $e_1, e_2 \in E_v^-$. Then for $i \in \{1, 2\}$ we have $e_i = (w, v)$ for some $w = \left(t_j^{(i)} \right)_{j \geq n-1}$ with $t_j^{(i)} = t_j^{(2)} = t_j$ for $j \geq n$, and $t_{n-1}^{(1)} \neq t_{n-1}^{(2)}$. Thus we set $e_1 <^v e_2$ if and

⁴Note that by slight abuse of notation we do not distinguish between the tree \mathbb{T} as a graph, which is the pair (V, E) consisting of the set of vertices and of edges $E \subset V^2$, and its simplicial realization, which is the topological space obtained by gluing a copy of the interval $[0, 1]$ for each edge in the obvious way and with the natural quotient topology.

only if $t_j^{(1)} < t_j^{(2)}$. Then the collection $\{<^v : v \in \mathbb{T}\}$ defines a planar order $<$ on (\mathbb{T}, ω) which is invariant under the action of G .

Finally note that the map $\mathfrak{t} \mapsto \alpha_{\mathfrak{t}}$ is G -equivariant and increasing with respect to the order on \mathbb{S} and the order on $\partial^*\mathbb{T}$ induced from the planar order. Thus the G -action on $(\mathbb{S}, <)$ can be identified with an action on an orbit in $\partial^*\mathbb{T}$. It follows that $\varphi_{\mathfrak{s}}$ is conjugate to the dynamical realization of the action on the planar directed tree $(\mathbb{T}, <, \omega)$.

Finally we note that the above discussion also shows that the action $\varphi_{\mathfrak{s}}$ can be increasingly horograded by the cyclic action of G on \mathbb{R} by integer translations defined by the homomorphism $\tau: G \rightarrow \mathbb{Z}$. Indeed it is enough to take the horofunction $\pi: \mathbb{T} \rightarrow \mathbb{R}$ associated with ω , which sends every vertex $v \in \mathbb{S}_{\geq n}$ to $n \in \mathbb{R}$, and maps the edge from v to its successor to the interval $[n, n+1]$. It follows from Proposition 8.32 that for every element $g \in G$ the image $\varphi_{\mathfrak{s}}(g)$ is a homothety if $\tau(g) \neq 0$, while it is totally bounded if $\tau(g) = 0$. In particular every pseudohomothety in the image of $\varphi_{\mathfrak{s}}$ is actually a homothety.

Remark 9.5. When a bi-infinite sequence $\mathfrak{s} = (s_n)_{n \in \mathbb{Z}}$ as in (9.1) also satisfies $\lim_{n \rightarrow -\infty} s_n = a$ (that is, when the defining element f_0 has no fixed point on X), then the dynamical realization $\varphi_{\mathfrak{s}}$ can also be increasingly horograded by the action of G on X .

To see this, we proceed to define an ultrametric kernel $\delta: \mathbb{S} \times \mathbb{S} \rightarrow X \cup \{a\}$. This is given, for any distinct $\mathfrak{t}, \mathfrak{t}' \in \mathbb{S}$, by $\delta(\mathfrak{t}, \mathfrak{t}') = t_{m+1} = t'_{m+1}$, where $m = m(\mathfrak{t}, \mathfrak{t}') + 1$ is the integer defined in (9.3), and declare $\delta(\mathfrak{t}, \mathfrak{t}) = a$. It follows directly from the definitions that δ is an ultrametric kernel, and that it is G -equivariant with respect to the actions of G on \mathbb{S} and X . Notice also that the δ -balls are convex with respect to the total order relation $<$ on \mathbb{S} and thus $<$ is δ -convex. In order to apply Corollary 8.41 we need to show that the action of G on \mathbb{S} expands δ -balls. As the action on \mathbb{S} is transitive, it is enough to consider δ -balls centered at \mathfrak{s} . Fix $x \in X$, and consider the δ -ball $B = B_{\delta}(\mathfrak{s}, x)$. Since \mathfrak{s} converges to a as $n \rightarrow -\infty$, we can find $n \in \mathbb{Z}$ such that $B_{\delta}(\mathfrak{s}, s_n) \subseteq B$. On the other hand, we have $f_0.B_{\delta}(\mathfrak{s}, s_n) = B_{\delta}(\mathfrak{s}, s_{n+1})$. Since the δ -balls $B_{\delta}(\mathfrak{s}, s_n)$ define an exhaustion of \mathbb{S} , as n runs over the integers, we deduce that the same holds for the collection of δ -balls $f_0^n.B$, which proves that the action of G on \mathbb{S} expands δ -balls as desired. Thus, from Corollary 8.41 we get that $\varphi_{\mathfrak{s}}$ is increasingly horograded by the standard action of G on X .

9.2. Orders of germ type and semidirect products. Here we explain a framework to construct minimal exotic actions of various classes of groups of homeomorphisms of intervals, including most groups of piecewise linear or projective homeomorphisms.

Set $X = (a, b)$ and let $G \subseteq \text{Homeo}_0(X)$ be a (countable) micro-supported group acting minimally on X . In order to run our construction, we require G to satisfy the following condition.

- (G1) The group $\text{Germ}(G, b)$ admits a *section* inside $\text{Homeo}_0(X)$, namely a subgroup $\Gamma \subseteq \text{Homeo}_0(X)$ such that $\text{Germ}(\Gamma, b) = \text{Germ}(G, b)$ and which projects bijectively to $\text{Germ}(G, b)$.⁵

Under assumption (G1), let $\Gamma \subseteq \text{Homeo}_0(X)$ be a section of $\text{Germ}(G, b)$, and consider the overgroup $\widehat{G} = \langle G, \Gamma \rangle$. We will proceed by describing an action of \widehat{G} and then restricting it to G . Note that $\text{Germ}(\widehat{G}, b) = \text{Germ}(G, b)$ induces the same group of left-germs at b as G . The advantage of passing to \widehat{G} is that it splits as a semidirect product

$$\widehat{G} = \widehat{G}_+ \rtimes \Gamma.$$

⁵The problem of when a group of germs admits a section as a group of homeomorphisms is very interesting. We refer the reader to [69] for an example of a finitely generated group of germs which does not admit any such section.

where as usual $\widehat{G}_+ \subseteq \widehat{G}$ is the subgroup of elements whose germ at b is trivial. Using this splitting, we can let the group \widehat{G} act “affinely” on \widehat{G}_+ : the subgroup \widehat{G}_+ acts on itself by left-multiplication, and Γ acts on it by conjugation. Explicitly, if $g \in \widehat{G}$ and $h \in \widehat{G}_+$, writing $g = g_+\gamma_g$ with $g_+ \in \widehat{G}_+$ and $\gamma_g \in \Gamma$, we set

$$(9.5) \quad g \cdot h = gh\gamma_g^{-1}.$$

Note that we actually have $g \cdot h = g_+(\gamma_g h \gamma_g^{-1})$, from which it is straightforward to check that this defines indeed an action on \widehat{G}_+ . We want to find an order \prec on \widehat{G}_+ which is invariant under the action of G , and then consider the dynamical realization of the action of \widehat{G} on (\widehat{G}_+, \prec) . For this we look for a left-invariant order \prec on \widehat{G}_+ which is also invariant under the conjugation action of Γ . We will say for short that such an order is Γ -invariant.

Good candidates are the orders of germ type on \widehat{G}_+ described in §5.3.1. Recall that an order of germ type is determined by a family of left orders $\{\prec^{(x)} : x \in X\}$, where for every $x \in X$, $\prec^{(x)}$ is a left-order on the group of germs $\text{Germ}(\widehat{G}_{(a,x)}, x)$: the associated order of germ type on \widehat{G}_+ is the order \prec whose positive cone is the subset

$$P = \left\{ g \in \widehat{G}_+ : \mathcal{G}_{p_g}(g) \succ^{(p_g)} \text{id} \right\},$$

where $p_g = \sup\{x \in X : g(x) \neq x\}$. However, not every order of germ type is Γ -invariant, and this is because for every $x \in X$ the stabilizer $\text{Stab}_{\widehat{G}}(x)$ of x acts on $\widehat{G}_{(a,x)}$ by conjugation, and this action descends to an action of $\text{Germ}(\widehat{G}, x)$ on $\text{Germ}(\widehat{G}_{(a,x)}, x)$. In light of this, we are able to produce a Γ -invariant order of germ type on \widehat{G}_+ if and only if the following condition is satisfied.

(G2) For every $x \in X$, the group $\text{Germ}(\widehat{G}_{(a,x)}, x)$ admits a left-invariant order $\prec^{(x)}$ which is invariant under conjugation by $\text{Germ}(\Gamma, x)$.

Indeed, suppose that $\{\prec^{(x)} : x \in X\}$ is a family such that the associated order of germ type is Γ -invariant, then each $\prec^{(x)}$ is as in (G2). Conversely if we choose such an order $\prec^{(x)}$ as in (G2) for x in a system of representatives of the Γ -orbits in X , then we can extend it uniquely by Γ -equivariance to a family $\{\prec^{(x)} : x \in X\}$ which defines a Γ -invariant order of germ type.

Remark 9.6. Here are two simple sufficient conditions for (G2).

(G2i) The group Γ acts freely on X .

(G2ii) For every $x \in X$, every non-trivial germ in $\text{Germ}(\widehat{G}_{(a,x)}, x)$ has no fixed point accumulating on x from the left (this does not depend on the choice of the element representing the germ).

The fact that (G2i) implies (G2) is clear since in this case $\text{Stab}_{\Gamma}(x)$ is trivial. In contrast when (G2ii) holds, we can define an order $\prec^{(x)}$ on $\text{Germ}(\widehat{G}_{(a,x)}, x)$ by setting $\mathcal{G}_x(g) \succ^{(x)} \text{id}$ if and only if $g(y) > y$ for every $y \neq x$ in some left-neighborhood of x . Then this is a left-order on $\text{Germ}(\widehat{G}_{(a,x)}, x)$ which is invariant under conjugation by the whole stabilizer of x in $\text{Homeo}_0(X)$.

Summing up, under conditions (G1) and (G2) we can consider a Γ -invariant order of germ type \prec on \widehat{G}_+ and let \widehat{G} act on (\widehat{G}, \prec) by (9.5). By passing to the dynamical realization we obtain an action of \widehat{G} , and thus of G , on the real line. This construction yields the following criterion for the existence of exotic actions.

Proposition 9.7. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a finitely generated, micro-supported group acting minimally on X . Assume that $\text{Germ}(G, b)$ admits a section $\Gamma \subseteq \text{Homeo}_0(X)$ (that is, condition (G1) holds) such that $\widehat{G} = \langle G, \Gamma \rangle$ satisfies (G2).*

Then there exists a minimal faithful action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ which is \mathbb{R} -focal and not conjugate to the standard action of G on X .

Proof. Choose a Γ -invariant order of germ type \prec on \widehat{G}_+ , defined from the family of orders $\{\prec^{(x)}: x \in X\}$. We let $\psi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be the dynamical realization of the action of G on (\widehat{G}_+, \prec) . Set $N = [G_c, G_c]$, which by Proposition 4.4 is the smallest non-trivial normal subgroup of G .

Claim. *For every $x \in X$ the group $\psi(N_{(x,b)})$ acts on \mathbb{R} without fixed points.*

Proof of claim. Let $\iota: (\widehat{G}_+, \prec) \rightarrow (\mathbb{R}, +)$ be an equivariant good embedding associated with ψ (Definition 2.37). Observe that the subgroups $\widehat{G}_{(a,y)}$, for $y \in X$, are bounded convex subgroups of (\widehat{G}_+, \prec) which form an increasing exhaustion of \widehat{G}_+ , thus the convex hull of every $\iota(\widehat{G}_{(a,y)})$ is a bounded interval $I_y \subset \mathbb{R}$, giving an increasing exhaustion of \mathbb{R} . Now given any x and y in X , every $g \in N_{(x,b)}$ with $p_g > y$ satisfies $g\widehat{G}_{(a,y)} \neq \widehat{G}_{(a,y)}$, which in turn implies that $\psi(g)(I_y) \cap I_y = \emptyset$. Since y is arbitrary, and the intervals I_y exhaust \mathbb{R} , this implies that $\text{Fix}^\psi(N_{(x,b)}) = \emptyset$. \square

As G is finitely generated, we can consider a canonical model $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ of ψ , which is thus either minimal or cyclic. Since φ is semi-conjugate to ψ , the claim gives $\text{Fix}^\varphi(N_{(x,b)}) = \emptyset$. Using Proposition 4.4, we deduce that φ is faithful, and thus minimal. Moreover we see that it cannot be conjugate to the standard action of X , since $N_{(x,b)}$ does have fixed points on X .

Finally, as in the proof of the claim, it is not difficult to see that the collection

$$\mathcal{S} := \{\psi(g)(I_y) : g \in G, y \in X\}$$

is an invariant CF-cover for ψ (thus ψ is \mathbb{R} -focal). Since φ is semi-conjugate to ψ , Proposition 7.6 implies that φ is \mathbb{R} -focal. \square

Remark 9.8. With some additional work, one could show that φ can be increasingly horograded by the natural action of G on X . In particular the dynamics of element in the image of φ can be determined by their dynamics on X using Proposition 8.32. We do not elaborate on this, as in Section 10 we will prove that this is the case for all exotic actions of a class of locally moving groups (although this class does not include all groups covered by Proposition 9.7, the arguments in its proof can be adapted to the action constructed here).

The criterion given by Proposition 9.7 applies to several classes of examples of micro-supported groups. Let us give a first illustration. We say that the group of germs $\text{Germ}(G, b)$ acts *freely near b* if every non-trivial germ has no fixed point accumulating on b (this condition does not depend on the choice of the representative; cf. (G2ii)).

Corollary 9.9. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a finitely generated, micro-supported group acting minimally on X . Assume that $\text{Germ}(G, b)$ is abelian and acts freely near b . Then there exists a faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(X)$ which is \mathbb{R} -focal and not conjugate to the standard action of G on X .*

Proof. We first check that if $\text{Germ}(G, b)$ is abelian and acts freely near b , then it admits a section $\Gamma \subset \text{Homeo}_0(X)$ which acts freely on X . To see this, using that G is finitely generated, we can take finitely many elements g_1, \dots, g_r in G whose germs at b are non-trivial and generate $\text{Germ}(G, b)$. Up to taking inverses, the condition that $\text{Germ}(G, b)$ acts freely near b allows to find $z \in X$ such that $g_i(x) > x$ and $g_i g_j(x) = g_j g_i(x)$ for every $x \in (z, b)$ and $i, j \in \{1, \dots, r\}$. Choose an element $\gamma_1 \in \text{Homeo}_0(X)$ which coincides with g_1 on (z, b) and has no fixed points in X . Take $x_0 \in (z, b)$ and consider the fundamental domain $I = [x_0, \gamma_1(x_0))$. As the elements g_2, \dots, g_r commute with $g_1 = \gamma_1$ on I , they induce an action of \mathbb{Z}^{r-1} on the circle $X/\langle \gamma_1 \rangle = [x_0, \gamma_1(x_0))/_{x_0 \sim \gamma_1(x_0)}$, which can be lifted to an action of \mathbb{Z}^{r-1} on X commuting with γ_1 . In simpler terms, writing $I_n = \gamma_1^n(I)$, so that $X = \bigsqcup_{n \in \mathbb{Z}} I_n$, for every $i \in \{2, \dots, r\}$ we can consider the homomorphism $\gamma_i \in \text{Homeo}_0(X)$ defined by

$$(9.6) \quad \gamma_i(x) = \gamma_1^n g_i \gamma_1^{-n}(x) \quad \text{for } x \in I_n \text{ and } n \in \mathbb{Z}.$$

The elements $\gamma_2, \dots, \gamma_r$ define exactly the action of \mathbb{Z}^{r-1} on X which commutes with γ_1 , as discussed above. From the definition (9.6), we see that every γ_i coincides with g_i on $[x_0, b)$, and in particular we have $\mathcal{G}_b(\gamma_i) = \mathcal{G}_b(g_i)$. This gives that $\Gamma = \langle \gamma_1, \dots, \gamma_r \rangle$ is a section of $\text{Germ}(G, b)$ acting freely on X . Thus conditions (G1) and (G2i) are satisfied, so that the conclusion follows from Proposition 9.7. \square

A case in which the previous criterion applies is when $b < \infty$ and the group of germs $\text{Germ}(G, b)$ coincides with a group of germs of linear homeomorphisms $x \mapsto \lambda(x - b) + b$. This is for instance the case whenever G is a subgroup of the group $\text{PL}(X)$ of finitary piecewise linear homeomorphisms of X . This case can be generalized as follows.

Recall from Definition 5.16 that given $X = (a, b) \subset \mathbb{R}$ an interval, we denote by $\text{PDiff}_{loc}^\omega(X)$ the group of all locally piecewise analytic, orientation-preserving homeomorphisms of X , with a countable discrete set of breakpoints in X . We also let $\text{PP}_0(X)$ be the subgroup of $\text{PDiff}_{loc}^\omega(X)$ of piecewise projective homeomorphisms of X with finitely many breakpoints, namely those that are locally of the form $x \mapsto \frac{px+q}{rx+s}$, with $ps - qr = 1$.

Corollary 9.10. *For $X = (a, b)$, let $G \subseteq \text{PDiff}_{loc}^\omega(X)$ be a finitely generated micro-supported group acting minimally on X . Assume that either of the following conditions is satisfied:*

- (a) G is contained in the group $\text{PP}_0(X)$ of piecewise projective homeomorphisms;
- (b) the group of germs $\text{Germ}(G, b)$ admits a section Γ contained in $\text{PDiff}_{loc}^\omega(X)$.

Then there exists a faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ which is \mathbb{R} -focal and not conjugate to the action of G on X .

Proof. First of all observe that (a) implies (b). To see this, assume first that $X = \mathbb{R}$; then $\text{Germ}(G, +\infty)$ is a subgroup of the group of germs of the affine group $\text{Aff}(\mathbb{R}) = \{x \mapsto ax + b\}$, and thus admits a section inside $\text{Aff}(\mathbb{R}) \subseteq \text{PP}_0(\mathbb{R})$. For general X , observe that if we fix $x_0 \in X$, we can find $A, B \in \text{PSL}(2, \mathbb{R})$ which fix x_0 and such that A maps the interval (a, x_0) to $(-\infty, x_0)$ and B maps (x_0, b) to $(x_0, +\infty)$. Then the map $H: X \rightarrow \mathbb{R}$, given by

$$H(x) = \begin{cases} A(x) & \text{if } x \leq x_0, \\ B(x) & \text{if } x > x_0, \end{cases}$$

conjugates $\text{PP}(X)$ to $\text{PP}(\mathbb{R})$, so that we can conclude from the previous case.

Now assume that (b) holds, and choose a section $\Gamma \subseteq \text{PDiff}_0^\omega(X)$ of $\text{Germ}(G, b)$. Then since non-trivial analytic maps have isolated fixed points, we see that \hat{G} satisfies (G2ii), thus (G2), and we can apply Proposition 9.7. \square

Remark 9.11. The conditions in Proposition 9.7 cannot be dropped: in §12.2 we will construct an example of a finitely generated locally moving group $G \subseteq \text{Homeo}_0(\mathbb{R})$ which admits no exotic action. Moreover this example satisfies (G1) (but not (G2)), and its natural action is by piecewise linear homeomorphisms with a countable set of singularities which admit a finite set of accumulation points inside X . This shows the sensitivity of Corollary 9.10 to its assumptions.

9.3. A construction of \mathbb{R} -focal actions for groups of PL homeomorphisms. Let $X = (a, b)$ be an interval. Recall that we denote by $G(X; A, \Lambda)$ the Bieri-Strebel group of finitary PL-homeomorphisms associated to a non-trivial multiplicative subgroups $\Lambda \subset \mathbb{R}_*$ and a Λ -submodule $A \subset \mathbb{R}$ (see Definition 2.44). By Corollary 9.10, we already know that the group $G(X; A, \Lambda)$ always admits an (exotic) minimal \mathbb{R} -focal action. In this subsection we present a generalization of that construction which is specific to the groups $G(X; \Lambda)$. The main interest of this generalisation is that in Section 12 we will show that for a vast class of Bieri-Strebel groups on $X = \mathbb{R}$ (including the groups $G(\lambda)$ from Theorem 1.19), all exotic actions arise through the construction presented here.

Throughout we fix (Λ, A) and set $G := G(X; A, \Lambda)$. We introduce the functions $j^\pm : G \times X \rightarrow \Lambda$ defined by

$$j^+(g, x) = \prod_{y \geq x} \frac{D^-g(y)}{D^+g(y)} \quad \text{and} \quad j^-(g, x) = \prod_{y \leq x} \frac{D^+g(y)}{D^-g(y)},$$

that we call the *right* (respectively, *left*) *jump cocycles* (cf. Example 4.7). Note that these are well defined as only finitely many terms in each product are different from 1, and it is immediate to verify the cocycle relation

$$(9.7) \quad j^\pm(g_2g_1, x) = j^\pm(g_2, g_1(x)) j^\pm(g_1, x) \quad \text{for } x \in X \text{ and } g_1, g_2 \in G.$$

Remark 9.12. For fixed $g \in G$, the function $j^+(g, -) : X \rightarrow \Lambda$ (respectively, $j^-(g, -) : X \rightarrow \Lambda$) has the following properties:

- (1) it is left (respectively, right) continuous,
- (2) it is piecewise constant, with discontinuity points in A ,
- (3) it takes the constant value 1 on a neighborhood of b (respectively, a).

We next fix a preorder $\leq_\Lambda \in \text{LPO}(\Lambda)$ and we denote by $\Lambda_0 = [1]_{\leq_\Lambda}$ its residue. We then get a partition $G = P \sqcup H \sqcup P^{-1}$, with

$$(9.8) \quad H = \left\{ g \in G : j^+(g, x) \in \Lambda_0 \ \forall x \in X \right\} \quad \text{and} \quad P = \left\{ g \in G \setminus H : j^+(g, x_{g, \Lambda_0}) \in P_{\leq_\Lambda} \right\},$$

where for given $g \in G \setminus H$, we are writing

$$(9.9) \quad x_{g, \Lambda_0} = \max \left\{ x \in X : j^+(g, x) \notin \Lambda_0 \right\}.$$

It follows from the cocycle relation (9.7) that P is a semigroup, H is a subgroup and $HPH \subseteq P$. Thus, P is the positive cone of a preorder of G whose residue is H (see Remark 2.34).

Definition 9.13. For a given preorder \leq_Λ on Λ , we define the *right jump preorder* on $G = G(X; A, \Lambda)$ as the preorder \preceq_+ whose positive cone is the semigroup P in (9.8). Similarly, we define the *left jump preorder* \preceq_- as the preorder obtained from the analogue construction considering the left jump cocycle.

(This should be compared with the construction appeared in in §5.3.3 for the group $\text{PL}_{\mathbb{Q}}((0, 1))$.)

The aim of this subsection is to show the following.

Proposition 9.14. *Let $X = (a, b)$ be an interval and let $G = G(X; A, \Lambda)$ be a countable Bieri–Strebel group. a preorder $\leq_\Lambda \in \text{LPO}(\Lambda)$, let $\preceq \in \text{LPO}(G)$ be either the right or left jump preorder associated with \leq_Λ . Assume in addition that one of the following conditions satisfied:*

- (1) $a, b \in A \cup \{\pm\infty\}$, or
- (2) the residue $\Lambda_0 = [1]_{\leq_\Lambda}$ is non-trivial.

Then, the dynamical realization of \preceq is a faithful minimal \mathbb{R} -focal action.

Moreover, for two distinct preorders on Λ , the dynamical realizations of their associated right (respectively, left) jump preorders are not positively conjugate. In particular, when $\text{LPO}(\Lambda)$ is uncountable, then G admits uncountably many conjugacy classes of minimal \mathbb{R} -focal actions.

In order to do this, we take a different approach to define the jump preorders. We only discuss the case of right jump preorder, the other case being totally analogous. To start with, we note that the right jump cocycle j^+ induces a fibered action of G on the trivial bundle $X \times \Lambda \rightarrow X$. Such action descends to a fibered action on the quotient $X \times \Lambda/\Lambda_0 \rightarrow X$ and we can consider the induced action Φ on the space of sections $\mathcal{S}(X, \Lambda/\Lambda_0) = \{\mathfrak{t} : X \rightarrow \Lambda/\Lambda_0\}$. More explicitly, for a given function $\mathfrak{t} : X \rightarrow \Lambda/\Lambda_0$, and element $g \in G$, the action $\mathfrak{t} \mapsto \Phi(g)(\mathfrak{t})$ is defined by the expression

$$(9.10) \quad \Phi(g)(\mathfrak{t}) : x \mapsto j^+(g, g^{-1}(x)) \mathfrak{t}(g^{-1}(x)) \pmod{\Lambda_0}.$$

It is routine to check from (9.7) that this is indeed an action of the group G . As usual, we will use the shorthand notation $g.\mathfrak{t}$ for $\Phi(g)(\mathfrak{t})$.

After the regularity properties of the cocycles (Remark 9.12), it is more appropriate to restrict such actions to the invariant subspace \mathcal{S} of functions $\mathfrak{t} : X \rightarrow \Lambda/\Lambda_0$ satisfying the analogous properties:

- (1) \mathfrak{t} is left continuous,
- (2) \mathfrak{t} is piecewise constant, with discontinuity points in A ,
- (3) \mathfrak{t} takes the constant value Λ_0 on a neighborhood of b .

The preorder \leq_Λ defines a total order $<_{\Lambda/\Lambda_0}$ on the quotient Λ/Λ_0 , which is invariant by multiplication by elements of Λ . From this, we introduce a lexicographic total order $<_{\mathcal{S}}$ on the subspace \mathcal{S} : for distinct sections $\mathfrak{s}, \mathfrak{t} \in \mathcal{S}$, say that $\mathfrak{s} <_{\mathcal{S}} \mathfrak{t}$ if

$$\mathfrak{s}(m) <_{\Lambda/\Lambda_0} \mathfrak{t}(m), \quad \text{where } m = \max \{x \in X : \mathfrak{s}(x) \neq \mathfrak{t}(x)\}.$$

Lemma 9.15. *With notation as above, the action Φ of the group $G = G(X; A, \Lambda)$ on the space of sections \mathcal{S} preserves the total order $<_{\mathcal{S}}$.*

Proof. The order $<_{\mathcal{S}}$ is invariant under pointwise multiplication by elements in the subspace \mathcal{S} , and also by precomposition of sections by homeomorphisms of the base X . Thus, from the expression (9.10) for the action Φ , we immediately get the result. \square

In what follows, we will denote by \mathfrak{e} the trivial section (namely, $\mathfrak{e}(x) = \Lambda_0$ for every $x \in X$). Note that \mathfrak{e} belongs to \mathcal{S} . We have the following.

Lemma 9.16. *With notation as above, the pull-back of $<_{\mathcal{S}}$ by the map $g \in G \mapsto g.\mathfrak{e} \in \mathcal{S}$ coincides with the right jump preorder.*

Moreover, the restriction of the action Φ to the orbit of the trivial section \mathfrak{e} is faithful.

Proof. For $g \in G$, we compute from (9.10) the image of the trivial section:

$$(9.11) \quad g.\mathfrak{e}(x) = j^+(g, g^{-1}(x)) \pmod{\Lambda_0}.$$

Thus the subgroup H and the semigroup P defined at (9.8) are respectively, the residue and the positive cone of the pull-back preorder.

In order to prove that the action is faithful, we will find an element $g \in [G_c, G_c]$ and $x \in X$ for which $j^+(g, x) \notin \Lambda_0$. From (9.11), this will give that g does not fix the trivial section e , and we can conclude using Proposition 4.4. To do this, take an element $g_1 \in G_c$ whose support is a interval $J = (c, d) \Subset X$, with $D^-g_1(d) \notin \Lambda_0$. Take an element $h \in G$ such that $h(d) < c$, and consider the commutator $g = [g_1, h] \in [G_c, G_c]$. Then $j^+(g, d) = D^-g_1(d) \notin \Lambda_0$, as desired. \square

We need further preliminary results on the dynamics of the action Φ .

Lemma 9.17. *With notation as above, for any two distinct sections $s, t \in \mathcal{S}$ with $s \prec_S t$, there exists $g \in G$ such that $s \prec_S g.e \prec_S t$.*

Proof. Write $m = \max\{x \in X : s(x) \neq t(x)\}$, choose a point $m' < m$ in A such that t is locally constant on $[m', m]$, and take an element $g \in G$ such that

- (1) $t(x) = D^-g^{-1}(x)^{-1} \pmod{\Lambda_0}$ for every $x \in (m', b)$,
- (2) $t(m') >_{\Lambda/\Lambda_0} D^-g^{-1}(m')^{-1} \pmod{\Lambda_0}$,
- (3) $D^-g(x) = 1$ for every x in a sufficiently small neighborhood of b .

Such an element (or better, its inverse g^{-1}) can be produced by integrating a representative of the section t^{-1} on (m', b) (details are left to the reader). Note that the last condition gives

$$g.e(x) = j^+(g, g^{-1}(x)) = D^-g^{-1}(x)^{-1} \pmod{\Lambda_0},$$

so that by the first two conditions we get $s \prec_S g.e \prec_S t$. \square

Lemma 9.18. *With notation as above, assume in addition $b \in A \cup \{+\infty\}$ or $\Lambda_0 \neq \{1\}$. Then, for every choice of four sections $s_1, s_2, t_1, t_2 \in \mathcal{S}$ with $t_1 \prec_S s_1 \prec_S s_2 \prec_S t_2$, there exists an element $g \in G$ such that $g.s_1 \prec_S t_1 \prec_S t_2 \prec_S g.s_2$.*

Proof. First of all, we argue that we can assume that $s_1 \prec_S e \prec_S s_2$. Indeed, for general $s_1, s_2, t_1, t_2 \in \mathcal{S}$ with $t_1 \prec_S s_1 \prec_S s_2 \prec_S t_2$, after Lemma 9.17 we can take an element $h \in G$ such that $s_1 \prec_S h.e \prec_S s_2$, or equivalently $h^{-1}.s_1 \prec_S e \prec_S h^{-1}.s_2$. Assume there exists an element $k \in G$ such that

$$kh^{-1}.s_1 \prec_S h^{-1}.t_1 \prec_S h^{-1}.t_2 \prec_S kh^{-1}.s_2,$$

then the conjugate element $g = hkh^{-1}$ will make the job for $s_1, s_2, t_1, t_2 \in \mathcal{S}$.

Now, for general $t \in \mathcal{S} \setminus \{e\}$, define $x_t = \max\{x \in X : t(x) \neq \Lambda_0\}$ and write

$$x_* = \min\{x_{s_1}, x_{s_2}\}, \quad y_* = \max\{x_{t_1}, x_{t_2}\}.$$

Take an element $g \in G$ with the following properties:

- (1) $g(x_*) > y_*$ and
- (2) $j^+(g, g^{-1}(x)) = 1$ for every $x \in [g(x_*), b)$.

To ensure the second condition, we can either choose $g \in G(X; A, \Lambda_0)$ (when the residue Λ_0 is non-trivial) or an element g with no breakpoint in $[x_*, b)$ (in which case we have to assume that $b \in A \cup \{+\infty\}$).

Remark 9.19. We point out, for further use, that in the former case we can actually choose $g \in G(X; A, \Lambda_0) \cap [G_c, G_c]$.

Then, for $i \in \{1, 2\}$ we get from (9.10) the expression

$$g \cdot s_i(x) = s_i(g^{-1}(x)) \quad \text{for every } x \in [g(x_*) , b).$$

In particular, we have

$$g \cdot s_i(x) = \Lambda_0 = t_i(x) \quad \text{for every } x \in (g(x_{s_i}), b),$$

and $g \cdot s_i(g(x_{s_i})) = s_i(x_{s_i})$, while $t_i(g(x_{s_i})) = \Lambda_0$. As we are assuming $s_2 \prec_S e \prec_S s_2$, this gives

$$g \cdot s_1(g(x_{s_1})) <_{\Lambda/\Lambda_0} \Lambda_0 = t_1(g(x_{s_1})) \quad \text{and} \quad g \cdot s_2(g(x_{s_2})) >_{\Lambda/\Lambda_0} \Lambda_0 = t_2(g(x_{s_2})),$$

so that we can conclude $g \cdot s_1 \prec_S t_1 \prec_S t_2 \prec_S g \cdot s_2$. \square

Proof of Proposition 9.14. We will focus on the right jump preorder, the case for the left jump preorder being analogous. For given $t \in S$ and $x \in X$, consider the subset

$$C(t, x) = \{s \in S : s(y) = t(y) \text{ for every } y > x\}.$$

Then, we define $\mathcal{C} := \{C(t, x) : t \in S, x \in X\}$. It is a straightforward verification that \mathcal{C} is a $\Phi(G)$ -invariant CF-cover of S by \prec_S -convex subsets.

If we denote by φ the dynamical realization of the action Φ on the orbit of the trivial section e (which is countable), and by ι its associated good embedding, we have that the family $\{\text{hull}(\iota(C)) : C \in \mathcal{C}\}$ is a $\varphi(G)$ -invariant CF-cover (here $\text{hull}(A)$ denotes the interior of the convex hull of $A \subset \mathbb{R}$). Considering Lemma 9.18, we get from the minimality criterion (Lemma 2.42) that φ is minimal. Proposition 7.7 then implies that φ is an \mathbb{R} -focal action. Lemma 9.16 gives that φ is faithful.

To prove the second part of the statement we show that the preorder \leq_{Λ} can be read from the positive conjugacy class of φ . For this, given $g \in G_+$, let $m(g)$ denote the leftmost point of the support of g (namely, $m(g) = \sup\{x \in X : g(x) \neq x\}$). Given $\lambda \in \Lambda$ we consider a sequence $(g_n) \subseteq G(X; A, \langle \lambda \rangle) \cap G_+$ satisfying:

- $D^- g_n(m(g_n)) = \lambda$ and
- $m(g_n) \rightarrow b$ as $n \rightarrow +\infty$.

Note that this sequence exists since $G = G(X; A, \Lambda)$ is a Bieri–Strebel group. Similarly as in the proof of Lemma 9.18, we compute that for a given section $t \in S$, for every $n \in \mathbb{N}$ we have

$$g_n \cdot t(x) = t(x) \quad \text{for every } x \in (m(g_n), b)$$

and

$$g_n \cdot t(m(g_n)) = \lambda t(m(g_n)) \pmod{\Lambda_0}.$$

Assume first that $\lambda \in \Lambda_0$, so that $(g_n) \subseteq H = [1]_{\geq_+}$. Since $e \in \text{Fix}^{\Phi}(H)$ it holds that $\iota(e)$ is a common fixed point of the family $\{\varphi(g_n) : n \in \mathbb{N}\}$. Consider now the case where $\lambda \geq_{\Lambda} 1$. Note that in this case, for every $t \in S$ and $n \in \mathbb{N}$ large enough, the above computation gives $x_{g_n \cdot t} = m(g_n)$ and $(g_n \cdot t)(x_{g_n \cdot t}) \geq_{\Lambda} 1$. In particular we have that for every $t \in S$, $g_n \cdot t \rightarrow +\infty$ as $n \rightarrow +\infty$ which implies that $\varphi(g_n)(\xi) \rightarrow +\infty$ for every $\xi \in \mathbb{R}$. Analogously, in the case $\lambda \leq_{\Lambda} 1$ it holds that $\varphi(g_n)(\xi) \rightarrow -\infty$ for every $\xi \in \mathbb{R}$. As such qualitative properties of the action φ are invariant under positive conjugacy, we deduce that the positive conjugacy class of φ determines the preorder \leq_{Λ} . \square

Remark 9.20. Note that the existence of \mathbb{R} -focal actions for Bieri–Strebel groups also follows from Corollary 9.9. However, the examples in which \leq_{Λ} has non-trivial residue cannot be obtained from Corollary 9.9. Indeed, in Theorem 12.3 we show that all exotic actions of Bieri–Strebel groups of the form $G(\mathbb{R}; A, \Lambda)$ (with some finitary assumption on A and Λ) arise from cocycle preorders.

Remark 9.21. Given a preorder $\leq \in \text{LPO}(\Lambda)$ we can consider its opposite preorder, namely the preorder $\leq^{op} \in \text{LPO}(\Lambda)$ such that $P_{\leq} = P_{\leq^{op}}^{-1}$. Note that in the case $\leq_1, \leq_2 \in \text{LPO}(\Lambda)$ are one opposite of the other, their associated right (respectively, left) jump preorders are also opposite to each other. Therefore their dynamical realizations are conjugate (although they are not positively conjugate).

Corollary 9.22. *Let $G = G(X; A, \Lambda)$ be a countable Bieri–Strebel group and let $\leq_{\Lambda} \in \text{LPO}(\Lambda)$ be a preorder with non-trivial residue. Consider $\preceq \in \text{LPO}(G)$ the right or left jump preorder associated to \leq_{Λ} and φ its dynamical realization. Then, the action of $\varphi([G_c, G_c])$ on \mathbb{R} is minimal.*

Proof. We focus on the right jump preorder, the other case being analogous. We keep notation as above. We write $\Omega = [G_c, G_c].e$ for the orbit of the trivial section. We argue similarly as in the proof of Proposition 9.14. After (the proof of) Lemma 9.16, the dynamical realization of the action of $[G_c, G_c]$ on Ω is faithful. Also, by Lemma 9.18 and Remark 9.19 therein, we have that the stabilizer of e in $\Phi([G_c, G_c])$ is of homothetic type. As the action on $\Omega = [G_c, G_c].e$ is transitive, we can apply Proposition 2.43 and get that the dynamical realization, which coincides with the restriction of $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ to $[G_c, G_c]$, is minimal. \square

Remark 9.23. We point out that, although Thompson’s group F does not satisfy the hypothesis of Corollary 9.22, we show in Theorem 11.35 how to construct exotic actions of F with similar dynamical properties.

Remark 9.24. Recall that after Proposition 8.19, every minimal \mathbb{R} -focal action of a countable group can be obtained as the dynamical realization of an action on a planar directed tree. The result above show that there are cases for which such action cannot be chosen to be *simplicial*. Indeed, assume that $G = G(X; A, \Lambda)$ and $\leq_{\Lambda} \in \text{LPO}(\Lambda)$ satisfy the assumptions of Corollary 9.22. Let φ denote the dynamical realization of a corresponding jump preorder. Then, by Proposition 4.4, every non-trivial normal subgroup of G contains $[G_c, G_c]$ and therefore, by Corollary 9.22, the restriction of φ to any non-trivial normal subgroup of G is minimal. We conclude using Proposition 8.33.

9.4. A construction of micro-supported \mathbb{R} -focal actions. Recall from Proposition 7.14 that if G is a minimal micro-supported subgroup of $\text{Homeo}_0(\mathbb{R})$, then either G is locally moving or its standard action is \mathbb{R} -focal. In this subsection we give general construction of minimal micro-supported subgroups of $\text{Homeo}_0(\mathbb{R})$ whose action is \mathbb{R} -focal. We will use this construction to illustrate that the class of micro-supported subgroups of $\text{Homeo}_0(\mathbb{R})$ is much more flexible than the class of locally moving groups: we can find groups that admit uncountably many pairwise non-semi-conjugate faithful micro-supported actions (in contrast with Rubin’s theorem for locally moving groups, see Corollary 5.15). Also many of these examples can even be chosen to be of class C^1 (in contrast with Theorem 6.9). These groups are described as groups of automorphisms of planar directed (simplicial) trees, by adapting the classical construction of Burger and Mozes [21], and the related groups defined by Le Boudec [58].

9.4.1. Groups with many micro-supported actions. We say that a pair (A, a_0) is a *marked alphabet* if A is a set and $a_0 \in A$. Then, denote by $S \subseteq A^{\mathbb{Z}}$ the set of sequences with values in A which take the constant value a_0 in all but finitely many terms. Following the notation as in §9.1.2, denote by S_n the truncations to $\mathbb{Z}_{j \geq n}$ of the elements in S . Also, given a sequence $(t_j)_{j \geq n}$ say that $(t_j)_{j \geq n+1}$ is its successor. This defines a simplicial directed tree $(\mathbb{T}_A, \triangleleft)$ whose focus is defined by the successive truncations of the constant path s with value a_0 . By abuse of

notation, we will identify \mathbb{T}_A with its set of vertices $\bigsqcup_{n \in \mathbb{Z}} \mathcal{S}_n$. Recall that for a vertex $v \in \mathbb{T}_A$, we denote by E_v^- the set of edges below v (opposite to ω). For $v = (t_j)_{j \geq n}$, the set E_v^- is naturally identified with the alphabet A , considering the labeling

$$j_v : \begin{array}{ccc} E_v^- & \rightarrow & A \\ [(t_j)_{j \geq n-1}, (t_j)_{j \geq n}] & \mapsto & t_{n-1} \end{array}$$

Note also that every $g \in \text{Aut}(\mathbb{T}, \triangleleft)$ induces a bijection between E_v^- and $E_{g.v}^-$. We write $\sigma_{g,v} := j_{g(v)} \circ g \circ j_v^{-1}$ for the induced permutation of A . Observe that we have the cocycle relation

$$(9.12) \quad \sigma_{g,h(v)} \sigma_{h,v} = \sigma_{gh,v}.$$

Definition 9.25. Let (A, a_0) be a marked alphabet, $G \subset \text{Sym}(A)$ be a group of permutations of A , and let $(\mathbb{T}_A, \triangleleft)$ be the directed tree defined above together with the labelings $j_v, v \in \mathbb{T}_A$. We define the *generalized Brin–Navas group* $\text{BN}(A; G)$ as the group of all elements $g \in \text{Aut}(\mathbb{T}, \triangleleft)$ such that $\sigma_{g,v} \in G$ for all $v \in \mathbb{T}$ and $\sigma_{g,v} = \text{id}$ for all but finitely many $v \in \mathbb{T}$.

Remark 9.26. The name comes from the fact that the Brin–Navas group considered in Example 4.6 is isomorphic to the group $\text{BN}(\mathbb{Z}; G)$ where $G \subset \text{Sym}(\mathbb{Z})$ is the group of translations of the integers.

We need to define a suitable generating set. For this, given a vertex $v \in \mathbb{T}_A$ denote by $G_v \subset \text{BN}(A; G)$ the subgroup of all $g \in \text{BN}(A; G)$ which fix v and such that $\sigma_{g,w} = \text{id}$ for $w \neq v$ (clearly $G_v \cong G$). We then consider an extra generator defined as follows. Note that the shift map $\sigma : \mathcal{S} \rightarrow \mathcal{S}$, which sends a bi-infinite sequence $(t_j)_{j \in \mathbb{Z}}$ to $(t_{j-1})_{j \in \mathbb{Z}}$ naturally acts on the set of truncated sequences $\bigsqcup_{n \in \mathbb{Z}} \mathcal{S}_n$ preserving the successor relation. Thus it defines an automorphism $f_0 \in \text{Aut}(\mathbb{T}_A, \triangleleft)$. It is direct to check that f_0 is a hyperbolic element in $\text{BN}(A; G)$ whose axis consists on the geodesic $(w_n)_{n \in \mathbb{Z}}$ with $w_n = (a_0)_{j \geq n}$, and that moreover $\sigma_{f_0,v} = \text{id}$ for every vertex $v \in \mathbb{T}_A$. With this notation set, we have the following.

Lemma 9.27. *Let (A, a_0) be a marked alphabet, and assume that $G \subset \text{Sym}(A)$ acts transitively on A . Then the group $\text{BN}(A; G)$ is generated by G_{w_0} and f_0 . In particular it is finitely generated as soon as G is so.*

Proof. Write $\Gamma = \langle G_{w_0}, f_0 \rangle$ for the subgroup of $\text{Aut}(\mathbb{T}_A, \triangleleft)$ generated by G_{w_0} and f_0 . We first observe the following.

Claim. *For every vertices $v_1, v_2 \in \mathbb{T}_A$, there exists $g \in \Gamma$ such that $g.v_1 = v_2$ and $\sigma_{g,v} = \text{id}$ for every vertex $v \triangleleft v_1$.*

Proof of claim. First notice that, by composing with powers of f_0 and using the cocycle relation (9.12), we can assume that both v_1 and v_2 belong to \mathcal{S}_0 . Similarly, we can also assume that $v_1 = w_0 = (a_0)_{n \geq 0}$; write $v_2 = (t_n)_{n \geq 0}$. Since G acts transitively on A there exists a sequence $(h_n)_{n \geq 0}$ in G such that $h_n(a_0) = t_n$. Moreover, since $t_n = a_0$ for n large enough, we can take h_n to be the identity for n large enough. By abuse of notation denote by $h_n \in G_{w_0}$ the element with $\sigma_{h_n, w_0} = h_n$. Thus, the product $g := \prod_{n \geq 0} (f_0^n h_n f_0^{-n})$ is actually a finite product and thus defines an element of Γ , which moreover satisfies $\sigma_{g,v} = \text{id}$ for every $v \in \bigsqcup_{n < 0} \mathcal{S}_n$. It follows directly from the choices that $g.w_0 = v_2$. This proves the claim. \square

Take a vertex $v_0 \in \mathbb{T}_A$, by the previous claim we can take $g \in \Gamma$ so that $g.v_0 = w_0$ and $\sigma_{g,v} = \text{id}$ for every $v \triangleleft v_0$. Then it is direct to check that $g^{-1}G_{w_0}g = G_{v_0}$. This shows that $G_{v_0} \subseteq \Gamma$ for every vertex $v_0 \in \mathbb{T}_A$. Finally, given $g \in \text{BN}(A; G)$ write

$$C(g) := |\{v \in \mathbb{T}_A : \sigma_{g,v} \neq \text{id}\}|.$$

Notice that when $C(g) = 0$ then g is a power of f_0 , hence it belongs to Γ . Take an element $g \in \text{BN}(G)$ with $C(g) > 0$, and a vertex $v \in \mathbb{T}_A$ so that $\sigma_{g,v} \neq \text{id}$. Then we can find $h \in G_v$ satisfying $\sigma_{h,v} = \sigma_{g,v}$. Then we have $C(gh^{-1}) = C(g) - 1$. By repeating this procedure finitely many times we can find $h' \in \Gamma$ so that $C(gh') = 0$. We deduce that g belongs to the group Γ . \square

Notice that for every order $<$ on the alphabet A there exists a unique planar order \prec on $(\mathbb{T}_A, \triangleleft)$ for which the maps j_v are order preserving. If in addition the subgroup $G \subset \text{Sym}(A)$ preserves $<$, the group $\text{BN}(A; G)$ preserves the associated planar order \prec . We call this action the induced *planar directed tree representation* associated with G and $<$.

Proposition 9.28. *Let (A, a_0) be a marked alphabet, assume that $G \subset \text{Sym}(A)$ acts transitively on A and let $<$ be an G -invariant total order on A . Then, the dynamical realization of the planar directed tree representation associated with G and $<$ is a minimal, faithful and micro-supported \mathbb{R} -focal action of $\text{BN}(A; G)$.*

Moreover, for two different G -invariant orders on A , the dynamical realizations of their corresponding planar directed tree representations are not positively semi-conjugate.

Proof. Let \prec be the planar order associated with $<$, and let $\Phi : \text{BN}(A; G) \rightarrow \text{Aut}(\mathbb{T}_A, \triangleleft, \prec)$ be the corresponding action. Since Φ acts transitively on the vertices of \mathbb{T}_A and $\text{BN}(A; G)$ contains a hyperbolic element, the focality of Φ follows.

Let $\xi \in \partial^* \mathbb{T}_A$ be the end defined by the sequence $(w_n)_{n \leq 0}$, and let \mathcal{O}_ξ be its orbit under the associated $\text{BN}(A; G)$ -action on $\partial^* \mathbb{T}_A$. Since f_0 is a hyperbolic element with axis $(w_n)_{n \in \mathbb{Z}}$, it acts on (\mathcal{O}_ξ, \prec) as a homothety fixing ξ . This allows us to apply Lemma 2.42 to deduce that the dynamical realization of $\text{BN}(G) \rightarrow \text{Aut}(\mathcal{O}_\xi, \prec)$ is minimal. In other words, the dynamical realization of Φ , that we denote by φ , is minimal. On the other hand, since G_v is supported on the set of points below v , its associated action on \mathcal{O}_ξ is supported on the shadow of v . This implies that the support of $\varphi(G_v)$ is relatively compact, and thus by Proposition 4.2 we deduce that φ is micro-supported. Finally, the faithfulness of φ follows from that of Φ .

Given $h \in G$ and vertex $v \in \mathbb{T}_A$ we denote by h_v the element of G_v satisfying $\sigma_{h_v, v} = h$. Also, let $p \in \mathbb{R}$ be the fixed point of $\varphi(f_0)$. Then, given $h \in G$ we have

$$\varphi(h_{w_0})(p) > p \quad \text{if and only if} \quad h(a_0) > a_0.$$

Since the action of G on A is transitive, this implies that we can read the total order $<$ from the action φ . In particular, dynamical realizations corresponding to different G -invariant orders on A give rise to non-positively-conjugate actions. Finally, since these dynamical realizations are minimal and not positively conjugate, they are not positively semi-conjugate. \square

Note that by the transitivity assumption in Proposition 9.28, the planar directed tree representation is in fact determined by a choice of a left-invariant preorder on G , as an abstract group, and the second part in the statement says that two different left-invariant preorders on G with same residue yield non-positively-semi-conjugate actions. As a particular case, one can consider a left-invariant order on G , in which case we can identify the marked alphabet with $(G, 1_G)$, and the subgroup $G \subset \text{Sym}(G)$ is the group of left translations. In such case, we will simply write $\text{BN}(G)$ for the group $\text{BN}(G; G)$.

Recall that if $G \subset \text{Homeo}_0(\mathbb{R})$ is a locally moving group, then every faithful locally moving action of G on \mathbb{R} is conjugate to its standard action (by Rubin's theorem, or by Corollary 5.15). The groups $\text{BN}(G)$ show that this is far from being true for micro-supported subgroups of $\text{Homeo}_0(\mathbb{R})$.

Corollary 9.29. *Let G be a finitely generated group whose space of left-invariant orders is uncountable. Then $\text{BN}(G)$ has uncountably many, pairwise-non-conjugate, faithful, minimal and micro-supported actions on the line.*

Proof. After the previous discussion, we can apply Proposition 9.28 to each left order in $\text{LO}(G)$, which gives the desired result. \square

9.4.2. *Groups with many differentiable micro-supported actions.* Here we extend the result given by Corollary 9.29 by showing that for some finitely generated groups G , we can actually get many faithful micro-supported actions of class C^1 of the group $\text{BN}(G)$ (compare with Theorem 6.9).

Theorem 9.30. *There exists a finitely generated group admitting uncountably faithful minimal micro-supported actions, which are pairwise not semi-conjugate (and not semi-conjugate to a non-faithful action), and each of which is semi-conjugate to a C^1 action. More precisely, the generalized Brin–Navas group $\text{BN}(\mathbb{Z}^2)$ satisfies such properties.*

Remark 9.31. In fact, it seems also possible to prove that the group $\text{BN}(\mathbb{Z}^2)$ admits minimal faithful micro-supported actions which are *conjugate* to a C^1 action and pairwise not semi-conjugate, but since this is more technical we will content ourselves of the previous statement (which suffices to disprove the analogue of Theorem 6.9 for micro-supported groups).

Theorem 9.30 will be obtained as a consequence of Proposition 9.35, which is a criterion to recognize faithful actions of $\text{BN}(G)$ on the line. As a preliminary result, which may also help to follow the rather technical proof of Proposition 9.35, we work out a presentation of $\text{BN}(G)$, analogue to the one for $\text{BN}(\mathbb{Z})$ appearing in Example 4.6. So let G be a finitely generated group, and fix a presentation

$$G = \langle g_1, \dots, g_n \mid r_\nu(g_1, \dots, g_n) (\nu \in \mathbb{N}) \rangle.$$

We will assume for simplicity that the generating set is symmetric and that for every distinct $i, j \in \{1, \dots, n\}$, the generators g_i and g_j represent distinct non-trivial elements in G . The free product $G * \mathbb{Z}$ admits the presentation

$$G * \mathbb{Z} = \langle f, g_1, \dots, g_n \mid r_\nu(g_1, \dots, g_n) (\nu \in \mathbb{N}) \rangle.$$

It is convenient to introduce the following more redundant presentation of $G * \mathbb{Z}$, which corresponds to applying a (multiple) Tietze transformation to the previous one. For $m \in \mathbb{Z}$, write $\mathcal{G}_m = \{g_{i,m} : i \in \{1, \dots, n\}\}$ and $\mathcal{G} = \bigcup_{m \in \mathbb{Z}} \mathcal{G}_m$. Then, we get the presentation

$$(9.13) \quad G * \mathbb{Z} = \left\langle f_0, \mathcal{G} \mid f g_{i,m} f^{-1} = g_{i,m+1} \ (i \in \{1, \dots, n\}, m \in \mathbb{Z}), r_\nu(g_{1,0}, \dots, g_{n,0}) (\nu \in \mathbb{N}) \right\rangle.$$

Write \mathcal{R}_0 for the set of relators in the previous presentation. As in the proof of Lemma 9.28, given $g \in G \subset \text{Sym}(G)$ and $v \in \mathbb{T}_G$ we denote by g_v the element in $\text{BN}(G)$ fixing v such that $\sigma_{g,v} = g$ and $\sigma_{g,w} = \text{id}$ for every $w \neq v$. Then, we denote by $\Psi_0 : G * \mathbb{Z} \rightarrow \text{BN}(G)$ the morphism such that $\Psi_0(f) = f_0$ and $\Psi_0(g_{i,m}) = f_0^m (g_i)_{w_0} f_0^{-m}$ for every $i \in \{1, \dots, n\}$, and $m \in \mathbb{Z}$ (as before, we write $w_0 = (1_G)_{n \geq 0}$).

In order to complete the set of relations for the desired presentation of $\text{BN}(G)$, we need to study the support of some elements.

Lemma 9.32. *With notation as above, fix $m \in \mathbb{Z}$. Then for every $g \in \langle \mathcal{G}_m \rangle \setminus \{1\}$ and $g_1, g_2 \in \bigcup_{q < m} \mathcal{G}_q$ we have that the commutator $[g_1, g g_2 g^{-1}]$ is in the kernel of Ψ_0 .*

Proof. We will show that for any such choices of elements, we have that g_1 and gg_2g^{-1} have disjoint support in \mathbb{T}_G and hence commute. On the one hand, for every $k \in \mathbb{Z}$, and $h \in \mathcal{G}_k$, the support of $\Psi_0(h)$ is contained in $\mathbb{T}_G^{w_k}$, where $w_k = (1_G)_{n \geq k}$. On the other hand, for any $h \in \bigcup_{q < m} \mathcal{G}_q$, we have that the support of $\Psi_0(ghg^{-1})$ is contained in the subtree $\mathbb{T}_G^{v_{g,m-1}}$, where $v_{g,m-1} \in \mathbb{T}_G$ is the vertex corresponding to the sequence $(t_n)_{n \geq m-1}$ such that $t_0 = 1_G$ for $n \geq m$ and $t_{m-1} = g$. In particular, these two remarks apply respectively to the elements g_1 and g_2 from the statement, so that the images $\Psi_0(g_1)$ and $\Psi_0(gg_2g^{-1})$ have disjoint supports. \square

Write \mathcal{R}_1 for the set of all the commutation relators from Lemma 9.32 and set

$$(9.14) \quad \Gamma = \langle f, \mathcal{G} \mid \mathcal{R}_0, \mathcal{R}_1 \rangle.$$

Then, we are ready to state the following:

Proposition 9.33. *The map $\Psi : \Gamma \rightarrow \text{BN}(G)$ induced by Ψ_0 is an isomorphism.*

Before proving Proposition 9.33 we need to fix some notation and state a technical lemma that will also be used later for Proposition 9.35, and whose proof is postponed. As in the proof of Lemma 9.32, given $g \in G$ and $m \in \mathbb{Z}$ denote by $v_{g,m} \in \mathbb{T}_G$ the vertex $(t_n)_{n \geq m}$ such that $t_n = 1_G$ for $n > m$ and $t_m = g$; in particular $w_m = v_{1,m}$. Note that after the conjugation relations in \mathcal{R}_0 , the subgroup $H := \langle \mathcal{G} \rangle$ is the normal closure of \mathcal{G}_0 in Γ , and the quotient Γ/H is generated by the image of f . Given $\gamma \in H$, we denote by $\|\gamma\|_{\mathcal{G}}$ its word-length with respect to the generating system \mathcal{G} .

Lemma 9.34. *Take a non-trivial element of H written as $\gamma = \gamma_1 \cdots \gamma_k$ with $\gamma_j \in \mathcal{G}_{m_j}$ for $j \in \{1, \dots, k\}$, and write $M = \max\{m_j : j \in \{1, \dots, k\}\}$. Assume that $\Psi(\gamma)(w_m) = w_m$ for some $m < M$. Then, there exist $h_1, \dots, h_l \in \langle \bigcup_{m < M} \mathcal{G}_m \rangle$ and pairwise distinct elements $f_1, \dots, f_l \in \langle \mathcal{G}_M \rangle$, such that*

$$(9.15) \quad \gamma = (f_1 h_1 f_1^{-1}) \cdots (f_l h_l f_l^{-1})$$

and $\|h_i\|_{\mathcal{G}} < k$ for every $i \in \{1, \dots, l\}$.

Proof of Proposition 9.33. First notice that, by Lemma 9.27, Ψ is surjective. To prove injectivity of Ψ , suppose by contradiction that $\ker \Psi$ is non-trivial. Consider the focal germ representation $\tau : \text{BN}(G) \rightarrow \text{Germ}(b) \cong \mathbb{Z}$ and notice that τ vanishes at $G_{w_0} = \Psi(\mathcal{G}_0)$ and that $\tau(f_0)$ is non-trivial. As the quotient of Γ by the normal closure H of \mathcal{G}_0 is generated by the image of f , we deduce that $\ker \Psi \subseteq H$. In particular, the word-length $\|\cdot\|_{\mathcal{G}}$ is defined on the kernel of Ψ , so that we can consider a non-trivial element $\gamma \in \ker \Psi$ of minimal word-length. We write $\gamma = \gamma_1 \cdots \gamma_k$ with $k = \|\gamma\|_{\mathcal{G}}$ and $\gamma_j \in \mathcal{G}_{m_j}$ for every $j \in \{1, \dots, k\}$. As $\Psi(\gamma)$ acts trivially, in particular it fixes every vertex of the form w_m , so that we can apply Lemma 9.34 to the factorization $\gamma = \gamma_1 \cdots \gamma_k$ and obtain a decomposition as in (9.15) with $\|h_i\|_{\mathcal{G}} < \|\gamma\|_{\mathcal{G}}$ for $i \in \{1, \dots, l\}$. Keeping the same notation as in Lemma 9.34, we observe that the support of $\Psi(f_i h_i f_i^{-1})$ is contained in $\mathbb{T}_G^{v_{f_i, M-1}}$ for $i \in \{1, \dots, l\}$. Thus, since $f_i \neq f_j$ for $i \neq j$, we get that different factors in the factorization of γ above have disjoint support. We deduce that every such factor must be in the kernel of Ψ , and therefore also every element h_i . However, as the word-length of every such element is less than $\|\gamma\|_{\mathcal{G}}$, the minimality assumption on γ implies that every h_i is trivial, contradicting the choice of γ . \square

Proof of Lemma 9.34. In order to simplify notation, from here and until the end of the proof, we will write $\prod_{j \in E} \alpha(j) := \alpha(i_1) \cdots \alpha(i_k)$, for any function $\alpha : E \rightarrow \Gamma$, with $E = \{i_1, \dots, i_k\} \subseteq \mathbb{N}$ with $i_1 < \cdots < i_k$. We will also write σ_g instead of $\sigma_{\Psi(g), w_M}$, for every $g \in \Gamma$. Note that $\sigma_g = \text{id}$ for every $g \in \langle \bigcup_{m < M} \mathcal{G}_m \rangle$. For given $j \in \{1, \dots, k\}$, we let i_j be the index such

that $\gamma_j = g_{i_j, m_j}$. Notice that $\Psi(\gamma_j)(w_M) = w_M$ for every $j \in \{1, \dots, k\}$, and in particular $\Psi(\gamma)(w_M) = w_M$. Using the cocycle relation (9.12), we have

$$(9.16) \quad \sigma_\gamma = \prod_{j=1}^k \sigma_{\gamma_j}.$$

On the other hand, since $\Psi(\gamma)(w_s) = w_s$ for some $s < M$ and the action of $G \subset \text{Sym}(G)$ on G is free, we conclude that $\sigma_\gamma = \text{id}$. Since only factors in \mathcal{G}_M give non-trivial factors in the product (9.16), writing $P = \{j \in \{1, \dots, k\} : \gamma_j \in \mathcal{G}_M\}$, we get the equality

$$\text{id} = \prod_{j \in P} \sigma_{\gamma_j}.$$

Note that for $j \in P$, the permutation σ_j corresponds to the left translation by g_{i_j} , whence we get $\prod_{j \in P} g_{i_j} = 1_G$. This implies

$$(9.17) \quad \prod_{j \in P} \gamma_j = 1_\Gamma.$$

Next, for every $j \in \{1, \dots, k\}$, consider the product in $\langle \mathcal{G}_M \rangle$ given by

$$f_j := \prod_{l \in P, l \leq j} \gamma_l.$$

Set $Q := \{1, \dots, k\} \setminus P$ and notice that after the relation (9.17) we can write

$$(9.18) \quad \gamma = \prod_{j \in Q} (f_j \gamma_j f_j^{-1}).$$

Note that as $f_j \in \langle \mathcal{G}_M \rangle$ and $\gamma_j \in \langle \mathcal{G}_{m_j} \rangle$ with $m_j < M$ for every $j \in Q$, we deduce from Lemma 9.32 that we can make commute any two factors $f_i \gamma_i f_i^{-1}$ and $f_j \gamma_j f_j^{-1}$ in the product in (9.18), as soon as $f_i \neq f_j$. Rearranging factors in this way, we can write

$$(9.19) \quad \gamma = \prod_{j \in Q_0} \left(f_j \left(\prod_{l \in Q_0^j} \gamma_l \right) f_j^{-1} \right),$$

where $Q_0 \subseteq Q$ is a section of the map $Q \rightarrow \Gamma$ given by $j \mapsto f_j$, and for $j \in Q_0$ we set $Q_0^j := \{l \in Q_0 : f_l = f_j\}$. We claim that the decomposition in (9.19) is the one that we are looking for. First notice that, by the choice of Q_0 , $f_i \neq f_j$ whenever i, j are different indices in Q_0 . Secondly for every $j \in Q_0$, by definition of Q_0 and Q_0^j , the element $h_j := \prod_{l \in Q_0^j} \gamma_l$ belongs to $\langle \bigcup_{m < M} \mathcal{G}_m \rangle$ and clearly satisfies $\|h_j\|_{\mathcal{G}} \leq |Q_0^j| < k$. \square

Consider now, for a left-invariant order $<$ on a group G , the directed tree representation $\Phi : \text{BN}(G) \rightarrow \text{Aut}(\mathbb{T}_G, \triangleleft, \prec)$ associated with G and $<$; let $\varphi : \text{BN}(G) \rightarrow \text{Homeo}_0(\mathbb{R})$ be its corresponding dynamical realization. Pursuing the discussion from the proof of Proposition 9.28, we extract some properties of φ , which characterize it up to (positive) semi-conjugacy.

First recall that $G_{w_0} \subseteq \text{BN}(G)$ is a subgroup supported on the subset of points below the vertex w_0 . This implies that $J_0 := \text{Supp}^\varphi(G_{w_0})$ is a relatively compact interval. Since $\Phi(f_0)$ is a hyperbolic element, the homeomorphism $\varphi(f_0)$ is a homothety (see Proposition 8.26). For $n \in \mathbb{Z}$, write $J_n := f_0^n . J_0$; since $f_0^{-1} . w_0 \triangleleft w_0$, we have the inclusion $J_{-1} = f_0^{-1} . J_0 \Subset J_0$. Finally, since the action of G on itself by left translations is free, we get that G_{w_0} acts freely on $E_{w_0}^-$, which implies $g . J_{-1} \cap J_{-1} = \emptyset$ for every $g \in G_{w_0} \setminus \{1\}$. Moreover, the total order $<$ on G (which coincides with the planar order \prec^{w_0} on $E_{w_0}^-$) can be read from the action:

$$g > 1 \Leftrightarrow g.x > x \text{ for some } x \in J_{-1} \text{ (equivalently } \forall x \in J_{-1}).$$

Summarizing we have the following:

- (a) $\text{Supp}^\varphi(G_{w_0}) =: J_0$ consists on a relatively compact interval;
- (b) $\varphi(f_0)$ is a homotethy satisfying $J_{-1} \Subset J_0$, with $J_{-1} := f_0^{-1}.J_0$;
- (c) $g.J_{-1} \cap J_{-1} = \emptyset$ for every $g \in G_{w_0} \setminus \{1\}$;
- (d) given $x \in J_{-1}$ and $g \in G_{w_0}$, it holds that $g.x > x$ if and only if $g > 1$.

For the next statement, recall that $\text{BN}(G)$ is isomorphic to the group Γ which, in turn, can be written as the quotient $\Gamma = G * \mathbb{Z} / \langle \langle \mathcal{R}_1 \rangle \rangle$ (following the notation in (9.13) and (9.14), see Proposition 9.33). We denote by $\pi : G * \mathbb{Z} \rightarrow \Gamma$ the corresponding projection. We also define the *height* of an element $\gamma \in H = \langle \mathcal{G} \rangle$ as

$$(9.20) \quad \text{ht}(\gamma) := \inf \left\{ n \in \mathbb{Z} : \gamma \in \left\langle \bigcup_{m \leq n} \mathcal{G}_m \right\rangle \right\}.$$

Notice that the $\text{ht}(\gamma) > -\infty$ for every $\gamma \in H \setminus \{1\}$. Indeed, if it were not the case, its image under the isomorphism Ψ in Proposition 9.33 would be trivial, which is not the case.

Proposition 9.35. *Let $(G, <)$ be a finitely generated left-ordered group. Consider the free product $G * \mathbb{Z}$, and denote by f a generator of its cyclic factor. Consider also an action $\varphi_0 : G * \mathbb{Z} \rightarrow \text{Homeo}_0(\mathbb{R})$ and assume that it satisfies conditions (a) to (d) above with $\langle \mathcal{G}_0 \rangle$ and f instead of G_{w_0} and f_0 , respectively. Then, φ_0 factors through the quotient $\pi : G * \mathbb{Z} \rightarrow \Gamma$ inducing an action $\varphi_1 : \Gamma \rightarrow \text{Homeo}_0(\mathbb{R})$ which is (positively) semi-conjugate to the dynamical realization of the directed tree representation associated with $G \subset \text{Sym}(G)$ and $<$.*

Proof. After Proposition 9.33, in order to show that φ_0 factors through the projection π we need to check that the elements in \mathcal{R}_1 belong to the kernel of φ_0 . That is, we need to check that the elements of the form $[g_1, gg_2g^{-1}]$ are in the kernel of φ_0 , whenever $g \in \langle \mathcal{G}_m \rangle \setminus \{1\}$ and $g_1, g_2 \in \bigcup_{q < m} \mathcal{G}_q$. We closely follow the proof of Lemma 9.32. On the one hand, for every $q < m$ and $h \in \mathcal{G}_q$, there exists $h' \in \mathcal{G}_0$ such that $h = f^q h' f^{-q}$, so that by condition (a), we have that the support of $\varphi_0(h)$ is contained in $J_q := f_0^q.J_0$, and thus in J_m after condition (b) (and the same argument for $q = m$). On the other hand, for any $g \in \langle \mathcal{G}_m \rangle \setminus \{1\}$, condition (c) gives that $g.J_{m-1} \cap J_{m-1} = \emptyset$. Thus, the support of $\varphi_0(gg_2g^{-1})$ is disjoint from J_m . Putting this all together we get that the support of $\varphi_0(g_1)$ and that of $\varphi_0(gg_2g^{-1})$ are disjoint, as desired.

As in the statement, denote by $\varphi_1 : \Gamma \rightarrow \text{Homeo}_0(\mathbb{R})$ the action induced by φ_0 , and let \mathcal{I} be the orbit of J_0 under $\varphi_1(\Gamma)$. Let also $\Phi : \Gamma \rightarrow \text{Aut}(\mathbb{T}_G, \triangleleft, \prec)$ be the directed tree representation associated with G and $<$ and denote by φ the dynamical realization of Φ . Our goal is to show that φ and φ_1 are semi-conjugate. This will be a direct consequence of the following statement.

Claim. *The family of intervals \mathcal{I} is a CF-cover, which determines a simplicial planar directed tree which is order-isomorphic to $(\mathbb{T}_G, \triangleleft, \prec)$, via a Γ -equivariant isomorphism.*

Proof of claim. We consider the map

$$F : \mathbb{T}_G \rightarrow \mathcal{I} \\ g.w_0 \mapsto g.J_0$$

and then prove that it gives the desired Γ -equivariant order-isomorphism. To simplify notation, given $I_1, I_2 \in \mathcal{I}$, we write $I_1 < I_2$ if $\sup I_1 \leq \inf I_2$. Also, given two vertices $v_1, v_2 \in \mathbb{T}_G$, we write $v_1 \prec v_2$ if for every points $\xi_1 \in \partial U_{v_1}$ and $\xi_2 \in \partial U_{v_2}$ in the shadows, we have $\xi_1 \prec \xi_2$.

We want to show that the map F is well-defined, Γ -equivariant, and satisfies the following conditions:

- (1) $v_1 \triangleleft v_2$ implies $F(v_1) \subset F(v_2)$,
- (2) $v_1 \prec v_2$ implies $F(v_1) < F(v_2)$

To see that F is well-defined, we need to check that $\text{Stab}^\Phi(w_0) \subseteq \text{Stab}^{\varphi_1}(J_0)$. Given $\gamma \in \langle \mathcal{G} \rangle$ consider its height $\text{ht}(\gamma)$ defined as in (9.20). Notice that if $\text{ht}(\gamma) \leq 0$ then the support of γ is contained in $\bigcup_{r \leq 0} J_r$ and therefore $\gamma \in \text{Stab}^{\varphi_1}(J_0)$. In order to show the inclusion between the stabilizers we claim that, every $\gamma \in \text{Stab}^\Phi(w_0)$ can be written as

$$(9.21) \quad \gamma = \gamma_1 \gamma_2 \quad \text{with } \gamma_1 \in \text{Stab}^{\varphi_1}(J_0) \text{ and } \text{ht}(\gamma_2) < \text{ht}(\gamma).$$

To obtain a decomposition as in (9.21), first note that, in the case $\text{ht}(\gamma) \leq 0$ we are done. Suppose it is not the case, and write $\gamma = \gamma_1 \cdots \gamma_k$ with

$$\max\{\text{ht}(\gamma_j) : j \in \{1, \dots, k\}\} = \text{ht}(\gamma) > 0.$$

Then, since $\gamma.w_0 = w_0$ and $0 < \text{ht}(\gamma)$ we are in condition to apply Lemma 9.34 to the decomposition $\gamma = \gamma_1 \cdots \gamma_k$. Thus, we can write $\gamma = (f_1 h_1 f_1^{-1}) \cdots (f_l h_l f_l^{-1})$ with $f_1, \dots, f_l \in \langle \mathcal{G}_{\text{ht}(\gamma)} \rangle$ such that $f_i \neq f_j$ for $i \neq j$, and $\text{ht}(h_i) < \text{ht}(\gamma)$ for $i \in \{1, \dots, l\}$. As discussed above, for every $i \in \{1, \dots, l\}$ such that $f_i \neq 1_\Gamma$, we have that the support of $\varphi_1(f_i h_i f_i^{-1})$ is disjoint from $J_{\text{ht}(\gamma)-1}$ and, as a consequence, disjoint from J_0 . Thus, if $f_i \neq 1_\Gamma$ for every $i \in \{1, \dots, l\}$ we are done. Suppose it is not the case, and that for some i we have $f_i = 1_\Gamma$. In this case, by applying the commutation relations in \mathcal{R}_1 from Lemma 9.32, we can assume that $f_l = 1_\Gamma$ and we set $\gamma_1 = (f_1 h_1 f_1^{-1}) \cdots (f_{l-1} h_{l-1} f_{l-1}^{-1})$ and $\gamma_2 = h_l$. Notice that, as we argued in the previous case, the element γ_1 fixes J_0 . Finally notice that, by the choice from Lemma 9.34, we have $\text{ht}(h_l) < \text{ht}(\gamma)$. This gives the desired decomposition as in (9.21).

By applying the decomposition as in (9.21) finitely many times, we get a factorization $\gamma = \delta_1 \cdots \delta_r$ where δ_i is in the stabilizer of J_0 for $i \in \{1, \dots, r-1\}$, and $\text{ht}(\delta_r) \leq 0$. Finally, since $\text{ht}(\delta_r) \leq 0$, we also have that δ_r is in the stabilizer of J_0 and therefore the inclusion between the stabilizers follows. This gives that the map F is well-defined, as wanted. Moreover, by definition of F , we also have that it is Γ -equivariant.

In order to prove condition (1), first recall that $\text{BN}(G)$ acts transitively on the vertices of \mathbb{T}_G (see the claim in the proof of Lemma 9.27). Thus, for every vertices $v_1 \triangleleft v_2$ in \mathbb{T}_G , there exists $\gamma \in \Gamma$ such that $\gamma.v_1 = w_s$ for some $s \in \mathbb{Z}$ and thus $\gamma.v_2 = w_r$ for some $r > s$. Since the partial order \triangleleft is preserved by the action Φ and the inclusion relation is preserved by the action induced by φ_1 on \mathcal{I} , using Γ -equivariance of F we only need to check that condition (1) holds when we take v_1, v_2 in the subset $\{w_n : n \in \mathbb{Z}\}$. For this consider $w_i \triangleleft w_j$, then $F(w_i) = J_i \subset J_j = F(w_j)$ as desired. To prove condition (2) first notice that, since condition (1) holds, it is enough to check the condition taking $v_1 \prec v_2$ adjacent to $v_1 \wedge v_2$. Notice that the relation \prec on \mathbb{T}_G is invariant under Φ , and the relation $<$ on \mathcal{I} is invariant under the action induced by φ_1 . Following the same reasoning as for condition (1), it is enough to check condition (2) taking v_1, v_2 adjacent to w_0 . In that case, condition (1) follows from condition (d) in the statement. Summarizing, conditions (1) and (2) are satisfied by the map F , as wanted. \square

After the claim, we have that the directed tree representation $\Phi : \Gamma \rightarrow \text{Aut}(\mathbb{T}_G, \triangleleft, \prec)$ is conjugate to a focal action representing φ_1 (technically speaking, its positive semi-conjugacy class, as φ_1 need not be minimal and we may be forced to consider a minimal action semi-conjugate to φ_1). This implies that φ and φ_1 are positively semi-conjugate. \square

Proof of Theorem 9.30. Given any irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, denote by $\tau^\alpha : \mathbb{Z}^2 \rightarrow \text{Homeo}_0(\mathbb{R})$ the action by translations so that $\tau^\alpha((1, 0))(x) = x + 1$ and $\tau^\alpha((0, 1))(x) = x + \alpha$. By the Pixton–Tsuboi examples (see [107]), for each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists an action $\varphi^\alpha : \mathbb{Z}^2 \rightarrow \text{Diff}_0^1(\mathbb{R})$ such that:

- φ^α is supported on $(0, 1)$;
- the restriction $\varphi^\alpha \upharpoonright_{(0,1)}$ is semi-conjugate to τ^α and
- $\varphi^\alpha \upharpoonright_{(0,1)}$ has an exceptional minimal set $\Lambda_\alpha \subseteq (0, 1)$.

Then, for any irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, consider an affine expanding homothety $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ with fixed point in $((0, 1) \setminus \Lambda_\alpha)$ so that $f_\alpha^{-1}((0, 1)) \cap \Lambda_\alpha = \emptyset$. Consider the free product $\mathbb{Z}^2 * \mathbb{Z}$ and denote by f_0 a generator of the cyclic factor. Then, we define the action $\varphi_0^\alpha : \mathbb{Z}^2 * \mathbb{Z} \rightarrow \text{Diff}_0^1(\mathbb{R})$ so that φ_0^α coincides with φ^α on the \mathbb{Z}^2 factor and $\varphi_0^\alpha(f_0) = f_\alpha$. It is direct to check that the action φ_0^α satisfies conditions (a) to (d) in the statement of Proposition 9.35 with $G = \mathbb{Z}^2$ and with $<$ being the left-order $<_\alpha$ induced by τ^α . Then, applying Proposition 9.35 we conclude that φ_0^α induces an action $\Psi^\alpha : \text{BN}(\mathbb{Z}^2) \rightarrow \text{Diff}_0^1(\mathbb{R})$ which is semi-conjugate to the dynamical realization of the planar directed tree representation associated with $\mathbb{Z}^2 \subseteq \text{Sym}(\mathbb{Z}^2)$ and $<_\alpha$, which is minimal, faithful, and micro-supported (in particular, any action sei-conjugate to Ψ^α must be faithful). On the other hand, by Proposition 9.28, different orders $<_{\alpha_1}$ and $<_{\alpha_2}$ give rise to planar directed tree representations with non-conjugate dynamical realizations, therefore, Ψ^{α_1} and Ψ^{α_2} are not semi-conjugate. \square

10. A STRUCTURE THEOREM FOR EXOTIC ACTIONS OF A CLASS OF LOCALLY MOVING GROUPS

10.1. The class \mathcal{F} . In this section we will focus on a class of locally moving groups, and prove a more precise structure theorem on their exotic continuous actions on the line. Such class, named \mathcal{F} , is defined in terms of a finite generation condition on the rigid stabilizers.

Definition 10.1. For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup. We say that G is in the class \mathcal{F} if it satisfies the following conditions:

- (1) G is locally moving,
- (2) for every $x \in X$ the group $G_{(a,x)}$ is contained in a finitely generated subgroup of G_+ , and the group $G_{(x,b)}$ is contained in a finitely generated subgroup of G_- .

Various examples of groups in the class \mathcal{F} arise as groups of piecewise linear and projective homomorphisms: Thompson’s group F and all Thompson–Brown–Stein groups F_{n_1, \dots, n_k} , many other Bieri–Strebel groups (see Section 12) and the groups of Lodha–Moore [65]. The following proposition shows that this class goes far beyond groups of piecewise linear or projective homeomorphisms.

Proposition 10.2. *For $X = (a, b)$, let $H \subset \text{Homeo}_0(X)$ be a countable subgroup. Then there exists a finitely generated subgroup $G \subset \text{Homeo}_0(X)$ which contains H , has an element without fixed points, and belongs to the class \mathcal{F} (hence to the class \mathcal{F}_0 defined later).*

Proof. We can assume without loss of generality that H is finitely generated, since every countable subgroup of $\text{Homeo}_0(X)$ is contained in a finitely generated one [63]. Assume also for simplicity that $X = (0, 1)$. Let $b_\ell : (0, 1) \rightarrow (0, 3/4)$ and $b_r : (0, 1) \rightarrow (1/4, 1)$ be the homeomorphisms defined respectively by

$$b_\ell(x) = \begin{cases} x & x \in (0, 1/2), \\ \frac{1}{2}x + \frac{1}{4} & x \in [1/2, 1), \end{cases} \quad b_r(x) = \begin{cases} \frac{1}{2}x + \frac{1}{4} & x \in (0, 1/2), \\ x & x \in [1/2, 1). \end{cases}$$

Set also $b_0 = b_\ell b_r$. That is, b_0 is the homothety of slope 1/2 centered at the point 1/2, while each of b_ℓ and b_r fixes half of the interval $(0, 1)$ and coincides with b_0 on the other half. See Figure 9.

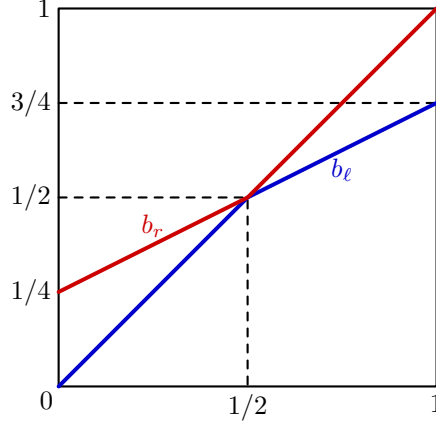


FIGURE 9. The maps b_r (red) and b_ℓ (blue).

Set $H_0 = b_0 H b_0^{-1}$, $H_\ell = b_\ell H b_\ell^{-1}$, and $H_r = b_r H b_r^{-1}$. Note that each of these groups is a group of homeomorphisms of a subinterval of $(0, 1)$, and we see them as groups of homeomorphisms of $(0, 1)$ by extending them to the identity outside their support. Set $G = \langle H, H_\ell, H_r, H_0, F \rangle$, where F is the standard copy of Thompson's group F acting on $(0, 1)$. As F contains elements without fixed points, so does G . Let us prove that G is in the class \mathcal{F} .

As F and H are finitely generated, so is G ; moreover G is locally moving since F is so. Let us show that $G_{(0,1/2)}$ is contained in a finitely generated subgroup of G_+ . To this end, consider the group $\Gamma = b_\ell G b_\ell^{-1}$. Then Γ is a finitely generated subgroup of $\text{Homeo}_0((0, 1))$ supported in $(0, 3/4)$ (we again extend it as the identity on $(3/4, 1)$). Moreover, Γ contains $G_{(0,1/2)}$: indeed since b_ℓ acts trivially on $(0, 1/2)$, for $g \in G_{(0,1/2)}$ we have $g = b_\ell g b_\ell^{-1} \in \Gamma$. Thus the desired conclusion follows if we show that Γ is contained in G (hence in G_+).

For this, note that $b_\ell F b_\ell^{-1} = F_{(0,3/4)} \subset F \subset G$ and $b_\ell H b_\ell^{-1} = H_\ell \subset G$; we also easily observe

$$b_\ell H_r b_\ell^{-1} = b_\ell b_r H (b_\ell b_r)^{-1} = b_0 H b_0^{-1} = H_0 \subset G.$$

Choose an element $f \in F$ which coincides with b_ℓ in restriction to $(0, 3/4)$. Then the conjugation action of f on the subgroups H_ℓ, H_0 coincides with the conjugation by b_ℓ , thus $b_\ell H_\ell b_\ell^{-1} = f H_\ell f^{-1}$ and $b_\ell H_0 b_\ell^{-1} = f H_0 f^{-1}$ are also contained in G . Since

$$\Gamma = b_\ell \langle H, H_0, H_\ell, H_r, F \rangle b_\ell^{-1},$$

we conclude that Γ is contained in G . Thus $\Gamma \subset G_+$ is a finitely generated subgroup that contains $G_{(0,1/2)}$. Since for every $x \in (0, 1)$ the group $G_{(0,x)}$ is conjugate to a subgroup of $G_{(0,1/2)}$, we have that $G_{(0,x)}$ is contained in a finitely generated subgroup of G_+ for every x . The case of the subgroups $G_{(x,1)}$ is analogous. \square

10.2. The main theorem for the class \mathcal{F} . The goal of this subsection is to show that all exotic actions of a group in the class \mathcal{F} must be tightly related to the standard action, through the notion of horograding of \mathbb{R} -focal actions (see §8.4).

Theorem 10.3. *For $X = (a, b)$, let $G \subset \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F} . Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is semi-conjugate to an action in one of the following families:*

- (i) (Standard) *the natural action on X ;*
- (ii) (Induced from a quotient) *an action induced from the largest quotient $G/[G_c, G_c]$*
- (iii) (\mathbb{R} -focal) *a minimal \mathbb{R} -focal action, which can be horograded by the natural action of G on X .*

Remark 10.4. In the \mathbb{R} -focal case, Proposition 8.32 allows to determine the dynamics of individual elements by looking at the standard action of G on X . For this, consider the case where φ is increasingly horograded by the natural action of G on X (the other case is analogous). Then, we have the following:

- if $\text{Fix}(g)$ accumulates at b , then $\varphi(g)$ is totally bounded;
- if $g(x) > x$ (respectively $g(x) < x$) in a neighborhood of b , then φ is an expanding (respectively contracting) pseudohomothety and
- if $g(x) > x$ (respectively $g(x) < x$) for every $x \in X$, then $\varphi(g)$ is an expanding (respectively contracting) homothety.

10.2.1. *Sketch of the proof and preliminary observations.* We will divide the proof into steps, which individually yield some additional information. As G is locally moving, by Theorem 5.3 it is enough to consider the case where φ is an exotic action. By symmetry, we will assume that the subgroup G_+ is locally dominated by commuting elements (within $[G_c, G_c]$). We first observe that we can assume that the action φ is minimal (Lemma 10.5) and then describe the set of fixed points for the images of the rigid stabilizers $\varphi(G_{(a,x)})$ (Lemma 10.6). This allows to introduce an invariant CF-cover, which will give that φ is \mathbb{R} -focal (Proposition 10.8). To determine the horograded action, we have to construct a focal action on a planar directed tree representing φ (this is done throughout §10.2.3). We then put all of this together and prove Theorem 10.3. Let us start.

Lemma 10.5. *For $X = (a, b)$, let $G \subset \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F} . Consider a faithful exotic action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$. Then φ is semi-conjugate to a minimal action.*

Proof. We will assume that G_+ is locally dominated by commuting elements, the other case being analogous. Then for every $x \in X$ the image $\varphi(N_{(x,b)})$ has no fixed points (see Proposition 5.12). Since G is in the class \mathcal{F} , for every $x \in X$ there exists a finitely generated subgroup $\Gamma \subset G_-$ containing $N_{(x,b)}$, so that $\varphi(\Gamma)$ has no fixed points. Thus, there exists a compact interval $I \subset \mathbb{R}$ which intersects every $\varphi(\Gamma)$ -closed invariant subset, and in particular every $\varphi(G)$ -closed invariant subset. As in the proof of Proposition 7.6, it follows that $\varphi(G)$ has a unique minimal invariant set, and that φ is semi-conjugate to a minimal action. \square

Thus without loss of generality, upon replacing φ with a semi-conjugate action, we will assume that φ is minimal. The key ingredient in the proof of Theorem 10.3 is the following result, which follows again from the trichotomy for locally moving groups (Theorem 5.3) together with the assumption that G is in \mathcal{F} .

Lemma 10.6 (Key lemma). *For $X = (a, b)$, let $G \subset \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F} . Consider a minimal faithful exotic action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ satisfying that G_+ is locally dominated by commuting elements. Then for every $x \in X$ the subset $\text{Fix}^\varphi(G_{(a,x)})$ is non-empty, accumulates on both $\pm\infty$, and has empty interior. The family of closed subsets*

$\{\text{Fix}^\varphi(G_{(a,x)})\}_{x \in X}$ is decreasing in x , and we have

$$\lim_{x \rightarrow a} \text{Fix}^\varphi(G_{(a,x)}) = \mathbb{R} \quad \text{and} \quad \lim_{x \rightarrow b} \text{Fix}^\varphi(G_{(a,x)}) = \emptyset,$$

where the limits are taken with respect to the Fell topology on closed subsets of \mathbb{R} .

Proof. Since G_+ is locally dominated by commuting elements, all its finitely generated subgroups are totally bounded. Moreover, since for every $x \in X$ the subgroup $G_{(a,x)}$ is contained in a finitely generated subgroup of G_+ , it is also totally bounded, i.e. the subset $C_x := \text{Fix}^\varphi(G_{(a,x)})$ is non-empty and accumulates on both $\pm\infty$. Let us postpone the proof that it has empty interior. Observe that since $\{G_{(a,x)} : x \in X\}$ is an increasing family of subgroups, the family of subsets $\{C_x : x \in X\}$ is decreasing. In particular, the two limits in the statement exist and are given by

$$\lim_{x \rightarrow a} C_x = \overline{\bigcup_{x \in X} C_x} \quad \text{and} \quad \lim_{x \rightarrow b} C_x = \bigcap_{x \in X} C_x.$$

From this, since the family $\{C_x\}$ satisfies the equivariant property $C_{g(x)} = g.C_x$, it is clear that the two limits are closed G -invariant subsets, thus by minimality each of them is equal either to \mathbb{R} or to \emptyset . Since the limit on the left-hand side contains every subset C_x , it is non-empty, thus it must be equal to \mathbb{R} . As for the second, it is contained in each subset C_x , which is a strict subset of \mathbb{R} (otherwise $\varphi(G_{(a,x)})$ would act trivially, contradicting faithfulness of the action), and therefore it must be equal to \emptyset .

Let us now show that every subset C_x must have empty interior. Assume by contradiction that $J \subset C_x$ is a non-empty open interval. Fix also an arbitrary $y \in X$ and write $H = G_{(a,y)}$. Let I be a connected component of $\text{Supp}^\varphi(H) = \mathbb{R} \setminus C_y$. Since φ is minimal, then it is proximal (Lemma 5.13), hence there exists $g \in G$ such that $g.J \supset I$. Thus $\varphi(G_{(a,g(x))})$ acts trivially on I , and since $G_{(a,x)} \cap H \neq \{\text{id}\}$, we deduce that the action of H on I is not faithful. Since H is a locally moving subgroup of $\text{Homeo}_0((a,y))$, by Proposition 4.4 we see that the kernel of this action must contain $[H_c, H_c]$. Since I was an arbitrary connected component of $\text{Supp}^\varphi(H)$, this implies that $[H_c, H_c]$ acts trivially on all $\text{Supp}^\varphi(H)$, and thus the φ -image of $[H_c, H_c]$ is trivial, contradicting the assumption that φ is faithful. \square

10.2.2. *An invariant CF-cover.* Lemma 10.6 allows to identify a natural invariant CF-family for the action φ , and to construct a planar directed tree. To this end, let us introduce the following notation.

Definition 10.7. Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be as in Lemma 10.6. For every $x \in X$ and $\xi \in \mathbb{R}$, with $\xi \notin \text{Fix}^\varphi(G_{(a,x)})$, we denote by $I^\varphi(x, \xi)$ the connected component of $\text{Supp}^\varphi(G_{(a,x)})$ which contains ξ .

Note that the family of open subsets $\text{Supp}^\varphi(G_{(a,x)}) = \mathbb{R} \setminus \text{Fix}^\varphi(G_{(a,x)})$ is increasing with respect to $x \in X$. Thus if $\xi \in \text{Supp}^\varphi(G_{(a,x)})$, then for every $y > x$ we have $\xi \in \text{Supp}^\varphi(G_{(a,y)})$ and $I^\varphi(x, \xi) \subset I^\varphi(y, \xi)$ for every $y > x$. Note also that Lemma 10.6 implies the subset $\text{Supp}^\varphi(G_{(a,x)})$ is open and dense for every $x \in X$. Thus, by Baire's theorem, the intersection

$$(10.1) \quad \Xi := \bigcap_{x \in X} \text{Supp}^\varphi(G_{(a,x)})$$

is a G_δ dense subset of \mathbb{R} , which is moreover invariant under the action of G . For $\xi \in \Xi$, the interval $I^\varphi(x, \xi)$ is well-defined and non-empty for every $x \in X$. In the following ξ will always denote a point in Ξ unless mentioned otherwise. With this notation, we have the following consequence of Lemma 10.6.

Proposition 10.8. *With notation as above, the collection*

$$\mathcal{S} = \{I^\varphi(x, \xi) : x \in X, \xi \in \Xi\}$$

defines a G -invariant CF-cover. In particular, φ is a minimal \mathbb{R} -focal action.

Proof. It is clear from the definition of the intervals $I^\varphi(x, \xi)$ that they verify the equivariance property $g \cdot I^\varphi(x, \xi) = I^\varphi(g(x), g \cdot \xi)$. Thus the family \mathcal{S} is invariant, and covers \mathbb{R} by the second limit in Lemma 10.6. Note also that Lemma 10.6 implies that every interval in \mathcal{S} is bounded. To see that it is cross-free, consider two intervals $I_1, I_2 \subset \mathbb{R}$ of the form $I_1 = I^\varphi(x_1, \xi_1)$ and $I_2 = I^\varphi(x_2, \xi_2)$, with $x_1 \leq x_2$. Observe first that if $x_1 = x_2$, then I_1 and I_2 are both connected components of $\text{Supp}^\varphi(G_{(a, x_1)})$, so they are either equal or disjoint. Otherwise, assume that $I_1 \cap I_2 \neq \emptyset$. Note that $I_1 \subset I^\varphi(x_2, \xi_1)$, so that $I_2 = I^\varphi(x_2, \xi_1)$. Thus $I_1 \subset I_2$. This shows that \mathcal{S} is an invariant CF-cover. As φ is minimal, Proposition 7.7 implies that φ is \mathbb{R} -focal. \square

10.2.3. *A planar directed tree.* We now explain how to use the invariant CF-cover \mathcal{S} to construct a planar directed tree with the horograding required for the conclusion of Theorem 10.3. For this, we need to examine some further properties of this family.

As already observed, for a fixed point $\xi \in \Xi$ the intervals $\{I^\varphi(x, \xi) : x \in X\}$ are a totally order family, non-decreasing in the variable $x \in X$, but in general they do not vary continuously with respect to x (in fact one could easily show that their endpoint *cannot* vary continuously at every $x \in X$). Thus we also introduce the outer and inner limits

$$I_{\text{out}}^\varphi(x, \xi) := \text{Int} \left(\bigcap_{y > x} I^\varphi(y, \xi) \right) \quad \text{and} \quad I_{\text{inn}}^\varphi(x, \xi) = \bigcup_{y < x} I^\varphi(y, \xi).$$

Notice that the fact that being crossed is an open condition together with Proposition 10.8 imply that both $\{I_{\text{out}}^\varphi(x, \xi)\}$ and $\{I_{\text{inn}}^\varphi(x, \xi)\}$ are also invariant CF-covers.

We define the directed tree \mathbb{T} as the family of intervals

$$\mathbb{T} := \{I_{\text{out}}^\varphi(x, \xi) : x \in X, \xi \in \Xi\},$$

with the partial order \triangleleft given by the inclusion of intervals. The group G acts on \mathbb{T} , by the expression $g \cdot I_{\text{out}}^\varphi(x, \xi) = I_{\text{out}}^\varphi(g(x), g \cdot \xi)$, and this action clearly preserves the inclusion relations. To show that this defines indeed a directed tree we first show the following lemma, which analyzes how these intervals vary as functions of (x, ξ) .

Lemma 10.9 (Injectivity in the variable x). *With notation as above, if $I^\varphi(x_1, \xi_1) = I^\varphi(x_2, \xi_2)$, then $x_1 = x_2$. As a consequence, the same conclusion holds for the families of intervals $I_{\text{out}}^\varphi(x, \xi)$ and $I_{\text{inn}}^\varphi(x, \xi)$.*

Proof. Assume by contradiction that $x_1 < x_2$. Then, since $G_{(a, x_2)}$ preserves the interval $I^\varphi(x_2, \xi_2)$, for every $g \in G_{(a, x_2)}$ we have

$$I^\varphi(x_2, \xi_2) = g \cdot I^\varphi(x_2, \xi_2) = g \cdot I^\varphi(x_1, \xi_1) = I^\varphi(g(x_1), g \cdot \xi_1).$$

By definition, the last interval is a connected component of $\text{Supp}^\varphi(G_{(a, g(x_1))})$, so in particular the image $\varphi(G_{(a, g(x_1))})$ has no fixed point in $I^\varphi(x_2, \xi_2)$ for every $g \in G_{(a, x_2)}$. However since $G_{(a, x_2)}$ acts without fixed points on (a, x_2) we can choose g such that $g(x_1)$ is arbitrarily close to a . This contradicts the fact that $\lim_{y \rightarrow a} \text{Fix}^\varphi(G_{(a, y)}) = \mathbb{R}$, established by Lemma 10.6. \square

The previous lemma implies that the function

$$\begin{aligned} \pi : \quad \mathbb{T} &\rightarrow X \\ I_{\text{out}}^\varphi(x, \xi) &\mapsto x \end{aligned}$$

is well defined, and clearly G -equivariant.

Lemma 10.10. *With notation as above, the set $(\mathbb{T}, \triangleleft)$ is a directed tree and the map $\pi: \mathbb{T} \rightarrow X$ is an increasing horograding.*

Proof. Let us check that all conditions in Definition 8.1 are satisfied. Given $I = I_{\text{out}}^\varphi(x, \xi) \in \mathbb{T}$, the set of intervals above I is $\{I_{\text{out}}^\varphi(y, x) : y \geq x\}$, which by Lemma 10.9 is order-isomorphic to the interval $[x, b)$, showing (T1). To check (T2) note that since intervals $\{I^\varphi(x, \xi)\}$ define a CF-cover, any two elements in \mathbb{T} have a common upper bound, and by definition of the outer approximations I_{out}^φ the infimum of the set of such upper bounds is again a common upper bound (thus the smallest possible). Finally to see (T3), note that for a given $x \in X$ there are countably many intervals of the form $I_{\text{out}}^\varphi(x, \xi)$ when ξ varies (since every two such intervals are either equal or disjoint). Thus by letting $X_0 \subset X$ be any countable dense set, the collection $\Sigma = \{I_{\text{out}}^\varphi(x, \xi) : x \in X_0, \xi \in \Xi\}$ satisfies (T3). The fact that π is an increasing horograding is clear from the construction. \square

Finally note that for a point $v = I_{\text{out}}^\varphi(x, \xi) \in \mathbb{T}$, the set E_v^- of directions below v is in one-to-one correspondence with the set of intervals $\{I_{\text{inn}}^\varphi(x, \xi') : \xi' \in I_{\text{out}}^\varphi(x, \xi)\}$. Since any two distinct intervals of this form are disjoint, this set of intervals inherits a natural order from the ordering of \mathbb{R} , which allows to define an order $<^v$ on E_v^- . The family of orders $\prec = \{<^v : v \in \mathbb{T}\}$ defined in this way is a planar order on \mathbb{T} which is invariant for the action of G .

Remark 10.11. We point out that the residual set Ξ defined in (10.1) is in natural correspondence with the π -complete boundary of $(\mathbb{T}, \triangleleft)$ (see Definition 8.8). However, we do not elaborate on this since it is not necessary for the rest of the discussion.

Proof of Theorem 10.3. After Theorem 5.3 we can suppose that φ is an exotic action, and we will assume that the subgroup G_+ is locally dominated by commuting elements. Then Lemma 10.5 and Proposition 10.8 together prove the first part in point (iii) of Theorem 10.3, namely that, up to semi-conjugacy, φ is \mathbb{R} -focal.

Consider then the action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$ of G on the planar directed tree defined above. Note that the fact that the action is proximal (Lemma 5.13) gives that the action is focal. The map $\pi: \mathbb{T} \rightarrow X$ is an increasing G -equivariant horograding, so we only need to show that the action φ is conjugate to the dynamical realization of Φ . To this end observe that every $\xi \in \Xi$ can be thought of as an end $\xi \in \partial_* \mathbb{T}$, namely the unique infimum of the maximal totally ordered subset $\{I_{\text{out}}^\varphi(x, \xi) : x \in X\}$. The order on the G -orbit of ξ with respect to the action of φ induced from \mathbb{R} coincides with the order induced by the planar order on $\partial^* \mathbb{T}$. This shows that φ is the dynamical realization of $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \omega, \prec)$, as desired. \square

10.3. Non-orientation-preserving actions. Let us record here a simple consequence of the previous results, which might be worth pointing out. Namely if we allow group actions on intervals that are not necessarily orientation-preserving, then a stronger rigidity phenomenon occurs for groups in the class \mathcal{F} , which rules out the existence of exotic actions altogether.

Corollary 10.12. *For $X = (a, b)$, let $G \subseteq \text{Homeo}(X)$ be a subgroup such that $G \not\subseteq \text{Homeo}_0(X)$ and $G \cap \text{Homeo}_0(X)$ is in the class \mathcal{F} . Then every action $\varphi: G \rightarrow \text{Homeo}(\mathbb{R})$, which does not fix any point or a pair of points, is semi-conjugate either to the natural action of G on X or to a non-faithful action.*

In particular every faithful minimal action $\varphi: G \rightarrow \text{Homeo}(\mathbb{R})$ is topologically conjugate to the natural action on X .

Note that the proof does not require Theorem 10.3, but only Lemma 10.6 (actually only the first part of its statement).

Proof of Corollary 10.12. Set $H = G \cap \text{Homeo}_0(X)$, and let $\varphi: G \rightarrow \text{Homeo}(\mathbb{R})$ be an action without fixed points. Note that the assumption implies that the image of H has no fixed points, since otherwise every point in $\text{Fix}^\varphi(H)$ has a G -orbit of order at most $2 = |G/H|$. Assume first that the action $\varphi \upharpoonright_H$ is exotic. Indeed, if this is not the case, then by Proposition 5.12 and Lemma 10.6 we can assume that $\text{Fix}^\varphi(H_{(a,x)}) \neq \emptyset$ for every $x \in X$, while $\text{Fix}^\varphi(H_{(x,b)}) = \emptyset$ for every $x \in X$ (or that the opposite condition holds). However, since G contains elements that reverse the orientation of X , every subgroup of the form $H_{(a,x)}$ is conjugate inside G to a subgroup of the form $H_{(y,b)}$, so this is not possible. Thus, $\varphi \upharpoonright_H$ is semi-conjugate either to an action of $H/[H_c, H_c]$ or to the standard action of H on X . The first case occurs if and only if $\text{Fix}^\varphi([H_c, H_c]) \neq \emptyset$. Since $[H_c, H_c]$ is normal in the whole group G , this implies that in fact the action of the whole group G is semi-conjugate to an action of $G/[H_c, H_c]$. If $\varphi \upharpoonright_H$ is semi-conjugate to the standard action, then there exists a unique minimal H -invariant set $\Lambda \subset \mathbb{R}$, which is invariant under the whole group G , and we can obtain a minimal action $\psi: G \rightarrow \text{Homeo}(\mathbb{R})$ semi-conjugate to φ by collapsing all connected components of $\mathbb{R} \setminus \Lambda$. Then by construction $\psi \upharpoonright_H$ is conjugate to the standard action of H on X , so let $q: \mathbb{R} \rightarrow X$ be a topological conjugacy. It is therefore enough to check that the map q is equivariant under the whole group G . This is easily done after observing that since H is locally moving, different points in X have distinct stabilizers in H , so both actions can be reconstructed from the conjugation action of G on the stabilizers in H . Namely, for every $\xi \in \mathbb{R}$ we have $\text{Stab}_H^\varphi(\xi) = \text{Stab}_H(q(\xi))$. Thus conjugating by $g \in G$ we obtain

$$\text{Stab}_H^\varphi(g.\xi) = g \text{Stab}_H^\varphi(\xi) g^{-1} = g \text{Stab}_H(q(\xi)) g^{-1} = \text{Stab}_H(g(q(\xi)))$$

Since on the other hand we have $\text{Stab}_H^\varphi(g.\xi) = \text{Stab}_H(q(g.\xi))$, this implies that $q(g.\xi) = g(q(\xi))$ for every $g \in G$ and $\xi \in \mathbb{R}$, i.e. that the map q is G -equivariant, as desired. \square

An example of a group satisfying the assumption of Corollary 10.12 is *Thompson's group with flips* $F^\pm \subset \text{Homeo}((0, 1))$, which is defined similarly to Thompson group F , but by allowing negative slopes in the definition.

10.4. An attractor in the Deroin space and local rigidity. In this subsection we consider the following subclass \mathcal{F}_0 of the class \mathcal{F} .

Definition 10.13. Let $X = (a, b)$. We say that a group $G \subseteq \text{Homeo}_0(X)$ is in the class \mathcal{F}_0 if it is finitely generated, it belongs to \mathcal{F} (Definition 10.1) and there exists $f \in G$ without fixed points in X .

For a group G in the class \mathcal{F}_0 we prove the following theorem, which corresponds to Theorem 1.14 from the introduction.

Theorem 10.14. *Let X be an open interval and let $G \subset \text{Homeo}_0(X)$ be a finitely generated group in the class \mathcal{F}_0 . Let $\mathcal{U} \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ be the set of irreducible actions that are not semi-conjugate to any action induced from the largest quotient $G/[G_c, G_c]$. Then \mathcal{U} is open and admits a subset $\mathcal{S} \subset \mathcal{U}$ satisfying the following.*

- (i) *The set \mathcal{S} is closed in \mathcal{U} and relatively compact in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$. In particular it is a locally compact Polish space.*
- (ii) *Every $\varphi \in \mathcal{S}$ is a minimal and faithful action.*

- (iii) Every $\varphi \in \mathcal{U}$ is positively semi-conjugate to a unique action $\bar{\varphi} \in \mathcal{S}$ and the map $\varphi \mapsto \bar{\varphi}$ is a continuous retraction from \mathcal{U} to \mathcal{S} . In particular the quotient of \mathcal{U} by the semi-conjugacy equivalence relation, with the quotient topology, is homeomorphic to \mathcal{S} .
- (iv) The unique action $\psi \in \mathcal{S}$ which is positively semi-conjugate to the standard action of G on X is an isolated point in \mathcal{S} .

In order to prove Theorem 10.14 we will proceed by analyzing the structure of the Deroin space $\text{Der}_\mu(G)$ of normalized harmonic G -actions (see §2.2).

Recall that the Deroin space carries a natural flow Φ which, given $\varphi \in \text{Der}_\mu(G)$ and $t \in \mathbb{R}$, gives the action $\Phi^t(\varphi)$ obtained by conjugating φ by the translation $x \mapsto x + t$ (whence the name *translation flow*). Throughout the subsection, we work in the following setting.

Assumption 10.15. For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F}_0 . We fix $f \in G$ such that $f(x) > x$ for every $x \in X$. We also fix once and for all a symmetric probability measure μ supported on a finite generating set of G .

10.4.1. *Description of the Deroin space.* Let us introduce some further notation. Given an action $\varphi \in \text{Der}_\mu(F)$, we denote by $\hat{\varphi}$ its reversed action (the conjugate under the reflection $x \mapsto -x$). Then, by Theorem 10.3 we have a decomposition of the Deroin space

$$(10.2) \quad \text{Der}_\mu(G) = \mathcal{N} \sqcup \mathcal{I} \sqcup \hat{\mathcal{I}} \sqcup \mathcal{P},$$

defined by what follows.

- We let $\mathcal{N} \subset \text{Der}_\mu(G)$ be the subset consisting of actions φ which are not faithful, or equivalently such that $\varphi([G_c, G_c]) = \{\text{id}\}$.
- We fix a representative $\iota \in \text{Der}_\mu(G)$ positively conjugate to the standard action on X , and let $\mathcal{I} = \{\Phi^t(\iota) : t \in \mathbb{R}\}$ be its Φ -orbit and $\hat{\mathcal{I}}$ be the Φ -orbit of $\hat{\iota}$.
- We let $\mathcal{P} = \mathcal{P}_+ \sqcup \mathcal{P}_-$ be the subset of $\text{Der}_\mu(G)$ of \mathbb{R} -focal actions, where \mathcal{P}_+ and \mathcal{P}_- are the subsets of those which are increasingly (respectively, decreasing) horograded by the standard action on X .

Proposition 8.27 implies that for every $\varphi \in \mathcal{P}$ the image $\varphi(f)$ is a homothety, which is expanding if $\varphi \in \mathcal{P}_+$ and contracting if $\varphi \in \mathcal{P}_-$. For $\varphi \in \mathcal{P}$, denote by ξ_φ the unique fixed point of $\varphi(f)$ in \mathbb{R} . We say that φ is *f-centered* if $\xi_\varphi = 0$. Let us denote by $\mathcal{P}^0 \subset \mathcal{P}$ the subset of *f-centered* \mathbb{R} -focal actions, with $\mathcal{P}_\pm^0 = \mathcal{P}^0 \cap \mathcal{P}_\pm$.

Note that each of the subsets of $\text{Der}_\mu(G)$ in (10.2) is Φ -invariant. We aim to analyze the topology of these subsets and the dynamics of the flow Φ on them. A first obvious observation is that the set \mathcal{N} is closed in $\text{Der}_\mu(G)$. Our main goal is to show that \mathcal{N} is a *uniform attractor* for the flow Φ , that is all Φ -orbits in the complement of \mathcal{N} spend a uniformly bounded amount of time away from any neighborhood of \mathcal{N} . This is formalized in Proposition 10.17 below. Whenever U is a Φ -invariant subset of $\text{Der}_\mu(G)$ we say that $K \subset U$ is a *cross section* of Φ inside U if it intersects every Φ -orbit contained in U exactly once. With this notation we have the following.

Lemma 10.16. *Under Assumption 10.15, the following hold.*

- (i) Both subsets \mathcal{P}_\pm are open and Φ -invariant, and each subset \mathcal{P}_\pm^0 is a cross section of Φ inside \mathcal{P}_\pm .
- (ii) The closures of the subsets \mathcal{P}_\pm^0 in $\text{Der}_\mu(G)$ satisfy $\overline{\mathcal{P}_\pm^0} \subset \mathcal{P}_\pm^0 \cup \mathcal{N}$.

Proof. It is clear that \mathcal{P}_+ and \mathcal{P}_- are both Φ -invariant. To show that \mathcal{P}_+ is open, fix $\varphi \in \mathcal{P}_+$. Since $\varphi(f)$ is an expanding homothety centered at ξ_φ , for every bounded open interval I

containing ξ_φ we have $I \Subset \varphi(f)(I)$. Thus φ has a neighborhood \mathcal{U} in $\text{Der}_\mu(G)$ such that every $\psi \in \mathcal{U}$ satisfies $I \Subset \psi(f)(I)$. This gives that for $\psi \in \mathcal{U}$, the map $\psi(f)$ cannot be a contracting homothety, hence $\mathcal{U} \cap \mathcal{P}_- = \emptyset$. Similarly, this gives $\mathcal{U} \cap (\mathcal{I} \cup \widehat{\mathcal{I}}) = \emptyset$, as in the standard action the map f has no fixed point. Moreover since \mathcal{N} is closed, we can restrict \mathcal{U} so that it is contained in \mathcal{P}_+ , showing that \mathcal{P}_+ is open. An analogous argument shows that \mathcal{P}_- is also open. Moreover for every $\varphi \in \mathcal{P}_\pm$ we have $\Phi^t(\varphi) \in \mathcal{P}_\pm^0$ if and only if $t = -\xi_\varphi$, showing that each \mathcal{P}_\pm^0 is a cross section for Φ inside \mathcal{P}_\pm . Finally suppose that $\psi \in \overline{\mathcal{P}_+^0}$. Since $\varphi(f)$ fixes 0 for every $\varphi \in \mathcal{P}_+^0$, so does $\psi(f)$, and in particular $\psi \notin \mathcal{I} \cup \widehat{\mathcal{I}}$. Since \mathcal{P}_- is open, the only possibility is that $\psi \in \mathcal{P}_+^0 \cup \mathcal{N}$, so that $\overline{\mathcal{P}_+^0} \subset \mathcal{P}_+^0 \cup \mathcal{N}$. Similarly $\overline{\mathcal{P}_-^0} \subset \mathcal{P}_-^0 \cup \mathcal{N}$. \square

In particular it follows that the restriction of the flow Φ to the complement of \mathcal{N} in $\text{Der}_\mu(G)$ has a cross section, namely $\mathcal{P}^0 \cup \{\iota, \widehat{\iota}\}$. The central point here is the following.

Proposition 10.17 (\mathcal{N} is a uniform attractor). *Under Assumption 10.15, let \mathcal{U} be an open neighborhood of \mathcal{N} in $\text{Der}_\mu(G)$. Then there exists $t_0 > 0$ such that for every $t \in \mathbb{R}$ with $|t| > t_0$ and every $\varphi \in \mathcal{P}^0 \cup \{\iota, \widehat{\iota}\}$ we have $\Phi^t(\varphi) \in \mathcal{U}$.*

Before proving Proposition 10.17, let us observe that it implies Theorem 10.14.

Proof of Theorem 10.14. Set $\mathcal{S} = \mathcal{P}^0 \cup \{\iota, \widehat{\iota}\}$, and $\mathcal{V} = \mathcal{P} \cup \mathcal{I} \cup \widehat{\mathcal{I}}$. Note also that the set \mathcal{U} in the statement of Theorem 10.14 is precisely the preimage of \mathcal{V} under the harmonic retraction $r: \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R})) \rightarrow \text{Der}_\mu(G)$ given by Theorem 3.2. By Lemma 10.16 the set \mathcal{S} is closed in \mathcal{V} (and hence in \mathcal{U}), its closure is contained in $\text{Der}_\mu(G)$ and thus is compact, and the point ι and $\widehat{\iota}$ are isolated in \mathcal{S} . Now the uniform convergence in Proposition 10.17 implies that the map $\sigma: \mathbb{R} \times \mathcal{S} \rightarrow \mathcal{V}$, given by $\sigma(t, \varphi) = \Phi^t(\varphi)$, is a homeomorphism from $\mathbb{R} \times \mathcal{S}$ to \mathcal{V} . It follows that there is a continuous retraction of $p: \mathcal{V} \rightarrow \mathcal{S}$ which preserves Φ -orbits. Thus the map $p \circ r$ is a continuous retraction of \mathcal{U} to \mathcal{S} which preserves semi-conjugacy classes as desired. \square

For the proof of Proposition 10.17 we need some preliminary lemmas. The case $\varphi \in \{\iota, \widehat{\iota}\}$ is actually straightforward and boils down to the following.

Lemma 10.18. *Under Assumption 10.15, for $\varphi \in \{\iota, \widehat{\iota}\}$ all limit points of $\Phi^t(\varphi)$ as $t \rightarrow \pm\infty$ are contained in \mathcal{N} .*

Proof. For every finite subset $S \subset G_c$ the image $\varphi(S)$ is supported in a compact interval. Since $\varphi_t := \Phi^t(\varphi)$ is the conjugate of φ by a translation by t , it follows that when $|t|$ is sufficiently large, the image $\varphi_t(S)$ acts trivially on an arbitrarily large compact interval, so that $\psi(S) = \{\text{id}\}$ for every limit action ψ of φ_t as $|t| \rightarrow +\infty$. Since S is arbitrary, we deduce that $G_c \subseteq \ker \psi$, and thus $\psi \in \mathcal{N}$. \square

We now turn to analyze the limit points of $\Phi^t(\varphi)$ for $\varphi \in \mathcal{P}^0$. In fact, a pointwise convergence towards \mathcal{N} for every $\varphi \in \mathcal{P}^0$ would be not difficult to obtain from the qualitative properties of \mathbb{R} -focal actions. To obtain a uniform convergence we will need a quantitative control on the size of the intervals in the CF-family of intervals associated with φ . Recall that given an \mathbb{R} -focal action $\varphi \in \mathcal{P}_+$, for given $x \in X$ and $\xi \in \text{Supp}^\varphi(G_{(a,x)})$ we denote by $I^\varphi(x, \xi)$ the connected component of $\text{Supp}^\varphi(G_{(a,x)})$ containing ξ (Definition 10.7), and that the intervals of this form allow to define an invariant CF-family for φ (see Proposition 10.8). In the case of \mathbb{R} -focal actions in \mathcal{P}_- , one can introduce the analogous notation for connected components of $\text{Supp}^\varphi(G_{(x,b)})$. The key point in the proof of Proposition 10.17 is the following uniform bound on the size of these intervals in harmonic coordinates.

Lemma 10.19 (Bound on the CF-family in harmonic coordinates). *Under Assumption 10.15, there exists a constant $C = C(\mu)$ such that the following holds. For every $x \in X$, there exists a constant $D = D(x, \mu)$ such that for every $\varphi \in \mathcal{P}_+$ and every $\xi \in \text{Supp}^\varphi(G_{(a,x)})$ with $|\xi - \xi_\varphi| \geq D$, we have $|\text{I}^\varphi(x, \xi)| \leq C$.*

The analogous result holds for actions $\varphi \in \mathcal{P}_-$.

Proof. For given $g \in G$, set

$$\Delta_g := \max \{ |\varphi(g)(\xi) - \xi| : \varphi \in \text{Der}_\mu(G), \xi \in \mathbb{R} \}.$$

Note that this quantity is positive and finite for every $g \in G$, by compactness of the space $\text{Der}_\mu(G)$. We will establish the lemma for actions $\varphi \in \mathcal{P}_+$, the other case $\varphi \in \mathcal{P}_-$ being totally symmetric.

Fix $\varphi \in \mathcal{P}_+$. First of all we observe that the unique fixed point ξ_φ of $\varphi(f)$ satisfies $\xi_\varphi \in \text{Supp}^\varphi(G_{(a,x)})$ for every $x \in X$. Indeed if there exists $y \in X$ such that $\xi_\varphi \in \text{Fix}^\varphi(G_{(a,y)})$ we would have $\xi_\varphi \in \text{Fix}^\varphi(G_{(a,f^n(y))})$ for every $n \in \mathbb{N}$, so that ξ_φ must be fixed by the normal subgroup $G_+ = \bigcup_{n \in \mathbb{N}} G_{(a,f^n(y))}$. Since φ is minimal, it follows that G_+ acts trivially, which is a contradiction with the fact that $\varphi \in \mathcal{P}$ is faithful. Thus, the interval $\text{I}^\varphi(x, \xi_\varphi)$ is well-defined and non-empty for every $x \in X$.

Fix now $x \in X$. Since G belongs to the class \mathcal{F} , we can fix a finite subset $S \subset G_{(x,b)}$ such that $\langle S \rangle$ contains $G_{(y,b)}$ for some $y \geq x$. Set $D = \max\{\Delta_s : s \in S\}$ (note that it depends on μ and on x).

Claim. *We have $\sup \{ |\text{I}^\varphi(x, \xi_\varphi)| : \varphi \in \mathcal{P}_+ \} \leq D$.*

Proof of claim. Assume by contradiction that for some $\varphi \in \mathcal{P}_+$, we have $|\text{I}^\varphi(x, \xi_\varphi)| > D$. Since every $s \in S$ fixes x , the interval $\varphi(s)(\text{I}^\varphi(x, \xi_\varphi)) = \text{I}^\varphi(x, \varphi(s)(\xi_\varphi))$ is a connected component of $\text{Supp}^\varphi(G_{(a,x)})$ and thus must be either equal to $\text{I}^\varphi(x, \xi_\varphi)$ or disjoint from it. But the assumption that $|\text{I}^\varphi(x, \xi_\varphi)| > \Delta_s$ rules out the second possibility. Thus we have that $\varphi(s)$ must preserve $\text{I}^\varphi(x, \xi_\varphi)$ for every $s \in S$, and hence $\text{I}^\varphi(x, \xi_\varphi)$ is $\varphi(G_{(y,b)})$ -invariant. In particular, the subgroup $N_{(y,b)} = [G_c, G_c]_{(y,b)}$ has fixed points, which contradicts Proposition 5.12. \square

Given $\varphi \in \mathcal{P}_+$, write $\text{I}^\varphi(x, \xi_\varphi) = (\alpha_0, \beta_0)$ and for $n \in \mathbb{N}$ set $\alpha_n = \varphi(f^n)(\alpha_0)$ and $\beta_n = \varphi(f^n)(\beta_0)$. On the one hand, note that for every $n \in \mathbb{N}$, the points α_n and β_n belong to $\text{Fix}^\varphi(G_{(a,f^n(x))})$ and hence to $\text{Fix}^\varphi(G_{(a,x)})$. Thus for every $\xi \in \text{Supp}^\varphi(G_{(a,x)}) \setminus (\alpha, \beta)$, there exists $n \in \mathbb{N}$ such that the corresponding connected component $\text{I}^\varphi(x, \xi)$ is contained in one of the two intervals (α_{n+1}, α_n) and (β_n, β_{n+1}) . In particular, after the claim, this happens whenever $|\xi - \xi_\varphi| \geq D$. On the other hand, setting $C = \Delta_f$ (note it only depends on μ only), we must have that both lengths $|\alpha_{n+1} - \alpha_n|$ and $|\beta_{n+1} - \beta_n|$ are upper-bounded by C . This gives the desired conclusion. \square

Proof of Proposition 10.17. Assume by contradiction that we can find sequences $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{P}^0$ and $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, with $|t_n| \rightarrow \infty$, such that $\psi_n := \Phi^{t_n}(\varphi_n)$ converges to a limit $\psi \notin \mathcal{N}$ as $n \rightarrow \infty$. Up to extracting a subsequence we can assume that $(\varphi_n) \subset \mathcal{P}_+^0$ (the case $(\varphi_n) \subset \mathcal{P}_-^0$ being analogous). Note that since $\xi_{\varphi_n} = 0$ we have $\xi_{\psi_n} = -t_n$. Fix $x \in X$ and the corresponding constants $C = C(\mu)$ and $D = D(x, \mu)$ given by Lemma 10.19. For n large enough we have $|t_n| \geq D$, which implies that $|\text{I}^{\varphi_n}(x, \xi_{\psi_n})| = |\text{I}^{\varphi_n}(x, 0)| \leq C$, so that there exists $\eta_n \in [-C, C]$ which is fixed by $\psi_n(G_{(a,x)})$. Since every accumulation point of (η_n) is fixed by $\psi(G_{(a,x)})$ we obtain that $\text{Fix}^\psi(G_{(a,x)}) \cap [-C, C] \neq \emptyset$. Since C does not depend on x , a compactness argument shows that $\text{Fix}^\psi(G_+) = \bigcap_{x \in X} \text{Fix}^\psi(G_{(a,x)}) \neq \emptyset$. As ψ is minimal and G_+ is normal, we deduce that ψ is not faithful, and this contradicts the assumption that $\psi \notin \mathcal{N}$. \square

10.4.2. *Application to local rigidity.* Theorem 10.14 has the following immediate consequence.

Theorem 10.20. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F}_0 . Then the set of actions in $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ which are semi-conjugate to the standard action of G on X is open. In particular the standard action of G on X is locally rigid.*

Remark 10.21. An arbitrary locally moving group $G \subseteq \text{Homeo}_0(X)$ is not always locally rigid: the construction of actions in §5.3.2 shows that this is never the case for any countable group $G \subseteq \text{Homeo}_0(X)$ consisting of elements of compact support. However the assumption that G belongs to \mathcal{F}_0 is probably not optimal. We are able to prove Theorem 10.20 rigidity under some variants of this assumptions, all of which involve the finite generation of G and some additional finite generation condition related to the subgroups G_I for proper intervals $I \subset X$. It would be interesting to know whether these assumptions can be completely removed, namely whether Theorem 10.20 holds for every finitely generated locally moving group.

Remark 10.22. In the setting of Theorem 10.20, the natural action of G on X is not always the *unique* locally rigid action of G : we will see in Section 12 that there are examples of groups in \mathcal{F}_0 that admit a finite number of faithful \mathbb{R} -focal actions up to conjugacy (i.e. for which the subset \mathcal{P} contains finitely many Φ -orbits), which are therefore locally rigid as well.

We point out the following consequence.

Corollary 10.23. *Let $H \subseteq \text{Homeo}_0(\mathbb{R})$ be a countable group. Then, there exists a finitely generated subgroup $G \subset \text{Homeo}_0(\mathbb{R})$ containing H such that the action of G on \mathbb{R} is locally rigid.*

Proof. By Proposition 10.2 there exists a finitely generated group $G \subseteq \text{Homeo}_0(\mathbb{R})$ that contains H and belongs to class \mathcal{F}_0 , and by Theorem 10.20, its action is locally rigid. \square

10.4.3. *Smoothness of the semi-conjugacy relation.* Recall from the discussion in §3.5 that the Deroin space can be used to study when the semi-conjugacy relation on $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is smooth, that is when there is a Borel complete invariant that classifies actions of G up to semi-conjugacy. When G is in \mathcal{F}_0 , an obvious obstruction for this to hold is that this may fail already for action of the largest quotient $G/[G_c, G_c]$. However our analysis of $\text{Der}_\mu(G)$ has the following straightforward consequence.

Corollary 10.24. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup in the class \mathcal{F}_0 . Let \mathcal{M} be the subset of $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ consisting of actions that are minimal and faithful. Then the following hold.*

- (i) *The topological conjugacy relation on \mathcal{M} is smooth.*
- (ii) *The semi-conjugacy relation on the space $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ is smooth if and only if the same holds true for the space $\text{Hom}_{\text{irr}}(G/[G_c, G_c], \text{Homeo}_0(\mathbb{R}))$.*

Proof. The first claim follows from Lemma 10.16 and from Theorem 3.2. The second is a direct consequence of the previous analysis of $\text{Der}_\mu(G)$ and of Corollary 3.22. \square

11. AN ILLUSTRATIVE EXAMPLE: THOMPSON'S GROUP F

11.1. **Main results in the case of F and outline of the section.** Perhaps the most basic example of finitely generated locally moving group is Thompson's group F (cf. Proposition 4.8), which also belongs to the class \mathcal{F}_0 . It turns out that actions of the group F on \mathbb{R} display a rich combination of rigidity and flexibility phenomena. The goal of this section is to summarize our results in this special case and to go more into detail.

On the one hand, Corollary 6.11 implies the following for its actions by C^1 diffeomorphisms.

Theorem 11.1 (Rigidity of actions by C^1 diffeomorphisms). *Thompson’s group F satisfies the following.*

- For every faithful action $\varphi: F \rightarrow \text{Diff}^1([0, 1])$ without fixed points in $[0, 1]$, the φ -action of F on $(0, 1)$ is semi-conjugate to the standard action of F on $(0, 1)$.
- Every faithful action $\varphi: F \rightarrow \text{Diff}^1(\mathbb{R})$ without closed discrete orbit is semi-conjugate to the standard action on $(0, 1)$.

Remark 11.2. Note that the conclusion is optimal: there exist C^1 actions (and even C^∞ actions) of F on closed intervals which are semi-conjugate, but not conjugate to its standard action: the existence of such actions was shown by Ghys and Sergiescu [45], or alternatively can be shown using the “2-chain lemma” of Kim, Koberda, and Lodha [55] (see Proposition 4.8).

On the other hand this rigidity does not hold for continuous actions: there exists minimal faithful actions of F on \mathbb{R} which are not semi-conjugate to the standard action (see for instance the constructions in §§9.1–9.2). Nevertheless, Theorem 10.3 implies that the dynamics of all exotic actions of F on \mathbb{R} is strongly reminiscent of the standard action on $(0, 1)$, although not via a semi-conjugacy but via the notion of *horogradings* of an \mathbb{R} -focal action (see §8.4). Indeed for the group F , Theorem 10.3 reads as follows (recall that the commutator subgroup $[F, F]$ is simple and coincides with the group F_c of compactly supported elements, so that the largest quotient $F/[F_c, F_c]$ coincides with the abelianization $F^{ab} \cong \mathbb{Z}^2$).

Theorem 11.3 (Structure theorem for actions by homeomorphisms). *Every action $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is semi-conjugate to one of the following.*

- (i) (Non-faithful) *An action by translations of $F^{ab} \cong \mathbb{Z}^2$.*
- (ii) (Standard) *The standard piecewise linear action of F on $(0, 1)$.*
- (iii) (Exotic) *A minimal \mathbb{R} -focal action which can be horograded by the standard action of F on $(0, 1)$.*

Theorem 11.3 implies serious constraints on the structure of actions of F on \mathbb{R} . For example, it implies that for all exotic actions, the individual elements of F satisfy a dynamical classification, and the type of each element can be read from the standard action on $(0, 1)$ (see Proposition 8.32). For ease of reference let us restate the dynamical classification in this special case. Given $g \in F$ and $x \in [0, 1]$ we denote by $D^-g(x) \in \{2^n, n \in \mathbb{Z}\}$ the left derivative of g at x . Note that germ of $g \in F$ inside $\text{Germ}(F, 1)$ is uniquely determined by $D^-g(1)$, and 1 is an attracting fixed point for g if and only if $D^-g(1) < 1$.

Proposition 11.4 (Dynamical classification of elements). *Let $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal \mathbb{R} -focal action, increasingly horograded by the standard action on $(0, 1)$. Then the following hold.*

- (i) *For every $x \in (0, 1)$, the image $\varphi(F_{(0,x)})$ is totally bounded. In particular the φ -image of every element $g \in F$ with $D^-g(1) = 1$ is totally bounded.*
- (ii) *For every $g \in F$ such that $D^-g(1) \neq 1$ the φ -image of g is a pseudohomothety, which is expanding if $D^-g(1) < 1$ and contracting otherwise. If moreover $g \in F$ has no fixed points in $(0, 1)$ then its image is a homothety.*

As in §10.4, Theorem 11.3 can also be used to analyze the structure of the Deroin space $\text{Der}_\mu(F)$ of normalized harmonic F -actions, a representation of which is given in Figure 10; a detailed description of this picture is provided in §11.2 below. This description implies in particular that a representative of the standard action in $\text{Der}_\mu(F)$ cannot be accumulated by

exotic actions, which in turns implies the following (which is a particular instance of Theorem 10.20).

Theorem 11.5 (Local rigidity of the standard action). *The standard piecewise linear action of F on $(0, 1)$ is locally rigid.*

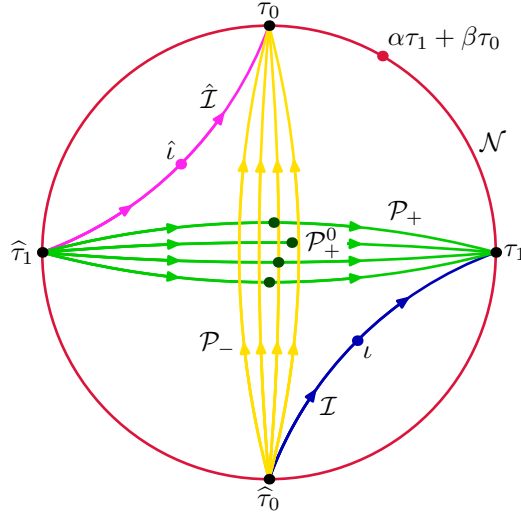


FIGURE 10. The Deroin space of Thompson’s group F . See §11.2 for a detailed explanation.

It also allows to say that the Borel complexity structure of the semi-conjugacy relation on $\text{Hom}_{\text{irr}}(F, \text{Homeo}_0(\mathbb{R}))$ is the simplest possible (see the discussion in §3.5). In other words, there exists a Borel invariant for actions of F on the real line without fixed points, which completely determine the semi-conjugacy class.

Theorem 11.6 (Smoothness of the semi-conjugacy relation). *The semi-conjugacy relation on the space $\text{Hom}_{\text{irr}}(F, \text{Homeo}_0(\mathbb{R}))$ of actions of F without fixed points is smooth. In particular, the conjugacy relation on the space of minimal actions of F on \mathbb{R} is smooth.*

Proof. Direct consequence of Corollary 3.22 and the description of the Deroin space of F in §11.2. □

Leaving temporary aside the setting of orientation-preserving actions, we may consider Thompson’s group with flips F^\pm , which is defined as the subgroup of $\text{Homeo}((0, 1))$ defined analogously to F but by allowing negative slopes (equivalently, the group generated by F and by the reflection along $1/2$). Then Theorem 11.3 implies the following (which is a special case of Corollary 10.12).

Theorem 11.7 (Rigidity in the non-orientation-preserving case). *Let F^\pm denote Thompson’s group with flips. Then every minimal action $\varphi: F^\pm \rightarrow \text{Homeo}(\mathbb{R})$ is conjugate to its standard action on $(0, 1)$.*

Remark 11.8. In Theorem 11.7, the assumption that the action be faithful is actually redundant. In fact by analyzing the proper quotients of F^\pm , one can check that the only ones that act non-trivially on the line by homeomorphisms are isomorphic to \mathbb{Z} or to the dihedral group D_∞ , so that none of them can act minimally.

While the above results can all be seen as rigidity properties for actions of F on the line, a large part of this section will be devoted to illustrate their considerable flexibility. Indeed within the constraints imposed by Theorem 11.3, it turns out the group F admits a wild zoo of exotic (hence \mathbb{R} -focal) actions. Some first examples of \mathbb{R} -focal actions of F can be obtained by the constructions in §§9.1–9.2, which already show that F admits an uncountable family of non-conjugate such actions. In §§11.4–11.6 we illustrate the abundance of \mathbb{R} -focal actions of the group F by providing various other constructions, and describing some subtle differences in their dynamical behavior. While we will focus more on some significant examples, the reader will notice that these constructions admit various variants involving several choices, which depend on the data and are not always compatible between them. Trying to take them all into account simultaneously would result in an obscure treatment.

Along the way we will observe that many examples of minimal \mathbb{R} -focal actions of F share an interesting property: the image of the commutator subgroup $[F, F]$ does not act minimally on the real line (equivalently it does not admit any closed minimal invariant subset, see Lemma 7.12). In §11.3 we show that this condition is particularly relevant and has many equivalent characterizations (see Proposition 11.13). In particular it characterizes the \mathbb{R} -focal actions of F which can be encoded by an action on a *simplicial* planar directed tree (cf. Proposition 8.33). Nevertheless it turns out that not *all* \mathbb{R} -focal actions of F have this property: in §11.6 we will construct a family of minimal \mathbb{R} -focal actions of F such that $[F, F]$ still acts minimally.

To conclude this discussion, we would like to emphasize that the abundance of exotic actions relies on some specific features of the group F . For instance in the next section (Section 12) we will see that a class of finitely generated groups of piecewise linear homeomorphisms of the whole *real line* (with finitely many breakpoints) defined analogously to F admit exactly *two* minimal faithful exotic actions up to conjugacy, and we will also construct a finitely generated locally moving group which does not admit any exotic action at all.

11.2. An analysis of the Deroin space of F . The goal of this subsection is to analyze the Deroin space of F and to explain and justify Figure 10. From now on, we fix a symmetric probability measure μ on F whose support is finite and generates F , and consider the associated Deroin space $\text{Der}_\mu(F)$ (see §2.2), with its translation flow Φ . We will follow closely the discussion in §10.4, and refine it when possible to obtain more precise information in this special case.

Remark 11.9. As mentioned in the introduction, the Deroin space is a continuous analogue of the space of *left-invariant* orders for a finitely generated group. We point out that the space of *bi-invariant* orders on F was described in [94].

We take notation analogous to Assumption 10.15. Namely we fix any element $f \in F$ which in the standard action satisfies $f(x) > x$ for every $x \in (0, 1)$. For definiteness we choose f to be the element of the standard generating pair of F given by

$$(11.1) \quad f(x) = \begin{cases} 2x & x \in [0, \frac{1}{4}], \\ x + \frac{1}{4} & x \in [\frac{1}{4}, \frac{1}{2}], \\ \frac{1}{2}x & x \in [\frac{1}{2}, 1]. \end{cases}$$

By Theorem 11.3, we have a decomposition of $\text{Der}_\mu(F)$ into Φ -invariant subspaces

$$\text{Der}_\mu(F) = \mathcal{N} \sqcup \mathcal{I} \sqcup \widehat{\mathcal{I}} \sqcup \mathcal{P},$$

defined as follows:

- we let $\mathcal{N} \subset \text{Der}_\mu(F)$ be the subset consisting of actions by translations of F^{ab} ;
- we fix a representative $\iota \in \text{Der}_\mu(F)$ of the standard action on $(0, 1)$, and let $\mathcal{I} = \{\Phi^t(\iota) : t \in \mathbb{R}\}$ denote its Φ -orbit, whilst $\widehat{\mathcal{I}}$ denotes the Φ -orbit of the reversed action $\widehat{\iota}$;
- we let $\mathcal{P} = \mathcal{P}_+ \sqcup \mathcal{P}_-$ be the subset of $\text{Der}_\mu(F)$ of \mathbb{R} -focal actions, where \mathcal{P}_+ (respectively, \mathcal{P}_-) is the subset of \mathbb{R} -focal actions which are increasingly (respectively, decreasingly) horograded by the standard action on $(0, 1)$. For $\varphi \in \mathcal{P}$ we let ξ_φ be the unique fixed point of $\varphi(f)$. We say that $\varphi \in \mathcal{P}$ is f -centered if $\xi_\varphi = 0$, and let $\mathcal{P}^0 \subset \mathcal{P}$ be the subset of f -centered \mathbb{R} -focal actions. Finally we set $\mathcal{P}_\pm^0 = \mathcal{P}^0 \cap \mathcal{P}_\pm$.

Let us now give a closer description of the topology of these subsets and of the dynamics of the flow Φ on them.

First of all, from the elementary structure of the maximal quotient $F/[F_c, F_c] \cong \mathbb{Z}^2$ we obtain an explicit description of the subset \mathcal{N} in this case. Indeed \mathcal{N} is homeomorphic to $\text{Der}_{\bar{\mu}}(\mathbb{Z}^2)$, where $\bar{\mu}$ is the projection of μ to $F^{ab} \cong \mathbb{Z}^2$. Every element of $\text{Der}_{\bar{\mu}}(\mathbb{Z}^2)$ corresponds to a \mathbb{Z}^2 -action by translations given by a non-trivial homomorphism $\varphi: \mathbb{Z}^2 \rightarrow (\mathbb{R}, +)$ up to rescaling by a positive real, so that $\text{Der}_{\bar{\mu}}(\mathbb{Z}^2)$ is homeomorphic to the circle \mathbb{S}^1 . We deduce that \mathcal{N} is a closed subset of $\text{Der}_\mu(F)$ homeomorphic to a circle, and it consists of points which are fixed by the translation flow Φ .

Let now $\tau_0, \tau_1: F \rightarrow \mathbb{Z}$ be the two natural homomorphisms obtained by identifying the groups of germs $\text{Germ}(F, 0)$ and $\text{Germ}(F, 1)$ with \mathbb{Z} , with the convention that $\tau_x(g) > 0$ if and only if the corresponding endpoint $x \in \{0, 1\}$ is an attracting fixed point of g . Explicitly,

$$(11.2) \quad \tau_0(g) = -\log_2 D^+g(0) \quad \text{and} \quad \tau_1(g) = -\log_2 D^-g(1).$$

We identify τ_0 and τ_1 with the elements of \mathcal{N} given by their corresponding cyclic action. One readily shows that the actions ι and $\widehat{\iota}$ satisfy the following (compare with Lemma 10.18):

$$\lim_{t \rightarrow +\infty} \Phi^t(\iota) = \tau_1, \quad \lim_{t \rightarrow -\infty} \Phi^t(\iota) = \widehat{\tau}_0, \quad \lim_{t \rightarrow +\infty} \Phi^t(\widehat{\iota}) = \tau_0, \quad \lim_{t \rightarrow -\infty} \Phi^t(\widehat{\iota}) = \widehat{\tau}_1.$$

Thus \mathcal{I} is a copy of \mathbb{R} inside $\text{Der}_\mu(F)$ which connects the points $\widehat{\tau}_0$ and τ_1 , while $\widehat{\mathcal{I}}$ connects τ_0 to τ_1 as shown in Figure 10.

Recall that by Lemma 10.16 each subset \mathcal{P}_\pm is open, and \mathcal{P}_\pm^0 is a cross section for Φ inside \mathcal{P}_\pm . Moreover, in this setting we have the following more precise version of Proposition 10.17.

Proposition 11.10. *The subsets \mathcal{P}_+^0 and \mathcal{P}_-^0 are closed in $\text{Der}_\mu(F)$. Moreover, for every $\varphi \in \mathcal{P}_\pm^0$ the limits $\lim_{t \rightarrow \pm\infty} \Phi^t(\varphi)$ exist and the following holds:*

(1) *if $\varphi \in \mathcal{P}_+^0$ then*

$$\lim_{t \rightarrow +\infty} \Phi^t(\varphi) = \tau_1 \quad \text{and} \quad \lim_{t \rightarrow -\infty} \Phi^t(\varphi) = \widehat{\tau}_1,$$

where the convergence is uniform on $\varphi \in \mathcal{P}_+^0$;

(2) *if $\varphi \in \mathcal{P}_-^0$ then*

$$\lim_{t \rightarrow +\infty} \Phi^t(\varphi) = \tau_0 \quad \text{and} \quad \lim_{t \rightarrow -\infty} \Phi^t(\varphi) = \widehat{\tau}_0,$$

where the convergence is uniform on $\varphi \in \mathcal{P}_-^0$.

Proof. We already know by Lemma 10.16 that $\overline{\mathcal{P}_\pm^0} \subset \mathcal{P}_\pm^0 \cup \mathcal{N}$. Let us show that in this case the sets \mathcal{P}_\pm^0 are actually closed. Assume by contradiction that $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{P}_+^0$ is a sequence converging to $\psi \in \mathcal{N}$. Since $\varphi_n(f)$ fixes 0 for every $n \in \mathbb{N}$, so does $\psi(f)$. Since ψ is an action by translations, this is possible only if $\psi(f) = \text{id}$. Now fix $x \in (0, 1)$ and choose any $h \in F_{(0,x)}$ with non-trivial germ at 0. Recall the notation $I^\varphi(x, \xi)$ from Definition 10.7. After the claim in

Lemma 10.19, there exists $D = D(x, \mu) > 0$ such that $|I^{\varphi_n}(x, 0)| \leq D$ for every $n \in \mathbb{N}$. Thus, upon extracting a subsequence we can suppose that $\eta_n := \sup I^{\varphi_n}(x, 0)$ converges to a limit $\eta \in \mathbb{R}$ as $n \rightarrow \infty$. The choice of h implies that $\varphi_n(h)$ fixes η_n for every n , so that $\psi(h)$ fixes η , and we deduce again that $\psi(h) = \text{id}$. However note that in the standard action the element f has non-trivial derivatives at both endpoints, while h has non-trivial germ only at 0. Thus h and f project to two linearly independent elements of $F^{ab} \cong \mathbb{Z}^2$. Since ψ is an action by translations and it vanishes on both, we have that ψ is trivial, which is a contradiction. This shows that \mathcal{P}_+^0 is closed, and the proof for \mathcal{P}_-^0 is closed is similar.

Let us now show that $\lim_{t \rightarrow +\infty} \Phi^t(\varphi) = \tau_1$ uniformly on $\varphi \in \mathcal{P}_+^0$. Assume that $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{P}_+^0$ and $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ are sequences such that t_n increases to $+\infty$, and let ψ be a cluster point of $\psi_n := \Phi^{t_n}(\varphi_n)$. Repeating the proof of Proposition 10.17 in this case, one sees that $\text{Fix}^\psi(F_+) \neq \emptyset$, hence $\psi(F_+) = \{\text{id}\}$. This is only possible when $\psi \in \{\tau_1, \widehat{\tau}_1\}$. However note that $\psi_n(f)$ is an expanding homothety with fixed point $-t_n$ for every $n \in \mathbb{N}$, so that $\psi_n(f)(0) > 0$ for sufficiently large n . Passing to the limit we obtain that $\psi(f)(0) \geq 0$. Thus necessarily $\psi = \tau_1$. \square

An immediate consequence of the uniform convergence in Proposition 11.10 is the following. Given a topological space K , we define the *double cone* over K to be the quotient $K \times [-1, 1] / \sim$ where $(x, -1) \sim (y, -1)$ and $(x, 1) \sim (y, 1)$ for every $x, y \in K$.

Corollary 11.11. *The closures of the subsets \mathcal{P}_+ and \mathcal{P}_- in $\text{Der}_\mu(F)$ are given by $\overline{\mathcal{P}}_+ = \mathcal{P}_+ \cup \{\tau_1, \widehat{\tau}_1\}$ and $\overline{\mathcal{P}}_- = \mathcal{P}_- \cup \{\tau_0, \widehat{\tau}_0\}$, and both are homeomorphic to the double cone over the corresponding cross section \mathcal{P}_\pm^0 .*

With this discussion in mind, illustration of $\text{Der}_\mu(F)$ is provided by Figure 10. Note that in particular it is visible from this description that $\overline{\mathcal{P}} \cap \mathcal{I} = \emptyset$, so that ι is transversely isolated. In particular the standard action of F is locally rigid (Theorem 11.5 above).

To conclude this discussion, we observe that it is a tantalizing problem to obtain further results on the topology of the compact cross sections \mathcal{P}_+^0 and \mathcal{P}_-^0 , which is at the moment quite mysterious. Note that these do not depend up to homeomorphism on the choice of the generator $f \in F$ made above, and by symmetry they are homeomorphic one to the other. The constructions of \mathbb{R} -focal actions which will appear in the rest of the section show that the spaces \mathcal{P}_+^0 and \mathcal{P}_-^0 are uncountable, and contain homeomorphic copies of a Cantor set. However we were not able to construct a non-trivial connected subset of \mathcal{P}_+^0 , and we do not know whether they are totally disconnected. We also do not know the answer to the following question.

Question 11.12. *Do the cross sections \mathcal{P}_+^0 and \mathcal{P}_-^0 admit isolated points?*

By Corollary 3.4 this is equivalent to the question whether F admits minimal \mathbb{R} -focal actions which are locally rigid.

11.3. Simplicial \mathbb{R} -focal actions of F and minimality of $[F, F]$. Before discussing examples of minimal \mathbb{R} -focal actions of F , we would like to single out an important question to address about any such action: whether the commutator subgroup $[F, F]$ acts minimally or not. By Lemma 7.12, the second possibility is equivalent to the fact that it does not admit any non-empty closed minimal invariant subset of \mathbb{R} . The \mathbb{R} -focal actions of F with this property turn out to be very special due to the following result, which elaborates on Proposition 8.33.

Proposition 11.13 (Simplicial \mathbb{R} -focal actions of F). *Let $\varphi : F \rightarrow \text{Homeo}_+(\mathbb{R})$ be a minimal \mathbb{R} -focal action which can be increasingly horograded by the standard action on $(0, 1)$. Then, the following are equivalent:*

- (i) The image $\varphi(F_+)$ admits no non-empty closed minimal invariant set.
- (ii) The image $\varphi([F, F])$ admits no non-empty closed minimal invariant set.
- (iii) φ is represented by a focal action on a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$, such that \mathbb{T} is a simplicial tree of countable degree, and the action of F on \mathbb{T} is by simplicial automorphisms.
- (iv) φ is represented by a focal action on a planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ such that F acts by isometries with respect to a compatible \mathbb{R} -tree metric on \mathbb{T} .
- (v) Every pseudohomothety in the image of φ is a homothety.
- (vi) There exist bounded open intervals $I, J \subset \mathbb{R}$ such that for every $g \in F_+$ the image $g.I$ does not contain J .

Remark 11.14. The fact that some minimal \mathbb{R} -focal actions of F can be encoded by a simplicial action on a simplicial tree can seem in contradiction with the fact that they can also be horograded by the standard action of F , which is highly not isometric. However the tree provided by the proof Proposition 11.13 is not the same directed tree as the one provided by the proof of Theorem 10.3 (it arises from a different CF-cover). This is an illustration of the fact that a planar directed tree encoding an \mathbb{R} -focal action is not unique, and identifying a tree with good properties may be important for some purposes. We note however that the focal germ representations associated to both trees are semi-conjugate to the germ homomorphism $F \rightarrow \text{Germ}(F, 1) \simeq \mathbb{Z}$ associated to the standard action (recall Lemma 8.25).

Proof of Proposition 11.13. Let us first show that (i) and (ii) are equivalent. If $[F, F]$ admits a closed minimal invariant subset, then by Lemma 7.12 it acts minimally on \mathbb{R} , and thus so does F_+ . Conversely, assume by contradiction that $[F, F]$ does not act minimally on \mathbb{R} , but F_+ does. Then we can apply Proposition 8.33 to $G = F_+$ and $N = [F, F]$ and we deduce that $\varphi \upharpoonright_{F_+}$ is \mathbb{R} -focal and is the dynamical realization of an action of F_+ on an planar directed simplicial tree $(\mathbb{T}, \triangleleft, \prec)$. Moreover, the argument in its proof shows that the focal germ representation of the action on $(\mathbb{T}, \triangleleft, \prec)$ coincides with the projection to $F_+/[F, F] \cong \mathbb{Z}$. Since the quotient $F_+/[F, F]$ is simply the group of germs $\text{Germ}(F_+, 0)$ we deduce that the φ -image of every $g \in F_+$ with a non-trivial germ at 0 must be a pseudohomothety. However we were assuming that φ is an \mathbb{R} -focal action of F increasingly horograded by the standard action on $(0, 1)$, so that the image of every element $g \in F_+$ must be totally bounded. This is a contradiction.

The implication (i) \Rightarrow (iii) follows from Proposition 8.33, and (iii) \Rightarrow (iv) is obvious. Let us prove that (iv) implies (v). Note that the image of an element $g \in F$ is a pseudohomothety if and only if g has a non-trivial germ at 1. As in the proof of Proposition 8.33 we can consider a horofunction $\pi: \mathbb{T} \rightarrow \mathbb{R}$ given by $\pi(v) = d_{\mathbb{T}}(v \wedge v_0, v_0) - d_{\mathbb{T}}(v \wedge v_0, v)$, where v_0 is some basepoint, and we see that this provides a horograding by an action by translations on \mathbb{R} coming from the germ homomorphism $F \rightarrow \text{Germ}(F, 1) \cong \mathbb{Z} \rightarrow (\mathbb{R}, +)$. Then from Proposition 8.27 we obtain that every element with a non-trivial germ at 1 must act as a homothety under φ .

To show that (v) implies (vi), we show that (v) actually implies the following more explicit condition, which clearly implies (vi).

- (vi') For every element $h \in F$ which in the standard action satisfies $h(x) > x$ for every $x \in (0, 1)$ (so that $\varphi(h)$ is an expanding homothety) there exists an open bounded interval $I \subset \mathbb{R}$ such that $I \subset h.I$ and such that for every $g \in F_+$, the interval $h.I$ is not contained in $g.I$.

Indeed assume that h is such that $h(x) > x$ for every $x \in X$ and let $\xi \in \mathbb{R}$ be the unique fixed point of $\varphi(h)$. Fix $x \in X$ and let $I := I^\varphi(x, \xi)$ (note that this connected component is

defined for every $x \in X$). As $\varphi(h)$ is an expanding homothety, we have $I \subset h.I$. Suppose by contradiction that there exists $g \in F_+$ such that $h.I \subset g.I$. Note that we have the equalities

$$g.I = g.I^\varphi(x, \xi) = I^\varphi(g(x), g.\xi) = I^\varphi(g(x), \xi)$$

(the last equality follows from the assumption that $\xi \in g.I$). Moreover we have that $h.I = I^\varphi(h(x), \xi)$ and therefore we conclude that $I^\varphi(h(x), \xi) \subset I^\varphi(g(x), \xi)$ implying that $g(x) \geq h(x)$. Since $g \in F_+$ this implies that for some $y \geq x$ it holds that $g(y) = h(y)$. Therefore,

$$g.I^\varphi(y, \xi) = I^\varphi(g(y), \xi) = I^\varphi(h(y), \xi) = h.I^\varphi(y, \xi)$$

(the first equality follows from the fact that $I^\varphi(y, \xi) \supset I$). Finally, this implies that $\varphi(g^{-1}h)$ preserves $I^\varphi(y, \xi)$. Since $g^{-1}h$ has the same germ as h at 1, it acts as a pseudohomothety, so that by (v) it is a homothety, and this is a contradiction with the fact that it preserves $I^\varphi(y, \xi)$.

Finally to show that (vi) implies (i), assume by contradiction (using Lemma 7.12) that the action of F_+ is minimal. As we are assuming (vi), the action cannot be proximal, so by Theorem 2.17 the centralizer of $\varphi(F_+)$ in $\text{Homeo}_0(\mathbb{R})$ must be infinite cyclic generated by an element T without fixed point. Since F_+ is normal, we deduce that the whole group $\varphi(F)$ must normalize $\langle T \rangle$ and thus centralize it, contradicting that a minimal \mathbb{R} -focal action is always proximal (Proposition 7.7). \square

Definition 11.15. We will say that a minimal \mathbb{R} -focal action $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ is simplicial if it satisfies one of the equivalent conditions in Proposition 11.13.

For many constructions of \mathbb{R} -focal actions of F discussed below one can easily check conditions (i) or (ii), and thus they turn out to be simplicial (although a simplicial tree does not always appear naturally in the construction, and it might also be not obvious to check directly condition (v)). It is more delicate to construct \mathbb{R} -focal actions of F which are *not* simplicial: this will be done in §11.6.

11.4. A plethora of \mathbb{R} -focal actions I: restriction preorders. Starting from now we will present various constructions of \mathbb{R} -focal actions of the group F and study some of their properties. We will focus on some examples, and indicate how they can be modified to obtain more.

Remark 11.16. The attentive reader will notice that a common feature of all our constructions of \mathbb{R} -focal actions of F is the choice of a closed subset $K \subset (0, 1)$, which is invariant under the generator f given by (11.1). These sets appear quite naturally with the point of view of focal actions. To understand this, take a minimal \mathbb{R} -focal action $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$. By Theorem 11.3 we know that it can be horograded by the standard action on $(0, 1)$, meaning that one can find a focal action $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \triangleleft)$ on a directed planar tree which is horograded by the standard action, and whose dynamical realization is (conjugate to) φ . By Proposition 11.4, the element f fixes a unique end $\xi_0 \in \partial^*\mathbb{T}$, so that it preserves the axis $]\xi_0, \omega[\subset \mathbb{T}$, which is naturally identified with the interval $(0, 1)$ via the horograding map $\pi: \mathbb{T} \rightarrow (0, 1)$. In particular, the π -image of the closure $\overline{]\xi_0, \omega[\cap \text{Br}(\mathbb{T})}$ of the subset of branching points on this axis defines an $\varphi(f)$ -invariant closed subset $K \subset (0, 1)$. Although for some choices of the action Φ (as the one in the proof of Theorem 10.3) the subset K is the whole interval $(0, 1)$, in most examples it is not the case with correct choice of Φ .

11.4.1. A reinterpretation in terms of preorders. We start with a simple observation which is useful to understand \mathbb{R} -focal actions of F . Recall that we write $\tau_1: F \rightarrow \mathbb{Z} \cong \text{Germ}(F, 1)$ for the germ homomorphism given by (11.2), and $f \in F$ for the element given by (11.1). We will

also write 1_F for the trivial element of F (and we will simply denote it by 1 when there is no risk of confusion). Since $\tau_1(f)$ is a generator of $\text{Germ}(F, 1)$, we have a splitting

$$F = F_+ \rtimes \langle f \rangle,$$

where as usual we write $F_+ = \ker \tau_1$. Then we can make F act on F_+ “affinely” by letting F_+ act on itself by left translations and f act on F_+ by conjugation. In formula, for $g = hf^n \in F$, with $h \in F_+$ and $n \in \mathbb{Z}$ and for $r \in F_+$, this action is given by $g \cdot r = hf^n r f^{-n}$.

Assume that \preceq is a left preorder on F_+ which is invariant under conjugation by f . In particular its residual subgroup $H = [1]_{\preceq}$ is normalized by f , so that the action of F on F_+ descends to an order-preserving action on $(F_+/H, \prec)$, where \prec is the total order induced by \preceq . Then, we can consider the dynamical realization $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ of this action. We have the following equivalence.

Proposition 11.17. *Let $\varphi: F \rightarrow \text{Homeo}_+(\mathbb{R})$ be an action. The following are equivalent.*

- (i) φ is a minimal \mathbb{R} -focal action increasingly horograded by the standard action on $(0, 1)$.
- (ii) There exists a preorder \preceq on F_+ invariant under conjugation by f , such that, writing $H = [1]_{\preceq}$, the map f acts as a homothety on $(F_+/H, \prec)$, and φ is conjugate to the dynamical realization of the action of F on $(F_+/H, \prec)$.

Moreover two distinct preorders as in (ii) give rise to non-conjugate minimal \mathbb{R} -focal actions.

Proof. Let us prove that (ii) implies (i). Assume that \preceq verifies the conditions, and let φ be the dynamical realization of the action of F on $(F_+/H, \prec)$. Since f is a homothety on $(F_+/H, \prec)$, Proposition 2.43 implies that φ is minimal, and $\varphi(f)$ is a homothety. Since moreover φ must fall into one of the cases of Theorem 11.3, the only possibility is that φ is \mathbb{R} -focal, increasingly horograded by the standard action on $(0, 1)$.

For the converse, let φ be as in (i). Then $\varphi(f)$ is an expanding homothety (see Proposition 11.4); let $\xi \in \mathbb{R}$ be its unique fixed point, and consider the preorder \preceq on F_+ associated with this point: $g \preceq h$ if and only if $g \cdot \xi < h \cdot \xi$. Using that ξ is fixed by f , we see that \preceq is invariant under conjugation by f , and that the natural action of F on $(F_+/[1]_{\preceq}, \prec)$ can be identified with the action of F on the orbit of ξ , showing the claim.

Finally note that these two constructions are inverse to each other, and since ξ is the unique fixed point of f , the preorder \prec is uniquely determined by the conjugacy class of the action. \square

11.4.2. *Restriction preorders on F_+ .* We now explain a concrete construction of preorders on F_+ satisfying (ii) in Proposition 11.17. This yields a family of \mathbb{R} -focal actions of F which contains as special cases the constructions in §§9.1–9.2.

Let $K \subseteq (0, 1)$ be a closed f -invariant subset. To the subset K we associate a preorder \preceq^K on F_+ which is obtained by looking at the restriction of elements of F_+ to K , as follows. We first consider the subgroup $H = \{g \in F_+ : g(x) = x \text{ for every } x \in K\}$ of elements which fix K pointwise, and for $g \in F_+$ define

$$x_g = \begin{cases} 0 & \text{if } g \in H, \\ \sup\{x \in K : g(x) \neq x\} & \text{if } g \in F_+ \setminus H. \end{cases}$$

We immediately observe that $x_g = x_{g^{-1}}$ for every $g \in F_+$. Moreover, we have the following behavior when considering compositions.

Lemma 11.18. *Let $K \subseteq (0, 1)$ be a non-empty closed subset, and take $g, h \in F_+$. Then we have the inequality $x_{gh} \leq \max\{x_g, x_h\}$, and when $x_h \neq x_g$ the equality $x_{gh} = \max\{x_g, x_h\}$ holds.*

Proof. Note that if $x \in K$ is such that $x > \max\{x_g, x_h\}$, then $gh(x) = g(x) = x$. This gives the inequality $x_{gh} \leq \max\{x_g, x_h\}$.

Assume now $x_h < x_g$ and assume first we are in the case $g(x_g) \neq x_g$. Then $gh(x_g) = g(x_g) \neq x_g$, proving that $x_g \leq x_{gh}$, hence $x_{gh} = x_g$ (using the previous inequality). When $g(x_g) = x_g$, then x_g is accumulated from the left by points of K which are moved by g ; in particular for every such point x with $x_h < x < x_g$, we have $gh(x) = g(x) \neq x$, giving $x \leq x_{gh}$. Taking the supremum we obtain the desired equality $x_g = x_{gh}$. Note also that the same assumption $x_h < x_g$ (which is equivalent to $x_{h^{-1}} < x_{g^{-1}}$) gives $x_{g^{-1}h^{-1}} = x_{g^{-1}} = x_g$. As $x_{hg} = x_{g^{-1}h^{-1}}$, we deduce from the previous case that $x_{hg} = x_g$. This concludes the proof. \square

We next introduce the subset

$$(11.3) \quad P_K = \{g \in F_+ \setminus H : \text{either } g(x_g) > x_g, \text{ or } g(x_g) = x_g \text{ and } D^-g(x_g) > 1\}$$

and observe the following.

Lemma 11.19. *For any non-empty closed subset $K \subseteq (0, 1)$, the subset P_K defines a positive cone in F_+ .*

Proof. We have to verify the conditions in Remark 2.34. Let us first prove that $F_+ = P_K \sqcup H \sqcup P_K^{-1}$. For this notice that, since $x_g = x_{g^{-1}}$, we have

$$P_K^{-1} = \{g \in F_+ \setminus H : \text{either } g(x_g) < x_g, \text{ or } g(x_g) = x_g \text{ and } D^-g(x_g) < 1\}.$$

Thus, we automatically get that $H \cap (P_K \cup P_K^{-1}) = \emptyset$ and $P_K^{-1} \cap P_K = \emptyset$. It only remains to show that $F_+ \subseteq P_K \sqcup H \sqcup P_K^{-1}$. For this, take $g \in F_+ \setminus H$, so that $x_g > 0$. If $x_g \neq g(x_g)$ we are done. In the complementary case, x_g must be accumulated from the left by points that are moved by g . Since g is piecewise linear we must have $D^-g(x_g) \neq 1$ showing that $g \in P_K \sqcup P_K^{-1}$. Next, let us check that P_K is a semigroup and $HP_KH \subseteq P_K$.

Take $g, h \in P_K$, and assume first $x_h < x_g$. Then Lemma 11.18 gives $x_{gh} = x_g$ and $gh(x_{gh}) = g(x_g)$. If $g(x_g) > x_g$, we deduce immediately $gh \in P_K$; otherwise x_g is accumulated from the left by points of K , which must be fixed by h , so that $D^-h(x_g) = 1$. Then $D^-(gh)(x_{gh}) = D^-g(x_g)D^-h(x_g) > 1$, and we conclude that $gh \in P_K$.

Assume now that $x_g < x_h$, so that $x_{gh} = x_h$ by Lemma 11.18. Consider first the case $h(x_h) = x_h$. Then $gh(x_{gh}) = g(x_h) = x_h = x_{gh}$, and as in the previous case we see that $D^-g(x_h) = 1$, so that $D^-(gh)(x_{gh}) = D^-h(x_h) > 1$. When $h(x_h) > x_h$, then $gh(x_{gh}) > g(x_h) = x_h$. In both cases we have $gh \in P_K$.

Note that the previous argument works also when one of the two elements is in the residue H , proving that $HP_KH \subseteq P_K$.

Finally, consider the case $x_g = x_h$. As $h(x_h) \geq x_h$ and $g(x_g) \geq x_g$, then if any of the two inequalities is strict we deduce $gh(x_g) > x_g$, and thus $x_{gh} = x_g$ (by the inequality of Lemma 11.18) and $gh \in P_K$. Otherwise, assume that both g and h fix $x_g = x_h$. Then we have the relation $D^-(gh)(x_{gh}) = D^-g(x_g)D^-h(x_h) > 1$, showing that $x_{gh} = x_g$ (again by Lemma 11.18) and $gh \in P_K$ also in this case. \square

The previous lemma leads to the following definition.

Definition 11.20. Given a closed subset $K \subseteq (0, 1)$, the preorder \preceq^K on F_+ defined by the positive cone P_K in (11.3) will be called the *restriction preorder* associated with K . We will always write $H = [1]_{\preceq^K}$ for its residue.

Let us describe some elementary properties related to the preorder \preceq^K that will be useful in the sequel.

Lemma 11.21. *Let $K \subseteq (0, 1)$ be a non-empty closed subset, and let \preceq^K be the corresponding restriction preorder on F_+ . Then the following hold.*

- (i) *For $g, h \in F_+$ with $1 \preceq^K g \preceq^K h$, we have $x_g \leq x_h$.*
- (ii) *For $x \in (0, 1)$, the subset $L_x := \{g \in F_+ : x_g \leq x\}$ is a \preceq^K -convex subgroup.*

Proof. We first prove (i). We can assume $g \in P_K$ otherwise $x_g = 0$ and the result follows. Assume for contradiction that $x_g > x_h$. Then from Lemma 11.18 we have $x_{g^{-1}h} = x_g$. Consider first the case $g(x_g) > x_g$, then $g^{-1}h(x_g) = g^{-1}(x_g) < x_g$, so that $g^{-1}h \not\preceq^K 1_F$, contradicting the assumption $g \preceq^K h$. Consider next the case $g(x_g) = x_g$, so that $D^-g(x_g) > 1$ and $D^-h(x_g) = 1$ (as in this case x_g is accumulated from the left by points of K). Then $g^{-1}h(x_g) = x_g$ and $D^-(g^{-1}h)(x_g) = D^-g(x_g)^{-1} < 1$, giving again the contradiction $g^{-1}h \not\preceq^K 1_F$.

The inequality $x_{gh} \leq \max\{x_g, x_h\}$ from Lemma 11.18 shows that the subset L_x in (ii) is a subgroup, whilst (i) proves that L_x is \preceq^K -convex. \square

Note that the coset space F_+/H can be identified with the set of restrictions $\{g \upharpoonright_K : g \in F_+\}$, so that two elements $g, h \in F_+$ are equivalent for \preceq^K if and only if their restrictions to K coincide.

Lemma 11.22. *For every f -invariant closed set $K \subseteq (0, 1)$, the restriction preorder \preceq^K on F_+ is invariant under conjugation by f , and the conjugacy induces a homothety on $(F_+/H, \prec^K)$ fixing H .*

Proof. The verification that \preceq^K is invariant under conjugation follows easily from f -invariance of K . Let us check that f acts a homothety on $(F_+/H, \prec^K)$. It is clear that it fixes the point corresponding to H . We next verify that conjugation by f preserves the positive cone P_K . Take $h \in P_K$, write $x_* = x_h$ and note that $f(x_h) = x_{fhf^{-1}}$; when $h(x_h) > x_h$, we have $fhf^{-1}(x_{fhf^{-1}}) = fh(x_h) > x_h$, otherwise we have $h(x_h) = x_h$ and

$$D^-(fhf^{-1})(x_{fhf^{-1}}) = D^-h(x_h) > 1.$$

Hence $fhf^{-1} \in P_K$, as wanted.

More generally, for $n \in \mathbb{N}$, consider $h_n = f^n h f^{-n}$ and observe that the point $x_{h_n} = f^n(x_h)$ tends to 1 as $n \rightarrow \infty$. Take $r \in P_K$ and let $y \in (0, 1)$ be such that r acts trivially on $(y, 1)$. If n is large enough so that $h_n(x_{h_n}) = f^n h(x_h)$ and x_{h_n} are greater than y , we have that $x_{r^{-1}h_n} = x_{h_n}$ and $r^{-1}h_n$ coincides with h_n on a neighborhood of x_{h_n} . Since $h_n \in P_K$, and this depends only on the behavior of h_n on some neighborhood of x_{h_n} , we must have $r^{-1}h_n \in P_K$ for n large enough, and thus $h_n \succeq^K r$. Since h and r were arbitrary \preceq^K -positive elements and we can repeat the same reasoning for arbitrary $h, r \in P_K^{-1}$, this shows that the conjugation by f is a homothety. \square

For every f -invariant subset $K \subseteq (0, 1)$ let us denote by $\psi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$ the dynamical realization of the action of F on $(F_+/H, \prec^K)$ defined above. By Proposition 11.17 this action is \mathbb{R} -focal and increasingly horograded by the standard action on $(0, 1)$. Note also that since the residue H is the fixator of K , and two distinct closed subsets of $(0, 1)$ have different fixators, then \preceq^K determines K completely. In particular, by the last part of Proposition 11.17, if $K_1 \neq K_2$ their associated \mathbb{R} -focal actions ψ_{K_1} and ψ_{K_2} are not conjugate.

11.4.3. *Some properties of the actions arising from restriction preorders.* Given a non-empty f -invariant closed subset $K \subseteq (0, 1)$, we keep denoting by $\psi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$ the action constructed above. We want to point out some dynamical properties of this family of actions. Recall that a minimal action of a group G on a locally compact space Y is *topologically free* if the set of fixed points $\text{Fix}(g)$ has empty interior for every $g \in G$. By Baire's theorem this is equivalent to the requirement that a dense G_δ -set of points in Y have a trivial stabilizer in G .

Proposition 11.23 (Freeness and non-freeness). *Let $K \subseteq (0, 1)$ be a non-empty f -invariant closed subset. Then the \mathbb{R} -focal action $\psi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$ defined above is topologically free if and only if $K = (0, 1)$. In particular, F admits both topologically free and non-topologically free minimal \mathbb{R} -focal actions.*

Proof. Assume $K = (0, 1)$. We claim that the action $\psi := \psi_K$ is topologically free. Indeed in this case the preorder \preceq^K is actually a total order on F_+ . Thus there is a dense subset of points in \mathbb{R} with a trivial stabilizer for $\psi(F_+)$, which implies that the action of F_+ is topologically free. Assume by contradiction that $g \in F$ is such that $\text{Fix}^\psi(g)$ has non-empty interior, and let I be a connected component of its interior. Note that $g \notin F_+$ so that by Propositions 11.17 and 11.4, the image $\psi(g)$ must be a pseudohomothety; in particular I is bounded. As the action ψ is proximal (see for instance Lemma 5.13), there exists $h \in F$ such that $\psi(h)(I) \Subset I$. Then it is not difficult to see that the commutator $[g, h] = ghg^{-1}h^{-1}$ is non-trivial, belongs to F_+ and fixes $\psi(h)(I)$ pointwise. This is a contradiction since we have already shown that the action of F_+ is topologically free.

Now consider the case $K \neq (0, 1)$. We can take a connected component $U = (y, z)$ of the complement $(0, 1) \setminus K$, and consider a non-trivial element $h \in F_+$ whose support is contained in U . Fix $x < y$ and consider the \preceq^K -convex subgroup L_x from Lemma 11.21. Take an element $g \in L_x$, and let us prove that the conjugate $g^{-1}hg$ belongs to H . For this, note that the condition $x_g < x$ implies $g^{-1}(U) = U$, so that the restriction of $g^{-1}hg$ to the complement $(0, 1) \setminus U$ is trivial. This immediately implies that $g^{-1}hg$ fixes every point of K , so that $g^{-1}hg$ belongs to the residue H . This proves that $hgH = gH$ for any element $g \in L_x$, so that the element h fixes the \prec^K -convex subset L_x/H pointwise. We deduce that $\psi_K(h)$, which is non-trivial as the action ψ_K is faithful, fixes a non-trivial interval. \square

Remark 11.24. Proposition 11.23 should be compared with the fact that many groups arising via a micro-supported action by homeomorphisms satisfy rigidity results for their non-topologically free actions on compact spaces, as shown in [60, 61, 82] using results on uniformly recurrent subgroups (URS) and confined subgroups. As an example tightly related to this setting consider Thompson's group F and its sibling T acting on the circle. Then every minimal action of T on any compact space is either topologically free or factors onto its standard action on the circle, while every faithful minimal action of F on a compact space is topologically free [60]. Proposition 11.23 shows that actions on the line behave very differently from this perspective, and the notion of topological freeness is much less relevant.

Another feature of this family of actions is that they are all simplicial in the sense of §11.3.

Proposition 11.25 (Simpliciality). *Let $K \subset (0, 1)$ be a non-empty f -invariant closed subset and consider the corresponding action $\psi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$, as constructed above. Then the image of F_+ does not act minimally on \mathbb{R} . In particular every action ψ_K is simplicial.*

Proof. Fix $x \in (0, 1)$ and consider the \preceq^K -convex subgroup $L_x = \{g \in F_+ : x_g \leq x\}$ (Lemma 11.21). In the dynamical realization ψ_K of the action $F \rightarrow \text{Aut}(F_+/H, \prec^K)$, the cosets of L_x

span an F_+ -invariant family of disjoint open intervals, showing that the ψ_K -action of F_+ is not minimal. In particular neither is the action of $[F, F] \subseteq F_+$. \square

One way to analyze finer properties of \mathbb{R} -focal actions of the group F is to apply Theorem 11.3 iteratively, by exploiting the self-similarity of F . Namely assume that $\varphi: F \rightarrow \text{Homeo}_+(\mathbb{R})$ is a minimal \mathbb{R} -focal action increasingly horograded by the standard action on $(0, 1)$. Recall that for every dyadic $x \in (0, 1)$ the group $F_{(0,x)}$ is isomorphic to F , and its image under φ is totally bounded, that is $\text{Fix}^\varphi(F_{(0,x)})$ accumulates on both $\pm\infty$. Thus we can apply Theorem 11.3 to the action of $F_{(0,x)}$ on every connected component J of $\text{Supp}^\varphi(F_{(0,x)})$. It follows that this action still falls into one of the three cases up to semi-conjugacy: action by translations, the standard action, and \mathbb{R} -focal actions. In the third case, this analysis can of course be iterated. We will speak of “sublevels” of the action φ to refer to the actions of the subgroups $F_{(0,x)}$ obtained in this way. From this point of view the actions ψ_K arising from restriction preorder are very special: indeed they are not exotic on any sublevel (in contrast with other \mathbb{R} -focal actions of F ; see Proposition 11.33 below).

Proposition 11.26 (Absence of exotic sublevels). *Let $K \subset (0, 1)$ be a non-empty f -invariant closed subset and consider the corresponding action $\psi_K: F \rightarrow \text{Homeo}_0(\mathbb{R})$, as constructed above. Let $x \in X$ be a dyadic point, and let J be a connected component of $\text{Supp}^\psi(F_{(0,x)})$. Then the ψ -action of $F_{(0,x)}$ on J is semi-conjugate either to its standard action on $(0, x)$, or to a cyclic action by translations induced from group of germs $\text{Germ}(F_{(0,x)}, x) \cong \mathbb{Z}$.*

Proof. Let ξ_0 be the unique fixed point of $\psi(f)$. Let us first show the claim for the action of $F_{(0,x)}$ on $J = I^\psi(x, \xi_0)$ (the connected component of $\text{Supp}^\psi(F_{(0,x)})$ containing ξ_0). The semi-conjugacy type of this action is determined by the preorder $\preceq_{\xi_0} \in \text{LPO}(F_{(0,x)})$ induced by the point ξ_0 on $F_{(0,x)}$, which coincides with the restriction of \preceq^K to $F_{(0,x)}$. Now we distinguish two cases.

First assume that $K \cap (0, x)$ does not accumulate on x . Write $y = \sup\{K \cap (0, x)\} < x$ and let $\preceq_y \in \text{LPO}(F_{(0,x)})$ be its induced preorder on $F_{(0,x)}$. By definition of \preceq^K , it follows that \preceq_{ξ_0} is dominated (in the sense of Definition 3.6) by \preceq_y . Since the dynamical realization of \preceq_y is the standard action of $F_{(0,x)}$, the conclusion in this case follows from Lemma 3.16.

Assume now that $\sup\{K \cap (0, x)\} = x$. In this case, by definition of \preceq^K we get that \preceq_{ξ_0} is dominated by a preorder obtained as the pull-back of one of the two non-trivial preorders on $\text{Germ}(F_{(0,x)}, x) \cong \mathbb{Z}$. This shows the conclusion for $\xi = \xi_0$.

If now $\xi \in \text{Supp}^\psi(F_{(0,x)})$ is arbitrary, then by minimality we can choose $h \in F$ such that $\psi(h)(\xi_0) \in I^\psi(x, \xi)$. Then the conclusion follows from the previous case applied to the action of $F_{(0, h^{-1}(x))} = h^{-1}F_{(0,x)}h$ on $I^\psi(h^{-1}(x), \xi_0)$. \square

11.4.4. *Some variations on the restriction preorder construction.* The restriction preorder construction can be modified in multiple ways to produce new families of minimal \mathbb{R} -focal actions, which are not conjugate to the actions ψ_K defined above. We indicate some of them, without detailed exploration nor attempting to include them all in a unified family.

- (1) *Twisting with sign choices.* In addition to the f -invariant set $K \subseteq (0, 1)$ consider an f -invariant choice of signs $u: K \rightarrow \{+1, -1\}$. We proceed to define a preorder $\preceq^{(K,u)}$ in F_+ . For this, given $g \in F_+$ we say that $g \succeq^{(K,u)} 1_F$ if either $u(x_g) = 1$ and $g \succeq^K 1_F$ or $u(x_g) = -1$ and $g \preceq^K 1_F$. It is direct to check (following the proof of Lemma 11.19) that $\preceq^{(K,u)}$ is an invariant preorder on F_+ and that the f -invariance of u makes $\preceq^{(K,u)}$ invariant under conjugation by f . Of course, when $u \equiv 1$ the preorders $\preceq^{(K,u)}$ and \preceq^K

coincide. There are some straightforward variations to this twist. For instance one may consider two different f -invariant functions $u, v: K \rightarrow \{\pm 1\}$ to determine the sign in the two different cases $g(x_g) \neq x_g$ and $g(x_g) = x_g$.

- (2) *Twisting with derivative morphisms.* In this case, in addition to the f -invariant set $K \subseteq (0, 1)$ consider a total left order $<_0$ on the abelian group

$$A = \{(2^n, 2^m) : n, m \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

(note that A can be thought as the set of derivatives that an element of F can take at a dyadic point). As before, we will define a preorder on F_+ which is invariant under conjugacy by f . For this, consider a different definition of x_g , namely define

$$x'_g := \sup \left\{ x \in K : g(x) \neq x, \text{ or } g(x) = x \text{ and } (D^-g(x), D^+g(x)) \neq (1, 1) \right\}.$$

Then, set $\preceq_0^K \in \text{LPO}(F_+)$ so that $g \succeq_0^K 1_F$ if either $g(x'_g) > x'_g$, or $g(x'_g) = x'_g$ and $(D^-g(x'_g), D^+g(x'_g)) >_0 (1, 1)$. Again, it is straightforward to check (following the proof of Lemma 11.19) that the preorder \preceq_0^K is left invariant and also invariant under conjugation by f .

To compare these preorders with the preorders of the form $\preceq^{(K,u)}$, consider $p \in (0, 1) \cap \mathbb{Z}[1/2]$ and the closed subset $K_p = \{f^n(p) : n \in \mathbb{Z}\}$. In this case all the twists $\preceq^{(K_p, u)}$ given by sign choices coincide with \preceq^{K_p} , while the preorder $\preceq_0^{K_p}$ just defined does not. The interested reader can also check that in this case, the dynamical realization of $\preceq_0^{K_p}$ has sublevels (in the sense of Proposition 11.26) semi-conjugate to $\varphi \circ \pi^{ab}$ where $\pi^{ab} : F_{(0,x)} \rightarrow F_{(0,x)}^{ab} \cong \mathbb{Z}^2$ is the abelianization of $F_{(0,x)}$ (i.e. x is dyadic) and φ is the dynamical realization of $<_0$.

- (3) *Twisting with new orderings of $(0, 1)$.* In the construction of the preorder \preceq^K one can modify the definition of the point x_g by taking the supremum with respect to an order $<_0$ on K which is different from the order induced from the embedding $K \subseteq (0, 1)$. The whole construction will still be well-defined provided $<_0$ is f -invariant and satisfies suitable assumptions, which are not difficult to figure out but are rather technical to state. Instead of discussing this in general, let us give an example.

Take $0 < x_0 < p_1 < p_2 < f(x_0) < 1$ and define K as the union of the orbits of p_1 and p_2 . Then, we define the total order $<_0$ on K so that $f^m(p_i) <_0 f^n(p_j)$ if either $m + i < n + j$, or $m + i = n + j$, $i = 1$ and $j = 2$. More explicitly we have

$$\cdots <_0 f^{-2}(p_2) <_0 p_1 <_0 f^{-1}(p_2) <_0 f(p_1) <_0 \cdots .$$

It is clear that $<_0$ is f -invariant. We can then define a preorder $\preceq^{K, <_0}$ in the same way as the restriction preorder \preceq^K , except that we replace the point x_g by the point x''_g consisting on the $<_0$ -greatest element of the subset $\{x \in K : g(x) \neq x\}$. It is straightforward to check that $\preceq^{K, <_0}$ is left invariant and also invariant under conjugation by f , inducing an order-preserving action $F \rightarrow \text{Aut} \left(F_+ / [1]_{\preceq^{K, <_0}}, \preceq^{K, <_0} \right)$ as above. Denote by $\Psi_0 = \Psi_{K, <_0}$ the dynamical realization of this action and assume that its associated good embedding satisfies $\iota([1]_{\preceq^{K, <_0}}) = 0$. It can be shown that different choices of p_1 and p_2 produce non-conjugate actions. On the other hand, the interested reader can check that the semi-conjugacy classes of the sublevels $F_{(0,x)} \curvearrowright \Gamma^{\Psi_0}(x, 0)$ only depend on the choice of p_2 but not of p_1 . This shows that exotic actions cannot be reconstructed with the information of the semi-conjugacy classes of its sub-levels as defined in Proposition 11.26.

Again there are some obvious variations of this, such as considering preorders on A instead of orders, and modifying the definition of the point x'_g accordingly.

Of course one can consider appropriate combinations of the variants defined above. However whether such combinations make sense or not depends on the choice of the parameters, and a unified treatment would be obscure and pointless. All constructions obtained using these methods yield simplicial actions.

11.5. A plethora of \mathbb{R} -focal actions II: ordering the orbit of a closed subset of $(0, 1)$. We now describe another method to construct \mathbb{R} -focal actions of F . The starting ingredient of this method is again a non-empty closed subset $K \subset (0, 1)$ which is invariant under the generator f given by (11.1). We assume now $K \neq (0, 1)$, and consider the F -orbit of K among closed subsets of $(0, 1)$, and denote it by

$$\mathcal{O}_K := \{g(K) : g \in F\}.$$

As $K \subset (0, 1)$ is a proper subset, we clearly have that the orbit \mathcal{O}_K is infinite. It is then natural to try to define an F -invariant order \prec on \mathcal{O}_K , and then consider its dynamical realization. While this may seem similar to the construction just discussed in §11.4.2, it turns out to be quite different and to produce actions with more exotic dynamical properties. Note that we are not aware of any general receipt to build F -invariant orders on \mathcal{O}_K which works for all K : the way such orders arise depend subtly on the properties of the subset K . However, there is a general strategy which is conveniently described in the language of directed trees. We will first describe this strategy in general, and then illustrate it in practice with a concrete choice of subset K . More examples of actions obtained using this method will appear later in §11.6.

11.5.1. *A strategy to order \mathcal{O}_K .* Assume that $K \subset (0, 1)$ is an f -invariant closed subset. Since the germ of f at 1 generates the group of germs $\text{Germ}(F, 1) \cong \mathbb{Z}$ and K is f -invariant, it follows that every $K_1 = g(K) \in \mathcal{O}_K$ must coincide with K on an interval of the form $(1 - \varepsilon, 1)$, with $\varepsilon > 0$. Thus, it follows that for every pair $K_1, K_2 \in \mathcal{O}_K$, the subsets K_1 and K_2 coincide on some interval of the form $(1 - \varepsilon, 1)$, so that we can define

$$(11.4) \quad \alpha(K_1, K_2) = \inf\{x \in (0, 1) : K_1 \cap [x, 1) = K_2 \cap [x, 1)\}.$$

As K is closed, we have $\alpha(K_1, K_2) \in K_1 \cap K_2$, unless $K_1 = K_2$ (in which case $\alpha(K_1, K_2) = 0$). Moreover, in light of the previous discussion, we get that $\alpha(K_1, K_2) < 1$ for every $K_1, K_2 \in \mathcal{O}_K$. It is clear from the definition that for every $K_1, K_2, K_3 \in \mathcal{O}_K$ with $\alpha(K_1, K_3) \leq \alpha(K_2, K_3)$, we have $\alpha(K_1, K_2) \leq \alpha(K_2, K_3)$ (indeed, when $K_2 \neq K_3$, the three intersections $K_i \cap [\alpha(K_2, K_3), 1)$ for $i \in \{1, 2, 3\}$ coincide). This gives the ultrametric inequality

$$\alpha(K_1, K_2) \leq \max\{\alpha(K_2, K_3), \alpha(K_1, K_3)\}.$$

Note also that for given $g \in F$ and $K_1, K_2 \in \mathcal{O}_K$, we have

$$\begin{aligned} \alpha(g(K_1), g(K_2)) &= \inf\{x \in (0, 1) : K_1 \cap [g^{-1}(x), 1) = K_2 \cap [g^{-1}(x), 1)\} \\ &= \inf\{g(y) \in (0, 1) : K_1 \cap [y, 1) = K_2 \cap [y, 1)\} = g(\alpha(K_1, K_2)). \end{aligned}$$

In the terminology of §8.6, we have just verified that the map $\alpha : \mathcal{O}_K \times \mathcal{O}_K \rightarrow [0, 1)$ is an F -equivariant ultrametric kernel with respect to the standard F -actions on \mathcal{O}_K and $(0, 1)$ respectively. Moreover, we have the following.

Lemma 11.27. *For every closed f -invariant subset $K \subsetneq (0, 1)$, the action of F on \mathcal{O}_K expands α -balls.*

Proof. As the action of F on \mathcal{O}_K is transitive, it is enough to check that there exists a sequence of elements $(g_n) \subset F$ such that the sequence of balls $g_n.B_\alpha(K, x)$ defines an increasing exhaustion of \mathcal{O}_K . For this, note that by f -invariance we have $f^n.B_\alpha(K, x) = B_\alpha(K, f^n(x))$ and thus $\mathcal{O}_K = \bigcup_{n \geq 0} f^n.B_\alpha(K, x)$, as desired. \square

As a consequence of the discussion above and Corollary 8.41, for every α -convex order \triangleleft on \mathcal{O}_K we get a minimal \mathbb{R} -focal action increasingly horograded by the standard F -action. However, it is not clear *a priori* that for a given subset K such an α -convex order exists, and this is why what we have just described is simply a *strategy*. We will see in §11.5.2 and §11.6, that for some choices of K , α -convex orders actually exist, although this is false in general (see Example 11.29 below).

Remark 11.28. In practice, α -convex orders on \mathcal{O}_K are such that the order relation between K_1 and K_2 only depends on how K_1, K_2 behave “right before” the point $\alpha(K_1, K_2)$, in an F -invariant manner. For a formal presentation of this correspondence see Proposition 8.40.

Example 11.29 (Non planarly orderable actions). Recall from Definition 8.36, that there is a natural construction of directed tree associated with the ultrametric kernel α on \mathcal{O}_K . More precisely, we obtain an action

$$\Phi : F \rightarrow \text{Aut}(\mathbb{T}_K, \triangleleft)$$

together with a F -equivariant injective map $i : \mathcal{O}_K \rightarrow \partial^* \mathbb{T}_K$. Roughly speaking, the directed tree $(\mathbb{T}_K, \triangleleft)$ is obtained by taking a copy of $(0, 1)$ for each $K_1 \in \mathcal{O}_K$, and by gluing the two copies corresponding to K_1 and K_2 along the interval $[\alpha(K_1, K_2), 1)$. We denote by $p : \mathcal{O}_K \times (0, 1) \rightarrow \mathbb{T}_K$ the quotient projection and $[K_1, x] := p(K_1, x)$. Then, two points $v, w \in \mathbb{T}_K$ satisfy $v \triangleleft w$ (that is, v lies below w) if and only there exists $K_1 \in \mathcal{O}_K$ and $x, y \in (0, 1)$ so that $v = [K_1, x]$, $w = [K_1, y]$, and $x < y$. The diagonal action of F on $\mathcal{O}_K \times (0, 1)$ descends to an action on $(\mathbb{T}_K, \triangleleft)$ and the projection to the second coordinate descends to an increasing F -equivariant horograding $\pi : \mathbb{T}_K \rightarrow (0, 1)$. Finally, the embedding $i : \mathcal{O}_K \rightarrow \partial^* \mathbb{T}_K$ is defined so that each $K_1 \in \mathcal{O}_K$ is sent to the infimum of the \triangleleft -chain $\{[K_1, x] : x \in (0, 1)\}$, which naturally belongs to $\partial^* \mathbb{T}_K$.

Although non-strictly necessary for the sequel, this point of view is well-suited for understanding in a more conceptual way whether for given K the action $\Phi : F \rightarrow \text{Aut}(\mathbb{T}_K, \triangleleft)$ admits a Φ -invariant planar order, and this is the same (after Proposition 8.40) to the condition that \mathcal{O}_K admit an α -convex order. This turns out to depend on the local geometry of K relatively to dilations by 2. We explain this with an example, but first we introduce some terminology to discuss the local geometry. We say that two closed sets $K_1, K_2 \subseteq (0, 1)$ have equivalent left-germs at x if for some $\varepsilon > 0$ it holds $K_1 \cap (x - \varepsilon, x] = K_2 \cap (x - \varepsilon, x]$. We denote by K_x^- the left-germ class of the subset K at x . Notice that the group $\text{Germ}_-(x)$ of left-germs of homeomorphisms fixing x naturally acts on the set of left-germs of closed sets at x . We denote by $h_x \in \text{Germ}_-(x)$ the germ of the homothety that fixes x and has derivative 2.

Recall from Remark 8.16 that the existence of such invariant planar ordering boils down to the existence, for each branching point $v \in \mathbb{T}_K$, of an ordering of the set of connected components E_v^- below v , which is invariant under the action of the stabilizer $\text{Stab}^\Phi(v)$. An obstruction for this is clearly given by finite orbits. With this in mind, consider an f -invariant closed subset $K \subseteq (0, 1)$ containing a dyadic point $x \in K \cap \mathbb{Z}[\frac{1}{2}]$ such that $h_x(K_x^-) \neq K_x^-$ but $h_x^n(K_x^-) = K_x^-$ for some $n > 1$. Write $v = [K, x]$ and let $e_v(K)$ be the component of E_v^- corresponding to the ray $\{[K, x] : x \in (0, 1)\}$. Since $\text{Germ}_-(F, x) = \langle h_x \rangle$, it holds that the component $e_v(K)$ has a finite orbit which is not a fixed point, so that there exists no $\text{Stab}^\Phi(v)$ -invariant total order on E_v^- .

11.5.2. *A concrete example.* We now illustrate the flexibility of the method described in §11.5.1, with an explicit example of subset K . More precisely we will construct a subset $K \subset (0, 1)$ with the following property: *there is an explicit (continuous) injective map from the set $\mathbb{O}(\mathbb{N})$*

of orders on the natural numbers \mathbb{N} to the set of F -invariant orders on the orbit \mathcal{O}_K . This will provide a family of \mathbb{R} -focal actions of F which are naturally indexed by orders on \mathbb{N} .

We start by choosing an irrational point $x_0 \in (0, 1)$, and consider the interval $I = (f^{-1}x_0, x_0]$, which is a fundamental domain for f . Next we choose a sequence of open intervals $(J_n)_{n \geq 1}$ with dyadic endpoints such that $J_n \subseteq J_{n+1} \subset I$ for every $n \geq 1$, and such that $\bigcup_{n \geq 1} J_n = (f^{-1}x_0, x_0)$. For every $n \geq 1$, write $y_n = \sup J_n$ and choose an element $h_n \in F_{J_n}$ with the following properties:

- $h_n(x) > x$ for every $x \in J_n$,
- $h_n(J_{n-1}) \cap J_{n-1} = \emptyset$ for $n \geq 2$.
- $D^-h_n(y_n) = 1/2$ (in other words, the germ of h_n at y_n generates the group of germs $\text{Germ}(F_{(0, y_n)}, y_n)$).

Choose now a dyadic point $z_0 \in J_0$, and let $\Sigma_0 = \{h_1^n(z_0) : n \in \mathbb{N}\}$ be its forward orbit under h_1 . By construction we have the inclusion $\Sigma_0 \subset J_0$ and equality $\overline{\Sigma_0} = \Sigma_0 \cup \{y_1\}$. Set $\Sigma_1 = \bigcup_{n \geq 0} h_2^n(\overline{\Sigma_0})$, so that $\overline{\Sigma_1} = \Sigma_1 \cup \{y_2\}$. Continue in this way by defining for every $i \geq 1$ a subset $\Sigma_i = \bigcup_{n \geq 0} h_{i+1}^n(\overline{\Sigma_{i-1}})$. Set $\Sigma_\omega = \bigcup_{i \in \mathbb{N}} \Sigma_i$, and note that $\overline{\Sigma_\omega} = \Sigma_\omega \cup \{x_0\}$. Note also that $\overline{\Sigma_\omega}$ is contained in the fundamental domain I on f . Thus we obtain an f -invariant closed subset K as

$$(11.5) \quad K = \bigcup_{n \in \mathbb{Z}} f^n(\overline{\Sigma_\omega}).$$

By construction the subset $\overline{\Sigma_\omega}$ is invariant under the semigroup $S := \langle h_n : n \geq 1 \rangle_+$, in the sense that $s(\overline{\Sigma_\omega}) \subset \overline{\Sigma_\omega}$ for every $s \in S$. See Figure 11

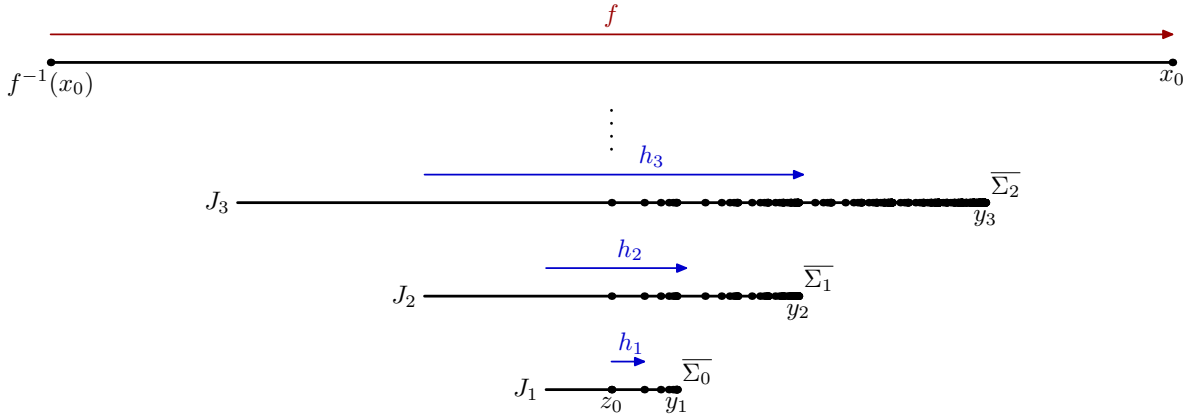


FIGURE 11. Construction of the compact set K .

The subset $\overline{\Sigma_\omega}$ is countable and compact, and its points can be classified according to their *Cantor–Bendixson rank* (see [53, §6]), as follows. Points of rank 0 are the isolated points: these are exactly points in the orbit of z_0 under the semigroup S . Points of rank 1 are those that are not isolated, but become isolated after removing the isolated points: these are exactly points in the S -orbit of y_1 . Continuing in this way, points of rank n are precisely points in the S -orbit of y_n . Finally there is a unique point whose rank is the first countable ordinal ω , namely the point x_0 . This discussion can be directly extended to the subset K . We write $\text{rk}_K(x)$ for the Cantor–Bendixson rank of a point $x \in K$. Note that for every $g \in F$ and $x \in K$, we have the relation $\text{rk}_{g(K)}(g(x)) = \text{rk}_K(x)$.

We next consider the F -equivariant ultrametric kernel $\alpha : \mathcal{O}_K \times \mathcal{O}_K \rightarrow [0, 1]$ defined as in (11.4) and the key observation is that the particular choice of the subset K allows to directly relate α with the Cantor–Bendixson rank.

Lemma 11.30. *Let $K \subset (0, 1)$ be the subset defined at (11.5). For every $K_1, K_2 \in \mathcal{O}_K$, the point $x = \alpha(K_1, K_2)$ is such that $\text{rk}_{K_1}(x)$ and $\text{rk}_{K_2}(x)$ are both finite, and moreover $\text{rk}_{K_1}(x) \neq \text{rk}_{K_2}(x)$ unless $\text{rk}_{K_1}(x) = \text{rk}_{K_2}(x) = 0$.*

Conversely, for every distinct $n, m \in \mathbb{N}$ there exist $K_1, K_2 \in \mathcal{O}_K$ such that the point $x = \alpha(K_1, K_2)$ satisfies $\text{rk}_{K_1}(x) = n$ and $\text{rk}_{K_2}(x) = m$.

Proof. We first need some observations.

Claim 1. *For every $x \in K$ and every $g \in F$ such that $g(x) = x$, there exists $\varepsilon > 0$ such that $g(K) \cap (x - \varepsilon, x] = K \cap (x - \varepsilon, x]$.*

Proof of claim. Up to replace g by its inverse, we can assume $D^-g(x) \leq 1$. Also, upon conjugating by powers of f , we can assume $x \in \overline{\Sigma_\omega}$. If $x = x_0$ then this follows from the fact that we chose x_0 to be irrational, so that every element of F that fixes x_0 must actually fix a neighborhood of it. If x is isolated in K the conclusion is obvious. Finally assume that $n := \text{rk}_K(x) \notin \{0, \omega\}$. Then x is in the S -orbit of the point y_n , so that it is fixed by a conjugate h of h_n , which has therefore the property that $D^-h(x) = 1/2$. Hence the restriction of g to a left-neighborhood of x must coincide with the restriction of some non-negative power of h , so that we can conclude from the fact that K is forward invariant under h . \square

Claim 2. *For every pair of points $x, y \in K$ with $\text{rk}_k(x) = \text{rk}_k(y)$, there exist an element $h \in F$ and $\varepsilon > 0$ such that $h(x) = y$ and $h(K) \cap (y - \varepsilon, y] = K \cap (y - \varepsilon, y]$.*

Proof of claim. Upon replacing x, y with $f^m(x), f^n(y)$ for suitable n, m we can assume that $x, y \in \overline{\Sigma_\omega}$. Then x and y are in the same S -orbit, and so it is enough to observe that elements of S and their inverses have this property. \square

With this in mind, let us prove the lemma. We can assume without loss of generality that $K_1 = K$. Take $g \in F$ such that $K_2 = g(K)$ and set $x = \alpha(K, K_2)$ and $y = g^{-1}(x) \in K$, so that $\text{rk}_{K_2}(x) = \text{rk}_K(y)$. Assume by contradiction that $\text{rk}_K(x) = \text{rk}_K(y) \geq 1$. After Claim 2, we can choose $h \in F$ such that $h(x) = y$ and $\varepsilon > 0$ such that $h(K) \cap (y - \varepsilon, y] = K \cap (y - \varepsilon, y]$. Then the element $g' = hg$ is such that $g'(y) = y$, so that upon taking a smaller ε , by Claim 1 we also have $g'(K) \cap (y - \varepsilon, y] = K \cap (y - \varepsilon, y]$. Applying h^{-1} we deduce that there is $\varepsilon' > 0$ such that $g(K) \cap (x - \varepsilon', x] = K \cap (x - \varepsilon', x]$, and the latter intersection is not reduced to $\{x\}$ since we assume that $\text{rk}_K(x) \geq 1$. This contradicts the definition of $x = \alpha(g(K), K)$. Thus $\text{rk}_K(x) \neq \text{rk}_{g(K)}(x)$ unless both ranks are 0. Finally this also implies that we cannot have $\text{rk}_K(x) = \omega$, indeed since points of rank ω are the only non-dyadic points in K this would imply that $\text{rk}_{g(K)}(x) = \omega$ as well, contradicting the previous reasoning. \square

Now let $\mathcal{O}(\mathbb{N})$ be the set of total orders on the natural numbers. To every order \prec in $\mathcal{O}(\mathbb{N})$ we associate an F -invariant order \prec^* on \mathcal{O}_K , as follows. Given distinct $K_1, K_2 \in \mathcal{O}_K$, set $n_1 = \text{rk}_{K_1}(\alpha(K_1, K_2))$ and $n_2 = \text{rk}_{K_2}(\alpha(K_1, K_2))$. If $n_1 \neq n_2$, then we declare $K_1 \prec^* K_2$ if and only if $n_1 \prec n_2$. Else, by Lemma 11.30 we have $n_1 = n_2 = 0$, namely the point $\alpha(K_1, K_2)$ is isolated in both K_1 and K_2 . In this case set

$$(11.6) \quad x_i = \max\{x \in K_i : x < \alpha(K_1, K_2)\} \quad \text{for } i \in \{1, 2\}.$$

Then for $i \in \{1, 2\}$ we must have $x_i < \alpha(K_1, K_2)$ and $x_1 \neq x_2$ by definition of $\alpha(K_1, K_2)$. In this case we declare $K_1 \prec^* K_2$ if and only if $x_1 < x_2$. It is routine to verify that this defines

indeed a total order relation, and it is clear from the construction, and F -equivariance of the ultrametric kernel α and the Cantor–Bendixson rank, that this order is F -invariant.

Denote by $\varphi_{\prec} : F \rightarrow \text{Homeo}_0(\mathbb{R})$ the dynamical realization of the action of F on (\mathcal{O}_K, \prec^*) . We want to prove that φ_{\prec} is a minimal \mathbb{R} -focal action, increasingly horograded by the standard action of F . After the discussion in §11.5.1, this is equivalent to the property that the α -balls are \prec^* -convex. This is what we verify next.

Lemma 11.31. *With notation as above, the α -ball*

$$B_{\alpha}(L, x) = \{L' \in \mathcal{O}_K : \alpha(L, L') \leq x\}$$

is \prec^ -convex for every $L \in \mathcal{O}_K$ and $x \in (0, 1)$.*

Proof. First notice that the \prec^* -order relation between $K_1, K_2 \in \mathcal{O}_K$ is determined by the intersections $K_1 \cap [x, 1)$ and $K_2 \cap [x, 1)$ for any $x \in (0, 1)$ such that these intersections do not coincide.

Now take elements $K_1, K_2 \in B_{\alpha}(L, x)$ for some $L \in \mathcal{O}_K$ and $x \in (0, 1)$. This is equivalent to the condition that

$$(11.7) \quad K_1 \cap [x, 1) = K_2 \cap [x, 1) = L \cap [x, 1).$$

Consider next an element K_3 between K_1 and K_2 (with respect to \prec^*) and assume by contradiction that $K_3 \notin B_{\alpha}(L, x)$. This means that $K_3 \cap [x, 1) \neq L \cap [x, 1)$ and therefore, considering the equalities (11.7), the \prec^* -order relation between K_i and K_3 is determined by the intersections $K_3 \cap [x, 1)$ and $L \cap [x, 1)$ for every $i \in \{1, 2\}$. Hence we conclude that the \prec^* -order relation between K_1 and K_3 coincides with that of K_2 and K_3 . As we are assuming that K_3 lies between K_1 and K_2 , we necessarily have $K_1 = K_2 = K_3$, but this contradicts the assumption $K_3 \notin B_{\alpha}(L, x)$. \square

As a conclusion of our discussion, we have the following.

Proposition 11.32. *With notation as above, for any $\prec \in \mathcal{O}(\mathbb{N})$, the dynamical realization $\varphi_{\prec} : F \rightarrow \text{Homeo}_0(\mathbb{R})$ of the action of F on (\mathcal{O}_K, \prec^*) is a minimal \mathbb{R} -focal action increasingly horograded by the standard action of F .*

Moreover, if \prec_1 and \prec_2 are distinct orders on \mathbb{N} , then the actions φ_{\prec_1} and φ_{\prec_2} are not conjugate.

Proof. After Lemma 11.31, the α -balls are \prec^* -convex, so the first statement is a consequence of Corollary 8.41.

For a given order $\prec \in \mathcal{O}(\mathbb{N})$, observe that by definition of dynamical realization, the F -action on (\mathcal{O}_K, \prec^*) can be identified with the φ_{\prec} -action on the orbit of the unique fixed point ξ of $\varphi_{\prec}(f)$ with the order induced by \mathbb{R} , so that the order \prec^* can be reconstructed from φ_{\prec} . Finally the order \prec on \mathbb{N} can be reconstructed from \prec^* by the last statement in Lemma 11.30. \square

We now point out a qualitative difference which distinguishes the family of \mathbb{R} -focal actions constructed here from the one obtained via the restriction preorder construction as in §11.4.2. Indeed, in this case the actions of the subgroups $F_{(0,x)} \cong F$ on the components of their support can remain exotic (compare this with Proposition 11.26).

Proposition 11.33 (Presence of exotic sublevels). *Fix an order \prec on \mathbb{N} and let $\varphi := \varphi_{\prec} : F \rightarrow \text{Homeo}_0(\mathbb{R})$ be the \mathbb{R} -focal action constructed above. Then there exist a dyadic point $x \in (0, 1)$ and a connected component J of $\text{Supp}^{\varphi}(F_{(0,x)})$ such that the action of $F_{(0,x)}$ on J is semi-conjugate to a faithful \mathbb{R} -focal action.*

Proof. Note first that for every $x \in (0, 1)$ and $g \in F$ such that $g(K \cap [x, 1)) = K \cap [g(x), 1)$ then $g.B_\alpha(K, x) = B_\alpha(K, g(x))$. In particular, the α -ball $B_\alpha(K, x)$ is preserved by the subgroup $F_{(0,x)}$. Let $\iota: (\mathcal{O}_K, \prec^*) \rightarrow \mathbb{R}$ be an equivariant good embedding associated with φ (in the terminology of Definition 2.37) and let I_x be the open interval spanned by $\iota(B_\alpha(K, x))$, namely I_x is the interior of the closure of $\iota(B_\alpha(K, x))$ (using minimality of the action and that the α -balls are \prec^* -convex after Lemma 11.31). Consider the element h_1 from the construction of K , and consider the points $z_0 \in J_1$ and $y_1 = \sup J_1$ as in the construction; for $n \geq 1$ set $z_n = h_1^n(z_0)$, which by construction is an increasing sequence converging to y_1 . For every $n \geq 0$ we have $h_1^n(K \cap [z_0, 1)) = K \cap [z_n, 1)$, so that $h_1^n.B_\alpha(K, z_0) = B_\alpha(K, z_n)$. The corresponding intervals I_{z_n} satisfy $I_{z_n} \subseteq I_{z_{n+1}}$ and $h_1.I_{z_n} = I_{z_{n+1}}$. Set $B := \bigcup_{n \geq 0} B_\alpha(K, h_1^n(z_0))$, and let $J = \bigcup_{n \geq 0} I_{z_n}$ be the interval spanned by $\iota(B)$. Then B is preserved by $F_{(0,y_1)}$ and the action of $F_{(0,y_1)}$ on B is cofinal (with respect to the order \prec^* restricted to B). As a consequence, $F_{(0,y_1)}$ preserves J and acts on it without fixed points, so that J is a connected component of $\text{Supp}^\varphi(F_{(0,y_1)})$. Since moreover $h_1.I_{z_n} = I_{z_{n+1}}$, we deduce that $\varphi(h_1)$ acts on J as a pseudohomothety. This cannot happen if the action of $F_{(0,y_1)}$ on $I^\varphi(y_1, \xi)$ is semi-conjugate to an action by translations, nor if it is semi-conjugate to the standard action on $(0, y_1)$. Thus by Theorem 11.3 the action of $F_{(0,y_1)}$ must be semi-conjugate to an \mathbb{R} -focal action. \square

Nonetheless, this family of examples still turns out to produce simplicial \mathbb{R} -focal actions in the sense of §11.3.

Proposition 11.34 (Simpliciality). *For an order \prec on \mathbb{N} , let $\varphi_\prec: F \rightarrow \text{Homeo}_0(\mathbb{R})$ be the \mathbb{R} -focal action constructed above. Then $\varphi_\prec(F_+)$ does not act minimally on \mathbb{R} . In particular each action φ_\prec is simplicial*

Proof. We keep the same notation as in the proof of Proposition 11.33. Let $x_0 \in K$ be the point as in the construction of K . We claim that the α -ball $B_\alpha(K, x_0) \subset \mathcal{O}_K$ has the property that for every $g \in F_+$ we have either $g(B_\alpha(K, x_0)) = B_\alpha(K, x_0)$ or $g(B_\alpha(K, x_0)) \cap B_\alpha(K, x_0) = \emptyset$. It then follows that the interval $I^\varphi(x_0, \xi)$ has the same property for $\varphi_\prec(F_+)$, so that the union of its translates defines a proper invariant open subset, contradicting minimality. Indeed, suppose that $g \in F_+$ and $K_1 \in B_\alpha(K, x_0)$ are such that $g(K_1) \in B_\alpha(K, x_0)$, namely we assume

$$K_1 \cap [x_0, 1) = g(K_1) \cap [x_0, 1) = K \cap [x_0, 1).$$

The key observation is that this implies that g must actually fix $K \cap [x_0, 1)$. First of all observe that g must send points of rank ω in K_1 to points of rank ω in $g(K_1)$, and the set of such points in both $K_1 \cap [x_0, 1)$ and $g(K_1) \cap [x_0, 1)$ consists precisely of the sequence $x_n := f^n(x_0)$ for $n \geq 0$. This is a discrete increasing sequence and $g(x_n) = x_n$ for n large enough, hence we deduce from the condition $g \in F_+$ that $g(x_n) = x_n$ for every $n \geq 0$. As a consequence the cyclic subgroup $\langle g \rangle$ must preserve every interval $[x_n, x_{n+1}]$, with $n \geq 0$, and thus every intersection $K \cap [x_n, x_{n+1}]$ for $n \geq 0$. Assume by contradiction that there exist $n \geq 0$ and a point $t \in K \cap [x_n, x_{n+1}]$ which is not fixed by g , and consider the orbit $\Omega = \{g^m(t) : m \in \mathbb{Z}\}$, which is a subset of K . As $K \cap [x_n, x_{n+1}]$ is compact, the point $\inf \Omega$ is in $K \cap [x_n, x_{n+1}]$, and it is accumulated by points of Ω (and hence of K) from the right. This is in contradiction with the choice of K , as by construction every point of K is isolated from the right hand side. Hence g fixes $K \cap [x_0, 1)$, which implies that $g.B_\alpha(K, x_0) = B_\alpha(K, x_0)$. \square

11.6. A plethora of \mathbb{R} -focal actions III: existence of non-simplicial \mathbb{R} -focal actions.

All the example of \mathbb{R} -focal actions of F discussed so far are simplicial in the sense of Definition 11.15, so that it is tempting to try to prove that all \mathbb{R} -focal actions of F must be simplicial.

However, we build an exotic action of Thompson's group F which fails to have this property. With Proposition 11.13 in mind, we will prove the following.

Theorem 11.35. *There exist faithful \mathbb{R} -focal actions $\varphi: F \rightarrow \text{Homeo}_0(\mathbb{R})$ such that $\varphi([F, F])$ acts minimally (and thus are not simplicial). More precisely, there exist uncountably many such actions which are pairwise non-conjugate, and whose restriction to $[F, F]$ yield pairwise non-conjugate actions of $[F, F]$.*

Note the following consequence of independent interest.

Corollary 11.36. *The group $[F, F]$ admits uncountably many, pairwise non-conjugate minimal actions $\varphi: [F, F] \rightarrow \text{Homeo}_0(\mathbb{R})$.*

This should be compared with the general constructions of exotic actions of groups of compactly supported homeomorphisms described in §5.3, which provide actions without any closed minimal invariant subset. The construction given here relies on the classical symbolic coding of the the standard action of F by binary sequences, which is specific to Thompson's groups.

For the proof it will be convenient to see F as a group of homeomorphisms of $X = \mathbb{R}$ rather than of the interval $(0, 1)$. Namely we realize $F \subseteq \text{Homeo}_0(\mathbb{R})$ as the group of piecewise linear maps of the line, with dyadic breakpoints, slopes in the group $\langle 2^n : n \in \mathbb{Z} \rangle$ and which coincide with integer translations near $\pm\infty$. It is well-known that this action is conjugate to the natural action of F on $(0, 1)$ (see e.g. [8, Lemma E18.4]).

From now and until the end of this subsection, the term standard action will refer to the action of F on \mathbb{R} described above. We will denote by $f \in F$ the translation $f(x) = x + 1$. (Note that the element f corresponds to the element given by (11.1) in the action on $(0, 1)$).

The proof of Theorem 11.35 employs the strategy described in §11.5.1, namely we will start with a closed f -invariant subset $K \subset \mathbb{R}$, and define an invariant order on its orbit $\mathcal{O}_K := \{g(K) : g \in F\}$. The main difficulty is that we need to construct a subset K satisfying a somewhat delicate combination of properties. We begin with a definition.

Definition 11.37. We say that a subset $K \subset \mathbb{R}$ has property (O) if it is proper, non-empty, closed, f -invariant, and moreover $g(K) \cap K$ is open in K for every $g \in F$.

Remark 11.38. Note that the last condition for property (O) is actually equivalent to that $K_1 \cap K_2$ be open in K_1 and K_2 for every $K_1, K_2 \in \mathcal{O}_K$.

Example 11.39. Property (O) is clearly satisfied when K is a non-empty f -invariant discrete subset, as for example the f -orbit of a point. However, this is not a good example for the construction described in this subsection, as the stabilizer of such K in $[F, F]$ is trivial (cf. Proposition 11.41).

Assume that K is a subset with property (O) (many examples are exhibited by Lemma 11.42). We consider the F -equivariant ultrametric kernel $\alpha : \mathcal{O}_K \times \mathcal{O}_K \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as in §11.5.1, namely

$$\alpha(K_1, K_2) := \inf \{x \in \mathbb{R} : [x, +\infty) \cap K_1 = [x, +\infty) \cap K_2\}.$$

Reasoning as in the example of §11.5.2, we proceed to construct an F -invariant order on \mathcal{O}_K and then prove that the dynamical realization of the action of F on (\mathcal{O}_K, \prec) is minimal and \mathbb{R} -focal. This is the content of the next result.

Proposition 11.40. *If a subset $K \subset \mathbb{R}$ has property (O), the relation \prec on \mathcal{O}_K defined by $K_1 \prec K_2$ if and only if*

$$\max \{x \in K_1 : x < \alpha(K_1, K_2)\} < \max \{x \in K_2 : x < \alpha(K_1, K_2)\},$$

is an F -invariant total order on \mathcal{O}_K . Moreover, the dynamical realization $\varphi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$ of the action of F on (\mathcal{O}, \prec) is a minimal \mathbb{R} -focal action.

Proof. Recall that $\alpha(K_1, K_2) \in K_1 \cap K_2$ whenever K_1 and K_2 are different elements of \mathcal{O}_K . As K satisfies property (O), we have that $K_1 \cap K_2$ is open inside both K_1 and K_2 and hence $K_1 \cap K_2$ is an open neighborhood of $\alpha(K_1, K_2)$ inside K_1 and K_2 . Therefore $\alpha(K_1, K_2)$ is isolated from the left hand side in both K_1 and K_2 . As for (11.6), we deduce that the points

$$x_i := \max \{x \in K_i : x < \alpha(K_1, K_2)\} \quad \text{for } i \in \{1, 2\}$$

are distinct, so we can declare $K_1 \prec K_2$ if and only if $x_1 < x_2$. As for the order \prec^* from §11.5.2, it is routine to check that this defines indeed an F -invariant total order on \mathcal{O}_K .

Similarly one proceeds as in §11.5.2 to check that φ_K is minimal and \mathbb{R} -focal. Namely, one verifies that the order \prec is α -convex, and the proof of Lemma 11.31 can be adapted *verbatim* to this case (just replacing \prec^* with \prec and $(0, 1)$ with \mathbb{R}). Then Corollary 8.41 gives the desired conclusion. \square

The main difference from the construction in §11.5.2 is the way that the commutator subgroup $[F, F]$ acts in the actions φ_K .

Proposition 11.41. *Given a subset $K \subset (0, 1)$ with property (O), let $\varphi_K : F \rightarrow \text{Homeo}_0(\mathbb{R})$ be the corresponding minimal \mathbb{R} -focal action from Proposition 11.40. Then the following hold.*

- (i) *If K has property (O), then $\varphi_K([F, F])$ acts minimally provided that the stabilizer of K in $[F, F]$ (with respect to the standard action) acts on K without fixed points. Moreover in this case the induced action of $[F, F]$ is minimal and \mathbb{R} -focal.*
- (ii) *If two distinct subsets $K, K' \subset \mathbb{R}$ have property (O), then the restrictions of φ_K and $\varphi_{K'}$ to $[F, F]$ are not conjugate actions of $[F, F]$. In particular φ_K and $\varphi_{K'}$ are not conjugate.*

Proof. To prove (i), assume that the stabilizer of K in $[F, F]$ acts without fixed points on K . Fix $x \in (0, 1)$ and choose a sequence of elements of $[F, F]$ which preserve K and such that $g_n(x)$ tends to $+\infty$. Then $g_n.B_\alpha(K, x) = B_\alpha(K, g_n(x))$, so that $\mathcal{O}_K = \bigcup_{n \geq 0} g_n.B_\alpha(K, x)$. Then by Proposition 7.6, the subgroup $[F, F]$ admits a unique minimal invariant set $\Lambda \subset \mathbb{R}$, which is preserved by F because $[F, F]$ is a normal subgroup. We deduce that the action of $[F, F]$ is also minimal.

To prove (ii) take $K \neq K'$ with property (O) and assume without loss of generality that $K' \not\subset K$. Write $\alpha : \mathcal{O}_K \times \mathcal{O}_K \rightarrow \mathbb{R} \cup \{-\infty\}$ and $\beta : \mathcal{O}_{K'} \times \mathcal{O}_{K'} \rightarrow \mathbb{R} \cup \{-\infty\}$ for the corresponding ultrametric kernels. Fix $x \in K$ and let D be the subgroup of $[F, F]_{(x, +\infty)}$ which fixes K pointwise. Then for every $g \in D$ we have $g.B_\alpha(K, x) = B_\alpha(K, x)$ and actually g fixes $B_\alpha(K, x)$ pointwise, so that the dynamical realization φ_K fixes a non-empty open interval pointwise. On the contrary, for any $y \geq x$ with $y \in K' \setminus K$, we can consider an element $h \in D$ such that $h(y) \notin K'$ and $h(y) > y$. Let us show that for such choices we have $h.B_\beta(K', y) \cap B_\beta(K', y) = \emptyset$. Indeed, assume there exists $L \in h.B_\beta(K', y) \cap B_\beta(K', y)$; then, as $h.B_\beta(K', y) = B_\beta(h(K'), h(y))$, we have

$$L \cap [h(y), +\infty) = h(K') \cap [h(y), +\infty)$$

and in particular $h(y) \in L$. However, if $L \in B(K', y)$, then $L \cap [y, +\infty) = K' \cap [y, +\infty)$ and thus $h(y) \notin L$, which is an absurd.

By f -invariance of $K' \setminus K$, we can find arbitrarily large points y , and thus elements $h \in D$, satisfying such properties. As $\mathcal{O}_{K'} = \bigcup_y B_\beta(K', y)$, this implies that D acts without fixed points, so that the actions φ_K and $\varphi_{K'}$ cannot be conjugate. \square

After the previous proposition, in order to prove Theorem 11.35 we need to show the existence of subsets $K \subset \mathbb{R}$ with property (O) and with the additional property that the stabilizer of K in $[F, F]$ does not have fixed points. For this, we are going to use the symbolic description of real numbers by binary expansions.

To each infinite sequence $(a_n)_{n \geq 1} \in \{0, 1\}^{\mathbb{N}}$ we associate the real number $\text{ev}((a_n)) := \sum_{n \geq 1} a_n 2^{-n} \in [0, 1]$. Note that this association is continuous if we endow $\{0, 1\}^{\mathbb{N}}$ with the product topology. If z, w_1, w_2 are finite binary sequences (*binary words* for short), we consider the *cylinder* over z , defined by

$$C_z = \{w \in \{0, 1\}^{\mathbb{N}} : z \text{ is a prefix of } w\},$$

and denote by $\tilde{K}_0(w_1, w_2) \subseteq \{0, 1\}^{\mathbb{N}}$ the subset of all infinite concatenations of w_1 's and w_2 's. Clearly, both images $\text{ev}(C_z)$ and $K_0(w_1, w_2) := \text{ev}(\tilde{K}_0(w_1, w_2))$ are closed subsets of $[0, 1]$, and the former is a closed interval with dyadic endpoints (a *dyadic interval* for short). Note that, conversely, any closed dyadic interval is the union of (the real numbers represented by) finitely many cylinders.

With this in mind, if z_1 and z_2 are binary words, the *substitution map* $S(z_1, z_2) : C_{z_1} \rightarrow C_{z_2}$ defined by $S(z_1, z_2)(z_1 w) = z_2 w$ represents an affine map $\overline{S(z_1, z_2)}$ between dyadic intervals of $[0, 1]$. Therefore, in the action of F on \mathbb{R} , every element of F locally coincides (except at breakpoints, which are finitely many dyadic rationals) with transformations of the form $f^n \circ \overline{S(z_1, z_2)} \circ f^m$, for some powers $n, m \in \mathbb{Z}$ and some finite sequences z_1, z_2 .

We say that a pair of binary words w_1, w_2 has the *cancellation property* if whenever $zw = w'$ for $w, w' \in \tilde{K}_0(w_1, w_2)$, it holds that z is a finite concatenation of w_1 's and w_2 's. As a concrete example of a pair of words with the cancellation property, we may take $w_1 = 0$ and $w_2 = 1$ but these are *constant* binary words (i.e. made of a single repeated bit). As a concrete example of non-constant binary words with the cancellation property we can take $w_1 = 10001$ and $w_2 = 01110$.

Lemma 11.42. *Let w_1 and w_2 be non-constant binary words satisfying the cancellation property and write $K_0 := K_0(w_1, w_2)$. Then, the subset $K := \bigcup_{n \in \mathbb{Z}} f^n(K_0)$ has property (O).*

Proof. Since K_0 is a closed subset of $[0, 1]$, the subset K is a closed and f -invariant subset of \mathbb{R} . Also, since w_1 and w_2 are non-constant, the set $\tilde{K}_0(w_1, w_2)$ has no eventually constant sequences, and so the subset K_0 contains no dyadic points. It follows that $\text{ev} : \tilde{K}_0(w_1, w_2) \rightarrow K_0$ is a homeomorphism onto its image, and that the set of intersections of the form $(p/2^n, (p+1)/2^n) \cap K$ with $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, forms a basis of its topology. The restriction of every element of F to K is locally given by maps of the form $f^n \circ \overline{S(z_1, z_2)} \circ f^m$, and since K is f -invariant, in order to check property (O) it is enough to check that $\overline{S(z_1, z_2)}(K_0 \cap \text{ev}(C_{z_1}))$ is open in K_0 for every pair of finite binary words z_1, z_2 .

For this, consider two binary finite words z_1, z_2 and also $w \in \tilde{K}_0(w_1, w_2) \cap C_{z_1}$ so that $S(z_1, z_2)(w) \in K_0(w_1, w_2)$. We need to check that $S(z_1, z_2)(\tilde{K}_0(w_1, w_2) \cap C_{z_1})$ contains a neighborhood of $S(z_1, z_2)(w)$. Since the pair w_1, w_2 has the cancellation property, we can write $w = z'_1 w'$ with $z'_1 = z_1 z''_1$ and $w' \in \tilde{K}_0(w_1, w_2)$. Since $S(z_1, z_2)(w)$ equals $z_2 z''_1 w'$ and

belongs to $\tilde{K}_0(w_1, w_2)$, again by the cancellation property we conclude that $z_2 z_1''$ is a finite concatenation of w_1 's and w_2 's. Therefore

$$S(z_1, z_2) \left(C_{z_1'} \cap \tilde{K}_0(w_1, w_2) \right) = C_{z_2 z_1''} \cap \tilde{K}_0(w_1, w_2),$$

showing that $S(z_1, z_2)(\tilde{K}_0(w_1, w_2))$ contains a neighborhood of $S(z_1, z_2)(w)$ inside $\tilde{K}_0(w_1, w_2)$. This concludes the proof. \square

In order to ensure that the stabilizer of K in $[F, F]$ has no fixed points, we need to impose one last extra condition on K .

Say that a map $h : I \rightarrow J$ is a *dyadic affine map* between intervals if I is a dyadic interval and h is of the form $x \mapsto ax + b$, where $a \in \{2^n : n \in \mathbb{Z}\}$, and $b \in \mathbb{Z}[1/2]$. Consider now a compact subset $K_0 \subseteq (0, 1)$. We say that K_0 admits a *self similar decomposition* if there exists a pair of dyadic affine maps $h_1, h_2 : I \rightarrow (0, 1)$ such that

- $h_1(I) \cap h_2(I) = \emptyset$,
- $h_1(K_0) \cup h_2(K_0) = K_0$.

For example, the subset $K_0(w_1, w_2)$ admits a self similar decomposition provided the words w_1 and w_2 are such that $\text{ev}(C_{w_1})$ and $\text{ev}(C_{w_2})$ are disjoint. Indeed in this case we have that for $i \in \{1, 2\}$, the symbolic maps $w \mapsto w_i w$ correspond to dyadic affine maps $h_i : [0, 1] \rightarrow [0, 1]$ with disjoint images and such that $K_0 = h_1(K_0) \sqcup h_2(K_0)$.

Lemma 11.43. *Let $K_0 \subset (0, 1)$ be a closed subset admitting a self similar decomposition, and let $K = \bigcup_{n \in \mathbb{Z}} f^n(K_0)$. Then, the action of $H = \{g \in [F, F] : g(K) = K\}$ on K has no fixed points.*

To show Lemma 11.43 we use its self-similarity to build elements in H moving points of the real line arbitrarily far away. But before giving the formal proof, let us see how this ends the proof of Theorem 11.35.

Proof of Theorem 11.35 given Lemma 11.43. Let $K \subsetneq \mathbb{R}$ be a subset satisfying property (O) and such that $K \cap (0, 1)$ admits a self similar decomposition. As concrete example we may take $K = \bigcup_{n \in \mathbb{Z}} f^n(K_0)$ with $K_0 = K_0(w_1, w_2)$ for $w_1 = 10001$ and $w_2 = 01110$. By Lemma 11.42 we may consider the \mathbb{R} -focal action φ_K , and by Lemma 11.43 and Proposition 11.41 we have that $\varphi_K([F, F])$ acts minimally. To finish the proof, we show that from the existence of one subset K with these properties, we can deduce the existence of uncountably many. Let K be one such subset. Clearly K is locally a Cantor set, so $\mathbb{R} \setminus K$ is a countable union of open intervals, that we call the *gaps* of K . Pick $\beta \in (0, 1)$. For each gap I of K , consider the point $p_I(\beta)$ where $p_I : (0, 1) \rightarrow I$ is the unique order-preserving affine map. We let K^β be the subset resulting from adding to K all the points of the form $p_I(\beta)$ where I runs over gaps of K . Clearly K^β is still closed and f -invariant. Moreover, K^β still admits a self similar decomposition since the maps involved in the definition of self similar decomposition are affine maps sending gaps of K to gaps of K , so in particular they preserve the proportion of the subdivision we have introduced in the gaps. We claim, that for uncountably many $\beta \in (0, 1)$, the subset K^β also satisfies (the last condition of) property (O), that is $g.K^\beta \cap K^\beta$ is open in K^β for every $g \in F$.

Fix $g \in F$. The only problem that may arise is that a point of the form $p_I(\beta)$ (which is an isolated point) might land inside K under the action of g . But if we fix a gap I of K , the set of parameters β such that $g(p_I(\beta))$ does not belong to K is open and dense in $(0, 1)$ since K has empty interior. In particular, since there are only countably many gaps and F is also

countable, with a Baire argument we obtain that K^β satisfies property (O) for a generic choice of $\beta \in (0, 1)$. \square

We conclude this subsection with the proof of Lemma 11.43. For this we need the following elementary interpolation lemma. Its proof follows from the transitivity of the action of $[F, F]$ on unordered n -tuples of dyadic numbers (see for instance [8]), and details are left to reader. To simplify the statement, given (possibly unbounded) intervals I, J , we write $I < J$ whenever $\sup I < \inf J$.

Lemma 11.44. *Consider intervals $I_1 < I_2 < \dots < I_k$ and $J_1 < J_2 < \dots < J_k$ with dyadic endpoints such that $I_1 = J_1 = (-\infty, p]$, $I_k = J_k = [q, +\infty)$, and such that $h_n : I_n \rightarrow J_n$ are dyadic affine maps for $n \in \{1, \dots, k\}$. Assume moreover that h_1 and h_k are restrictions of the identity. Then, there exists $g \in [F, F]$ such that $g \upharpoonright_{I_n} = h_n$ for $n \in \{1, \dots, k\}$.*

Proof of Lemma 11.43. Consider the dyadic affine maps $h_1, h_2 : I_0 \rightarrow (0, 1)$ given by the self similar decomposition of K_0 . Since K_0 is a closed subset of $(0, 1)$, we can assume that I_0 is a closed dyadic interval inside $(0, 1)$. For $i \in \{1, 2\}$, write $K_0^i = h_i(K_0)$ and $I_0^i = h_i(I_0)$. Note that $I_0^1 \cap I_0^2 = \emptyset$ and $K_0 = K_0^1 \sqcup K_0^2$.

Now, for $n \in \mathbb{Z}$ and $i \in \{1, 2\}$, write $I_n = f^n(I_0)$, $K_n^i = f^n(K_0^i)$, and $I_n^i = f^n(I_0^i)$. Then consider the following locally dyadic affine maps.

- $a : I_0^1 \sqcup I_0^2 \rightarrow I_0 \sqcup I_1$ defined by

$$a(x) = \begin{cases} h_1^{-1}(x) & \text{if } x \in I_0^1, \\ f \circ h_2^{-1}(x) & \text{if } x \in I_0^2, \end{cases}$$

- $b : I_3 \sqcup I_4 \rightarrow I_4^1 \sqcup I_4^2$ defined by

$$b(x) = \begin{cases} f^4 \circ h_1 \circ f^{-3}(x) & \text{if } x \in I_3, \\ f^4 \circ h_2 \circ f^{-4} & \text{if } x \in I_4, \end{cases}$$

- $c : [1, 2] \rightarrow [2, 3]$ defined by $c(x) = x + 1$.

Then, we can apply Lemma 11.44 to construct $h \in [F, F]$ which simultaneously extends a, b, c and $\text{id} \upharpoonright_{(-\infty, 0] \cup [4, +\infty)}$. By construction h preserves K and has no fixed points in $[1, 2]$. Thus the subgroup $H = \{h \in [F, F] : h(K) = K\}$ has no fixed points inside $[1, 2]$. Finally, note that f normalizes H , so that it preserves its set of fixed points. Since $\bigcup_{n \in \mathbb{Z}} f^n([1, 2]) = \mathbb{R}$, we deduce that H has no fixed points on \mathbb{R} , whence on K . \square

12. FINITELY GENERATED LOCALLY MOVING GROUPS WITH FEW EXOTIC ACTIONS

12.1. Bieri–Strebel groups over the real line. In this section we study actions of Bieri–Strebel groups $G(X; A, \Lambda)$ in the case $X = \mathbb{R}$ (see Definition 2.44). These are close relatives of Thompson’s group F , yet their minimal \mathbb{R} -exotic actions turn out to be much more rigid (and, in some cases, there are only finitely many such actions). Here we denote by $\langle S \rangle_*$ the multiplicative group generated by a subset $S \subset \mathbb{R}_+^*$.

The results of [8, §§A–B] allow to characterize under which conditions the group $G(\mathbb{R}; A, \Lambda)$ is in the class \mathcal{F} . (Note that since they all contain translation, this is the same as asking whether they belong to \mathcal{F}_0 .) As for many properties of the groups $G(X; A, \Lambda)$, this depends on the $\mathbb{Z}[\Lambda]$ -submodule

$$I\Lambda \cdot A := \langle (\lambda - 1)a : \lambda \in \Lambda, a \in A \rangle.$$

Indeed, we have the following.

Lemma 12.1. *The Bieri–Strebel group $G(\mathbb{R}; A, \Lambda)$ is in the class \mathcal{F} if and only if the following conditions are satisfied:*

- (BS1) Λ is finitely generated as a group,
- (BS2) A is finitely generated as $\mathbb{Z}[\Lambda]$ -module,
- (BS3) the quotient $A/I\Lambda \cdot A$ is finite.

Proof. Conditions (BS1)–(BS2) characterize when the group $G(\mathbb{R}; A, \Lambda)$ is finitely generated [8, Theorem B7.1]. On the other hand, Bieri–Strebel groups of the form $G((-\infty, x]; A, \Lambda)$ and $G([x, +\infty); A, \Lambda)$ are finitely generated if and only if $x \in A$ and all three conditions (BS1)–(BS3) are satisfied (this is the statement of [8, Theorem B8.2]); this immediately implies that $G(\mathbb{R}; A, \Lambda)$ is in the class \mathcal{F} . For the converse, recall that two points $p, q \in A \cap (-\infty, x)$ are in the same orbit under the group $G((-\infty, x]; A, \Lambda)$ if and only if $p - q \in I\Lambda \cdot A$ [8, Corollary A.5.1]. If G is in the class \mathcal{F} there exists a finitely generated subgroup $H \subseteq G((-\infty, x]; A, \Lambda)$ which contains a subgroup of the form $G((-\infty, y]; A, \Lambda)$ with $y < x$. Since all points of $A \cap (-\infty, y)$ occur as breakpoints for elements in $G((-\infty, y]; A, \Lambda)$, they must all belong to the H -orbit of one of the finitely many breakpoints of a finite generating set of H (see [8, §B6]). Thus condition (BS3) is also necessary. \square

As an important special case, the reader can have in mind the following example.

Example 12.2. For $\lambda > 1$, we denote by $G(\lambda)$ the Bieri–Strebel group $G(\lambda) := G(\mathbb{R}; A, \Lambda)$ corresponding to the cyclic group $\Lambda = \langle \lambda \rangle_*$ and to $A := \mathbb{Z}[\lambda, \lambda^{-1}]$. One has $I\Lambda \cdot A = (\lambda - 1)\mathbb{Z}[\lambda, \lambda^{-1}]$. It is not difficult to see that the quotient $A/I\Lambda \cdot A$ is finite if and only if λ is algebraic (see [8, Illustration A4.3]). For instance, for $\lambda = p/q$ rational (with p and q coprime), one has $|A/I\Lambda \cdot A| = p - q$. Therefore, by Lemma 12.1, the group $G(\lambda)$ is in the class \mathcal{F} exactly for algebraic λ .

Thus if $G(\mathbb{R}; A, \Lambda)$ satisfies conditions (BS1)–(BS3), its actions on the real line satisfy Theorem 10.3, so that all its exotic actions are semi-conjugate to a minimal \mathbb{R} -focal action horograded by the standard action on \mathbb{R} . Our goal is to give an explicit description of all such minimal \mathbb{R} -focal actions, thus yielding a complete classification of the faithful actions of such groups up to semi-conjugacy. For this, recall that in §9.3 we gave a construction of actions of Bieri–Strebel groups $G(X; A, \Lambda)$ parameterized by preorders on the group Λ . In what follows, we will denote by $\varphi_{+, \leq \Lambda}$ (respectively, $\varphi_{-, \leq \Lambda}$) the dynamical realization of the right (respectively, left) jump preorder associated with $\leq \Lambda \in \text{LPO}(\Lambda)$ (see Definition 9.13). The main result is that these are the only exotic actions of G .

Theorem 12.3. *Let $G = G(\mathbb{R}; A, \Lambda)$ be a Bieri–Strebel group satisfying conditions (BS1)–(BS3). Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points, is semi-conjugate to an action in one of the following families.*

- (1) (Non-faithful) An action induced from the quotient $G/[G_c, G_c]$.
- (2) (Standard) The standard piecewise linear action of G on \mathbb{R} .
- (3) (\mathbb{R} -focal) An action of the form $\varphi_{\pm, \leq \Lambda}$, obtained as the dynamical realization of a jump preorder (see Definition 9.13).

In particular, this provides a classification of all minimal faithful actions of G up to conjugacy.

Corollary 12.4. *Under the assumptions of Theorem 12.3, every minimal faithful action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is topologically conjugate either to the standard action on \mathbb{R} or to the dynamical realization of a jump preorder.*

The following special case gives Theorem 1.19 from the introduction.

Example 12.5. Let us consider the special case $G = G(\lambda)$ as in Example 12.2, with $\lambda > 1$ algebraic. Since in this case the group Λ is infinite cyclic, it admits only two non-trivial preorders, namely the usual order $<_\Lambda$ and its opposite. Thus, the jump preorder construction gives exactly two actions $\varphi_+ := \varphi_{+,<_\Lambda}$ and $\varphi_- := \varphi_{-,<_\Lambda}$, which are the dynamical realizations of the right and left jump preorders associated with $<_\Lambda \in \text{LPO}(\Lambda)$. Note indeed, that after Remark 9.21, the dynamical realization of the jump preorder corresponding to $<_\Lambda^{\text{op}}$ is conjugate to that for $<_\Lambda$. Thus in this case the group G admits finitely many (more precisely, two) \mathbb{R} -focal actions. Note that such actions φ_\pm are both locally rigid, which shows that groups in the class \mathcal{F}_0 may admit locally rigid actions other than the standard action.

Remark 12.6. In the setting of Theorem 12.3, let us comment on the actions coming from the largest quotient $G/[G_c, G_c]$. The structure of this quotient highly depends on A and Λ . Note that G/G_c embeds in the product $\text{Aff}(A, \Lambda) \times \text{Aff}(A, \Lambda)$ via the product $\mathcal{G}_{-\infty} \times \mathcal{G}_{+\infty}$ of the germ homomorphisms, and thus $G/[G_c, G_c]$ is solvable of derived length at most 3. More precisely (see [8, Corollary A5.5]), writing $\tau_\pm(f) \in A$ for the translation part of the germ of $f \in G$ at $\pm\infty$, respectively, the image $(\mathcal{G}_{-\infty} \times \mathcal{G}_{+\infty})(G)$ in $\text{Aff}(A, \Lambda) \times \text{Aff}(A, \Lambda)$ is the subgroup

$$\{(f, g) \in \text{Aff}(A, \Lambda) : \tau_+(g) - \tau_-(f) \in I\Lambda \cdot A\}.$$

The investigation of the abelianization $G_c/[G_c, G_c]$ is much less understood; partial results are discussed in [8, §C12]. If $G_c/[G_c, G_c]$ is trivial, then one must have $I\Lambda \cdot A = A$ [8, Theorem C12.14], but the converse is not true in general.

In the case $G = G(\lambda)$ for $\lambda > 1$ algebraic (Example 12.2) one can check that the image of G/G_c under the map induced by $\mathcal{G}_{-\infty} \times \mathcal{G}_{+\infty}$ has finite index in $\text{Aff}(A, \Lambda) \times \text{Aff}(A, \Lambda)$ (see Example 12.2). Moreover if $\lambda = p/q$ is rational (with p and q coprime), the abelianization $G_c/[G_c, G_c]$ is free of rank $p - q - 1$ (see [8, Corollary C12.12]). For other algebraic values of λ , up to our knowledge, only the case $\lambda = \frac{\sqrt{5}+1}{2}$ has been studied in detail: after work of Burillo, Nucinkis, and Reeves [22], we know that $G_c/[G_c, G_c] \cong \mathbb{Z}_2$ (note that in this case $I\Lambda \cdot A = A$). Note also that for any algebraic $\lambda > 1$, the abelianization $G/[G, G]$ is free of rank $|A/I\Lambda \cdot A| + 1$ (see [8, Corollary C12.2]).

The situation is particularly simple in the case $\lambda = 2$, where we have

$$G/[G_c, G_c] \cong \text{Aff}(\mathbb{Z}[1/2], \langle 2 \rangle_*) \times \text{Aff}(\mathbb{Z}[1/2], \langle 2 \rangle_*) \cong \text{BS}(1, 2) \times \text{BS}(1, 2),$$

where $\text{BS}(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ is the solvable Baumslag–Solitar group (see Proposition 12.17 below).

Before getting to the proof of Theorem 12.3, we need further preliminary discussions. Recall that the group G contains the affine subgroup $\text{Aff}(A, \Lambda) \cong A \rtimes \Lambda$ of transformations of the form $x \mapsto \lambda x + a$, with $\lambda \in \Lambda$ and $a \in A$. Given $a \in A$ and $\lambda \in \Lambda$, we will denote by $g(a, \lambda)$ the affine transformation $x \mapsto \lambda x + (1 - \lambda)a$, which is the unique element of $\text{Aff}(A, \Lambda)$ which fixes a and has slope λ . We will also consider the elements

$$(12.1) \quad g_+(a, \lambda) : x \mapsto \begin{cases} x & x \in (-\infty, a], \\ g(a, \lambda)(x) & x \in [a, +\infty), \end{cases}$$

and $g_-(a, \lambda) := g(a, \lambda) g_+(a, \lambda)^{-1}$. Note that $g_+(a, \lambda) \in G_{(a, +\infty)}$ and $g_-(a, \lambda) \in G_{(-\infty, a)}$. For $a \in A$ we also denote by t_a the translation $x \mapsto x + a$.

For every $a \in A$ and $\lambda \in \Lambda$, if $h \in G(\mathbb{R}; A, \Lambda)$ is an element with no breakpoint on $(a, +\infty)$ (respectively on $(-\infty, a)$), we have

$$hg_+(a, \lambda)h^{-1} = g_+(h(a), \lambda) \quad (\text{respectively, } hg_-(a, \lambda)h^{-1} = g_-(h(a), \lambda)).$$

In particular we have the following relations for such elements (see [8, §B7]):

$$(12.2) \quad hg_{\pm}(a, \lambda)h^{-1} = g_{\pm}(h(a), \lambda) \quad \text{for every } h \in \text{Aff}(A, \Lambda),$$

as well as

$$(12.3) \quad \begin{aligned} g_+(a, \lambda)g_+(b, \mu)g_+(a, \lambda)^{-1} &= g_+(g(a, \lambda)(b), \mu) \quad \text{for every } a > b, \\ g_-(a, \lambda)g_-(b, \mu)g_-(a, \lambda)^{-1} &= g_-(g(a, \lambda)(b), \mu) \quad \text{for every } a < b. \end{aligned}$$

We also remark that the subset

$$\{t_a\}_{a \in A} \cup \{g(0, \lambda), g_+(0, \lambda)\}_{\lambda \in \Lambda}$$

is generating for $G(\mathbb{R}; A, \Lambda)$ (see [8, Theorem B7.1]).

For what follows, the reader can keep in mind the following example.

Example 12.7. For $\lambda > 1$, the Bieri–Strebel group $G(\lambda)$ is generated by the finite subset $\{g(0, \lambda), g_+(0, \lambda), t_1\}$.

Remark 12.8. The group $\langle g(0, \lambda), g_+(0, \lambda), t_1 \rangle$ appears in [15] (denoted G_λ), where it was shown that, for certain algebraic numbers $\lambda > 1$ (called Galois hyperbolic *ibid.*) it admits no faithful C^1 action on the closed interval. The fact that this group coincides with the Bieri–Strebel group $G(\lambda)$ was unnoticed in [15].

We also need some preliminary results, stated in the following more flexible setting.

Assumption 12.9. Fix a non-trivial multiplicative subgroup $\Delta \subseteq \mathbb{R}_*$, and a Δ -submodule $A \subseteq \mathbb{R}$, and let $H = G(\mathbb{R}; A, \Delta)$ be the corresponding Bieri–Strebel group. Moreover we let $G \subseteq \text{Homeo}_0(\mathbb{R})$ be a subgroup in the class \mathcal{F} which contains H as a subgroup. Finally we assume that $\mathcal{L} = \{L_x\}_{x \in \mathbb{R}}$ is a family of subgroups of G with the following properties.

- (C1) For each $x \in \mathbb{R}$ we have $\bigcup_{y < x} G_{(-\infty, y)} \subseteq L_x \subseteq G_{(-\infty, x)}$.
- (C2) For every $x \in \mathbb{R}$ and every $g \in G$ we have $gL_xg^{-1} = L_{g(x)}$.
- (C3) For every $x \in A$ and every $\delta \in \Delta$ we have $g_-(x, \delta) \in L_x$.

We write $L_+ = \bigcup_{x \in \mathbb{R}} L_x$ which, by (C2), defines a normal subgroup of G . Finally we assume that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is a faithful minimal action of G which is \mathbb{R} -focal and increasingly horograded by its standard action on \mathbb{R} .

Remark 12.10. For the proof of Theorem 12.3, the reader can have in mind the case where $G = G(\mathbb{R}; A, \Lambda)$ is itself a Bieri–Strebel group, H is a subgroup corresponding to some $\Delta \subseteq \Lambda$, and the L_x are subgroups of $G_{(-\infty, x)}$ consisting of elements whose left-derivative at x belongs to some intermediate group $\Delta \subseteq \Lambda_1 \subseteq \Lambda$. However a different choice of G will be used later in §12.2.

We will write $X = \mathbb{R}$ for the real line on which the standard action of G is defined. We resume notation of §10.2.3. As introduced in Definition 10.7, for every $x \in X$ and $\xi \in \mathbb{R}$ with $\xi \notin \text{Fix}^\varphi(G_{(-\infty, x)})$, we write $I^\varphi(x, \xi)$ for the connected component of $\text{Supp}^\varphi(G_{(-\infty, x)})$ containing ξ . Recall that intervals of this form define a CF-cover. Moreover we will consider the planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ constructed in §10.2.3, whose dynamical realization is φ . Recall that vertices of \mathbb{T} are intervals of the form $I_{\text{out}}^\varphi(x, \xi) = \text{Int}\left(\bigcap_{y > x} I^\varphi(y, \xi)\right)$ and that the

map $\pi: \mathbb{T} \rightarrow \mathbb{R}$ given by $\pi(I_{\text{out}}^\varphi(x, \xi)) = x$ is an equivariant horograding. Recall also that we write $I_{\text{inn}}^\varphi(y, \xi) = \bigcup_{x < y} I^\varphi(x, \xi)$.

Lemma 12.11. *Under Assumption 12.9, the group $\varphi(\text{Aff}(A, \Delta))$ has a unique fixed point $\eta \in \mathbb{R}$. This point satisfies $\eta \in \text{Supp}^\varphi(G_{(-\infty, x)})$ for every $x \in X$.*

Moreover, the map

$$\begin{aligned} q_+ : \mathbb{R} &\rightarrow (\eta, +\infty) \\ x &\mapsto \sup I^\varphi(x, \eta) \end{aligned}$$

is monotone increasing and $\text{Aff}(A, \Delta)$ -equivariant. In particular the standard affine action of $\text{Aff}(A, \Delta)$ on $X = \mathbb{R}$ is positively semi-conjugate to its action φ on $(\eta, +\infty)$. Similarly the map

$$\begin{aligned} q_- : \mathbb{R} &\rightarrow (-\infty, \eta) \\ x &\mapsto \inf I^\varphi(x, \eta) \end{aligned}$$

defines a negative semi-conjugacy.

Proof. Every translation t_a , with $a \in A$, acts without fixed points, therefore by Proposition 8.32, its φ -image is a homothety. As the subgroup of translations is abelian, its φ -image has a unique fixed point $\eta \in \mathbb{R}$. Moreover, as such subgroup is normal in $\text{Aff}(A, \Delta)$, this is a fixed point for $\varphi(\text{Aff}(A, \Delta))$.

Assume first that there exists $x \in X$ such that $\eta \in \text{Fix}^\varphi(G_{(-\infty, x)})$. Then we would have $\eta = g.\eta \in \text{Fix}^\varphi(G_{(-\infty, x)})$ for every $g \in \text{Aff}(A, \Delta)$, so that by minimality of the standard action of $\text{Aff}(A, \Delta)$ on X we get $\eta \in \text{Fix}^\varphi(G_+)$. Since G_+ is a normal subgroup of G , and φ is minimal, this implies that $\varphi(G_+)$ acts trivially, contradicting faithfulness of φ in Assumption 12.9. Thus $\eta \in \text{Supp}^\varphi(G_{(-\infty, x)})$, and the interval $I^\varphi(x, \eta)$ is well-defined for every $x \in X$.

The second statement is now a direct consequence of the properties of these two families of intervals: the family of $\{I^\varphi(x, \eta)\}_{x \in X}$ is increasing with respect to $x \in X$ and moreover one has the equivariance relation $g.I^\varphi(x, \eta) = I^\varphi(g(x), g.\eta)$ for every $x \in X$ and $g \in G$ (see §10.2.2). \square

In what follows we will always denote by η the unique fixed point of $\varphi(\text{Aff}(A, \Delta))$ provided by Lemma 12.11. For $x \in X$ and $\xi \in \text{Supp}^\varphi(L_x)$ we will denote by $I^\varphi(\mathcal{L}, x, \xi)$ the connected component of $\text{Supp}^\varphi(L_x)$ containing ξ . Note that condition (C1) implies that $I^\varphi(\mathcal{L}, x, \xi)$ is increasing with respect to $x \in \mathbb{R}$, and moreover

$$I_{\text{inn}}^\varphi(x, \xi) \subset I^\varphi(\mathcal{L}, x, \xi) \subset I^\varphi(x, \xi).$$

The key point is to establish the following strict inclusion when $x \in A$.

Lemma 12.12. *Under Assumption 12.9, assume that there exists $g \in H$ such that $g.\eta \neq \eta$. Then for every $x \in A$ we have a strict inclusion $I_{\text{inn}}^\varphi(x, \eta) \subsetneq I^\varphi(\mathcal{L}, x, \eta)$.*

Proof. Assume by contradiction that $I_{\text{inn}}^\varphi(x, \eta) = I^\varphi(\mathcal{L}, x, \eta)$ for some $x \in A$. Note that then this is automatically true for every $x \in A$, since the group $\text{Aff}(A, \Delta)$ acts transitively on A (it contains all translations by elements in A) and fixes η , so that for $h \in \text{Aff}(A, \Delta)$ we have $h.I_{\text{inn}}^\varphi(x, \eta) = I_{\text{inn}}^\varphi(h(x), \eta)$ and

$$(12.4) \quad h.I^\varphi(\mathcal{L}, x, \eta) = I^\varphi(\mathcal{L}, h(x), \eta)$$

(after condition (C2)).

Fix $a \in A$ and choose $\delta \in \Delta$ with $\delta > 1$ such that the element $g_-(a, \delta).\eta \neq \eta$. Such a δ exists because the elements $g_-(a, \delta)$ together with $\text{Aff}(A, \Delta)$ generate H , and we assume that

$\varphi(H)$ does not fix η . Note also that once such an element $g_-(a, \delta)$ is found, it follows that $g_-(x, \delta) \cdot \eta \neq \eta$ for every $x \in A$, since these elements are all conjugate to each other by elements of $\text{Aff}(A, \Delta)$.

Note that by condition (C3) the image $\varphi(g_-(x, \delta))$ must preserve $I^\varphi(\mathcal{L}, x, \eta)$ for every $x \in A$. We claim that it acts on it as an expanding homothety. To this end, we look at the action of H on the planar directed tree $(\mathbb{T}, \triangleleft, \prec)$ constructed in §10.2.3. Consider the subset \mathbb{T}_0 of \mathbb{T} given by

$$\mathbb{T}_0 = \{I_{\text{out}}^\varphi(y, \xi) : y < x, \xi \in I^\varphi(\mathcal{L}, x, \eta)\}.$$

Then the equality $I_{\text{inn}}^\varphi(x, \eta) = I^\varphi(\mathcal{L}, x, \eta)$ implies that \mathbb{T}_0 is a directed planar subtree of \mathbb{T} invariant under the subgroup L_x (note that \mathbb{T}_0 is equal to the direction below the vertex $v = I_{\text{out}}^\varphi(x, \eta)$ corresponding to $I_{\text{inn}}^\varphi(x, \eta)$). Moreover the restriction of the horograding $\pi : \mathbb{T} \rightarrow \mathbb{R}$ to \mathbb{T}_0 takes values in $(-\infty, x)$ and is L_x -equivariant. Finally since the intervals $I_{\text{out}}^\varphi(y, \eta)$ for $y < x$ are relatively compact inside $I_{\text{inn}}^\varphi(x, \eta)$ and every $I \in \mathbb{T}_0$ is contained in one such interval, the restriction of the planar order on \mathbb{T}_0 is proper. Thus we can apply Proposition 8.26 to the action of L_x on \mathbb{T}_0 , and since the element $g_-(x, \delta)$ satisfies $g_-(x, \delta)(y) > y$ for every $y \in (-\infty, x)$, it follows that it has no fixed points in \mathbb{T}_0 and thus acts on $(\partial\mathbb{T}_0, \prec)$ as an expanding homothety, and we conclude as in the proof of Proposition 8.32 that its φ -image is an expanding homothety on $I^\varphi(\mathcal{L}, x, \eta)$, as desired.

Now for $x \in A$ let us denote by $\xi_x \in I^\varphi(\mathcal{L}, x, \eta)$ the unique fixed point of $\varphi(g_-(x, \delta)) \upharpoonright_{I^\varphi(\mathcal{L}, x, \eta)}$. Note that $\xi_x \neq \eta$, by the choice of δ . Without loss of generality, we assume that $\xi_b > \eta$ for some $b \in A$. Then we have the following.

Claim. *For every $x \in A$ we have $\xi_x > \eta$ and the map $x \mapsto \xi_x$ is monotone increasing.*

Proof of claim. The relations (12.2) and (12.4) give that the map $x \mapsto \xi_x$ is $\text{Aff}(A, \Delta)$ -equivariant. The conclusion follows using that $\text{Aff}(A, \Delta)$ acts transitively on A and that, after Lemma 12.11, we know that the action of $\text{Aff}(A, \Delta)$ on $(\eta, +\infty)$ is positively semi-conjugate to the standard affine action. \square

Now fix $x \in X$. By the assumption that $I_{\text{inn}}^\varphi(x, \eta) = I^\varphi(\mathcal{L}, x, \eta)$ and by (C1) we can find $y \in A$ with $y < x$ such that $I^\varphi(\mathcal{L}, y, \eta)$ contains ξ_x . After the claim, we have $\eta < \xi_y < \xi_x$. Since ξ_x is a repelling fixed point for $\varphi(g_-(x, \delta))$, we have the inclusion $g_-(x, \delta) \cdot I^\varphi(\mathcal{L}, y, \eta) \supset I^\varphi(\mathcal{L}, y, \eta)$, and since the latter contains η we have $g_-(x, \delta) \cdot I^\varphi(\mathcal{L}, y, \eta) = I^\varphi(\mathcal{L}, g_-(x, \delta)(y), \eta)$. Then (12.3) implies that $g_-(x, \delta) \cdot \xi_y = \xi_{g_-(x, \delta)(y)}$. However, $g_-(x, \delta) \cdot \xi_y < \xi_y$ since ξ_y lies to the left of ξ_x . On the other hand $\xi_{g_-(x, \delta)(y)} > \xi_y$ since $g_-(x, \delta)(y) > y$, and we know that the map $y \mapsto \xi_y$ is increasing after the claim. This is a contradiction, giving end to the proof of the lemma. \square

Proposition 12.13. *Under the same assumptions as in Lemma 12.12, for every $x \in A$ the action of $\varphi(L_x)$ on $I^\varphi(\mathcal{L}, x, \eta)$ is semi-conjugate to a non-faithful action induced from an action of the group of left germs $\text{Germ}(L_x, x)$.*

Proof. Lemma 12.12 implies that the normal subgroup $(G_{(-\infty, x)})_+ = \bigcup_{y < x} G_{(-\infty, y)}$ of L_x has fixed points in $I^\varphi(\mathcal{L}, x, \eta)$ (namely the endpoints of $I_{\text{inn}}^\varphi(x, \eta)$). Thus the action of $\varphi(L_x)$ on $I^\varphi(\mathcal{L}, x, \eta)$ is semi-conjugate to an action induced from the quotient $L_x / (G_{(-\infty, x)})_+ \cong \text{Germ}(L_x, x)$. \square

The next result is the only place where a particular choice of the family $\mathcal{L} = \{L_x\}$ is needed.

Corollary 12.14. *Let $G = G(\mathbb{R}; A, \Lambda)$ be a Bieri–Strebel group satisfying conditions (BS1)–(BS3). Let $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ be a minimal faithful \mathbb{R} -focal action, increasingly horograded by the standard action on \mathbb{R} . Assume there exists an element $g = g_-(x, \lambda) \in G$ such that $g.\eta \neq \eta$. Then $g.\mathbb{I}_{\text{inn}}^\varphi(x, \eta) \cap \mathbb{I}_{\text{inn}}^\varphi(x, \eta) = \emptyset$.*

Proof. Take $g = g_-(x, \lambda)$ not fixing η and consider the Bieri–Strebel group $H = G(\mathbb{R}; A, \langle \lambda \rangle_*)$, which is a subgroup of G . We consider the family $\mathcal{L} = \{L_x : x \in X\}$ defined by

$$L_x = \left\{ h \in G_{(-\infty, x)} : D^-h(x) \in \langle \lambda \rangle_* \right\}.$$

It is straightforward to verify that Assumption 12.9 is fulfilled by such choices. Therefore, by Proposition 12.13, the action of L_x on $\mathbb{I}^\varphi(\mathcal{L}, x, \eta)$ is semi-conjugate to an action that factors through the germ homomorphism $\mathcal{G}_x : L_x \rightarrow \text{Germ}(L_x, x)$. Since in this case $\text{Germ}(L_x, x)$ is generated by $\mathcal{G}_x(g)$, we conclude that $\text{Fix}^\varphi(g) \cap \mathbb{I}^\varphi(\mathcal{L}, x, \eta) = \emptyset$.

On the other hand, by Lemma 12.12, we get that $\mathbb{I}^\varphi(\mathcal{L}, x, \eta)$ strictly contains $\mathbb{I}_{\text{inn}}^\varphi(x, \eta)$. Then, since $\{\mathbb{I}_{\text{inn}}^\varphi(x, \xi) : x, \xi \in \mathbb{R}\}$ is a CF-cover preserved by the action, we must have $g.\mathbb{I}_{\text{inn}}^\varphi(x, \eta) \cap \mathbb{I}_{\text{inn}}^\varphi(x, \eta) = \emptyset$ as desired. \square

The next two lemmas analyse properties of the jump preorders. The first one gives decompositions for elements in $G_+ = \bigcup_{x < \infty} G_{(-\infty, x)}$ which are well suited for our purposes. The second one allows to identify dynamical realizations of jump preorders.

Lemma 12.15. *Let $G = G(\mathbb{R}; A, \Lambda)$ be a Bieri–Strebel group and let \leq_Λ be a preorder on Λ . Let \preceq be the corresponding right jump preorder and take $g \in G_+$ with $\text{id} \preceq g$. Then, there exist elements $h, k, g_-(y, \lambda) \in G_+$ such that*

- (1) $g = kg_-(y, \lambda)h$,
- (2) $h \in [1]_{\preceq}$,
- (3) $1 \preceq_\Lambda \lambda$, and
- (4) $k \in G_{(-\infty, z)}$ for some $z < y$.

The analogous result holds for the left jump preorder.

Proof. The condition $g \in G_+$ gives the equality $j^+(g, y) = D^-g(y)$ for every $y \in X$. Write $\Lambda_0 = [1]_{\preceq_\Lambda}$. Since we are assuming $\text{id} \preceq g$, we can consider $x = x_{g, \Lambda_0}$ as in (9.9). Thus the restriction of g to $(x, +\infty)$ coincides with some element $h \in G(\mathbb{R}; A, \Lambda_0)_+$. Consider the product $f = gh^{-1}$. The rightmost point of $\text{Supp}(f)$ is $y = g(x) = h(x)$, and by the chain rule we have

$$D^-f(y) = D^-g(x) D^-h^{-1}(y) = D^-g(x) D^-h(x)^{-1}.$$

Since $D^-h(x) \in \Lambda_0$ and $D^-g(x) \in \Lambda \setminus \Lambda_0$, we get $1 \preceq_\Lambda D^-f(y)$. Write $\lambda = D^-f(y)$. As before, we have that the rightmost point of the support of $fg_-(y, \lambda)^{-1}$ coincides with $g_-(y, \lambda)(z)$ where z is the second largest breakpoint of f (the one before y). Then write $k = fg_-(y, \lambda)^{-1}$. It is direct to check that the decomposition $g = kg_-(y, \lambda)h$ satisfies conditions (1–4) in the statement. \square

Lemma 12.16. *Consider a Bieri–Strebel group $G = G(\mathbb{R}; A, \Lambda)$ and a preorder $\preceq \in \text{LPO}(G)$ containing $\text{Aff}(A, \Lambda)$ in its residue subgroup. Assume further that $\preceq' \in \text{LPO}(G)$ is a right (respectively, left) jump preorder coinciding with \preceq over G_+ (respectively, G_-). Then \preceq and \preceq' are the same preorder.*

Proof. Assume that \preceq' is the right jump preorder associated with the preorder $\leq_\Lambda \in \text{LPO}(\Lambda)$, the case where \preceq' is a left jump preorder is analogous. Denote by Λ_0 the residue of \leq_Λ and recall from §9.3, that in this case the residue of \preceq' is the subgroup

$$H := \{g \in G : j^+(g, x) \in \Lambda_0 \ \forall x \in \mathbb{R}\}$$

(see (9.8)). Since elements of $\text{Aff}(A, \Lambda)$ have constant derivative it holds that $j^+(g, x) = 1$ for every $g \in \text{Aff}(A, \Lambda)$ and $x \in \mathbb{R}$. In particular we have $\text{Aff}(A, \Lambda) \subseteq H$.

Note that G decomposes as $G = G_+ \rtimes \text{Aff}(A, \Lambda)$. Then, for every $g \in G$ we can write $g = g_+ a_g$ with $g_+ \in G_+$ and $a_g \in \text{Aff}(A, \Lambda)$. Denote by P and P' the positive cones of \preceq and \preceq' respectively. Since $\text{Aff}(A, \Lambda)$ is contained in the residue of both \preceq and \preceq' , it holds that $g \in P$ if and only if $g_+ \in G_+ \cap P$ and also that $g \in P'$ if and only if $g_+ \in G_+ \cap P'$. Finally, since by hypothesis it holds $G_+ \cap P = G_+ \cap P'$, the lemma follows. \square

Proof of Theorem 12.3. The assumptions on $G = G(\mathbb{R}; A, \Lambda)$ ensure that G is in the class \mathcal{F} (see Lemma 12.1). After Theorem 10.3, we only need to show that a minimal faithful \mathbb{R} -focal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$, increasingly (respectively, decreasingly) horograded by its standard action on \mathbb{R} , is conjugate to an action of the form $\varphi_{+, \leq_\Lambda}$ (respectively, $\varphi_{-, \leq_\Lambda}$) for some preorder \leq_Λ . We will only discuss the case of increasing horograded by the standard action, the decreasing case being totally analogous.

We write $\eta \in \mathbb{R}$ for the unique fixed point of $\varphi(\text{Aff}(A, \Lambda))$ given by Lemma 12.11 (applied to the case $\Delta = \Lambda$). In what follows, let \preceq be the preorder on G induced by η , namely by declaring $g \preceq h$ if and only if $g.\eta < h.\eta$. We will show that this preorder coincides with a right jump preorder associated with some $\leq_\Lambda \in \text{LPO}(\Lambda)$. Let us first identify such preorder. For $x \in A$, the set of elements $T_x := \{g_-(x, \lambda) : \lambda \in \Lambda\}$ is a subgroup of G isomorphic to Λ , which is a section inside $G_{(-\infty, x)}$ of the group of germs $\text{Germ}(G_{(-\infty, x)}, x)$. We put on Λ the preorder \leq_Λ given by restricting \preceq to this subgroup, namely by setting $\lambda \leq_\Lambda \mu$ if $g_-(x, \lambda).\eta < g_-(x, \mu).\eta$. Note that this preorder does not depend on the choice of $x \in A$, as for $x, y \in A$ the groups T_x and T_y are conjugate by an element of $\text{Aff}(A, \Lambda)$ (see (12.2)), which fixes η . Denote \preceq' the right jump preorder in G associated with \leq_Λ . We proceed to show that \preceq' and \preceq coincide on G_+ which, by Lemma 12.16 will conclude the proof.

Denote by Λ_0 the residue of the preorder \leq_Λ and recall that in this case the residue of \preceq' equals $H = \{g \in G : j^+(g, x) \in \Lambda_0 \ \forall x \in \mathbb{R}\}$ (see (9.8)). As observed in the proof of Lemma 12.15, for $g \in G_+$ we have $j^+(g, x) = D^-g(x)$ for every $x \in X$, so that we have the equality $H \cap G_+ = G(\mathbb{R}; A, \Lambda_0)_+$. Note that $H \cap G_+$ fixes η , since it is generated by $\{g_-(x, \lambda) : x \in A, \lambda \in \Lambda_0\}$ (this can be easily checked from [8, §8.1]). Thus, we have

$$(12.5) \quad G_+ \cap [1]_{\preceq'} \subseteq G_+ \cap [1]_{\preceq}.$$

Assume now that $g \in G_+$ satisfies $\text{id} \preceq' g$. We proceed to show that in this case $\eta < g.\eta$ which implies that $\text{id} \preceq g$. For this, consider the decomposition $g = kg_-(y, \lambda)h$ given by Lemma 12.15. Then, by Lemma 12.15.(2) and (12.5), we get that $h.\eta = \eta$ and therefore $g.\eta = kg_-(y, \lambda).\eta$. On the other hand, Lemma 12.15.(3) together with the definition of \leq_Λ imply that $g_-(y, \lambda).\eta > \eta$. Then, by Corollary 12.14, we get that $g_-(y, \lambda) \cdot I_{\text{inn}}^\varphi(y, \eta) \cap I_{\text{inn}}^\varphi(y, \eta) = \emptyset$. Finally, by Lemma 12.15.(4) we have that $k \cdot I_{\text{inn}}^\varphi(y, \eta) = I_{\text{inn}}^\varphi(y, \eta)$ which, in light of what we have already done, shows that $g.\eta = kg_-(y, \lambda).\eta > \eta$ as desired. Analogously one shows that if $g \preceq' \text{id}$ then $g \preceq \text{id}$. This shows that the preorders \preceq' and \preceq coincide over G_+ and concludes the proof. \square

12.2. A finitely generated group in the class \mathcal{F}_0 with no \mathbb{R} -focal actions. Here we construct an example of a finitely generated locally moving group in the class \mathcal{F}_0 , which admits

no faithful \mathbb{R} -focal actions. In particular every minimal faithful action on \mathbb{R} is conjugate to the standard action.

The starting point of the construction is the Bieri–Strebel group $G(2) = G(\mathbb{R}; \mathbb{Z}[1/2], \langle 2 \rangle_*)$ of all finitary dyadic PL homeomorphisms of \mathbb{R} , which we already proved to admit only two \mathbb{R} -focal actions (Theorem 12.3 and Example 12.5). Moreover, as discussed in Remark 12.6, we have that its largest proper quotient is isomorphic to the direct product of two solvable Baumslag–Solitar groups $\text{BS}(1, 2)$. Therefore also the non-faithful case in Theorem 12.3 is very restrictive. Indeed, we have the following result (compare with [15, Theorem 6.12]).

Proposition 12.17. *Let Γ_1 and Γ_2 be two groups isomorphic to the Baumslag–Solitar group $\text{BS}(1, 2)$, and consider their direct product $\Gamma := \Gamma_1 \times \Gamma_2$. Then every action $\varphi: \Gamma \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is semi-conjugate either to an action by translation of the abelianization $\Gamma^{ab} = \mathbb{Z}^2$, or to an action obtained by composing the projection to one of the factors Γ_i , $i \in \{1, 2\}$, with the action of Γ_i on \mathbb{R} by affine dyadic maps.*

Proof. The following proof is based on the classification of actions of the Baumslag–Solitar group $\text{BS}(1, 2)$ on the real line, without fixed points: up to semi-conjugacy, it is either an action of the abelianization \mathbb{Z} , or the standard affine action (see e.g. [16, 98]). Consider an action $\varphi: \Gamma \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points. After Proposition 5.4, without loss of generality, we can assume that the φ -image $[\Gamma_1, \Gamma_1]$ has fixed points. Thus the action φ is semi-conjugate to an action of the quotient $\Gamma_1^{ab} \times \Gamma_2$. Assume that the factor $\Gamma_1^{ab} \cong \mathbb{Z}$ has no fixed point and that the action φ is not semi-conjugate to any action of the abelianization $\Gamma^{ab} = \Gamma_1^{ab} \times \Gamma_2^{ab}$. This means that the action of Γ_2 is semi-conjugate to the standard affine action. This is however not possible, because elements in Γ_2 corresponding to homotheties under the semi-conjugacy have a compact set of fixed points, which must be fixed by Γ_1 , an absurd. \square

As a consequence of Corollary 9.10, if we want to avoid \mathbb{R} -focal actions, we must leave the setting of *finitary* PL transformations. We will consider groups whose elements are PL with a countably many breakpoints that accumulate at some finite subset of “higher order” singularities (with some control on these).

Given an open interval $X \subset \mathbb{R}$ we say that a homeomorphism $f \in \text{Homeo}_0(X)$ is *locally* PL if there is a finite subset $\Sigma \subset X$ such that f is (finitary) PL in $X \setminus \Sigma$. For such an f , we denote by $\text{BP}^2(f) \subset X$ the minimal subset such that f is PL in $X \setminus \text{BP}^2(f)$. The set $\text{BP}^2(f)$ is the set of breakpoints of *second order* of f . Points $x \in X \setminus \text{BP}^2(f)$ where f has discontinuous derivative are called breakpoints of *first order*, and we denote them by $\text{BP}^1(f)$. Also, we write $\text{BP}(f) = \text{BP}^1(f) \cup \text{BP}^2(f)$ for the set of breakpoints of f . Clearly, when $\text{BP}^2(f) = \emptyset$ we have that f is PL. We will silently use a couple of times the observation that for f and g locally PL, we have that $\text{BP}^2(fg) \subset \text{BP}^2(g) \cup g^{-1} \text{BP}^2(f)$.

Definition 12.18. Let $X \subset \mathbb{R}$ be an open interval. We write $G(X) = G(X; \mathbb{Z}[1/2], \langle 2 \rangle_*)$ for the Bieri–Strebel group (see Definition 2.44). We also denote by $G_\omega(X)$ the group of all locally PL homeomorphisms of X with the following properties:

- f is locally *dyadic* PL, that is at each $x \in X \setminus \text{BP}(f)$ the map f is locally an affine map of the form $x \mapsto 2^n x + b$ for $n \in \mathbb{Z}$ and $b \in \mathbb{Z}[1/2]$;
- breakpoints of f are contained in a compact subset of X : $\text{BP}(f) \Subset X$;
- breakpoints of f and their images are dyadic rationals: $\text{BP}(f) \cup f(\text{BP}(f)) \subset \mathbb{Z}[1/2]$.

The group $G_\omega(X)$ is uncountable, so too big for our purposes. We will instead consider some subgroups defined in terms of the local behaviour at the breakpoints of second order. Here we keep the notation from the previous subsection, such as $g(a, \lambda)$, $g_\pm(a, \lambda)$ (see 12.1), and

$t_a : x \mapsto x + a$ which denote elements in $\text{PL}(\mathbb{R})$. For $r \in \mathbb{R}$, we also write $h_r = g(r, \frac{1}{2})$, which corresponds to the homothety of ratio $1/2$ centered at r , and similarly we write $h_{r\pm} = g_{\pm}(r, \frac{1}{2})$ for shorthand notation.

Definition 12.19. Let $g: I \rightarrow J$ be a homeomorphism between two open intervals. We say that g has a 2^n -scaling germ at $r \in I$, if there exists a neighborhood U of r such that $gh_r^n \upharpoonright_U = h_{g(r)}^n g \upharpoonright_U$.

Remark 12.20. Note that when $g(r) = r$ this simply means that the germ of g at r commutes with the germ of h_r^n . More generally, if $g(r) \neq r$, and if h is any PL map such that $hg(r) = r$, then g has a 2^n -scaling germ at r if and only if the germs of hg and h_r at r commute: this does not depend on the choice of h , since every PL map has 2^n -scaling germ (and more generally k -scaling germ for any $k > 0$, with the obvious extension of the definition) at every point, including breakpoints.

Definition 12.21. Given an open interval $X \subset \mathbb{R}$ and $n \geq 1$, we let $G_{\omega}^{(n)}(X)$ be the subgroup of $G_{\omega}(X)$ consisting of elements that have 2^n -scaling germs at every breakpoint of second order (and thus at all points $x \in X$).

For every dyadic point $x \in X$, let $\mathcal{D}_x^{(n)}$ be the group of germs at x of elements in $G_{\omega}^{(n)}(X)$ which fix x , that is, the group of germs of homeomorphisms that are locally dyadic PL away from $\{x\}$ and that commute with h_x^n . We denote by $\mathcal{D}_{x-}^{(n)}$ and $\mathcal{D}_{x+}^{(n)}$ the corresponding groups of left and right germs, respectively, so that $\mathcal{D}_x^{(n)} \cong \mathcal{D}_{x-}^{(n)} \times \mathcal{D}_{x+}^{(n)}$. The groups $\mathcal{D}_{x-}^{(n)}$ and $\mathcal{D}_{x+}^{(n)}$ are isomorphic to a well-known group, namely the lift $\tilde{T} \subseteq \text{Homeo}_0(\mathbb{R})$ of Thompson's group T acting on the circle. Explicitly, \tilde{T} is the group of all dyadic PL homeomorphisms of \mathbb{R} which commute with the unit translation $t_1: x \mapsto x + 1$. The point is that for every $n \geq 1$ and $x \geq 1$ dyadic, the map $h_x^n \upharpoonright_{(-\infty, x)}$ can be conjugated to the translation t_1 by a dyadic PL homeomorphism $f: (-\infty, x) \rightarrow \mathbb{R}$. This establishes an isomorphism of $\mathcal{D}_{x-}^{(n)}$ with the group of germs of \tilde{T} at $+\infty$, which is isomorphic to \tilde{T} itself. Similarly one argues for $\mathcal{D}_{x+}^{(n)}$. This fact will be constantly used in what follows. A first consequence is that the groups $\mathcal{D}_{x-}^{(n)}$ and $\mathcal{D}_{x+}^{(n)}$ are finitely generated, since \tilde{T} is so. This leads to the following:

Proposition 12.22. *For every dyadic open interval $X = (a, b) \subset \mathbb{R}$ and every $n \geq 1$, the group $G := G_{\omega}^{(n)}(X)$ satisfies the following.*

- (1) G is finitely generated, and belongs to the class \mathcal{F} .
- (2) the subgroup G_c of compactly supported elements is perfect (and thus simple). In particular the largest proper quotient of G is $G/G_c = \text{Germ}(G, a) \times \text{Germ}(G, b)$.

Note that, as breakpoints of every element in $G_{\omega}^{(n)}(X)$ are contained in a compact subset of X , the group $\text{Germ}(G, a)$ is infinite cyclic if $a > -\infty$, and isomorphic to $\text{BS}(1, 2)$ if $a = -\infty$, and similarly for $\text{Germ}(G, b)$.

Proof of Proposition 12.22. Fix a dyadic point $x \in (a, b)$. Since the group of germs $\mathcal{D}_x^{(n)} = \mathcal{D}_{x-}^{(n)} \times \mathcal{D}_{x+}^{(n)}$ is finitely generated, we can find a finite subset $S \subset G_{\omega}^{(n)}(X)$ which fix x and whose germs generate $\mathcal{D}_x^{(n)}$, and that have no breakpoints of second order apart from x .

Claim. *We have $G_{\omega}^{(n)}(X) = \langle G(X), S \rangle$.*

Proof of claim. Let $g \in G_\omega^{(n)}$, and let us show that $g \in \langle G(X), S \rangle$ by induction on the number $k = |\text{BP}^2(g)|$ of breakpoints of g of second order. If $k = 0$, then $g \in G(X)$. Assume that $k \geq 1$, and let $y \in \text{BP}^2(g)$ be a breakpoint of second order of g . Since $G(X)$ acts transitively on dyadic rationals, we can choose $h_1 \in G(X)$ such that $h_1(g(x)) = x$. As $\text{BP}^2(h_1) = \emptyset$, we have that the element $g' = h_1 g$ satisfies $|\text{BP}^2(g')| = k$, and moreover x belongs to $\text{BP}^2(g')$ and is fixed by g' . Choose $h_2 \in \langle S \rangle$ whose germ at x is equal to the germ of g . By the choice of S , we have $\text{BP}^2(g) = \{x\}$, so that for the element $g'' = h_2^{-1} g'$ we have $\text{BP}^2(g'') = \text{BP}^2(g') \setminus \{x\}$, and thus $|\text{BP}^2(g'')| = k - 1 < k$. By induction, we have $g'' \in \langle G(X), S \rangle$, and it follows that $g = h_1^{-1} h_2 g'' \in \langle G(X), S \rangle$. \square

Since the Bieri–Strebel group $G(X) = G(X; \mathbb{Z}[1/2], \langle 2 \rangle_*)$ is finitely generated for X dyadic, from the claim we get that G is finitely generated as well. The fact that $G_\omega^{(n)}(X)$ belongs to the class \mathcal{F} follows from finite generation of the subgroups $G_\omega^{(n)}(Y)$ for $Y \subset X$ dyadic.

Finally the same argument for the claim shows that the group $G_\omega^{(n)}(X)_c$ of compactly supported elements is generated by $\langle G(X)_c, S \rangle$. Since $\mathcal{D}_x^{(n)}$ is a perfect group we can choose the set S consisting of commutators. And since $G(X)_c$ is perfect as well, the group $G_\omega^{(n)}(X)_c$ is perfect. The last statement follows from Proposition 4.4. \square

Here is the main result of this subsection, whose proof will need some preliminary lemmas.

Theorem 12.23. *For $n \geq 2$, every action $\varphi: G_\omega^{(n)}(\mathbb{R}) \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is semi-conjugate to one of the following.*

(1) *A non faithful action induced by an action of the groups of germs*

$$\text{Germ}(G_\omega^{(n)}(\mathbb{R}), -\infty) \times \text{Germ}(G_\omega^{(n)}(\mathbb{R}), +\infty) \cong \text{BS}(1, 2) \times \text{BS}(1, 2)$$

(these are classified in Proposition 12.17).

(2) *The standard action.*

In particular every faithful minimal action of $G_\omega^{(n)}(\mathbb{R})$ on the real line is conjugate to its standard action.

Until the end of the subsection, for fixed $n \geq 1$ we write $G = G_\omega^{(n)}(\mathbb{R})$ and $H = G(\mathbb{R}) = G(2)$, so that $H \subseteq G$.

Lemma 12.24 (Upgrading fixed points). *With the notations as above, let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action on the real line. Then every fixed point of $\varphi(H)$ must be fixed by $\varphi(G)$. In other words, $\text{Fix}^\varphi(H) = \text{Fix}^\varphi(G)$.*

Proof. Consider the subgroups

$$K_l = \left\{ g \in G_{(-\infty, 0)} : \text{BP}^2(g) \subset \{0\} \right\} \quad \text{and} \quad K_r = \left\{ g \in G_{(0, +\infty)} : \text{BP}^2(g) \subset \{0\} \right\},$$

and set $K = \langle K_l, K_r \rangle \cong K_l \times K_r$. Note that group K realizes the group of germs $\mathcal{D}_0^{(n)}$ and after the assumption on breakpoints of second order, the claim in the proof of Proposition 12.22 gives that $G = \langle H, K \rangle = \langle H, K_l, K_r \rangle$.

Consider the subgroup $H_l \subseteq K_l$ consisting of all elements whose germ at 0 is given by a power of h_{0-}^n . In particular, every $g \in H_l$ satisfies $\text{BP}^2(g) = \emptyset$, hence H_l is a subgroup of H . Since the germ of h_{0-}^n is central in $\mathcal{D}_{0-}^{(n)}$, we have that H_l is normal in K_l , with quotient $K_l/H_l \cong \mathcal{D}_{0-}^{(n)}/\langle h_{0-}^n \rangle$ which is isomorphic to Thompson’s group T acting on the circle. The same considerations hold for the subgroup $H_r \subseteq K_r$ defined analogously.

Assume now that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is an action such $\text{Fix}^\varphi(H) \neq \emptyset$, so that $\text{Fix}^\varphi(H_l)$ is non-empty and contains $\text{Fix}^\varphi(H)$. Then $\varphi(K_l)$ preserves $\text{Fix}^\varphi(H_l)$, and the φ -action of K_l on $\text{Fix}^\varphi(H_l)$ factors through the quotient $K_l/H_l \cong T$. Since T is a simple group, and it contains elements of finite order, every order-preserving action on a totally ordered set is trivial. Thus the action of K_l on $\text{Fix}^\varphi(H_l)$ is actually trivial, and in particular it fixes $\text{Fix}^\varphi(H)$. Similarly so does K_r . Since $G = \langle H, K_l, K_r \rangle$ this implies that every point in $\text{Fix}^\varphi(H)$ is fixed by $\varphi(G)$. \square

The next lemma makes use of the assumption that $n \geq 2$ in Theorem 12.23, and leverages the fact that the group \tilde{T} admits only one action on the real line up to semi-conjugacy. In the statement, with abuse of notation, we identify h_{x-} with its germ in $\mathcal{D}_{x-}^{(n)}$

Lemma 12.25. *For every $n \geq 2$, the group $\mathcal{D}_{x-}^{(n)}$ admits no non-trivial left-invariant preorder which is invariant under conjugation by the element h_{x-} .*

Proof. The natural isomorphism $\mathcal{D}_{x-}^{(n)} \cong \tilde{T}$ maps h_{x-} to an element $h \in \tilde{T}$ which is an n th root of the translation t_1 , i.e. $h^n = t_1$. So it is enough to show that \tilde{T} admits no non-trivial left-invariant preorder \prec which is invariant under conjugation by such an h . Assume by contradiction that \prec is such a preorder. By [81, Theorem 8.7] the dynamical realization of \prec is semi-conjugate to the standard action of \tilde{T} on the real line, so that the maximal \prec -convex subgroup K must be equal to the stabilizer \tilde{T}_y of some point $y \in \mathbb{R}$ for the standard action. On the other hand K must be normalized by h , so that we must have $\tilde{T}_y = \tilde{T}_{h(y)}$. However since $h^n = t_1$ we have $h(y) \neq y$ and $|h(y) - y| < 1$, so that y and $h(y)$ have different projections to the circle \mathbb{R}/\mathbb{Z} . But any two distinct points in the circle have different stabilizers in Thompson's group T , and thus y and $h(y)$ have different stabilizers in \tilde{T} , which is a contradiction. \square

Proof of Theorem 12.23. Let $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ be an action without fixed points. Since G is in the class \mathcal{F} (Proposition 12.22), we can apply Theorem 10.3. We then assume that φ is faithful minimal and, by symmetry, it is enough to exclude that φ is \mathbb{R} -focal, increasingly horograded by the standard action of h on \mathbb{R} . Note that by Lemma 12.24, we know that $\varphi(H)$ has no fixed point. In order to fulfill Assumption 12.9, we will consider the family of subgroups $\mathcal{L} = \{L_x\}_{x \in \mathbb{R}}$, where $L_x = G_{(-\infty, x)}$, and with this choice we will simply have $I^\varphi(\mathcal{L}, x, \xi) = I^\varphi(x, \xi)$. We apply Lemma 12.11: let η be the unique fixed point of $\varphi(\text{Aff}(A, \Lambda))$. Fix a dyadic rational $x \in \mathbb{R}$ and consider the preorder \prec_η on $G_{(-\infty, x)}$ associated with the action of $G_{(-\infty, x)}$ on $I^\varphi(x, \eta)$. By Proposition 12.13, this preorder descends to a non-trivial preorder $\bar{\prec}_\eta$ on $\text{Germ}(G_{(-\infty, x)}, x) = \mathcal{D}_{x-}$. Consider now the element $h_x \in \text{Aff}(A, \Lambda)$. Since h_x fixes x , it normalizes $G_{(-\infty, x)}$; moreover fixes η and thus $\varphi(h_x)$ preserves $I^\varphi(x, \eta)$. We also see that the preorder \prec_η is invariant under the automorphism induced by h_x on $G_{(-\infty, x)}$. But this automorphism coincides with the inner automorphism defined by conjugation by h_{x-} , so that the preorder \prec_η , and thus $\bar{\prec}_\eta$, must be invariant under conjugation by h_{x-} . This is in contradiction with Lemma 12.25. \square

13. UNCOUNTABLE GROUPS

Locally moving groups also contain several natural “large” groups, such as the group $\text{Homeo}_0(\mathbb{R})$ or $\text{Diff}_0^r(\mathbb{R})$ or the subgroups $\text{Homeo}_c(\mathbb{R})$ and $\text{Diff}_c^r(\mathbb{R})$ of compactly supported elements. Actions of such groups on the line are well understood thanks to work of Militon [85], and of the recent work of Chen and Mann [24]. In fact, such results fit in a program started by Ghys [43], asking when the group of all diffeomorphisms (or homeomorphisms) of a

manifold may act on another manifold; in the recent years, very satisfactory results have been obtained, and we refer to the survey of Mann [72] for an overview.

In this section we provide a rigidity criterion for locally moving groups whose standard action has uncountable orbits and satisfies an additional condition. We then explain how this criterion recovers some of the results in [24, 85], and unifies them with the setting of the other results of this paper. Recall from the introduction (Definition 1.21), that for a group G and subgroup $H \subset G$, we say that the pair (G, H) has *relative Schreier property* if every countable subset of H is contained in a finitely generated subgroup of G . We have the following result, which is a more general version of Theorem 1.22 from the introduction.

Theorem 13.1. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving, such that for every open subinterval $I \subset X$, all G_I -orbits in I are uncountable. Suppose that for every subinterval $I \Subset X$, the pairs $(G_+, [G_I, G_I])$ and $(G_-, [G_I, G_I])$ have the relative Schreier property. Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is either topologically conjugate to the standard action of G on X , or semi-conjugate to an action that factors through $G/[G_c, G_c]$.*

Remark 13.2. Note that if the pair $(G_c, [G_I, G_I])$ has the relative Schreier property, then so do the pairs $(G_+, [G_I, G_I])$ and $(G_-, [G_I, G_I])$, since G_c is contained in G_+ and G_- .

For the proof we need the following lemmas.

Lemma 13.3. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup such that every $x \in X$ has an uncountable G -orbit. Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ which is semi-conjugate to the standard action on X is topologically conjugate to it.*

Proof. Note that if orbits are uncountable, then the action is minimal. Indeed if by contradiction $\Lambda \subset X$ is a closed invariant subset, then the set $\partial\Lambda$ is countable and G -invariant, thus $\partial\Lambda = \emptyset$, hence either $\Lambda = \emptyset$ or $\Lambda = X$. Assume that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is semi-conjugate to the standard action on X , by a monotone equivariant map $q: \mathbb{R} \rightarrow X$. As the action on X is minimal, the semi-conjugacy q is continuous. If it is not injective, there exists points $x \in X$ for which $q^{-1}(x)$ is a non-trivial interval. But the set of such points is G -invariant and at most countable, which is a contradiction. \square

Lemma 13.4. *Let G be a group of homeomorphisms of a second countable Hausdorff space. Then there exists a countable subgroup $G_0 \subseteq G$ such that $\text{Fix}(G) = \text{Fix}(G_0)$.*

Proof. The statement is non-empty only when G is uncountable, and we will assume so. Let \mathcal{U} be a countable basis of open subsets of the space. For every $z \in \text{Supp}(G)$, we can find a neighborhood $U \in \mathcal{U}$ of z and $g_U \in G$ such that $g_U(U) \cap U = \emptyset$. Thus we can cover $\text{Supp}(G)$ with countably many subsets with this property and the subgroup G_0 generated by the corresponding $g_U \in G$ is countable and satisfies the desired condition. \square

Proof of Theorem 13.1. Assume by contradiction that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ is an action which is not topologically conjugate to the action on X and such that $\varphi([G_c, G_c])$ has no fixed point. As G is locally moving, by Theorem 5.12 we can assume, say, that every finitely generated subgroup Γ of G_+ is totally bounded.

Claim. *For every subinterval $I \Subset X$, the subgroup $[G_I, G_I]$ is totally bounded.*

Proof of claim. By Lemma 13.4, we can find a countable subgroup $H \subseteq [G_I, G_I]$ with $\text{Fix}^\varphi([G_I, G_I]) = \text{Fix}^\varphi(H)$. By the relative Schreier property of $(G_+, [G_I, G_I])$, the subgroup H is contained in a finitely generated subgroup of G_+ , and thus is totally bounded. Hence $[G_I, G_I]$ is also totally bounded. \square

Fix $I = (c, d) \in X$. Let \mathcal{A} be the collection of connected components of $\text{Supp}^\varphi([G_I, G_I])$ and fix $L \in \mathcal{A}$. For every $x \in (d, b)$ we have the inclusion $[G_I, G_I] \subseteq [G_{(c,x)}, G_{(c,x)}]$, so that L is contained in some connected component L_x of $\text{Supp}^\varphi([G_{(c,x)}, G_{(c,x)}])$. Consider the function

$$F_L : (d, b) \rightarrow \mathbb{R} \\ x \mapsto \sup L_x,$$

which is well-defined after the claim, monotone increasing, and which tends to $+\infty$ as x tends to b . Moreover for every $x \in (d, b)$ the point $F_L(x)$ belongs to $\text{Fix}^\varphi([G_{(c,x)}, G_{(c,x)}])$, thus to $\text{Fix}^\varphi([G_I, G_I])$, and thus the function F_L cannot be continuous at every point. The group $G_{(d,b)}$ centralizes $[G_I, G_I]$, so it permutes the intervals in \mathcal{A} , hence for every $g \in G_{(d,b)}$, the family of functions $\{F_L : L \in \mathcal{A}\}$ is equivariant in the following sense:

$$g.F_L(x) = F_{g.L}(g(x)).$$

In particular, a point x is a discontinuity point for F_L if and only if $g(x)$ is a discontinuity point for $F_{g.L}$. But since all functions F_L are monotone, and there are countably many of them, there are at most countably many points $x \in (d, b)$ which are discontinuity points of some F_L , for $L \in \mathcal{A}$. Thus the $G_{(d,b)}$ -orbit of every such point must be countable, which is in contradiction with our assumption. \square

Let us now give some examples of groups that satisfy the hypotheses of Theorem 13.1. In several situations the relative Schreier property can be established using an embedding technique for countable groups due Neumann and Neumann [96], based on unrestricted permutational wreath product. This method has been exploited by Le Roux and Mann [63] to show that many homeomorphisms and diffeomorphisms groups of manifolds have the Schreier property. In order to run this method, it is enough that the group G be closed under certain infinitary products, in the following sense.

Definition 13.5. For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be a subgroup. Let $(I_n)_{n \in \mathbb{N}}$ be a collection of disjoint open subintervals of X . For every $n \in \mathbb{N}$, take an element $g_n \in G_{I_n}$. We denote by $\prod g_n$ the homeomorphism of X defined by

$$\prod g_n : x \mapsto \begin{cases} g_n(x) & \text{if } x \in I_n, \\ x & \text{if } x \notin \bigcup_n I_n. \end{cases}$$

We say that the group G is closed under *monotone infinitary products* if for every monotone sequence $(I_n)_{n \in \mathbb{N}}$ of disjoint open subintervals (in the sense that the sequence $(\inf I_n)_{n \in \mathbb{N}}$ is monotone), and every choice of $g_n \in G_{I_n}$, we have $\prod g_n \in G$. We also say that G is closed under *discrete infinitary products* if the latter holds for every sequence of intervals $(I_n)_{n \in \mathbb{N}}$ such that $(\inf I_n)_{n \in \mathbb{N}}$ has no accumulation point in the interior of X .

The following lemma provides a criterion for the relative Schreier property in our setting.

Lemma 13.6. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Suppose that G satisfies one of the following conditions:*

- i) G is closed under monotone infinitary products;*
- ii) G is closed under discrete infinitary products and contains elements acting without fixed points on a neighborhood of a and trivially on a neighborhood of b , and viceversa.*

Then for every open subinterval $I \Subset X$ the pairs $(G_+, [G_I, G_I])$ and $(G_-, [G_I, G_I])$ have the relative Schreier property.

Proof. Fix $I = (x, y)$ and let us show that the pair $(G_+, [G_I, G_I])$ has relative Schreier property (the other case is analogous). It is enough to show that for every sequence $(f_n)_{n \in \mathbb{N}} \subset G_I$, there exists a finitely generated subgroup $\Gamma \subset G_+$ such that $[f_n, f_m] \in \Gamma$ for all $n \neq m$. So let (f_n) be such a sequence. Choose $t \in G_+$ such that $t(y) < x$, and if we are in case **ii**) suppose additionally that t has no fixed point in (a, y) , so that $\lim_{n \rightarrow \infty} t^n(y) = a$. Also choose an increasing sequence of positive integers (k_n) which is *parallelogram-free*, that is, if $k_{n_1} - k_{n_2} = k_{m_1} - k_{m_2} \neq 0$, then $n_1 = m_1$ and $n_2 = m_2$ (for example the sequence $k_n = 2^n$ has this property). Set $g_n = t^{k_n} f_n t^{-k_n}$ and note that $g_n \in G_{I_n}$ with $I_n := t^{k_n}(I)$. By construction, in either case **i**) or **ii**) the element $h := \prod g_n$ belongs to G . It is not difficult to check that for every $n \in \mathbb{N}$ the element $t^{-k_n} h t^{k_n}$ coincides with f_n on I , and the fact that (k_n) is parallelogram-free implies that for $n \neq m$ the intersection of the supports of $t^{-k_m} h t^{k_m}$ and of $t^{-k_n} h t^{k_n}$ is contained in I . Thus we have $[t^{-k_n} h t^{k_n}, t^{-k_m} h t^{k_m}] = [h_n, h_m]$. It follows that the finitely generated subgroup $\Gamma := \langle h, t \rangle \subset G_+$ satisfies the desired conclusion. \square

Combining this with Theorem 13.1, one can show that various sufficiently "large" locally moving groups do not admit any exotic action at all. For example, we have the following criterion.

Corollary 13.7. *For $X = (a, b)$, let $G \subseteq \text{Homeo}_c(X)$ be a locally moving, perfect subgroup of compactly supported homeomorphisms. Assume that for every open subinterval $I \subset X$, all G_I -orbits in I are uncountable, and that G is closed under monotone infinitary products. Then every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is topologically conjugate to the standard action of G on X .*

This criterion is clearly satisfied by the group $G = \text{Homeo}_c(\mathbb{R})$ of all compactly supported homeomorphisms of \mathbb{R} (it is well-known that it is perfect, see for instance [44, Proposition 5.11] for a very short proof). Thus, Corollary 13.7 recovers the following result due to Milton [85].

Corollary 13.8 (Milton). *Every action $\varphi: \text{Homeo}_c(\mathbb{R}) \rightarrow \text{Homeo}(\mathbb{R})$, without fixed points is topologically conjugate to the standard action.*

Let us now consider the group $G = \text{Diff}_c^r(\mathbb{R})$ of compactly supported diffeomorphisms of \mathbb{R} of class C^r , with $r \in [1, \infty]$. This case is more subtle since the group G is not closed under monotone infinitary products: if $g_n \in G_{I_n}$, the element $\prod g_n$ need not be a diffeomorphism on a neighborhood of any accumulation point of the intervals I_n . A way to go around this is provided by the arguments of Le Roux and Mann in [63, §3], where they show that the group $\text{Diff}^r(M)$ has Schreier property for every closed manifold M whenever $r \neq \dim(M) + 1$. Note that this result does not apply to the group $\text{Diff}_c^r(\mathbb{R})$, which in fact does not have Schreier property, since it can be written as countable strictly increasing union of subgroups (see [63]). However, the same argument of their proof can be adapted to show the following.

Proposition 13.9. *For every $r \in [1, \infty] \setminus \{2\}$ and interval $I \Subset \mathbb{R}$, the pair $(\text{Diff}_c^r(\mathbb{R}), \text{Diff}_c^r(I))$ has the relative Schreier property.*

The reason for the assumption $r \neq 2$ is that in this case the group $\text{Diff}_c^r(\mathbb{R})$ is known to be simple, by famous results of Thurston [104] (for $r = \infty$) and Mather [78] (for $r \neq 2$ finite). Whether this holds for $r = 2$ remains an open question.

Proof of Proposition 13.9. We outline the steps, and refer to [63, §3] for details. First of all, observe that in order to prove the proposition it is enough to show find a generating set S of $\text{Diff}_c^r(I)$ with the property that every sequence (b_n) of elements of S is contained in a finitely

generated subgroup of $\text{Diff}_c^r(\mathbb{R})$ (see Lemma 3.6 in [63]). Set $I = (x, y)$. Following the same strategy as in the proof of Proposition 13.6, let $t \in \text{Diff}_c^r(\mathbb{R})$ be such that $t(y) < x$, and choose a parallelogram-free increasing sequence $(k_n) \subset \mathbb{Z}_+$, so that the intervals $I_n := t^{k_n}(I)$ are pairwise disjoint. The main difference with the proof of Proposition 13.6 is that if we choose $g_n \in \text{Diff}_c^r(I_n)$ arbitrarily, the element $\prod g_n$ does not necessarily belong to $\text{Diff}_c^r(\mathbb{R})$. However, if the elements (g_n) are such that their C^r norms satisfy $\|g_n\|_r \leq 2^{-n}$ then the sequence of truncated products $\prod_{n=1}^m g_n$ is a Cauchy sequence, so that the infinite product $\prod_{n=1}^\infty g_n$ belongs to $\text{Diff}_c^r(\mathbb{R})$. Since conjugation by t^{k_n} is continuous in the C^r topology, this implies that there exists a sequence (ε_n) such that whenever the elements $f_n \in \text{Diff}_c^r(I)$ are such that $\|f_n\|_r \leq \varepsilon_n$, then the product $\prod(t^{k_n} f_n t^{-k_n})$ is indeed in $\text{Diff}_c^r(\mathbb{R})$ (compare [63, Lemma 3.7]).

With these preliminary observations in mind, the key idea of [63, §3] is to consider a well-chosen generating set of $\text{Diff}_c^r(I)$, consisting of elements belonging to suitable copies of the affine group. By an affine group inside $\text{Diff}_c^r(I)$ we mean a subgroup generated by two one-parameter subgroups $\{a^t\}_{t \in \mathbb{R}}$ and $\{b^t\}_{t \in \mathbb{R}}$ of $\text{Diff}_c^r(\mathbb{R})$, varying continuously in the C^r topology, which satisfy the relations $a^s b^t a^{-s} = b^{e^{st}}$. Existence of affine subgroups in $\text{Diff}_c^r(I)$ can be obtained by applying the trick of Muller and Tsuboi described in Section 6 to the two vector fields generating the affine group (see [63, Lemma 3.3]). Let now $S \subset \text{Diff}_c^r(I)$ be the set of all time-one maps $b := b^1$ of a flow b^t belonging to an affine subgroup. Since the set S is non-empty and stable under conjugation in $\text{Diff}_c^r(I)$, it is a generating set by simplicity of $\text{Diff}_c^r(I)$. Now let $(b_n) \subset S$ be a sequence, where each $b_n = b_n^1$ belongs to an affine subgroup $\langle a_n^t, b_n^t \rangle$. The relations in the affine subgroups imply that for every $t, s \in \mathbb{R}$ we have $[a_n^s, b_n^t] = b_n^{(e^s - 1)t}$. This equality implies that for $\delta > 0$ small enough, the element b_n^δ can be written as a commutator of elements with arbitrarily small C^r norm (see [63, Corollary 3.4]). Thus, if for every $n \in \mathbb{N}$ we choose a sufficiently large positive integer $\ell_n > 0$, we have $b_n^{1/\ell_n} = [f_{2n}, f_{2n+1}]$ for some sequence $(f_n) \subset \text{Diff}_c^r(I)$ such that $\|f_n\|_r \leq \varepsilon_n$. By the choice of the sequence (ε_n) made above, the element $h := \prod_{n=1}^\infty t^{k_n} f_n t^{-k_n}$ is in $\text{Diff}_c^r(\mathbb{R})$. The same argument in the proof of Lemma 13.6 then implies that $[t^{-k_{2n}} h t^{k_{2n}}, t^{-k_{2n+1}} h t^{k_{2n+1}}] = [f_{2n}, f_{2n+1}] = b_n^{1/\ell_n}$, so that the subgroup $\Gamma = \langle h, t \rangle$ contains the sequence (b_n) . By the remark made at the beginning of the proof, this proves the proposition. \square

Combined with Proposition 13.9, Theorem 13.1 provides an alternative proof of the following recent result of Chen and Mann [24].

Corollary 13.10 (Chen–Mann). *For $r \in [1, \infty] \setminus \{2\}$, every action $\varphi: \text{Diff}_c^r(\mathbb{R}) \rightarrow \text{Homeo}_0(\mathbb{R})$ without fixed points is topologically conjugate to its standard action.*

14. A RESULTS FOR ACTIONS ON THE CIRCLE

To conclude this paper we consider the problem of studying actions of locally moving groups on the circle. This question turns out to be much simpler than the question studied so far, essentially due to the compactness of \mathbb{S}^1 . In fact, we can actually prove a result for actions on \mathbb{S}^1 of a group of homeomorphisms $G \subseteq \text{Homeo}(X)$ where X is an arbitrary locally compact space, provided the action of G on X satisfies suitable dynamical conditions.

Given a group G of homeomorphisms of a space X and an open subset $U \subset X$, we let G_U be the pointwise fixator of $X \setminus U$. Similarly to Definition 4.1, we say that a subgroup $G \subseteq \text{Homeo}(X)$ is *micro-supported* if for every non-empty subset $U \subset X$, the subgroup G_U is non-trivial. The action of G on X is *extremely proximal* if for every compact subset $K \subsetneq X$

there exists $y \in X$ such that for every open neighborhood V of y , there exists $g \in G$ with $g(K) \subset V$.

When $G \subset \text{Homeo}(X)$ is a micro-supported group, we denote by $M_G \subset G$ the subgroup of G generated by the subgroups $[G_U, G_U]$ where U varies over relatively compact non-dense open subsets of X . Note that M_G is non-trivial and normal in G . In fact, standard arguments similar to the proof of Proposition 4.4 imply the following (see e.g. [59, Proposition 4.6]).

Proposition 14.1. *Let X be a locally compact Hausdorff space, and $G \subset \text{Homeo}(X)$ be a micro-supported group acting minimally and strongly proximally on X . Then every non-trivial normal subgroup of G contains M_G . In particular if $G = M_G$, then G is simple.*

Thus when $G \neq M_G$ then the group G/M_G is the largest non-trivial quotient of G .

For $x \in X$ we also denote by G_x^0 the subgroup of elements that fix pointwise a neighborhood of x , and call it the *germ-stabilizer* of x . Further we say that G has the *independence property for pairs of germs* if for every distinct $x_1, x_2 \in X$ and for every elements $g_1, g_2 \in G$ such that $g_1(x_1) \neq g_2(x_2)$, there exists $g \in G$ and open neighborhoods $U_i \ni x_i$ such that g coincides with g_i in restriction to U_i for $i \in \{1, 2\}$.

Theorem 14.2. *Let X be a locally compact Hausdorff space, and let $G \subseteq \text{Homeo}(X)$ be a minimal micro-supported subgroup of homeomorphisms of X satisfying the following conditions:*

- i) the action of G on X is extremely proximal;*
- ii) for every $x \in X$ the germ-stabilizer G_x^0 acts minimally on $X \setminus \{x\}$.*

Assume that $\varphi: G \rightarrow \text{Homeo}_+(\mathbb{S}^1)$ is a faithful minimal action.

Then there exists a continuous surjective map $\pi: \mathbb{S}^1 \rightarrow X$ which is G -equivariant with respect to the φ -action on \mathbb{S}^1 and the natural action on X . Moreover, if the action of G on X has the independence property for pairs of germs, then X is homeomorphic to \mathbb{S}^1 and the map $\pi: \mathbb{S}^1 \rightarrow X$ is a covering map.

Before giving the proof we give some comment on the statement.

Remark 14.3. The condition that φ be minimal is not so restrictive, since every group action on \mathbb{S}^1 either has a finite orbit (and thus is semi-conjugate to an action factoring through a finite cyclic group), or is semi-conjugate to a minimal action. The condition that φ is faithful cannot be avoided, since in this generality little can be said about actions of the largest quotient G/M_G (which could even be a non-abelian free group, see [59, Proposition 6.11]). However if $G = M_G$, then theorem implies that every non-trivial action $\varphi: G \rightarrow \text{Homeo}(\mathbb{S}^1)$ factors onto its standard action on X .

Remark 14.4. It is likely that the assumptions on G in Theorem 14.2 may be relaxed or modified, and we did not attempt to identify the optimal ones. In particular we do not know whether the assumption that the action of G has independent germs is needed in the last statement. However, note that even with this assumption, the map π may be a non-trivial self-cover of \mathbb{S}^1 (thus it is not necessarily a semi-conjugacy to the standard action). Indeed the groups of homeomorphisms of the circle constructed by Hyde, Lodha, and the third named author in [50, Section 3] satisfy all assumptions in Theorem 14.2 and their action lifts to an action on the universal cover $\mathbb{R} \rightarrow \mathbb{S}^1$, thus also lifts to an action under all self-coverings of \mathbb{S}^1 .

The proof of Theorem 14.2 follows a similar approach the proofs of its special cases that appeared in [60, Theorem 4.17] for Thompson's group T and in [81, Theorem D] for the groups $\mathbb{T}(\varphi)$ of piecewise linear homeomorphisms of suspension flows defined there. The main difference is that some arguments there make crucial use on specific properties of those groups,

such as the absence of free subgroups in the group of piecewise linear homeomorphisms of an interval, while here we get rid of these arguments using Proposition 5.4, and this allows for a generalization to a much broader class of groups.

Proof of Theorem 14.2. Assume that $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ is a faithful minimal action. First of all, note that there is no loss of generality in supposing that φ is extremely proximal. Namely since G is non-abelian, then by Theorem 2.16 every minimal faithful action of $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ has a finite centralizer C_φ and the action φ descends via the quotient map $\mathbb{S}^1 \rightarrow \mathbb{S}^1/C_\varphi \cong \mathbb{S}^1$ to an extremely proximal action φ_{ep} . Thus by replacing φ with φ_{ep} we can assume that φ is extremely proximal to begin with.

Given $x \in X$ we will denote by $K_x := (G_x^0)_c$ the subgroup of all elements whose support is a relatively compact subset of $X \setminus \{x\}$, i.e. the union of the groups G_U , where U varies over open subsets with $U \Subset X \setminus \{x\}$. Note that K_x is a normal subgroup of G_x^0 .

By extreme proximality and minimality, for every $U \Subset X$ there exists $g \in G$ such that $g(\overline{U}) \cap \overline{U} = \emptyset$, so that G_U is conjugate to a subgroup of its centralizer. Thus by Proposition 5.4 we have $\text{Fix}^\varphi([G_U, G_U]) \neq \emptyset$. By compactness of \mathbb{S}^1 , we deduce that for every $x \in X$ we have

$$\text{Fix}^\varphi([K_x, K_x]) = \bigcap_{U \Subset X, x \in U} \text{Fix}^\varphi([G_U, G_U]) \neq \emptyset.$$

Note also that $\text{Fix}^\varphi([K_x, K_x]) \neq \mathbb{S}^1$, otherwise the subgroup $[K_x, K_x]$ would act trivially, contradicting that the action is faithful. Now fix $x \in X$ and consider the subset $C := \text{Fix}^\varphi([K_x, K_x])$, which is G_x^0 -invariant by normality of $[K_x, K_x]$ in G_x^0 . Since we are assuming that φ is extremely proximal, we can find a sequence (g_n) in G such that $g_n.C$ tends to a point $\xi \in \mathbb{S}^1$ in the Hausdorff topology. Upon extracting a subnet from (g_n) , we can suppose that $g_n(x)$ tends to a limit in the Alexandroff compactification $\hat{X} := X \cup \{\infty_X\}$. Suppose that this limit is ∞_X , that is, that $(g_n(x))$ escapes every compact subset of X . Then every element $g \in G_c$ belongs to $G_{g_n(x)}^0$ for n large enough, so that $\varphi(g)$ preserves $g_n.C$ for every n large enough, and thus fixes the point ξ . Since $g \in G_c$ is arbitrary, we deduce that $\varphi(G_c)$ fixes ξ , and by normality of G_c in G and minimality of φ we deduce that $\varphi(G_c) = \{\text{id}\}$, contradicting that the action is faithful. Thus the limit of $(g_n(x))$ is a point $y \in X$. The same argument as above then shows that $\varphi(G_y^0)$ fixes ξ .

Once the existence of such points $y \in X$ and $\xi \in \mathbb{S}^1$ has been proven, the rest of the proof is essentially the same as in [60] or [81], but we outline a self-contained argument for completeness. We claim that for every $\zeta \in \mathbb{S}^1$ there exists a unique point $\pi(\zeta) \in X$ such that $\varphi(G_{\pi(\zeta)}^0)$ fixes ζ , and that the map $\pi: \mathbb{S}^1 \rightarrow X$ defined in this way is continuous.

Let us first show that if such a point exists, it must be unique. Namely assume that $x_1, x_2 \in X$ are distinct points such that $\varphi(G_{x_i}^0)$ fixes ζ for $i \in \{1, 2\}$, so that the φ -image of $H := \langle G_{x_1}^0, G_{x_2}^0 \rangle$ fixes y . Let $U \subset X$ be any non-dense open subset of X . After the assumption of minimality of the action of $G_{x_1}^0$ on $X \setminus \{x_1\}$, we can find $g \in G_{x_1}^0$ such that $g(x_2) \notin \overline{U}$, so that $g^{-1}G_U g = G_{g^{-1}(U)} \subset G_{x_2}^0 \subset H$. Since we also have $g \in H$, we obtain that $G_U \subset H$. Thus H contains the non-trivial normal subgroup N of G generated by G_U where U varies over all non-dense open subsets of X . Then $\varphi(N)$ fixes ζ , which is a contradiction using again minimality and normality of N .

To show existence and continuity, one first checks the following fact (*): if (z_i) is a net of points in X converging to a limit $z \in \hat{X} := X \cup \{\infty_X\}$, and if (ζ_i) is a net of points in \mathbb{S}^1 converging to some limit $\zeta \in \mathbb{S}^1$ such that $\varphi(G_{z_i}^0)$ fixes ζ_i for every i , then $z \in X$ and $\varphi(G_z^0)$ fixes ζ . The proof of (*) follows similar arguments as above; more precisely, the fact that z

belongs to X is shown by arguing as above, to prove that otherwise the group $\varphi(G_c)$ would fix ζ , and the same argument also shows that $\varphi(G_z^0)$ fixes ζ .

Now let $\zeta \in \mathbb{S}^1$ be arbitrary, and choose $y \in X$ and $\xi \in \mathbb{S}^1$ such that $\varphi(G_y^0)$ fixes ξ (whose existence has already been proven). By minimality of φ we can find a net (g_i) in G such that $\zeta_i := g_i \cdot \zeta$ converges to ζ , and upon extracting a subnet we can suppose that $y_i := g_i(y)$ converges to some $z \in \hat{X}$. Then by $(*)$ we have $z \in X$ and $\varphi(G_z^0)$ fixes ζ . This shows the existence of a point $\pi(\zeta) := z$ as desired, and $(*)$ also implies the continuity of the map $\pi: \mathbb{S}^1 \rightarrow X$ defined in this way. The map is clearly G -equivariant, so its image $\pi(\mathbb{S}^1)$ is a compact G -invariant subset of X , so by minimality of the standard action of G on X , the map π must be surjective.

Suppose now that G has the independence property for pairs of germs and let us show that the map π must be injective (and thus a homeomorphism). As a preliminary observation, note that for every $x \in X$ the fiber $\pi^{-1}(x)$ must have empty interior, since otherwise by G -equivariance, the open subset $\text{Int}(\pi^{-1}(x))$ would be a wandering domain in \mathbb{S}^1 , contradicting minimality. Assume by contradiction that there exist $\xi_1 \neq \xi_2$ in \mathbb{S}^1 such that $\pi(\xi_1) = \pi(\xi_2) =: x$, and let $I := (\xi_1, \xi_2)$ be the arc between them (with respect to the clockwise orientation of \mathbb{S}^1). Since $I \not\subset \pi^{-1}(x)$, we can choose $\zeta \in I$ with $z := \pi(\zeta) \neq x$. By minimality of φ , there exists $g \in G$ such that $g \cdot \zeta \notin I$.

Assume first that $g(x) \neq z$. Using the independence property for pairs of germs, we can find $h \in G$ which coincides with the identity on some neighborhood of z , and with g^{-1} on some neighborhood of x . Then $h^{-1} \in G_z^0$, so $h^{-1} \cdot \zeta = \zeta$. On the other hand $gh \in G_x^0$, so that $\varphi(gh)$ fixes ξ_1 and ξ_2 and preserves the arc I . Writing $g = (gh)h^{-1}$, we see that $g \cdot \zeta = gh \cdot \zeta \in I$, contradicting the choice of g .

Assume now that $g(x) = z$. In this case choose an open subarc $J \subset I$ containing ζ such that $g \cdot J \cap I = \emptyset$. Since $\pi^{-1}(z)$ has empty interior, we can find a point $\zeta' \in J$ such that the point $z' := \pi(\zeta')$ is different from z . Then we have $g(y) \neq z'$, so we can repeat the previous reasoning using the points ζ', z' instead of ζ, z . This provides the desired contradiction and shows that the map π is injective. \square

This implies a rigidity result for groups that are given by a locally moving action on the circle (in the sense that for every proper interval $I \subset \mathbb{S}^1$, the rigid stabilizer acts on I without fixed points).

Corollary 14.5. *For $X = \mathbb{S}^1$, let $G \subseteq \text{Homeo}_0(X)$ be locally moving. Then for every faithful minimal action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$, there exists a continuous surjective equivariant map $\pi: \mathbb{S}^1 \rightarrow X$. Moreover, if the standard action of G on X has the independence property for pairs of germs, then π is a covering map.*

Example 14.6. Let us explain how Corollary 14.5 recovers the result of Matsumoto that every non-trivial action of $G = \text{Homeo}_0(\mathbb{S}^1)$ on \mathbb{S}^1 is conjugate to its standard action. Note that G clearly satisfies all assumptions of Corollary 14.5 and has the independence property for pairs of germs. Since G has elements of finite order, it cannot act non-trivially on the circle with a fixed point, and since moreover it is simple it cannot act with a finite orbit, so that every non-trivial action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ must be semi-conjugate to the minimal action φ_{\min} (which is automatically faithful). Corollary 14.5 then shows that φ_{\min} is the lift of its natural action via a self-cover, and again it is not difficult to see (using elements conjugate to rotations) that this is possible only if φ_{\min} is conjugate to the standard action, and in particular all its orbits are uncountable. If φ had an exceptional minimal set $\Lambda \subsetneq \mathbb{S}^1$, then every connected component of the complement would be mapped to a point with a countable

orbit, so that this is not possible. We deduce that φ is minimal, thus conjugate to φ_{min} and thus to the standard action.

Example 14.7. In a similar fashion, Corollary 14.5 recovers the result of Ghys [44] that every action of Thompson’s group T on the circle is semi-conjugate to its standard action. Indeed T satisfies all assumptions in Corollary 14.5. As above, one uses its simplicity to show that every action $\varphi: T \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ is semi-conjugate to a faithful minimal action, and thus by the corollary, to the lift of the standard action through a self-cover $\pi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, and then argue that this is possible only if π is a homeomorphism.

By considering other kind of spaces, Theorem 14.2 can be used to construct groups that cannot admit interesting actions on the circle.

Corollary 14.8. *Let X be a locally compact Hausdorff space which is not a continuous image of \mathbb{S}^1 (e.g. if X is not compact, or not path-connected), and let $G \subseteq \text{Homeo}(X)$ be a subgroup as in Theorem 14.2. Then G has no faithful minimal action on \mathbb{S}^1 . In particular, if G is simple, every action $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{S}^1)$ has a fixed point.*

Example 14.9. Examples of groups to which Corollary 14.8 are the groups of piecewise linear homeomorphisms of flows $\mathbb{T}(\varphi)$ from [81]. For every homeomorphism φ of the Cantor set X , the group $\mathbb{T}(\varphi)$ is a group of homeomorphisms of the mapping torus Y^φ of (X, φ) defined analogously to Thompson’s group T (see [81] for details). When φ is a minimal homeomorphism, the group $\mathbb{T}(\varphi)$ is simple [81, Theorem B] and its action on Y^φ satisfies all assumptions in Theorem 14.2. Since however the space Y^φ is not path-connected, Corollary 14.8 recovers the fact proven in [81, Theorem D] that every action of the group $\mathbb{T}(\varphi)$ on the circle has a fixed point. Moreover, it allows to extend the conclusion to many groups defined similarly but not by PL homeomorphisms, for instance any simple overgroup of $\mathbb{T}(\varphi)$ in $\text{Homeo}(Y^\varphi)$ (see Darbyan and Steenbock [29] for a vast family of such groups).

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