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Marc Chardin, Navid Nemati

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EQUATIONS OF SOME EMBEDDINGS OF A PROJECTIVE SPACE INTO ANOTHER ONE

MARC CHARDIN AND NAVID NEMATI

Abstract. In [8], Eisenbud, Huneke and Ulrich conjectured a result on the Castelnuovo-Mumford regularity of the embedding of a projective space $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{r-1}$ determined by generators of a linearly presented $\mathfrak{m}$-primary ideal. This result implies in particular that the image is scheme defined by equations of degree at most n. In this text we prove that the ideal of maximal minors of the Jacobian dual matrix associated to the input ideal defines the image as a scheme; it is generated in degree n. Showing that this ideal has a linear resolution would imply that the conjecture in [8] holds. Furthermore, if this ideal of minors coincides with the one of the image in degree n - what we hope to be true - the linearity of the resolution of this ideal of maximal minors is equivalent to the conjecture in [8].

1. Introduction and Preliminaries

Let $\phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ be a rational map defined by homogeneous forms $f_1, \ldots, f_r$ and I be the ideal generated by these forms. The Rees algebra associated to the ideal I, $\mathcal{R}(I)$, gives the bihomogeneous coordinate ring of the closure of the graph of ϕ inside $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$. The special fiber ring $\mathcal{F}(I)$ is the coordinate ring of the closed image of ϕ. The rings $\mathcal{R}(I)$ and $\mathcal{F}(I)$ are *blowup rings* associated to I. It is a fundamental problem to find the defining equations of the Rees ring from generators of I, for significant classes of ideals. This problem has been studied extensively by commutative algebraists, algebraic geometers, and, more recently, by applied mathematicians in geometric modeling (see [1, 4, 10, 11, 12]). Rees algebras provide an algebraic realization for the concept of blowing up a variety along a subvariety.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and I be an ideal of S. The Rees algebra $\mathcal{R}(I)$ of an ideal I is defined as

$$\mathcal{R}(I) = S[It] = S \oplus It \oplus I^2t^2 \oplus \cdots .$$

The equations of the Rees algebra are elements in the kernel of the epimorphism $\Phi : S[T_1, \ldots, T_r] \to \mathcal{R}(I)$ given by $\Phi(T_i) = f_i t$ where $I = (f_1, \ldots, f_r)$. These equations do depend on the choice of generators of I; however, their number and degrees do not, if one chooses minimal generators of a graded ideal I.

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An important object for the study of Rees algebras is the symmetric algebra, $\mathrm{Sym}(I)$, of an ideal I. It could be defined as

$$\mathrm{Sym}(I) = S[T_1, \dots, T_r]/\mathcal{L},$$

where $\mathcal{L} = ([T_1, \dots, T_r] \cdot M)$ and M is a presentation matrix of I. The map Φ above factors through the symmetric algebra. To determine equations of $\mathcal{R}(I)$ of degrees at least two in the T_i's is equivalent to study the kernel of the map $\mathrm{Sym}(I) \to \mathcal{R}(I)$. Traditionally, techniques for computing the defining ideal of $\mathcal{R}(I)$ often revolved around the notion of *Jacobian dual*.

For a polynomial ring S and an ideal I with a presentation $S^N \xrightarrow{M} S^r \to I \to 0$, the *Jacobian dual* of M is defined to be a matrix $\Theta(M)$ (we use Θ if the context makes no confusion possible) with linear entries in $S[T_1, \dots, T_r]$ such that

$$[T_1, \dots, T_r] \cdot M = [x_1, \dots, x_n].\Theta(M).$$

In this article, we focus on the Jacobian dual matrix of linearly presented $\mathfrak{m}$-primary ideals. The main theorem of this paper is the following:

Main Theorem. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, $I = (f_1, \dots, f_r)$ be a linearly presented $\mathfrak{m}$-primary ideal of S and Θ be the Jacobian dual matrix of a presentation matrix of I. Then the image of the map $\mathbb{P}^{n-1} \to \mathbb{P}^{r-1}$ defined by the forms $f_1, \dots f_r$ is scheme-theoretically defined by the ideal of maximal minors of Θ.*

It is worth mentioning that it is not hard to show that the image is defined as a set by the ideal of maximal minors (see Lemma 1); the more delicate point is to show that the image is scheme-theoretically defined by the ideal of maximal minors. As a consequence, the image of Φ is scheme-theoretically defined by equation of degree n. We finish the section by presenting two conjectures about the ideal of maximal minors of Θ.

In the last section, we present the connection between our work and a conjecture by Eisenbud, Huneke, and Ulrich in [8].

Conjecture. [8, Conjecture 1.1] *Let $S = k[x_1, \dots, x_n]$ and I be an ideal of S. If I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d, then $I^{n-1} = \mathfrak{m}^{d(n-1)}$. Equivalently,*

$$\mathrm{reg}\,(I^{n-1}) = (n-1)d.$$

In particular, we show that this conjecture holds if the ideal of maximal minors of Θ has a linear resolution (see Theorem 5).

2. Jacobian dual matrix

For the rest of the article we adopt the following notations.

Notation 1. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and $I = (f_1, \dots, f_r)$ be a linearly presented $\mathfrak{m}$-primary ideal, $\phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ be the map defined by $\phi(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_r(\boldsymbol{x}))$ and denote the graph of ϕ by $\Gamma \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$. Set $\pi_1 : \Gamma \to \mathbb{P}^{n-1}$ and*

$\pi_2 : \Gamma \to \mathbb{P}^{r-1}$ for the two natural projections and $W := \mathrm{Im}(\pi_2)$. We write $B = k[T_1, \ldots, T_r]$. The defining ideal J of the symmetric algebra $Sym(I)$ is generated by the bilinear forms $L_i(\boldsymbol{x}, \boldsymbol{T}) = \sum a_i(\mathbf{x})T_i$ where $\sum a_i(\mathbf{x})f_i = 0$ is a linear syzygy of I. The Jacobian dual Θ is given by

$$\Theta \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} L_1 \\ \vdots \\ L_N \end{bmatrix},$$

where the L_i's are minimal generators of J (equivalently, the corresponding syzygies are minimal generators of the module of syzygies). Thus Θ is a matrix whose entries are linear forms in the T_i's and we set $W' := \mathrm{proj}(\bar{B})$ where $\bar{B} = B/I_n(\Theta)$ and $I_n(\Theta)$ is the ideal of maximal minors of Θ (the $n \times n$ minors). Finally, if $p \in W \subseteq W'$ and $\mathfrak{p}$ is the corresponding homogeneous prime in $\mathrm{proj}(B)$, we denote by M_p the image of a matrix M by the natural map obtained by applying the map $B \to B_{\mathfrak{p}}/I_r(M)_{\mathfrak{p}}$ to its entries, where r is the rank of M.

Definition 1. *Let M be a $p \times q$ matrix over B. For a prime ideal $\mathfrak{p} \supseteq I_q(M)$, consider the natural map $B \to (B/I_q(M))_{\mathfrak{p}}$ and its natural extension $\mathrm{Mat}_{p\times q}(B) \to \mathrm{Mat}_{p\times q}((B/I_q(M))_{\mathfrak{p}})$. We write $M_{\mathfrak{p}}$ for the image of M under this last map.*

Lemma 1. *With the above notations, $\sqrt{I_n(\Theta)} = I(W)$.*

Proof. As I is $\mathfrak{m}$-primary, $\Gamma \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$ is scheme defined by the symmetric algebra, and is in particular, set defined by J. Consider the projection $\pi_2 : \mathbb{P}^{n-1} \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ and $p = (t_1 : \cdots : t_r) \in \mathbb{P}^{r-1}$. Let $\Theta(p)$ be the image of Θ by the specialization $T_i \mapsto t_i$. Then,

$$\begin{aligned} p \in W &\Leftrightarrow \exists x = (x_1, \ldots, x_n); (x, p) \in \Gamma \\ &\Leftrightarrow \exists x; L_i(x, t) = 0 \, \forall i \\ &\Leftrightarrow \exists x; \Theta(p) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0 \\ &\Leftrightarrow t \in I_n(\Theta). \end{aligned}$$

□

Lemma 2. *Assume that $\psi : \mathbb{P}^{n-1} \setminus X \to \mathbb{P}^{r-1}$ is defined by $I = (f_1, \ldots, f_r)$ where $X = V(I)$. Let $\Gamma \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$ be the closure of the graph of ψ. If $\pi : \Gamma \to \overline{\mathrm{Im}(\psi)}$ is the projection map, then $\forall p \in \overline{\mathrm{Im}(\psi)}$, setting $\dim \emptyset := -1$,*

$$\dim \pi^{-1}(p) \leq \dim X + 1.$$

Proof. Let $p = (p_1 : \cdots : p_r) \in \overline{\mathrm{Im}\, \psi}$ and let V be an irreducible component of $\pi^{-1}(p)$. If V is contained in X it has dimension at most the one of X. Else $\psi|_V : V \setminus X \to \mathbb{P}^{r-1}$ is a constant map with image p. Hence for all $1 \leq i, j \leq r$, $(f_i p_j - f_j p_i) \in I(V)$. Choose i with $p_i \neq 0$, then $f_j - p_j f_i/p_i \in I(V)$ for all j, and therefore $I \subseteq I(V) + (f_i)$, and the height of $I(V) + (f_i)$ is the one of $I(V)$ plus one. □

Lemma 3. *Adopt Notation 1. For a point $p \in W \subseteq W'$, the rank of $\Theta_{\mathfrak{p}}$ is $n-1$.*

Proof. By Lemma 2, $\pi^{-1}(p)$ is a zero dimensional scheme. The assertion follows from the fact that this fiber is a linear space defined by the system of equations

$$\Theta_{\mathfrak{p}}(p) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

□

Remark 1. *Since $\operatorname{reg} I^t = dt + 0$ for $t \gg 0$, the regularity of fibers of the projection $\pi_2 : \Gamma \to \mathbb{P}^{r-1}$ at any point is zero* [7]. *The only finite set of point(s) with zero regularity is a single reduced point. More generally, the regularity of stalks of π_2 at any point is zero too* [2].

Definition 2. *Let N be a $m \times n$ matrix. For $1 \leq r \leq n$ and $1 \leq s \leq m$, define $N^{(i_1,\dots i_r)}_{(j_1,\dots,j_s)}$ be the matrix obtained by deleting $i_1, \dots, i_r$-th columns and $j_1, \dots j_s$-th rows of N. Let N be a $(n-1) \times n$ matrix, denote $\Delta_i(N)$ to be the $(-1)^i \det N^{(i)}$, with $N^{(i)}$ the matrix obtained by deleting i-th column of N. For simplicity we write Δ_i instead of $\Delta_i(N)$ if it is clear what matrix we consider.*

Lemma 4. *Let N be a $(n-1) \times n$ matrix, then*

$$(-1)^i \operatorname{adj}(N^{(i)})N = \left[\begin{array}{ccc|c|ccc} \Delta_i & & & -\Delta_1 & & & \\ & \ddots & & \vdots & & & \\ & & \Delta_i & -\Delta_{i-1} & & & \\ \hline & & & -\Delta_{i+1} & \Delta_i & & \\ & & & \vdots & & \ddots & \\ & & & -\Delta_n & & & \Delta_i \end{array}\right],$$

where only the non zero terms are displayed.

Proof. The (j,ℓ)-entry of $\operatorname{adj}(N^{(i)})N$ is $\sum_k b_{j,k}a_{k,\ell}$ where $b_{j,k}$ and $a_{k,\ell}$ are the entries of $\operatorname{adj}(N^{(i)})$ and N.

Notice that $b_{j,k} = (-1)^{j+k} \det N^{(i,j)}_{(k)}$ if $j < i$ and $b_{j,k} = (-1)^{j+k} \det N^{(i,j+1)}_{(k)}$ else.

Hence then the (j,ℓ)-entry of $\operatorname{adj}(N^{(i)})N$ is

$$\sum_k (-1)^{j+k} \det N^{(i,j)}_{(k)} a_{k,\ell} \text{ if } j < i \quad \text{and} \quad \sum_k (-1)^{j+k} \det N^{(i,j+1)}_{(k)} a_{k,\ell} \text{ if } j \geq i.$$

First let $\ell < i$. If $\ell = j$, then it is equal to $\det N^{(i)} = (-1)^i \Delta_i$. If $\ell \neq j < i$, replace the j-th column of N with its ℓ-th column and call it N'. By expanding $\Delta_i(N')$ along the j-th

column we get

$$0 = \Delta_i(N') = (-1)^i \sum_k (-1)^{j+k} \det N^{(i,j)}_{(k)} a_{k,\ell},$$

and similarly if $j \geq i$, by relacing the $(j+1)$-th column of N with its ℓ-th column.

Second, if $\ell > i$, similar arguments as in the case $\ell < i$ show that if $j = \ell - 1$, the $(j, j+1)$-entry is equal to Δ_i, and the (j, ℓ)-entry is equal to 0 if $j \neq \ell - 1$.

Thirdly, if $j < \ell = i$, the (j, i)-entry of $\mathrm{adj}(N^{(i)})N$ is

$$\sum_k (-1)^{j+k} \det N^{(i,j)}_{(k)} a_{k,i}.$$

By expanding $\Delta_j = (-1)^j \det N^{(j)}$ along the i-th column we get

$$\Delta_j = (-1)^j \sum_k (-1)^{i-1+k} a_{k,i} \det N^{(i,j)}_{(k)} = (-1)^{i-1} \sum_k (-1)^{j+k} a_{k,i} \det N^{(i,j)}_{(k)}.$$

Finally, if $j \geq \ell = i$ then $b_{j,k} = (-1)^{j+k} \det N^{(i,j+1)}_{(k)}$ so (j, i)-th entry $\mathrm{adj}(N^{(i)})N$ is

$$\sum_k (-1)^{j+k} \det N^{(i,j+1)}_{(k)} a_{k,i}.$$

By expanding $\Delta_{j+1} = (-1)^{j+1} \det N^{(j+1)}$ on the i-th column we get

$$\Delta_{j+1} = (-1)^{j+1} \sum_k (-1)^{i+k} a_{k,i} \det N^{(i,j+1)}_{(k)} = (-1)^{i-1} \sum_k (-1)^{j+k} a_{k,i} \det N^{(i,j+1)}_{(k)}.$$

□

Lemma 5. *For a $1 \times n$ row vector R define L as a product of R and column vector $(x_1, \ldots, x_n)$. Let N be a $(n-1) \times n$ matrix with row vectors $R_1, \ldots R_{n-1}$ and consider corresponding linear forms $L_1, \ldots, L_{n-1}$. Let M obtained by bordering N with a row vector R_n and corresponding linear form L_n. Then for $1 \leq i, j \leq n$*

1. $\Delta_i L_n + (-1)^{n+1} \det(M) x_i \in (L_1, \ldots, L_{n-1})$
2. $(x_i \Delta_j - x_j \Delta_i) \in (L_1, \ldots, L_{n-1})$.
3. *If $j \neq n$, $\Delta_i L_j \in (x_1 \Delta_i - x_i \Delta_1, \ldots, x_n \Delta_i - x_i \Delta_n)$.*

Proof. (1) Consider

$$\det(M) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{adj}(M) M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathrm{adj}(M) \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}.$$

The $(i, 1)$-entries in the above matrix are $\det(M) x_i$. On the other, it is equal to

$$\sum_{1 \leq k \leq n} (-1)^{k+i} \det M^{(i)}_{(k)} L_k = \sum_{1 \leq k \leq n-1} (-1)^{k+i} \det M^{(i)}_{(k)} L_k + (-1)^n \Delta_i L_n.$$

(2) By choosing $L_n = e_j$ in (1) we get

$$\begin{aligned}\Delta_i x_j + (-1)^{n+1}\det(M)x_i &= \Delta_i x_j + (-1)^{n+1}(-1)^{n+j}(-1)^j \Delta_j x_i \\ &= \Delta_i x_j - x_i\Delta_j \in (L_1,\dots,L_{n-1}).\end{aligned}$$

(3) By Lemma 4

$$\Delta_i \begin{bmatrix} L_1 \\ \vdots \\ L_{n-1}\end{bmatrix} = (-1)^i N^{(i)} \operatorname{adj}(N^{(i)}) N \begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix} = N^{(i)} \begin{bmatrix} x_1\Delta_i - x_i\Delta_1 \\ \vdots \\ x_{i-1}\Delta_i - x_i\Delta_{i-1} \\ x_{i+1}\Delta_i - x_i\Delta_{i+1} \\ \vdots \\ x_n\Delta_i - x_i\Delta_n \end{bmatrix}.$$

□

Corollary 1. *For any $(n-1)\times n$ submatrix of Θ which corresponds to $L_{i_1},\dots,L_{i_{n-1}}$ define Δ_i for $1\le i\le n$ as in Definition 2. If $\Delta_i \neq 0$ for some i, then*

(1) *$\Delta_i J \subseteq (L_{i_1},\dots,L_{i_{n-1}})$ modulo $I_n(\Theta)$,*
(2) *$(L_{i_1},\dots,L_{i_{n-1}})B_{(\Delta_i)} = I_2 \begin{bmatrix} x_1 & \cdots & x_n \\ \Delta_1 & \cdots & \Delta_n\end{bmatrix} B_{(\Delta_i)}$.*

Proof. Choose for N the submatrix of Θ such that

$$N \begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix} L_{i_1} \\ \vdots \\ L_{i_{n-1}}\end{bmatrix}.$$

(1) follows from Lemma 5 (1), by choosing for M the matrix corresponding to add any minimal generator L_j of J.

(2) follows from Lemma 5 (2) and (3). □

Theorem 1. *Let $S = k[x_1,\dots,x_n]$ be a polynomial ring over a field k, $I = (f_1,\dots,f_r)$ be a linearly presented $\mathfrak{m}$-primary ideal of S and $\Phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ be the map defined by the forms $f_1,\dots f_r$. Let Θ be the Jacobian dual matrix of a presentation matrix of I, then*

$$I_n(\Theta)^{sat} = I(W).$$

Proof. By Lemma 1, $\sqrt{I_n(\Theta)} = I(W)$. Let $\mathfrak{p} \in \operatorname{spec}(B)$ containing $I_n(\Theta)$. We need to show that $(B/I_n(M))_{\mathfrak{p}} \cong (B/I(W))_{\mathfrak{p}}$. For this to hold, it suffices to prove that $\bar{B}_{\mathfrak{p}} = (B/I_n(M))_{\mathfrak{p}}$ is a domain. By Lemma 3, there exist $L_{i_1},\dots L_{i_{n-1}}$ and i with $\Delta_i \notin \mathfrak{p}$, with notations as in Corollary 1. Now

$$\begin{aligned}\bar{B}_{\mathfrak{p}}[x_1,\dots,x_n]/J \otimes_B \bar{B}_{\mathfrak{p}} &= \bar{B}_{\mathfrak{p}}[x_1,\dots,x_n]/(L_{i_1},\dots L_{i_{n-1}}) \otimes_B \bar{B}_{\mathfrak{p}} && \text{by Corollary 1 (1)} \\ &= \bar{B}_{\mathfrak{p}}[x_i] && \text{by Corollary 1 (2).}\end{aligned}$$

As J scheme defines Γ, $\operatorname{proj}(\bar{B}_{\mathfrak{p}}[x_1,\dots,x_n]/J\otimes_B \bar{B}_{\mathfrak{p}})$ is the stalk of the isomorphism $\pi:\Gamma\to W$ over $V(\mathfrak{p})$; in particular, it is reduced and irreducible. This shows that $\operatorname{proj}(\bar{B}_p[x_i])\cong \operatorname{spec}(\bar{B}_{\mathfrak{p}})$ is reduced and irreducible. Hence $\bar{B}_{\mathfrak{p}}$ is a domain. □

We finish this section with two conjectures.

Conjecture 1. *Let $S=k[x_1,\dots,x_n]$ be a polynomial ring and $I=(f_1,\dots,f_r)$ be a linearly presented $\mathfrak{m}$-primary ideal of S. Suppose that $\Phi:\mathbb{P}^{n-1}\dashrightarrow\mathbb{P}^{r-1}$ is a rational map defined by forms $f_1,\dots f_r$. Let Θ be the Jacobian dual matrix of a presentation matrix of I, then*

$$I_n(\Theta)=I(W)_{\geq n}.$$

Conjecture 2. *Let $S=k[x_1,\dots,x_n]$ be a polynomial ring and $I=(f_1,\dots,f_r)$ be a linearly presented $\mathfrak{m}$-primary ideal of S. Let Θ be the Jacobian dual matrix of a presentation matrix of I then $I_n(\Theta)$ has a linear resolution.*

3. Asymptotic behavior of regularity

In this section, we study the conjecture of Eisenbud, Huneke, and Ulrich [8, Conjecture 1.1] on the asymptotic behavior of regularity of linearly presented $\mathfrak{m}$-primary ideals. We will show the relation between this conjecture and the ideal of maximal minors of Jacobian dual matrices. We start this section by stating two equivalent definition of Castelnuovo-Mumford regularity. Before that, we need to mention the definition of graded Betti numbers. For a finitely generated graded S-module M, a minimal graded free resolution of M is an exact sequence

$$0\to F_p\to F_{p-1}\to\cdots\to F_0\to M\to 0,$$

where every F_i is a graded free S-module of the form $F_i=\oplus_{j\in\mathbb{Z}}S(-j)^{\beta_{i,j}(M)}$ with the minimal number of basis elements, and every map is graded. The value $\beta_{i,j(M)}$ is called the ith graded Betti number of M of degree j.

Theorem 2. *Let M be a finitely generated S-module, the regularity of M is given by:*

$$\operatorname{reg}(M)=\max_i\{\operatorname{end}(H^i_{\mathfrak{m}}(M))+i\}.$$

Although the regularity is a measure of vanishing of local cohomologies, it is also a measure of vanishing of graded Betti numbers. Eisenbud and Goto in [6] *proved that*

$$\operatorname{reg}(M)=\max_i\{j-i|\beta_{i,j}(M)\neq 0\}.$$

The most significant simple result on the regularity of powers of graded ideals is the following one, due independently to Kodiyalam [9] and to Cutkosky, Herzog and Trung [5].

Theorem 3. *Let I be an ideal of $S=k[x_1,\dots,x_n]$. There exists t_0 and b such that*

$$\operatorname{reg}(I^t)=td+b,\quad \forall t\geq t_0$$

with

$$d:=\min\{\mu\mid \exists t\geq 1,\ (I_{\leq\mu})I^{t-1}=I^t\}.$$

Notice that when I is equigenerated the number d in the above theorem is the degree of the generators of I. We call smallest such t_0 as the stabilization index of I and denote it by $\mathrm{Stab}(I)$. In [8], authors showed that if I is a linearly presented $\mathfrak{m}$-primary ideal, then the powers of I eventually have a linear resolution.

Theorem 4 ([8])**.** *Let $S = k[x_1, \dots, x_n]$ and I be an ideal of S. If I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d, then $\operatorname{reg}(I^t) = td$ for $t \gg 0$.*

In addition, they conjectured an upper bound for the stabilization index of I.

Conjecture 3. [8, Conjecture 1.1] *Let $S = k[x_1, \dots, x_n]$ and I be an ideal of S. If I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d, then $I^{n-1} = \mathfrak{m}^{d(n-1)}$. In other word,*

$$\operatorname{reg}(I^{n-1}) = (n-1)d.$$

Remark 2. *Eisenbud, Huneke and Ulrich in* [8] *proved this conjecture when $n = 3$ or I is a monomial ideal.*

Remark 3. *Since the only $\mathfrak{m}$-primary ideals with linear resolution are the powers of $\mathfrak{m}$, for an $\mathfrak{m}$-primary ideal I generated in degree d,*

$$\mathrm{Stab}(I) = \min\{t \,|h_{I^t}(dt) = h_{\mathfrak{m}^t}(dt)\},$$

where $h_M(d) := \dim_k M_d$ is the Hilbert funtion of M at degree d.

The following proposition provide a connection between regularity of powers of an ideal $I = (f_1, \dots, f_r)$ and the image of $\phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ defined by $f_1, \dots, f_r$.

Proposition 1. *Let $S = k[x_1, \dots, x_n]$ and I be an ideal of S. Adopting the Notation 1, If I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d, then*

1. *W is smooth of dimension $n-1$, and $e(W) = d^{r-1}$.*
2. $\operatorname{reg} I(W) = \max\{\mathrm{Stab}(I) + 1, \operatorname{reg} I(V_n^{(d)})\} = \max\{\mathrm{Stab}(I) + 1, n - \lceil \frac{n}{d} \rceil\}$

where $V_n^{(d)}$ is the Veronese embedding of degree d.

Proof. Part (1) follows from [3, Proposition 1.7(b)]. For proving part (2) we use the definition of regularity via local cohomologies. Let $I(W) \subset B = k[\mathbf{T}]$ and $I(V_n^{(d)}) \subset B' = k[\mathbf{T}']$ with $B \subset B'$ and $I(W) = I(V_n^{(d)}) \cap B$. As the natural map $B/I(W) \to B'/I(V_n^{(d)})$ is an isomorphism locally on the punctured spectrum, $H^i_{\mathbf{T}}(B/I(W)) \cong H^i_{\mathbf{T}}(B'/I(V_n^{(d)})) \cong H^i_{\mathbf{T}'}(B'/I(V_n^{(d)}))$ for $i \geq 2$ and there is an exact sequence:

$$0 \to B/I(W) \to B'/I(V) \to H^1_{\mathbf{T}}(B/I(W)) \to 0.$$

Since $h_{B/I(W)}(a) = h_{I^a}(da)$ and $h_{B/I(V)}(a) = h_{\mathfrak{m}^{da}}(da)$, by Remark 3 and above exact sequence, $H^1_{\mathbf{T}}(B/I(W)) = 0$ if and only if $i \geq \mathrm{Stab}(I)$. In other words, end$H^1_{\mathbf{T}}(B/I(W)) = \mathrm{Stab}(I) - 1$. The assertion follows from the definition of regularity and the fact that $\operatorname{reg} I(W) = \operatorname{reg} B/I(W) + 1$. □

Conjecture 3 was the initial motivation of this work, we end this paper by showing that how our conjectures are related to Conjecture 3.

Theorem 5. *Let $S = k[x_1, \ldots, x_n]$ and I be an ideal of S. Adopting the Notation 1, If I is a linearly presented $\mathfrak{m}$-primary ideal generated in degree d, then*

(1) *Conjecture 3 follows from Conjecture 2.*
(2) *If Conjecture 1 holds, then Conjecture 2 and Conjecture 3 are equivalent.*

Proof. (1) Assume $I_n(\Theta)$ has a linear resolution, i.e. $\operatorname{reg}(I_n(\Theta)) = n$. By Theorem 1, $I_n(\Theta)^{sat} = I(W)$. Hence

$$\operatorname{reg} I(W) \leq \operatorname{reg}(I_n(\Theta)) \leq n.$$

As $\operatorname{reg}(I(V_n^{(d)}) = n$, Proposition 1 implies that $\operatorname{Stab}(I) \leq n-1$.

(2) First we should note that by Proposition 1, $\operatorname{Stab}(I) \leq n-1$ if and only if $\operatorname{reg}(I(W)) \leq n$. As we assume $I_n(\Theta) = I(W)_{\geq n}$, $\operatorname{reg}(I(W)) \leq n$ if and only if $I_n(\Theta)$ has a linear resolution. □

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Institut de mathématiques de Jussieu, CNRS & Sorbonne Université, 4 place Jussieu, 75005 Paris , France

Email address: marc.chardin@imj-prg.fr

Université Côte d'Azur, Inria, 2004 route des Lucioles, 06902 Sophia Antipolis, France

Email address: navid.nemati@inria.fr