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Exponential stabilization of waves for the Zaremba boundary condition.

Pierre Cornilleau* & Luc Robbiano†

October 11, 2021

Abstract

In this article we prove, under some geometrical condition on geodesic flow, exponential stabilization of wave equation with Zaremba boundary condition. We prove an estimate on the resolvent of semigroup associated with wave equation on the imaginary axis and we deduce the stabilization result. To prove this estimate we apply semiclassical measure technics. The main difficulties are to prove that support of measure is in characteristic set in a neighborhood of the jump in the boundary condition and to prove results of propagation in a neighborhood of a boundary point where Neumann boundary condition is imposed. In fact if a lot of results applied here are proved in previous articles, these two points are new.

Keywords

Stabilization of Waves, Zaremba problem, pseudo-differential calculus, controllability, semiclassical measure, boundary propagation

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1 Introduction and results

1.1 Framework

In this article we are interested by stabilization of wave equation with Zaremba boundary condition. To be precise we have to introduce some notation. Let Ω be a bounded open set in \mathbb{R}^d , with \mathcal{C}^∞ boundary. Let $\partial\Omega_D$ and $\partial\Omega_N$ two open sets in $\partial\Omega$ such that $\partial\Omega_D \cap \partial\Omega_N = \emptyset$ and $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \Gamma$, where Γ is a smooth manifold of dimension $d-2$. The manifold Γ is not necessary a connected set. Let P be a second order differential operator. We have $P = \sum_{1 \leq j, k \leq d} D_{x_j} p_{jk}(x) D_{x_k} + \sum_{1 \leq j \leq d} p_j(x) D_{x_j} + p_0(x)$, where $p_{jk}(x)$, $p_j(x)$ are in $\mathcal{C}^\infty(V)$ where V is a neighborhood of $\overline{\Omega}$. The matrix $(p_{jk}(x))_{jk}$ is assumed positive definite for every $x \in V$. We assume that the operator defined by Pu for $u \in \mathcal{D} = \{u \in H^1(\Omega), Pu \in L^2(\Omega), u|_{\partial\Omega_D} = 0, (\partial_\nu u)|_{\partial\Omega_N} = 0\}$, is self-adjoint and non negative. Here ∂_ν is the exterior normal derivative. Let $a \in \mathcal{C}^\infty(V)$ be such that $a(x) \geq 0$ for every $x \in \Omega$. We associate with P the following wave equation

$$\begin{cases} \partial_t^2 u + Pu + a(x)\partial_t u = 0 & \text{in } \Omega \times (0, \infty) \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^1(\Omega) \oplus L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega_D \times (0, \infty) \\ \partial_\nu u = 0 & \text{on } \partial\Omega_N \times (0, \infty) \end{cases} \quad (1)$$

We associated the energy that is $E(t, u_0, u_1) = (Pu|u)_{L^2(\Omega)} + \int_\Omega |\partial_t u|^2 dx$, where $(v|w)_{L^2(\Omega)} = \int_\Omega v(x)\overline{w(x)}dx$. Under assumptions on flows associated with P and a (see sections 1.2, and 1.3) we obtain that the energy satisfies $E(t) \leq Ce^{-ct}$ for some constants $c > 0$ and $C > 0$. We obtain this result by an estimate on the resolvent associated with this problem and by the Gearhart-Huang-Prüss theorem [20, 38, 27]. To prove the resolvent estimate we use semiclassical measures. The method is well-known, since the seminal work by Bardos-Lebeau-Rauch [8], and was applied in different contexts and different variants, defect measures, Wigner measures, H -measures, see Aloui-Khenissi-Robbiano [2], Anantharaman-Léautaud-Macià [3, 4], Burq [11], Burq-Lebeau [12], Dehman-Le Rousseau-Léautaud [16], Gérard-Leichtnam [23], Gérard [21, 22], Le Rousseau-Lebeau-Terpolilli-Trélat [29], Lebeau [30], Miller [37], Robbiano-Zuily [40], Tartar [43] for instance.

We can find an introduction to this subject in Zworski [46].

Problems for the Zaremba boundary condition was studied by several authors. In particular, for elliptic problem, Shamir [42] and Savaré [41] proved that the regularity of solution is not as for the Dirichlet boundary condition, there is a lack of regularity, $s = 3/2$ is critical in the Sobolev spaces H^s if the datum is in L^2 . The problem is related with boundary problem in non smooth domain, with corner for instance. There is a large literature on this subject. For damping wave equation with Zaremba Boundary condition, the problem was studied by Bey and al. [7], Cornilleau and al. [13] where they prove exponential decay with multiplier method, and in [14] where we only prove logarithmic decay but without geometric condition on the support of the damping. Same kind of result was proven by Fu [19] for mixed boundary condition of Robin type.

In the following we described the geometry in section 1.2. This allows to give the precise assumption and the result in section 1.3. At the end of this section we give a description of proofs.

1.2 Geometry

Here we give the geometrical notion we use in this article. This framework comes from Melrose and Sjöstrand [35, 36] and the reader may also find in Hörmander [26, Chapter 24] more informations and proofs. The characterisation of symplectic sub-manifold is probably classical and more details can be found in Grigis [24].

Assumption on the symbol. We define the symbol of P by

$$p(x, \xi) = \sum_{1 \leq j, k \leq d} p_{jk}(x) \xi_j \xi_k - 1, \quad (2)$$

Where p_{jk} are $\mathcal{C}^\infty(\bar{\Omega})$. Locally in a neighborhood of the boundary we can define Ω by $\varphi > 0$ with $d\varphi \neq 0$. We can also choose coordinates (i.e. *normal geodesic coordinates*) such that $\varphi(x) = x_d$ and $p(x, \xi) = \xi_d^2 + R(x, \xi') - 1$ where $x = (x', x_d)$ and $\xi = (\xi', \xi_d)$.

Symplectic sub-manifold Σ . We can define a symplectic manifold Σ , contained into $T^*\mathbb{R}^d \cap \{\varphi = 0\}$. We set $\Sigma = \{(x, \xi), \varphi(x) = 0 \text{ and } \{\varphi, p\}(x, \xi) = 0\}$. The set Σ is a symplectic manifold as $\{\varphi, \{\varphi, p\}\} \neq 0$. In coordinates (x', x_d) , we have $\Sigma = \{x_d = 0, \xi_d = 0\}$, this manifold is isomorphic to $T^*\partial\Omega$ and described by coordinates (x', ξ') .

The Hamiltonian vector field H_p is not a vector field on Σ , but for all X a vector field on $T^*\mathbb{R}^d$, we can find unique fonctions α and β such $X + \alpha H_\varphi + \beta H_{\{\varphi, p\}}$ is a vector field on Σ . For H_p we denote the associated vector H'_p and an elementary computation leads to

$$H'_p = H_p + \frac{\{p, \{\varphi, p\}\}}{\{\varphi, \{\varphi, p\}\}} H_\varphi.$$

In coordinates (x', x_d) , H'_p only depends on R and we have

$$H'_p = H'_R = \sum_{j=1}^{d-1} (\partial_{\xi_j} R(x', 0, \xi') \partial_{x_j} - \partial_{x_j} R(x', 0, \xi') \partial_{\xi_j}).$$

In particular the integral curves associated with H'_p starting from a point into Σ stay into Σ . In coordinates (x', x_d) , we denote the integral curve starting from (x', ξ') , either $\gamma_g(s; x', \xi')$, either $\gamma_g(x', \xi')$, if s is implicit or $\gamma_g(s)$, if (x', ξ') is implicit.

Symplectic sub-manifold Σ' . The manifold Γ can be locally defined by $\{\varphi = 0, \psi = 0\}$, where $d\varphi \wedge d\psi \neq 0$. We define the manifold Σ' by

$$\Sigma' = \{(x, \xi), \varphi(x) = \psi(x) = \{\varphi, p\}(x, \xi) = \{\psi, p\}(x, \xi) = 0\}.$$

This manifold is symplectic. It suffices to prove that the following matrix

$$\begin{pmatrix} 0 & 0 & \{\varphi, \{\varphi, p\}\} & \{\varphi, \{\psi, p\}\} \\ 0 & 0 & \{\psi, \{\varphi, p\}\} & \{\psi, \{\psi, p\}\} \\ \{\{\varphi, p\}, \varphi\} & \{\{\varphi, p\}, \psi\} & 0 & \{\{\varphi, p\}, \{\psi, p\}\} \\ \{\{\psi, p\}, \varphi\} & \{\{\psi, p\}, \psi\} & \{\{\psi, p\}, \{\varphi, p\}\} & 0 \end{pmatrix},$$

is invertible. Clearly this is true if the matrix

$$\begin{pmatrix} \{\varphi, \{\varphi, p\}\} & \{\varphi, \{\psi, p\}\} \\ \{\psi, \{\varphi, p\}\} & \{\psi, \{\psi, p\}\} \end{pmatrix}, \quad (3)$$

is invertible. The quadratic form in (t, z) , $\{t\varphi + z\psi, \{t\varphi + z\psi, p\}\}$ is positive definite as $p''_{\xi\xi}$, is a positive definite matrix and $d\varphi$ and $d\psi$ are independent as $d\varphi \wedge d\psi \neq 0$. The matrix associated with $\{t\varphi + z\psi, \{t\varphi + z\psi, p\}\}$ is the one given by (3), then this matrix is invertible.

We can choose local coordinates (x_1, x'', x_d) such that $\phi(x) = x_d$, $\psi(x) = x_1$ and

$$p(x, \xi) = \xi_d^2 + \xi_1^2 + R_1(x', \xi'') + x_d R_2(x, \xi') - 1 = \xi_d^2 + R(x, \xi') - 1.$$

In these local coordinates, the manifold $\Sigma' = \{x_1 = x_d = \xi_1 = \xi_d = 0\}$ which is isomorphic to $T^*\Gamma$, and described by the coordinates (x'', ξ'') .

For all vector field X defined on $T^*\mathbb{R}^d$, we can find unique functions $\alpha, \beta, \gamma, \zeta$ such that $X + \alpha H_\varphi + \beta H_\psi + \gamma H_{\{\varphi, p\}} + \zeta H_{\{\psi, p\}}$ is a vector field on Σ' . For $X = H_p$, we denote the associated vector field H_p'' and we have $H_p'' = H_p + \alpha H_\varphi + \beta H_\psi$, as $H_p\varphi = H_p\psi = 0$ on Σ' and $H_\psi\varphi = 0$. We can compute α and β but the precise values are not useful for general functions φ and ψ . In coordinates (x_1, x'', x_d) , we have $H_p'' = H_p + \alpha\partial_{\xi_1} + \beta\partial_{\xi_d}$. The equations $H_p''\xi_1 = H_p''\xi_d = 0$, on Σ' , give $\alpha = -\partial_{x_1}R(0, x'', 0, 0, \xi'')$ and $\beta = -\partial_{x_d}R_1(0, x'', \xi'')$. In particular H_p'' only depends on R_1 , and we have $H_p'' = H_R'' = H_{R_1}'' = \sum_{j=2}^{d-1} (\partial_{\xi_j} R_1(0, x'', \xi'') \partial_{x_j} - (\partial_{x_j} R_1(0, x'', \xi'') \partial_{\xi_j}))$. The integral curves starting from Σ' stay on Σ' . We denote the curves starting from (x'', ξ'') , $\gamma_{\text{sing}}(s; x'', \xi'')$, $\gamma_{\text{sing}}(x'', \xi'')$, if s is implicit and $\gamma_{\text{sing}}(s)$, if (x'', ξ'') is implicit.

Description and topology of $T_b^*\Omega$. Let $T_b^*\Omega = T^*\partial\Omega \cup T^*\Omega$, this set is equipped with the following topology.

First if $\rho \in T^*\Omega$, a set V is a neighborhood of ρ if V contains an open set W of $T^*\Omega$ such that $\rho \in W$.

Second if $\rho = (x'_0, \xi'_0) \in T^*\partial\Omega$, a set V is a neighborhood of ρ if V contains a set

$$\begin{aligned} & \{(x', \xi') \in T^*\partial\Omega, |x'_0 - x'| + |\xi'_0 - \xi'| \leq \varepsilon\} \\ & \cup \{(x, \xi) \in T^*\Omega, |x'_0 - x'| + |\xi'_0 - \xi'| \leq \varepsilon \text{ and } (x_d, \xi_d) \in U \cap \{x_d > 0\}\}, \end{aligned}$$

where $\varepsilon > 0$ and U is a neighborhood of $\{(x_d, \xi_d) \in \mathbb{R}^2, x_d = 0\}$ in \mathbb{R}^2 .

In local coordinates where Ω is define by $x_d > 0$, we define $j : T^*\overline{\Omega} \rightarrow T_b^*\Omega$ by $j(x, \xi) = (x, \xi)$ if $x \in \Omega$, and $j(x, \xi) = (x', \xi')$ if $x_d = 0$. The map j is continuous for the topology given above. We can define more intrinsically j with the previous notation where Ω is given by $\varphi(x) > 0$. For $(x, \xi) \in T^*\overline{\Omega}$, $j(x, \xi) = (x, \xi)$ if $x \in \Omega$ and $j(x, \xi) = (x, \xi dx - (\{p, \varphi\}/H_\varphi^2 p)d\varphi)$, if $\varphi(x) = 0$. We verify, in this last case that $j(x, \xi) \in \{\varphi = \{p, \varphi\} = 0\}$, as $\{p, \varphi\}(x, d\varphi) = \{\varphi, \{\varphi, p\}\}$.

As usually we define the map $\pi : T_b^*\Omega \rightarrow \overline{\Omega}$, in local coordinates, as $\pi(x, \xi) = x$, if $(x, \xi) \in T^*\Omega$ and $\pi(x', \xi') = x'$, if $(x', \xi') \in T^*\partial\Omega$.

Bicharacteristic and generalized flow. For $(x, \xi) \in T^*\mathbb{R}^d$, we denote by $\gamma(s; x, \xi)$ the integral curve of H_p starting from (x, ξ) . We use the same short notations $\gamma(s)$ and $\gamma(x, \xi)$ as above.

Now we define the generalized bicharacteristic denoted by $\Gamma(s, \rho)$ for $\rho \in T_b^*\Omega$. To describe this curve in a neighborhood of the boundary we use the coordinates (x', x_d, ξ', ξ_d) and we identify Σ' and $T^*\partial\Omega$ and locally $\Omega = \{x \in \mathbb{R}^d, x_d > 0\}$. Moreover, we assume $\rho \in \text{char}(P) = \{(x, \xi) \in T^*\Omega, p(x, \xi) = 0\} \cup \{(x', \xi') \in T^*\partial\Omega, R(x', 0, \xi') - 1 \leq 0\}$.

Now we define the curve $\Gamma(s, \rho)$ locally for each (s_0, ρ) and we use the group property of the flow, namely $\Gamma(s + t, \rho) = \Gamma(s, \Gamma(t, \rho))$ to extend this function for every $s \in \mathbb{R}$.

For $\Gamma(s_0, \rho) \in T^*\Omega$, $\Gamma(s, \rho) = \gamma(s - s_0; \Gamma(s_0, \rho))$ if $\gamma(s - s_0; \Gamma(s_0, \rho)) \in T^*\Omega$. In particular, this defines $\Gamma(s, \rho)$ at least for s in a neighborhood of s_0 as $\gamma(s - s_0; \Gamma(s_0, \rho))$ stay in $T^*\Omega$ for small $|s - s_0|$. Observe that $p(\gamma(s - s_0; \Gamma(s_0, \rho))) = 0$.

For $\rho = (x'_0, \xi'_0) \in T^*\partial\Omega$, we have to distinguish different cases, first if $R(x'_0, 0, \xi'_0) < 1$ and second if $R(x'_0, 0, \xi'_0) = 1$, $\Gamma(s, \rho)$ depends on the properties of $\gamma(s; x'_0, 0, \xi'_0, 0)$. In what follows we only define the flow in a neighborhood of $s = 0$. We can extend the flow by the group property.

If $R(x'_0, 0, \xi'_0) < 1$, let $\xi^\pm = \pm\sqrt{1 - R(x'_0, 0, \xi'_0)}$. Let $\gamma(s; x_0, \xi_0) = (x(s; x_0, \xi_0), \xi(s; x_0, \xi_0))$, as $\dot{x} = 2\xi_d$, we have $x_d(s; x'_0, 0, \xi'_0, \xi^+) > 0$ for $s > 0$ sufficiently small, and $x_d(s; x'_0, 0, \xi'_0, \xi^-) > 0$ for $s < 0$ sufficiently small. Then we set $\Gamma(0, \rho) = \rho$, $\Gamma(s, \rho) = \gamma(s; x'_0, 0, \xi'_0, \xi^+)$ for $s > 0$ sufficiently small and $\Gamma(s, \rho) = \gamma(s; x'_0, 0, \xi'_0, \xi^-)$ for $s < 0$ sufficiently small. Observe that for $s \neq 0$ sufficiently small, $\Gamma(s, \rho) \in T^*\Omega$ and $p(\Gamma(s, \rho)) = 0$.

Such points are called hyperbolic points and we set $\mathcal{H} = \{(x'_0, \xi'_0) \in T^*\partial\Omega, R(x'_0, 0, \xi'_0) < 1\}$.

Definition 1.1 (Finite contact with the boundary). *Let (x'_0, ξ'_0) be such that $R(x'_0, 0, \xi'_0) = 1$. We say that the bicharacteristic $\gamma(s; x'_0, 0, \xi'_0, 0) = \gamma(s)$ does not have an infinite contact with the boundary if there exists $k \in \mathbb{N}$, $k \geq 2$, $\alpha \neq 0$ such that $x_d(s; x'_0, 0, \xi'_0, 0) = x_d(s) = \alpha s^k + \mathcal{O}(s^{k+1})$ in a neighborhood of $s = 0$. We denote by \mathcal{G}^k the set of such points.*

For $k = 2$ we distinguish two cases.

- The diffractive points, and we denote

$$\mathcal{G}_d = \{(x', \xi') \in T^*\partial\Omega, R(x', 0, \xi') = 1, \partial_{x_d}R(x', 0, \xi') < 0\}.$$

- The gliding points, and we denote

$$\mathcal{G}_g = \{(x', \xi') \in T^*\partial\Omega, R(x', 0, \xi') = 1, \partial_{x_d}R(x', 0, \xi') > 0\}.$$

Remark 1. *By Taylor's theorem and as $x_d(0) = 0$ and $\dot{x}_d(0) = 2\xi_d(0) = 0$, we always have $x_d(s) = \mathcal{O}(s^2)$.*

We have four cases to treat.

- k even, $\alpha > 0$. In this case $x_d(s) > 0$ for $s \neq 0$ sufficiently small. We define $\Gamma(0, \rho) = \rho$ and $\Gamma(s, \rho) = \gamma(s; x'_0, 0, \xi'_0, 0) \in T^*\Omega$ for $s \neq 0$ sufficiently small.
- k even, $\alpha < 0$. In this case $x_d(s) < 0$ for $s \neq 0$ sufficiently small. We define $\Gamma(s) = \gamma_g(s, \rho) \in T^*\partial\Omega$ for s sufficiently small.
- k odd, $\alpha > 0$. In this case $x_d(s) > 0$ for $s > 0$ sufficiently small and $x_d(s) < 0$ for $s < 0$ sufficiently small. We define $\Gamma(s, \rho) = \gamma_g(s, \rho) \in T^*\partial\Omega$ for $s \leq 0$ sufficiently small, and $\Gamma(s, \rho) = \gamma(s; x'_0, 0, \xi'_0, 0) \in T^*\Omega$ for $s > 0$, sufficiently small.
- k odd, $\alpha < 0$. In this case $x_d(s) < 0$ for $s > 0$ sufficiently small and $x_d(s) > 0$ for $s < 0$ sufficiently small. We define $\Gamma(s, \rho) = \gamma(s; x'_0, 0, \xi'_0, 0) \in T^*\Omega$ for $s < 0$ sufficiently small, and $\Gamma(s, \rho) = \gamma_g(s, \rho) \in T^*\partial\Omega$ for $s \geq 0$, sufficiently small.

This local description of $\Gamma(s)$ allows to prolongate $\Gamma(s)$ for every $s \in \mathbb{R}$. The function $\Gamma(s, \rho)$ defined on $\mathbb{R} \times \text{char}(P)$ is continuous for the topology of $\mathbb{R} \times T_b^*\Omega$, where the topology of $T_b^*\Omega$ is defined above.

1.3 Statement of Theorems

In the following we give the assumptions on the flows and these assumptions depend on the starting points. The assumptions also depend on the damping a and we assume $a(x) \geq 0$ for every $x \in \bar{\Omega}$. We denote by $\omega = \{x \in \bar{\Omega}, a(x) > 0\}$.

Definition 1.2. *We say that P , $a(x)$ and Ω satisfy the modified Geometric Control Condition (mGCC) if the bicharacteristic only has finite contact with the boundary (Definition 1.1) and the following assumptions are verified. Let $\rho_0 \in \text{char}(P)$.*

- *If $\pi(\rho_0) \notin \Gamma$ we assume there exist $s_0 \in \mathbb{R}$ be such that $\pi\Gamma(s_0, \rho_0) \in \omega$ and for every $s \in [0, s_0]$, if $\pi\Gamma(s, \rho_0) \in \Gamma$ then $\Gamma(s, \rho_0) \in \mathcal{H}$.*
- *If $\pi(\rho_0) \in \Gamma$ we assume there exist $s_0 \in \mathbb{R}$ be such that $\pi\gamma_{\text{sing}}(s_0, \rho_0) \in \omega$.*

Remark 2. *This definition of mGCC is different from the usual GCC. We are not able to prove propagation of support of measure for generalized bicharacteristic hitting Γ except for hyperbolic points. It is possible that singularities can be create at Γ but we do not know what can happen. For points on Γ we are only able to prove propagation on Γ for integral curve of H_R'' if we already know that measure is supported in the fiber above a point of Γ .*

We recall the assumptions on P , a second order differential operator. We have

$$P = \sum_{1 \leq j, k \leq d} D_{x_j} p_{jk}(x) D_{x_k} + \sum_{1 \leq j \leq d} p_j(x) D_{x_j} + p_0(x), \quad (4)$$

where $p_{jk}(x)$, $p_j(x)$ are real valued and in $\mathcal{C}^\infty(V)$ where V is a neighborhood of $\bar{\Omega}$. We assume that P is formally self-adjoint. The domain of P is given by $\mathcal{D}(P) = \{u \in H^1(\Omega), Pu \in L^2(\Omega), u|_{\partial\Omega_D} = 0, (\partial_\nu u)|_{\partial\Omega_N} = 0\}$, where ∂_ν is the exterior normal derivative. With this domain P is self-adjoint, and

$$(Pu|v)_{L^2(\Omega)} = \sum_{1 \leq j, k \leq d} (p_{jk}(x) D_{x_k} u | D_{x_j} v)_{L^2(\Omega)} + \left(\sum_{1 \leq j \leq d} p_j(x) D_{x_j} u + p_0(x) u | v \right)_{L^2(\Omega)}, \quad (5)$$

where u and v are in $\mathcal{D}(P)$. Moreover we assume P positive definite, there is $\delta > 0$ such that

$$(Pu|u)_{L^2(\Omega)} \geq \delta \|u\|_{H^1(\Omega)}^2, \text{ for every } u \in \mathcal{D}(P). \quad (6)$$

To give a precise formulation of the wave equation we introduce $H = H^1(\Omega) \oplus L^2(\Omega)$, we denote by $U = (u_0, u_1)$ an element of H and the operator A is given by

$$A = \begin{pmatrix} 0 & 1 \\ -P & -a(x) \end{pmatrix}, \quad (7)$$

associated with the domain

$$\mathcal{D}(A) = \{U = (u_0, u_1) \in H, Pu_0 \in L^2(\Omega), u_1 \in H^1(\Omega), u_0 = 0 \text{ on } \partial\Omega_D, \partial_\nu u_0 = 0 \text{ on } \partial\Omega_N\}.$$

Let U be the solution of $\partial_t U = AU$ satisfying $U(0) = (u_0, u_1) \in H$, we have $U(t) = (u(t), \partial_t u(t)) = e^{tA}(u_0, u_1)$, where e^{tA} is the semigroup associated with A . Then u satisfies the wave equation

$$\begin{cases} \partial_t^2 u(x) + Pu(x) + a(x)\partial_t u(x) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega_D, \\ \partial_\nu u = 0 \text{ on } \partial\Omega_N, \\ (u(0), \partial_t u(0)) = (u_0, u_1). \end{cases}$$

To $U(t) = (u_0(t), u_1(t))$, we associate the energy $E(t, u_0, u_1) = (Pu_0(t)|u_0(t))_{L^2(\Omega)} + \int_\Omega |u_1(t)|^2 dx$. We have $\partial_t E(t, u_0, u_1) = -2(a u_1 | u_1)_{L^2(\Omega)} \leq 0$. This implies $E(t, u_0, u_1) \leq E(0, u_0, u_1)$.

Remark 3. We have assumed that P is positive definite for simplicity but if P is non negative we can introduce $\tilde{H} = H/\ker P$, working on \tilde{H} instead of H we can obtain same results. For instance see [32] where this reduction is used.

The main result of the paper is the following

Theorem 1. We assume that P , $a(x)$ and Ω satisfy the modified Geometric Control Condition given in Definition 1.2. We assume that P has the form given in (4) and P is self-adjoint positive definite. Let A be defined by (7), we have

- 1) $\exists M > 0$, $\|e^{tA}\|_{\mathcal{L}(H)} \leq M$,
- 2) $A - i\mu I$, is invertible for all $\mu \in \mathbb{R}$,
- 3) $\exists M > 0$, $\|(A - i\mu I)^{-1}\|_{\mathcal{L}(H)} \leq M$.

Remark 4. There are several results when the third item is replaced by other estimates as $e^{C|\mu|}$, $|\mu|^\alpha$ in these cases the energy decay with a speed depending of the estimate on the resolvent. For this kind of results we refer to Batty-Duyckaerts [6], Borichev-Tomilov [10], Burq [11], Lebeau [31].

The first item is a consequence of the energy decay. The second is given by unique continuation theorem and also by the result given in [14, Proposition 1.1]. The goal of this article is to prove the third item. From the Gearhart-Huang-Prüss test for the exponential stability (see [20], [38], [27, Theorem 3], [18]), the three items of the previous theorem imply that the semigroup generated by A is exponentially stable and this implies the following theorem.

Theorem 2. We assume that P , $a(x)$ and Ω satisfy the modified Geometric Control Condition given in Definition 1.2. We assume that P has the form given in (4) and P is self-adjoint positive definite. Let A defined by (7), there exist $C, c > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq Ce^{-ct}.$$

The outline of the proof is the following. In Section 2 we recall some tools on semiclassical pseudo-differential calculus (Section 2.1), we reduce the third item of Theorem 1 to a semiclassical estimate (Section 2.2 and Proposition 2.2), and we prove a basic estimate on the trace at the boundary (Section 2.3 and Proposition 2.6).

In Section 3 we construct a semiclassical measure and we prove that this measure is supported on the characteristic set. To do that at the boundary -for interior point the result is classical- we have to distinguish three kinds of points, hyperbolic points (see Section 3.1.2 and Proposition 3.2) and glancing points (see Section 3.1.3 and Proposition 3.6). Only for elliptic points (see Section 3.1.4) we need to consider the boundary conditions. In a neighborhood of boundary where we impose Dirichlet or Neumann boundary condition, we prove Proposition 3.11 and we deduce Proposition 3.12. The proof is delicate in a neighborhood of Γ (see Proposition 3.16). These estimates on trace allow us to prove Proposition 3.1 in Section 3.2. In Section 3.3 we prove that semiclassical measure is not identically zero and in Section 3.4 we prove that semiclassical measure is null on support of the damping.

We shall reach a contradiction if we also prove that the measure is identically null. This is done in the next sections.

In Section 4 we prove some properties of semiclassical measure. In Section 4.1, we obtain the action of Hamiltonian vector field on the semiclassical measure up to the boundary. The interior result is stated in Proposition 4.1. Propositions 4.7 and 4.8 are analogous results at boundary. In Section 4.2, we deduce from that a decomposition of the semiclassical measure in two measure, the measure restricted in interior and a boundary measure (Lemma 4.9). The action of Hamiltonian vector field allows us to deduce some properties of these measures. Lemma 4.10 describes the action of Hamiltonian on the interior measure, Lemma 4.11 gives precisions in neighborhood of hyperbolic points, Lemma 4.12 and Lemma 4.13 give properties of boundary measure in a neighborhood of Dirichlet and Neumann boundary, and Lemma 4.14 is the analogous in neighborhood of jump between Dirichlet and Neumann boundary conditions.

In Section 5 we prove the propagation of support of semiclassical measure. We have to distinguish the different cases, if a bicharacteristic hits boundary transversally or tangentially, $\partial\Omega_D \cup \partial\Omega_N$ or Γ . This allows to prove the main theorem.

In Appendix A we prove some regularity measure needed to prove some estimate in a neighborhood of Γ .

In Appendix B we prove some estimates on boundary trace in the case of Neumann boundary condition. This is useful to prove properties of semiclassical measure in a neighborhood of a diffractive point in a neighborhood of $\partial\Omega_N$. Appendix C is devoted to prove some technical results stated in the previous section.

This work is based on previous results, mention particularly, the course given by Patrick Gérard at IHP in 2015, the articles of Burq and Lebeau [12] and Luc Miller [37]. We thank Claude Zuily for the first step in this kind of problem (see [40]), Belhassen Dehman, Matthieu Léautaud and Jérôme Le Rousseau for the working group where we have together studied this subject, Nicolas Burq to draw our attention on the Tataru paper [44]. That allowed us to achieve the propagation of measure at boundary in the case of Neumann boundary condition.

2 Semiclassical formulation

2.1 Notations and pseudo-differential calculus

Here we summarize some result on pseudo-differential calculus. More details, results and extension are given in the Hörmander book [26, Chapter 18], Martinez [34] and Le Rousseau-Lebeau [28]. Essentially we follow here this last article.

To a smooth function $a(x, \xi)$, and $h \in (0, 1)$ (a may depend on h but the constants, in the estimates given below, does not depend on h), we associate an operator by the following formula

$$\text{Op}_{sc}(a)u = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} a(x, h\xi) \hat{u}(\xi) d\xi, \text{ where } \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} u(x) dx.$$

This formula make sense under some assumption on a and u . In this paper we mainly use symbols in S^k . We say that $a \in S^k$ if for every $\alpha, \beta \in \mathbb{N}^d$ there exists $C = C_{\alpha, \beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C \langle \xi \rangle^{k-\beta}, \text{ where } \langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

For $a \in S^k$, $\text{Op}_{sc}(a)u$ has a sense for $u \in \mathcal{S}(\mathbb{R}^d)$, and can be extended for $u \in H^s(\mathbb{R}^d)$ for every $s \in \mathbb{R}$.

In a neighborhood of $\partial\Omega$ it is useful to use a tangential calculus. For a smooth function $a(x, \xi')$ we associate a tangential operator by the following formula

$$\text{op}_{sc}(a)u = (2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} a(x, h\xi') \tilde{u}(\xi', x_d) d\xi', \text{ where } \tilde{u}(\xi', x_d) = \int_{\mathbb{R}^{d-1}} e^{-ix'\xi'} u(x', x_d) dx'.$$

This formula make sense if $a \in S_{\text{tan}}^k$, that is, for every $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{d-1}$, there exists $C = C_{\alpha, \beta}$ such that

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi')| \leq C \langle \xi' \rangle^{k-\beta}, \text{ where } \langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}.$$

We also use this notation for pseudo-differential operator on the boundary $x_d = 0$. In this case, a and u does not depend on x_d .

For technical reason we also have to use other classes of symbols. In these cases we use the Hörmander's notations, for instance, $S(\langle \xi \rangle^m, (dx)^2 + \langle \xi' \rangle^{-2} (d\xi')^2)$, $S(\langle \xi' \rangle^m, (dx)^2 + (d\xi')^2)$. In this case we keep the notations $\text{Op}_{sc}(a)$, if the symbol depends on ξ_d and $\text{op}_{sc}(a)$, if the symbol does not depend on ξ_d . We also use the notation $\text{op}_{sc}(b) = b(x, hD')$, in particular when we restrict a function on $x_d = 0$, this allows to distinguish $b(x, hD')$ and $b(x', 0, hD')$.

The main interest of pseudo-differential operators are the calculus of products, commutators, adjoints. We have for $a \in S^k$ and $b \in S^m$,

$$\text{Op}_{sc}(a) \text{Op}_{sc}(b) = \text{Op}_{sc}(c), \text{ where } c \in S^{m+k},$$

and c admits an asymptotic expansion, $c(x, \xi) = a(x, \xi)b(x, \xi) + hd(x, \xi)$, where $d \in S^{m+k-1}$.

$$[\text{Op}_{sc}(a), \text{Op}_{sc}(b)] = h \text{Op}_{sc}(c), \text{ where } c \in S^{m+k-1},$$

and c admits an asymptotic expansion, $c(x, \xi) = -i\{a, b\}(x, \xi) + hd(x, \xi)$, where $d \in S^{m+k-2}$, and $\{a, b\}(x, \xi) = \sum_{j=1}^d (\partial_{\xi_j} a(x, \xi) \partial_{x_j} b(x, \xi) - \partial_{x_j} a(x, \xi) \partial_{\xi_j} b(x, \xi))$ is the Poisson bracket. At some point, it is useful to use that the commutator between an operator and a derivative admits an exact formula, we have $[hD_{x_j}, \text{Op}_{sc} a] = -ih \text{Op}_{sc}(\partial_{x_j} a)$.

For $a \in S^k$, we have

$$\text{Op}_{sc}(a)^* = \text{Op}_{sc}(b) \text{ and } b(x, \xi') = \bar{a}(x, \xi) + hc(x, \xi) \text{ where } b \in S^k, c \in S^{k-1}.$$

The asymptotic expansions may be extended to all power in h . Analogous formulas exist for $\text{op}_{sc} a$.

Associated with this semiclassical calculus, we introduce the semiclassical Sobolev spaces. For $u \in \mathcal{S}'(\mathbb{R}^d)$, we define $\|u\|_{H_{sc}^s} = \|\text{Op}_{sc}(\langle \xi \rangle^s)u\|_{L^2(\mathbb{R}^d)}$, if $\text{Op}_{sc}(\langle \xi \rangle^s)u \in L^2(\mathbb{R}^d)$. On the boundary we define for $u \in \mathcal{S}'(\mathbb{R}^{d-1})$, $|u|_{H_{sc}^s(x_d=0)} = |\text{op}_{sc}(\langle \xi' \rangle^s)u|_{L^2(\mathbb{R}^{d-1})}$, if $\text{op}_{sc}(\langle \xi' \rangle^s)u \in L^2(\mathbb{R}^{d-1})$. On the boundary we define $(u|v)_0 = \int_{\mathbb{R}^{d-1}} u(x')\bar{v}(x')dx'$.

We keep the same notation for a general Ω , namely $(u|v)_0 = \int_{\partial\Omega} u(x')\bar{v}(x')d\sigma(x')$, where σ is the superficial measure on $\partial\Omega$.

Pseudo-differential operators act on Sobolev spaces. For $a \in S^k$, there exists $C > 0$ such that

$$\|\text{Op}_{sc}(a)u\|_{H_{sc}^{s-k}} \leq C\|u\|_{H_{sc}^s}, \text{ for every } u \in H_{sc}^s.$$

For $a \in S_{\text{tan}}^k$, there exists $C > 0$ such that

$$|\text{op}_{sc}(a)u|_{H_{sc}^{s-k}} \leq C|u|_{H_{sc}^s(x_d=0)}, \text{ for every } u \in H_{sc}^s(x_d=0).$$

When we consider Sobolev spaces on $x_d > 0$, it is useful to consider op_{sc} and distinguish variable $x_d \in (0, \infty)$ and variables $x' \in \mathbb{R}^{d-1}$. Let $L^2((0, \infty), H_{sc}^s)$ be the space such that $u \in L^2((0, \infty), H_{sc}^s)$ if $\int_{\mathbb{R}^{d-1}} \int_0^\infty |\text{op}_{sc}(\langle \xi' \rangle^s)u(x', x_d)|^2 dx_d dx' = \|u\|_{L^2((0, \infty), H_{sc}^s)}^2 < \infty$. We have the following estimate, let $a \in S_{\text{tan}}^k$, there exists $C > 0$, such that

$$\|\text{op}_{sc}(a)u\|_{L^2((0, \infty), H_{sc}^{s-k})} \leq C\|u\|_{L^2((0, \infty), H_{sc}^s)}, \text{ for every } u \in L^2((0, \infty), H_{sc}^s).$$

In the context of semiclassical Sobolev spaces we have the following trace formula. Let $s > 0$, there exists $C > 0$, such that

$$|u|_{x_d=0}|_{H_{sc}^s} \leq Ch^{-1/2}\|u\|_{H_{sc}^{s+1/2}}, \text{ for every } u \in H_{sc}^{s+1/2}. \quad (8)$$

We recall the Gårding inequality for semiclassical Sobolev spaces. Let $a \in S^0$ be such that $a(x, \xi) \geq 0$, there exists $C > 0$, such that

$$\text{Re}(\text{Op}_{sc}(a)u|u)_{L^2(\mathbb{R}^d)} + Ch\|u\|_{L^2(\mathbb{R}^d)}^2 \geq 0. \quad (9)$$

Here and in what follows $(w|v)_{L^2(K)}$ means the inner product in K .

For tangential symbol we have the analogous result. Let $a \in S_{\text{tan}}^0$ be such that $a(x, \xi') \geq 0$, there exists $C > 0$, such that

$$\text{Re}(\text{op}_{sc}(a)u|u)_{L^2((0, \infty) \times \mathbb{R}^{d-1})} + Ch\|u\|_{L^2((0, \infty) \times \mathbb{R}^{d-1})}^2 \geq 0. \quad (10)$$

We use consequences of this result. Let $a \in S_{\text{tan}}^0$, such that there exists $K > 0$ such that $|a(x, \xi')| \leq K$, then

$$\|\text{op}_{sc}(a)u\|_{L^2((0, \infty) \times \mathbb{R}^{d-1})} \leq 2K\|u\|_{L^2((0, \infty) \times \mathbb{R}^{d-1})} + Ch\|u\|_{L^2((0, \infty) \times \mathbb{R}^{d-1})}, \quad (11)$$

where $C > 0$, depends on a finite number of seminorm of a . We have the same estimate at the boundary, if $a \in S(1, (dx')^2 + (d\xi')^2)$, and $|a(x', \xi')| \leq K$, we have

$$|\text{op}_{sc}(a)u|_{L^2(\mathbb{R}^{d-1})} \leq 2K|u|_{L^2(\mathbb{R}^{d-1})} + Ch|u|_{L^2(\mathbb{R}^{d-1})}, \quad (12)$$

where $C > 0$, depends on a finite number of seminorm of a . In particular we use this estimate if a depends on a parameter but K is uniform with respect this parameter. In this case, in the previous estimate C depends on the parameter. For $w \in L^2(\Omega)$, we extend w for $x \in \mathbb{R}^d \setminus \Omega$ by 0, and we use the following notations

$$\underline{w}(x) = 1_\Omega w(x) = \begin{cases} w(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases} \quad (13)$$

If $w \in L^2(\mathbb{R}^{d-1} \times (0, \infty))$, we extend w by 0 for $x_d < 0$ and we use the notation $\underline{w}(x) = 1_{x_d > 0} w(x)$.

In this article we use the symbol \lesssim : $A \lesssim B$ means, there exists $C > 0$, $A \leq CB$, where C is independent of parameters.

We denote $z^s = \exp(s \log(z))$, where $\log z$ is the principal value of the logarithm, where $z \in \mathbb{C} \setminus \mathbb{R}^-$.

2.2 Evolution equation and resolvent estimate

We begin the proof of the third item of Theorem 1. We may assume $|\mu| \geq 1$ as $A - i\mu I$ is invertible for all $\mu \in \mathbb{R}$, by second item and $\mu \mapsto (A - i\mu I)^{-1}$ is continuous from \mathbb{R} to $\mathcal{L}(H, H)$.

Let $F = (f_0, f_1) \in H$ and let $U = (u_0, u_1) \in \mathcal{D}(A)$ be such that $AU - i\mu U = F$, we have

$$\begin{cases} u_1 - i\mu u_0 = f_0 \\ -Pu_0 - au_1 - i\mu u_1 = f_1. \end{cases} \quad (14)$$

Lemma 2.1. *Assume that there exists $C_1 > 0$ such that*

$$|\mu| \|u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega)} \leq C_1 (\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}) \text{ for all } (f_0, f_1) \in H, \quad (15)$$

where (u_0, u_1) are the solution of (14). Then there exists $C_2 > 0$ such that

$$\|U\|_H \leq C \|F\|_H, \text{ for all } F \in H,$$

where $AU - i\mu U = F$.

Proof. By (14), $u_1 = i\mu u_0 + f_0$, so that

$$\|u_1\|_{L^2(\Omega)} \leq \|f_0\|_{L^2(\Omega)} + |\mu| \|u_0\|_{L^2(\Omega)} \lesssim \|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)},$$

and (15) gives $\|u_0\|_{H^1(\Omega)} \lesssim \|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}$, for $|\mu| \geq 1$. This gives the result. \square

Formula (14) implies the following equation on u_0

$$-Pu_0 + \mu^2 u_0 - i\mu a u_0 = a f_0 + i\mu f_0 + f_1.$$

To use semiclassical tools, we set $h = 1/\mu$, we multiply (15) by h^2 , we obtain the following equation on u_0

$$-h^2 Pu_0 + u_0 - i h a u_0 = a h^2 f_0 + i h f_0 + h^2 f_1,$$

and (15) is equivalent to

$$\|u_0\|_{L^2(\Omega)} + \|h \nabla u_0\|_{L^2(\Omega)} \leq Ch (\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}). \quad (16)$$

We shall prove this inequality by contradiction. If (16) is false, up to a normalization, there exist a sequence $h_n \rightarrow 0$ as $n \rightarrow \infty$ denoted for sake of simplicity by h , $(u_h)_h \in H^1(\Omega)$ and $(f_0^h, f_1^h)_h \in H$ satisfying

$$\begin{aligned} -h^2 Pu_h + u_h - i h a u_h &= a h^2 f_0^h + i h f_0^h + h^2 f_1^h \\ \|u_h\|_{L^2(\Omega)} + \|h \nabla u_h\|_{L^2(\Omega)} &= 1 \\ h (\|f_0^h\|_{H^1(\Omega)} + \|f_1^h\|_{L^2(\Omega)}) &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned} \quad (17)$$

Let $g_0^h = ahf_0^h + hf_1^h$ and $g_1^h = ihf_0^h$, (17) is equivalent to

$$\begin{aligned} & -h^2Pu_h + u_h - iha u_h = hg_0^h + g_1^h \\ & \|u_h\|_{L^2(\Omega)} + \|h\nabla u_h\|_{L^2(\Omega)} = 1 \\ & \|g_0^h\|_{L^2(\Omega)} + \|g_1^h\|_{H^1(\Omega)} \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned} \quad (18)$$

Proposition 2.2. *There exist $\beta > \alpha > 0$, there exists $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$, supported in $[\alpha, \beta]$, there exists $(\tilde{u}_h)_h$, satisfying $\|\tilde{u}_h\|_{L^2(\Omega)} + \|h\nabla\tilde{u}_h\|_{L^2(\Omega)} \leq C$, for some $C > 0$, such that*

$$\begin{aligned} & -h^2Pv_h + v_h - ihav_h = hq_h, \\ & \|v_h\|_{L^2(\Omega)} = 1 \text{ and } \|h\nabla v_h\|_{L^2(\Omega)} \leq 2, \\ & \|q_h\|_{L^2(\Omega)} \rightarrow 0 \text{ as } h \rightarrow 0, \end{aligned} \quad (19)$$

where $v_h = \theta(h^2P)\tilde{u}_h$.

Proof. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$ such that

$$\psi(s) = \begin{cases} 1 & \text{if } s \leq 1 \\ 0 & \text{if } s \geq 2. \end{cases}$$

Let $A > 0$ be sufficiently large to be fixed below. We have

$$1 = \lim_{n \rightarrow \infty} \psi(2^{-n}A^{-1}s) = \psi(sA^{-1}) + \sum_{k=1}^{\infty} (\psi(2^{-k}A^{-1}s) - \psi(2^{-k+1}A^{-1}s)).$$

Setting $\phi(s) = \psi(s) - \psi(2s)$, we have

$$1 = \psi(sA^{-1}) + \sum_{k=1}^{\infty} \phi(2^{-k}A^{-1}s) \text{ and } \phi \text{ is supported in } [1/2, 2]. \quad (20)$$

By functional calculus for auto-adjoint operators, we have

$$I = (\psi(A^{-1}h^2P) - \psi(Ah^2P)) + \psi(Ah^2P) + \sum_{k=1}^{\infty} \phi(2^{-k}A^{-1}h^2P). \quad (21)$$

Lemma 2.3. *There exists $C > 0$ such that*

$$\|\psi(Ah^2P)u_h\|_{L^2(\Omega)} \leq C(A^{-1} + h + h\|g_0^h\|_{L^2(\Omega)} + \|g_1^h\|_{L^2(\Omega)}), \quad (22)$$

$$\left\| \sum_{k=1}^{\infty} \phi(2^{-k}A^{-1}h^2P)u_h \right\|_{L^2(\Omega)} \leq CA^{-1}(1 + h\|g_0^h\|_{L^2(\Omega)} + \|g_1^h\|_{L^2(\Omega)}). \quad (23)$$

Proof. We apply $\psi(Ah^2P)$ to equation (18), we obtain

$$-h^2P\psi(Ah^2P)u_h + \psi(Ah^2P)u_h - ih\psi(Ah^2P)(au_h) = h\psi(Ah^2P)g_0^h + \psi(Ah^2P)g_1^h.$$

Let $\tilde{\psi}(s) = s\psi(s)$ we have

$$\psi(Ah^2P)u_h = A^{-1}\tilde{\psi}(Ah^2P)u_h + ih\psi(Ah^2P)(au_h) + h\psi(Ah^2P)g_0^h + \psi(Ah^2P)g_1^h.$$

As $|\psi(s)| \leq 1$ and $|\tilde{\psi}(s)| \leq C$, we obtain (22).

To prove (23) first we estimate $\phi(2^{-k}A^{-1}h^2P)u_h$. Let $\tilde{\phi} \in \mathcal{C}_0^\infty((0, \infty))$ to be fixed below. We apply $\tilde{\phi}(2^{-k}A^{-1}h^2P)$ to equation (18), we obtain

$$\begin{aligned} -h^2P\tilde{\phi}(2^{-k}A^{-1}h^2P)u_h &= -\tilde{\phi}(2^{-k}A^{-1}h^2P)u_h + ih\tilde{\phi}(2^{-k}A^{-1}h^2P)(au_h) \\ &\quad + h\tilde{\phi}(2^{-k}A^{-1}h^2P)g_0^h + \tilde{\phi}(2^{-k}A^{-1}h^2P)g_1^h. \end{aligned}$$

Let $\tilde{\phi}(s) = -s^{-1}\phi(s)$ be supported in $[1/2, 2]$. We obtain

$$\begin{aligned} 2^k A\phi(2^{-k}A^{-1}h^2P)u_h &= -\tilde{\phi}(2^{-k}A^{-1}h^2P)u_h + ih\tilde{\phi}(2^{-k}A^{-1}h^2P)(au_h) \\ &\quad + h\tilde{\phi}(2^{-k}A^{-1}h^2P)g_0^h + \tilde{\phi}(2^{-k}A^{-1}h^2P)g_1^h. \end{aligned}$$

This yields

$$2^k A\|\phi(2^{-k}A^{-1}h^2P)u_h\|_{L^2(\Omega)} \leq C(1 + h\|g_0^h\|_{L^2(\Omega)} + \|g_1^h\|_{L^2(\Omega)}).$$

Summing over k we obtain (23). \square

Let $\theta(s) = \psi(A^{-1}s) - \psi(As)$, by Lemma 2.3 and (21) choosing A sufficiently large and $h \in (0, h_0]$ for $h_0 > 0$ sufficiently small, we can have $\|u_h - \theta(h^2P)u_h\|_{L^2(\Omega)}$ as small as we want. From equation (18) multiplying by \bar{u}_h and integrating by parts, we obtain

$$-(Pu_h|u_h)_{L^2(\Omega)} + \int_{\Omega} |u_h|^2 dx = ih(au_h|u_h)_{L^2(\Omega)} + h(g_0^h|u_h)_{L^2(\Omega)} + (g_1^h|u_h)_{L^2(\Omega)}.$$

Taking h_0 sufficiently small, we have

$$h^2(Pu_h|u_h)_{L^2(\Omega)} = \int_{\Omega} |u_h|^2 dx + \varepsilon_h \text{ where } \varepsilon_h \rightarrow 0 \text{ as } h \rightarrow 0. \quad (24)$$

We now observe that $h^2(Pu_h|u_h)_{L^2(\Omega)}$ is equivalent to $H_{sc}^1(\Omega)$ -norm, uniformly with respect to $h \in (0, 1)$. From (18), the assumption on norms consequently gives $\|u_h\|_{L^2(\Omega)} \geq C_0 + \tilde{\varepsilon}_h$, where $C_0 > 0$ and $\tilde{\varepsilon}_h \rightarrow 0$ as $h \rightarrow 0$.

Let $v_h = \theta(h^2P)u_h / \|\theta(h^2P)u_h\|_{L^2(\Omega)}$. Now we prove that v_h satisfies the equation. We apply $\theta(h^2P)$ to (18), we have

$$h^2P\theta(h^2P)u_h + \theta(h^2P)u_h - ih\theta(h^2P)(au_h) = h\theta(h^2P)g_0^h + \theta(h^2P)g_1^h$$

which is equivalent to

$$h^2Pv_h + v_h - ihav_h = \|\theta(h^2P)u_h\|_{L^2(\Omega)}^{-1} (ih[\theta(h^2P), a]u_h + h\theta(h^2P)g_0^h + \theta(h^2P)g_1^h) = hq_h. \quad (25)$$

To obtain the estimate on q_h we have to prove

$$\|[\theta(h^2P), a]u_h\|_{L^2(\Omega)} \rightarrow 0 \text{ as } h \rightarrow 0, \quad (26)$$

$$h^{-1}\|\theta(h^2P)g_1^h\|_{L^2(\Omega)} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (27)$$

To do that we need the following result proved below.

Lemma 2.4. *Let $\phi \in \mathcal{C}_0^\infty(0, \infty)$, and $a \in \mathcal{C}_0^\infty(\Omega)$, there exists $C > 0$ such that*

$$\|[\phi(h^2P), a]w\|_{L^2(\Omega)} \leq Ch\|w\|_{L^2(\Omega)},$$

$$\|[\partial_{x_j}, \phi(h^2P)]w\|_{L^2(\Omega)} \leq C\|w\|_{L^2(\Omega)},$$

$$\|\partial_{x_j}\phi(h^2P)w\|_{L^2(\Omega)} \leq C\|w\|_{H^1(\Omega)},$$

for $j = 1, \dots, d$.

Estimate (26) is a direct consequence of Lemma 2.4. To prove (27), let $\tilde{\theta}(s) = s^{-1}\theta(s) \in \mathcal{C}_0^\infty(0, \infty)$, we compute

$$\begin{aligned} h^{-2}\|\theta(h^2P)g_1^h\|_{L^2(\Omega)}^2 &= h^{-2}(\theta(h^2P)g_1^h|\theta(h^2P)g_1^h)_{L^2(\Omega)}, \\ &= h^{-2}(h^2P\tilde{\theta}(h^2P)g_1^h|\theta(h^2P)g_1^h)_{L^2(\Omega)}, \\ &= \sum_{1 \leq j, k \leq d} (p_{jk}\partial_{x_j}\tilde{\theta}(h^2P)g_1^h|\partial_{x_k}\theta(h^2P)g_1^h)_{L^2(\Omega)} \\ &\quad + \sum_{1 \leq j \leq d} (-ip_j\partial_{x_j}\tilde{\theta}(h^2P)g_1^h|\theta(h^2P)g_1^h)_{L^2(\Omega)} + (p_0\tilde{\theta}(h^2P)g_1^h|\theta(h^2P)g_1^h)_{L^2(\Omega)} \end{aligned}$$

as $\tilde{\theta}(h^2P)g_1^h$ and $\theta(h^2P)g_1^h$ are in domain of P we can apply (5). By Lemma 2.4, we obtain

$$h^{-2}\|\theta(h^2P)g_1^h\|_{L^2(\Omega)}^2 \leq C\|g_1^h\|_{H^1(\Omega)}^2.$$

This completes the proof of Proposition 2.2 because arguing as we did to obtain Formula (24), we get $\|v_h\|_{L^2(\Omega)} = 1$ so that $\|hDv_h\|_{L^2(\Omega)} = 1 + \varepsilon_h$ with $\varepsilon_h \rightarrow 0$ as $h \rightarrow 0$. \square

Proof. Lemma 2.4. We start with the following lemma which will be proved below.

Lemma 2.5. *Let $C_1 > 0$. There exists $C > 0$ such that*

$$\begin{aligned} \|(-h^2P + z)^{-1}f\|_{L^2(\Omega)} &\leq C|\operatorname{Im} z|^{-1}\|f\|_{L^2(\Omega)}, \\ \|h\partial_{x_j}(-h^2P + z)^{-1}f\|_{L^2(\Omega)} &\leq C|\operatorname{Im} z|^{-1}\|f\|_{L^2(\Omega)}, \\ \|(-h^2P + z)^{-1}h\partial_{x_j}f\|_{L^2(\Omega)} &\leq C|\operatorname{Im} z|^{-1}\|f\|_{L^2(\Omega)}, \end{aligned}$$

for all $|z| \leq C_1$ and $f \in L^2(\Omega)$.

To prove Lemma 2.4 we use the Helffer-Sjöstrand formula,

$$\phi(h^2P) = -\frac{1}{\pi} \int \bar{\partial}_z \tilde{\phi}(x, y) (-h^2P + z)^{-1} dx dy,$$

where $z = x + iy \in \mathbb{C}$ and $\tilde{\phi}$ is an almost analytic extension of ϕ (see [25, Proposition 7.2] and [15]): $\tilde{\phi}$ is compactly supported and satisfies

$$\begin{aligned} \tilde{\phi}(x, 0) &= \phi(x), \\ |\bar{\partial}_z \tilde{\phi}(x, y)| &\leq C_N |y|^N, \text{ for every } N. \end{aligned}$$

We recall that $\bar{\partial}_z = (1/2)(\partial_x + i\partial_y)$. The Helffer-Sjöstrand formula gives

$$\begin{aligned} [\phi(h^2P), a] &= -\frac{1}{\pi} \int \bar{\partial}_z \tilde{\phi}(x, y) [(-h^2P + z)^{-1}, a] dx dy \\ &= \frac{1}{\pi} \int \bar{\partial}_z \tilde{\phi}(x, y) (-h^2P + z)^{-1} [-h^2P, a] (-h^2P + z)^{-1} dx dy. \end{aligned}$$

As $[-h^2P, a]$ is a sum of terms of following type $bh^2\partial_{x_j}$ and ch^2 , where b and c are into $\mathcal{C}^\infty(\bar{\Omega})$, the first estimate of the lemma is given by the two following estimates

$$\begin{aligned} \|(-h^2P + z)^{-1}ch^2(-h^2P + z)^{-1}w\|_{L^2(\Omega)} &\lesssim h^2|\operatorname{Im} z|^{-1}\|(-h^2P + z)^{-1}w\|_{L^2(\Omega)} \\ &\lesssim h^2|\operatorname{Im} z|^{-2}\|w\|_{L^2(\Omega)}, \end{aligned} \tag{28}$$

by the first estimate of Lemma 2.5.

$$\begin{aligned} \|(-h^2P + z)^{-1}bh^2\partial_{x_j}(-h^2P + z)^{-1}w\|_{L^2(\Omega)} &\lesssim h|\operatorname{Im} z|^{-1}\|h\partial_{x_j}(-h^2P + z)^{-1}w\|_{L^2(\Omega)} \\ &\lesssim h|\operatorname{Im} z|^{-2}\|w\|_{L^2(\Omega)}, \end{aligned} \tag{29}$$

by the two first estimates of Lemma 2.5.

For the second estimate we have by the Helffer-Sjöstrand formula

$$\begin{aligned} [\partial_{x_j}, \phi(h^2P)] &= -\frac{1}{\pi} \int \bar{\partial}_z \tilde{\phi}(x, y) [\partial_{x_j}, (-h^2P + z)^{-1}] dx dy \\ &= \frac{1}{\pi} \int \bar{\partial}_z \tilde{\phi}(x, y) (-h^2P + z)^{-1} [\partial_{x_j}, -h^2P + z] (-h^2P + z)^{-1} dx dy, \end{aligned}$$

and $[\partial_{x_j}, -h^2P + z]$ is a sum of term of type $h^2\partial_{x_j}b\partial_{x_k}$, $ch^2\partial_{x_j}$ and dh^2 where b, c and d are in $\mathcal{C}^\infty(\bar{\Omega})$. The terms with $ch^2\partial_{x_j}$ and dh^2 were estimated in (28) and (29). For the term $h^2\partial_{x_j}b\partial_{x_k}$, we have

$$\begin{aligned} \|(-h^2P + z)^{-1}h^2\partial_{x_j}b\partial_{x_k}(-h^2P + z)^{-1}w\|_{L^2(\Omega)} &\lesssim |\operatorname{Im} z|^{-1}\|h\partial_{x_j}(-h^2P + z)^{-1}w\|_{L^2(\Omega)}, \\ &\lesssim |\operatorname{Im} z|^{-2}\|w\|_{L^2(\Omega)}, \end{aligned}$$

By the second and third estimate of Lemma 2.5. This gives the second estimate of Lemma 2.4.

To prove the third estimate of Lemma 2.4 we write

$$\partial_{x_j}\phi(h^2P) = \phi(h^2P)\partial_{x_j} + [\partial_{x_j}, \phi(h^2P)].$$

The first term is clearly estimated by H^1 -norm and second term is estimated by the second inequality of Lemma 2.4. \square

Proof. Lemma 2.5. Let $u = (-h^2P + z)^{-1}f$, we have $u \in H^1(\Omega)$ and u satisfies $(-h^2P + z)u = f$ and the Zaremba boundary condition. Multiplying the equation by \bar{u} , integrating over Ω and an performing integration by parts, we get

$$-(h^2Pu|u)_{L^2(\Omega)} + z\|u\|_{L^2(\Omega)}^2 = (f|u). \quad (30)$$

Taking the imaginary part of equation we have $|\operatorname{Im} z|\|u\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$, which gives the first estimate.

Taking the real part of (30) and from (6) we have $\|\nabla u\|_{L^2(\Omega)}^2 \lesssim (Pu|u)_{L^2(\Omega)}$, we obtain

$$\|h\nabla u\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}^2.$$

This gives the second estimate with the previous result.

It is sufficient to prove the third estimate of Lemma 2.5 for $f \in \mathcal{C}_0^\infty(\Omega)$ as $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. We have

$$\begin{aligned} \|(-h^2P + z)^{-1}h\partial_{x_j}f\|_{L^2(\Omega)}^2 &= ((-h^2P + z)^{-1}h\partial_{x_j}f|(-h^2P + z)^{-1}h\partial_{x_j}f) \\ &= -(f|h\partial_{x_j}(-h^2P + z)^{-1}(-h^2P + z)^{-1}h\partial_{x_j}f) \\ &\leq C|\operatorname{Im} z|^{-1}\|f\|_{L^2(\Omega)}\|(-h^2P + z)^{-1}h\partial_{x_j}f\|_{L^2(\Omega)}, \end{aligned}$$

by the previous result. This gives the third estimate of Lemma 2.5. \square

2.3 A priori estimate on traces

In this section we assume that Ω is locally given near a point of the boundary by $x_d > 0$ and we denote the variables by $x = (x', x_d)$ where $x' \in \mathbb{R}^{d-1}$, and we set $\mathbb{R}_+^d = \{(x', x_d) \in \mathbb{R}^d, x_d > 0\}$. In these local coordinates¹ we have $h^2D_{x_d}^2v_h + (R(x, hD') - 1 + iha)v_h = hq_h$. When there are no ambiguity we only write R instead of $R(x, hD')$.

Proposition 2.6. *Let v_h given in Proposition 2.2. Then, there exists $C > 0$ such that*

$$|(v_h)|_{x_d=0}|_{H_{sc}^{1/2}} \leq Ch^{-1/2}, \quad (31)$$

$$|(hD_{x_d}v_h)|_{x_d=0}|_{H_{sc}^{-1/2}} \leq Ch^{-1/2}. \quad (32)$$

Proof. As $v_h \in H_{sc}^1(\Omega)$, the trace formula (8) gives the first estimate.

¹Rigorously, the laplace-Beltrami operator has a term $h^2D_{x_d}$. We can eliminate this term after a conjugaison of operator by a function non null everywhere but this has no influence on the proof given here. For simplicity we choose to keep v_h instead of the conjugated function.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\chi(x_d) = 1$ if $x_d \leq \delta$, $\chi(x_d) = 0$ if $x_d \geq 2\delta$ and $0 \leq \chi \leq 1$. Firstly

$$\begin{aligned}
& h |\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', 0)|_{L^2}^2 = -i \int_0^\infty h D_{x_d} (\chi(x_d) |\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d)|^2) dx_d \\
& = - \int_0^\infty h \chi'(x_d) |\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d)|^2 dx_d \\
& \quad - 2i \int_0^\infty \chi(x_d) \text{Re} \left(\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h^2 D_{x_d}^2 v_h(x', x_d) | \text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d) \right) dx_d \\
& = - \int_0^\infty h \chi'(x_d) |\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d)|^2 dx_d \\
& \quad - 2i \int_0^\infty \chi(x_d) \text{Re} \left(\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) (-R + 1) v_h(x', x_d) | \text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d) \right) dx_d \\
& \quad - 2i \int_0^\infty \chi(x_d) \text{Re} \left(\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h (q_h(x', x_d) - i(av_h)(x', x_d)) | \text{op}_{sc}(\langle \xi' \rangle^{-1/2}) h D_{x_d} v_h(x', x_d) \right) dx_d \\
& = I_1 + I_2 + I_3.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
|I_1| & \lesssim h \|h D_{x_d} v_h\|_{L^2(\Omega)}^2 \leq C, \\
|I_3| & \lesssim h (\|q_h\|_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)}) \|h D_{x_d} v_h\|_{L^2(\Omega)} \leq C.
\end{aligned}$$

Finally, we obtain

$$I_2 = -2i \int_0^\infty \chi(x_d) \text{Re} \left(\text{op}_{sc}(\langle \xi' \rangle^{-1}) (-R + 1) v_h(x', x_d) | h D_{x_d} v_h(x', x_d) \right) dx_d.$$

By tangential semiclassical pseudo-differential calculus, $\text{op}_{sc}(\langle \xi' \rangle^{-1}) (-R + 1)$ is of order 1, then

$$|I_2| \lesssim \|\text{op}_{sc}(\langle \xi' \rangle) v_h\|_{L^2(\Omega)} \|h D_{x_d} v_h\|_{L^2(\Omega)} \leq C.$$

This achieves the proof of the proposition. \square

3 Semiclassical measure and the characteristic set

3.1 Support of the semiclassical measure

We now define a semiclassical measure associated with $(v_h)_h$. It is classical, as $(v_h)_h$ is bounded in $L^2(\Omega)$, that there exists μ a measure on $T^*(\mathbb{R}^d)$ such that, up to extraction of subsequence of $(v_h)_h$, we have for all $b(x, \xi) \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$(\text{Op}_{sc}(b) \underline{v_h} | \underline{v_h})_{L^2(\mathbb{R}^d)} \rightarrow \langle \mu, b \rangle \text{ as } h \rightarrow 0. \quad (33)$$

For first expositions on microlocal defect measure or H-measure see [22, 43]. For semiclassical measure or Wigner measure see [11, 21, 37, 40]. The goal of this section is to prove the following result.

Proposition 3.1. *The measure μ is supported in $\overline{\Omega} \times \mathbb{R}^d$ and $p\mu = 0$, where p is the semiclassical symbol of $h^2 P - 1$ (see (2)).*

To prove this proposition we consider four sets in $T^*(\mathbb{R}^d)$, exterior or interior points (i.e. $x \in \mathbb{R}^d \setminus \overline{\Omega}$ or $x \in \Omega$), hyperbolic points (i.e., $x \in \partial\Omega$ and $R(x, \xi') - 1 < 0$), glancing points (i.e., $x \in \partial\Omega$ and $R(x, \xi') - 1 = 0$), and elliptic points (i.e., $x \in \partial\Omega$ and $R(x, \xi') - 1 > 0$).

The proofs for exterior and interior points are classical, we give the proofs for the sake of completeness. The proofs for hyperbolic and glancing points are similar to the proofs given by Burq and Lebeau [12] in the context of defect measures. The proof for elliptic points is specific to Zaremba boundary condition. Of course in $\partial\Omega_D \cap \partial\Omega_N$ the proof is also classical.

3.1.1 Interior and exterior points

Clearly, for $\chi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \overline{\Omega})$ and $\phi(\xi) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $(\text{Op}_{sc}(\chi(x)\phi(\xi))\underline{v}_h|\underline{v}_h) \rightarrow 0$ as $h \rightarrow 0$. Then μ is supported in $\overline{\Omega} \times \mathbb{R}^d$.

Let $\chi(x) \in \mathcal{C}_0^\infty(\Omega)$, $\phi(\xi) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\chi_1(x) \in \mathcal{C}_0^\infty(\Omega)$ be such that $\chi_1\chi = \chi$, we have, by symbol calculus as the symbol $p(x, \xi) - 1$ is in $S^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $\chi(x)\phi(\xi) \in S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned} \text{Op}_{sc}(\chi(x)\phi(\xi))(h^2P - 1) &= \text{Op}_{sc}(\chi(x)\phi(\xi)p(x, \xi)) + h \text{Op}_{sc}(r_0) \\ &= \text{Op}_{sc}(\chi(x)\phi(\xi))\chi_1(x)(h^2P - 1) + h \text{Op}_{sc}(r'_0), \end{aligned}$$

where $r_0, \tilde{r}_0, r'_0 \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$. As χ_1 is compactly supported in Ω and P is a local operator $\chi_1(x)(h^2P - 1)\underline{v}_h = \chi_1(x)(h^2P - 1)v_h$, we have

$$\begin{aligned} (\text{Op}_{sc}(\chi(x)\phi(\xi))(h^2P - 1)\underline{v}_h|\underline{v}_h) &= (\text{Op}_{sc}(\chi(x)\phi(\xi))\chi_1(x)(h^2P - 1)v_h|\underline{v}_h) + h\mathcal{O}(\|v_h\|_{L^2(\Omega)}^2) \\ &= (\text{Op}_{sc}(\chi(x)\phi(\xi))\chi_1(x)(-iha - hq_h)v_h|\underline{v}_h) + h\mathcal{O}(\|v_h\|_{L^2(\Omega)}^2) \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned} \tag{34}$$

We also have

$$\begin{aligned} (\text{Op}_{sc}(\chi(x)\phi(\xi))(h^2P - 1)\underline{v}_h|\underline{v}_h) &= (\text{Op}_{sc}(\chi(x)\phi(\xi)p(x, \xi))\underline{v}_h|\underline{v}_h) + h\mathcal{O}(\|v_h\|_{L^2(\Omega)}^2) \\ &\rightarrow \langle \mu, p(x, \xi)\chi(x)\phi(\xi) \rangle \text{ as } h \rightarrow 0. \end{aligned}$$

This and (34) give that $\langle \mu, p(x, \xi)\chi(x)\phi(\xi) \rangle = 0$. This proves Proposition 3.1 for the interior points, as the space spanned by functions $\chi(x)\phi(\xi)$ is dense in $\mathcal{C}_0^\infty(\Omega \times \mathbb{R}^d)$.

Before proving Proposition 3.1 in a neighborhood of the boundary, we have to prove estimates more precise than Proposition 2.6. We have to distinguish microlocally hyperbolic, glancing and elliptic points. For hyperbolic and glancing points the boundary condition play no role. For elliptic points we have to distinguish points in $\partial\Omega_D \cup \partial\Omega_N$, and points in Γ . The results are stated in Proposition 3.12 and in Proposition 3.16.

3.1.2 Hyperbolic points

Let $\delta > 0$ be sufficiently small, let $\chi = \chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ be such that

$$\chi(x, \xi') = \begin{cases} 1 & \text{if } R(x, \xi') - 1 \leq -\varepsilon \text{ and } x_d \leq \delta \\ 0 & \text{if } R(x, \xi') - 1 \geq -\varepsilon/2 \text{ or } x_d \geq 2\delta, \end{cases} \tag{35}$$

and $0 \leq \chi \leq 1$. Observe that $\chi \in S_{\text{tan}}^{-\infty}$ as supported for $|\xi'| \leq C$, where C depends on R .

Proposition 3.2. *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\begin{aligned} |\text{Op}_{sc}(\chi|_{x_d=0})(v_h)|_{x_d=0}|_{H_{sc}^1} &\leq C_\varepsilon \\ |\text{Op}_{sc}(\chi|_{x_d=0})(hD_{x_d}v_h)|_{x_d=0}|_{L^2} &\leq C_\varepsilon, \end{aligned} \tag{36}$$

for all $h \in (0, 1]$.

Remark 5. *The estimate on traces, in the hyperbolic region, are better than the one proved in Proposition 2.6.*

We begin the proof by giving a localization result which is useful in each region defined in what follows.

Lemma 3.3. *Let $\chi(x, \xi'), \chi_1(x, \xi') \in S_{\text{tan}}^0$, be such that $\chi_1(x, \xi') = 1$, for (x, ξ') in the support of χ . We assume $0 \leq \chi \leq 1$ and $0 \leq \chi_1 \leq 1$. Let $w_h = \text{Op}_{sc}(\chi)v_h$, there exists q_2^h such that*

$$(h^2D_{x_d}^2 + \text{Op}_{sc}(\chi_1^2(R(x, \xi') - 1)))w_h = hq_2^h, \text{ where } \|q_2^h\|_{L^2(\Omega)} \leq C, \tag{37}$$

for some $C > 0$ depending on semi-norms of χ and χ_1 .

Proof. We have, for $x_d > 0$,

$$\begin{aligned} (h^2 D_{x_d}^2 + R(x, hD') - 1)w_h &= h \operatorname{op}_{sc}(\chi)(q_h - iav_h) + [R(x, hD'), \operatorname{op}_{sc}(\chi)]v_h + [h^2 D_{x_d}^2, \operatorname{op}_{sc}(\chi)] \\ &= hq_1^h. \end{aligned}$$

The symbol of $[R(x, hD'), \operatorname{op}_{sc}(\chi)]$ is in hS_{\tan}^1 and by exact symbol calculus with D_{x_d} , we have

$$\begin{aligned} [h^2 D_{x_d}^2, \operatorname{op}_{sc}(\chi)] &= -ih^2 D_{x_d} \operatorname{op}_{sc}(\partial_{x_d} \chi) - ih \operatorname{op}_{sc}(\partial_{x_d} \chi) hD_{x_d} \\ &= -2ih \operatorname{op}_{sc}(\partial_{x_d} \chi) hD_{x_d} + h^2 \operatorname{op}_{sc}(r_0), \end{aligned}$$

where $r_0 \in S_{\tan}^0$. By the properties of v_h given in Proposition 2.2 and the previous formulas, there exists $C > 0$ such that $\|q_1^h\|_{L^2} \leq C$.

We have

$$R(x, hD') - 1 = \operatorname{op}_{sc}(\chi_1^2(R(x, \xi') - 1)) + \operatorname{op}_{sc}((1 - \chi_1^2)(R(x, \xi') - 1)),$$

thus

$$(h^2 D_{x_d}^2 + \operatorname{op}_{sc}(\chi_1^2(R(x, \xi') - 1)))w_h = hq_1^h - \operatorname{op}_{sc}((1 - \chi_1^2)(R(x, \xi') - 1)) \operatorname{op}_{sc}(\chi)v_h = hq_2^h.$$

As $(1 - \chi_1^2)\chi = 0$ by assumption, we deduce from symbol calculus that

$$\|\operatorname{op}_{sc}((1 - \chi_1^2)(R(x, \xi') - 1)) \operatorname{op}_{sc}(\chi)v_h\|_{L^2(\Omega)} \leq Ch.$$

This gives (37). \square

Proof of Proposition 3.2. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ be such that

$$\chi_1(x, \xi') = \begin{cases} 1 & \text{if } R(x, \xi') - 1 \leq -\varepsilon/2 \text{ and } x_d \leq 2\delta \\ 0 & \text{if } R(x, \xi') - 1 \geq -\varepsilon/4 \text{ or } x_d \geq 3\delta, \end{cases}$$

in particular χ_1 is 1 on the support of χ . Let $b(x, \xi') = \chi_1(x, \xi')(1 - R(x, \xi'))^{1/2}$, observe that $1 - R(x, \xi') > 0$ on the support of χ_1 . We denote $w_h = \operatorname{op}_{sc}(\chi)v_h$, where χ , is defined by (35).

Lemma 3.4. *There exist $C_\varepsilon > 0$, q_3^h and q_4^h such that*

$$\begin{aligned} (hD_{x_d} - \operatorname{op}_{sc}(b))(hD_{x_d} + \operatorname{op}_{sc}(b))w_h &= hq_3^h \\ (hD_{x_d} + \operatorname{op}_{sc}(b))(hD_{x_d} - \operatorname{op}_{sc}(b))w_h &= hq_4^h, \end{aligned}$$

where $\|q_j^h\|_{L^2(\Omega)} \leq C_\varepsilon$ for $j = 3, 4$.

As the semi-norms of χ, χ_1 depend on ε the estimates depend on ε and to keep in mind that, we denote these constants by C_ε .

Proof. Let $k = 1, 2$. We have

$$\begin{aligned} (hD_{x_d} - (-1)^k \operatorname{op}_{sc}(b))(hD_{x_d} + (-1)^k \operatorname{op}_{sc}(b)) &= h^2 D_{x_d}^2 + (-1)^k hD_{x_d} \operatorname{op}_{sc}(b) \\ &\quad - (-1)^k \operatorname{op}_{sc}(b) hD_{x_d} - \operatorname{op}_{sc}(b)^2 \\ &= h^2 D_{x_d}^2 + \operatorname{op}_{sc}(\chi_1^2(R(x, \xi') - 1)) + \pm h \operatorname{op}_{sc}(r_1), \end{aligned}$$

as $\operatorname{op}_{sc}(b)^2 = \operatorname{op}_{sc}(b^2) + h \operatorname{op}_{sc}(\tilde{r}_1)$ and exact symbol calculus, we have $r_1 = (-1)^k D_{x_d} b + \tilde{r}_1$. As χ_1 is compactly supported, we have $r_1, \tilde{r}_1 \in S_{\tan}^0$. Then $\|\operatorname{op}_{sc}(r_1)w_h\|_{L^2(\Omega)} \leq C_\varepsilon \|\operatorname{op}_{sc}(\chi)v_h\|_{L^2(\Omega)} \leq C_\varepsilon$. With (37), this proves the lemma. \square

Let $z_k^h = (hD_{x_d} - (-1)^k \operatorname{op}_{sc}(b))w_h$, by Lemma 3.4, we have, for $k = 1, 2$,

$$(hD_{x_d} + (-1)^k \operatorname{op}_{sc}(b))z_k^h = h\tilde{q}_k^h,$$

where $\tilde{q}_1^h = q_3^h$ and $\tilde{q}_2^h = q_4^h$. By Proposition 2.2 and Lemma 3.4, we have $\|z_k^h\|_{L^2(\Omega)} \leq C_\varepsilon$ and $\|\tilde{q}_k^h\|_{L^2(\Omega)} \leq C_\varepsilon$ for $k = 1, 2$.

Lemma 3.5. *There exists $C_\varepsilon > 0$ such that $|(z_k^h)|_{x_d=0}|_{L^2} \leq C_\varepsilon$.*

Proof. We have

$$\begin{aligned} h\partial_{x_d} \int_{\mathbb{R}^{d-1}} |z_k^h(x', x_d)|^2 dx' &= 2 \operatorname{Re} \int_{\mathbb{R}^{d-1}} ihD_{x_d}(z_k^h(x', x_d)) \overline{z_k^h(x', x_d)} dx' \\ &= 2 \operatorname{Re} \int_{\mathbb{R}^{d-1}} i(-1)^{k+1} \operatorname{op}_{sc}(b)(z_k^h(x', x_d)) \overline{z_k^h(x', x_d)} dx' \\ &\quad + 2 \operatorname{Re} \int_{\mathbb{R}^{d-1}} ih\tilde{q}_k^h \overline{z_k^h(x', x_d)} dx'. \end{aligned}$$

Integrating with respect to x_d from 0 to ∞ , we obtain

$$h \int_{\mathbb{R}^{d-1}} |z_k^h(x', 0)|^2 dx' \lesssim |\operatorname{Re}(i \operatorname{op}_{sc}(b) z_k^h|_{z_k^h}|_{L^2(\Omega)})| + h \|\tilde{q}_k^h\|_{L^2(\Omega)} \|z_k^h\|_{L^2(\Omega)}. \quad (38)$$

As $(i \operatorname{op}_{sc}(b) z_k^h|_{z_k^h})_{L^2(\Omega)} = (z_k^h|_{z_k^h} - i \operatorname{op}_{sc}(b)^* z_k^h)_{L^2(\Omega)}$ and $\operatorname{op}_{sc}(b)^* = \operatorname{op}_{sc}(b) + h \operatorname{op}_{sc}(r_0)$ where $r_0 \in S_{\tan}^0$, we have $|\operatorname{Re}(i \operatorname{op}_{sc}(b) z_k^h|_{z_k^h})_{L^2(\Omega)}| \leq C_\varepsilon h \|z_k^h\|_{L^2(\Omega)}^2$. Then (38) implies Lemma 3.5. \square

The definition of z_k^h , implies

$$\begin{aligned} (hD_{x_d} w_h)|_{x_d=0} + \operatorname{op}_{sc}(b_0)(w_h)|_{x_d=0} &= (z_1^h)|_{x_d=0} \\ (hD_{x_d} w_h)|_{x_d=0} - \operatorname{op}_{sc}(b_0)(w_h)|_{x_d=0} &= (z_2^h)|_{x_d=0}, \end{aligned} \quad (39)$$

where $b_0 = b|_{x_d=0}$. Then we have

$$2(hD_{x_d} w_h)|_{x_d=0} = (z_1^h)|_{x_d=0} + (z_2^h)|_{x_d=0}.$$

As

$$(hD_{x_d} \operatorname{op}_{sc}(\chi) v_h)|_{x_d=0} = (\operatorname{op}_{sc}(\chi)|_{x_d=0} (hD_{x_d} v_h)|_{x_d=0} + h \operatorname{op}_{sc}(D_{x_d} \chi)|_{x_d=0} (v_h)|_{x_d=0}),$$

and from (31) we deduce that $|(hD_{x_d} w_h)|_{x_d=0}|_{L^2} \leq C_\varepsilon$ by Lemma 3.5. This gives the second estimate of (36).

From (39) we also have

$$|\operatorname{op}_{sc}(b_0)(w_h)|_{x_d=0}|_{L^2} \leq C_\varepsilon. \quad (40)$$

Let $\chi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, be such that $(\chi_2)|_{x_d=0} = 1$, on the support of $\chi|_{x_d=0}$ and $\operatorname{supp} \chi_2 \subset \{\chi_1 = 1\}$. By symbol calculus we have

$$\operatorname{op}_{sc}(\chi_2(1 - R(x, \xi'))|_{x_d=0})^{-1/2} \operatorname{op}_{sc}(b_0) = \operatorname{op}_{sc}(\chi_2)|_{x_d=0} + h \operatorname{op}_{sc}(r_0),$$

where $r_0 \in S_{\tan}^{-1}$, and

$$\operatorname{op}_{sc}(\chi_2)|_{x_d=0} \operatorname{op}_{sc}(\chi|_{x_d=0}) = \operatorname{op}_{sc}(\chi|_{x_d=0}) + h \operatorname{op}_{sc}(r_1),$$

where $r_1 \in S_{\tan}^{-1}$. We can write

$$\begin{aligned} (w_h)|_{x_d=0} &= \operatorname{op}_{sc}(\chi)|_{x_d=0} (v_h)|_{x_d=0} = \operatorname{op}_{sc}(\chi_2)|_{x_d=0} (w_h)|_{x_d=0} - h \operatorname{op}_{sc}(r_1)(w_h)|_{x_d=0} \\ &= \operatorname{op}_{sc}(\chi_2(1 - R(x, \xi'))|_{x_d=0})^{-1/2} \operatorname{op}_{sc}(b_0)(w_h)|_{x_d=0} - h \operatorname{op}_{sc}(r_0)(w_h)|_{x_d=0} \\ &\quad - h \operatorname{op}_{sc}(r_1)(w_h)|_{x_d=0}. \end{aligned}$$

As χ_2 is compactly supported, $\operatorname{op}_{sc}(\langle \xi' \rangle) \operatorname{op}_{sc}(\chi_2(1 - R(x, \xi'))|_{x_d=0})^{-1/2}$ has a symbol in S_{\tan}^0 and $\operatorname{op}_{sc}(\langle \xi' \rangle) \operatorname{op}_{sc}(r_k)$, $k = 0, 1$ have symbols in S_{\tan}^0 , then

$$|\operatorname{op}_{sc}(\chi|_{x_d=0})(v_h)|_{x_d=0}|_{H_{sc}^1} \leq C_\varepsilon |\operatorname{op}_{sc}(b_0)(w_h)|_{x_d=0}|_{L^2} + h |\operatorname{op}_{sc}(\chi|_{x_d=0})(v_h)|_{x_d=0}|_{L^2} \leq C_\varepsilon,$$

applying (40) and (31). This achieves the proof of Proposition 3.2. \square

3.1.3 Glancing points

Let $\chi = \chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ be such that

$$\chi(x, \xi') = \begin{cases} 1 & \text{if } |R(x, \xi') - 1| \leq 2\varepsilon \text{ and } x_d \leq 2\delta \\ 0 & \text{if } |R(x, \xi') - 1| \geq 3\varepsilon \text{ or } x_d \geq 3\delta, \end{cases} \quad (41)$$

and $0 \leq \chi \leq 1$. Observe that $\chi \in S_{\tan}^{-\infty}$ as supported for ξ' in a compact set, as $R(x, \xi')$ is bounded. Let $w_h = \text{op}_{sc}(\chi)v_h$. We have the following estimate on the traces on w_h .

Proposition 3.6. *There exists $C > 0$ such that, for any $\varepsilon > 0$,*

$$|\text{op}_{sc}(\chi|_{x_d=0})(v_h)|_{x_d=0}|_{H_{sc}^1} \leq C\varepsilon^{1/4}h^{-1/2} + C_\varepsilon h^{-3/8} \quad (42)$$

$$|\text{op}_{sc}(\chi|_{x_d=0})(hD_{x_d}v_h)|_{x_d=0}|_{L^2} \leq C\varepsilon^{3/4}h^{-1/2} + C_\varepsilon h^{-3/8}, \quad (43)$$

for $C_\varepsilon > 0$.

Remark 6. *Compared with the estimates stated in Proposition 2.6, we have the same power of h but with a power of ε in front of $h^{-1/2}$. The term $C_\varepsilon h^{-3/8} = C_\varepsilon h^{-1/2}h^{1/4}$ is a remainder. This gives a gain in this microlocal region.*

Proof. In this proof C is a constant independent of ε and we denote by C_ε a constant depending on ε . Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ be such that

$$\chi_1(x, \xi') = \begin{cases} 1 & \text{if } |R(x, \xi') - 1| \leq \varepsilon \text{ and } x_d \leq \delta \\ 0 & \text{if } |R(x, \xi') - 1| \geq -\varepsilon/2 \text{ or } x_d \geq \delta, \end{cases}$$

and $0 \leq \chi_1 \leq 1$. The symbol $\chi_1 \in S_{\tan}^{-\infty}$ since it is supported in $|\xi'| \leq 2$. We have by Lemma 3.3

$$(h^2 D_{x_d}^2 + \text{op}_{sc}(\chi_1^2(R(x, \xi') - 1)))w_h = hq_2^h, \text{ where } \|q_2^h\|_{L^2(\Omega)} \leq C_\varepsilon,$$

for some $C_\varepsilon > 0$.

Then we have

$$\int_{\mathbb{R}_+^d} |h^2 D_{x_d}^2 w_h|^2 dx \leq 2 \int_{\mathbb{R}_+^d} |\text{op}_{sc}(\chi_1^2(R(x, \xi') - 1))w_h|^2 dx + 2 \int_{\mathbb{R}_+^d} |hq_2^h|^2 dx. \quad (44)$$

First, by symbol calculus, we have

$$\begin{aligned} \text{op}_{sc}(\chi_1^2(R(x, \xi') - 1))w_h &= \text{op}_{sc}(\chi_1^2(R(x, \xi') - 1))\text{op}_{sc}(\chi)v_h \\ &= \text{op}_{sc}(\chi(R(x, \xi') - 1))v_h + h\text{op}_{sc}(r_0)v_h, \end{aligned}$$

where $r_0 \in S_{\tan}^0$. Observe that the semi-norms of r_0 depend on ε . This gives

$$\|\text{op}_{sc}(\chi_1^2(R(x, \xi') - 1))w_h\|_{L^2(\Omega)} \leq \|\text{op}_{sc}(\chi(R(x, \xi') - 1))v_h\|_{L^2(\Omega)} + C_\varepsilon h\|v_h\|_{L^2(\Omega)}. \quad (45)$$

On the support of χ , we have $|R(x, \xi') - 1| \leq 2\varepsilon$ then $4\varepsilon^2 - \chi^2(x, \xi')(R(x, \xi') - 1)^2 \geq 0$ and $\chi^2(x, \xi')(R(x, \xi') - 1)^2 \in S_{\tan}^0$ as χ is compactly supported. By Gårding inequality (10) and by symbol calculus, we have

$$4\varepsilon^2\|v_h\|_{L^2(x_d>0)}^2 - \|\text{op}_{sc}(\chi(R(x, \xi') - 1))v_h\|_{L^2(x_d>0)}^2 \geq -C_\varepsilon h\|v_h\|_{L^2(x_d>0)}^2.$$

We deduce from this equation, (44) and (45)

$$\|h^2 D_{x_d}^2 w_h\|_{L^2(x_d>0)} \leq (C\varepsilon + C_\varepsilon h^{1/2})\|v_h\|_{L^2(x_d>0)} + Ch\|q_2^h\|_{L^2(x_d>0)}.$$

By estimates on v_h and q_2^h , we obtain

$$\|h^2 D_{x_d}^2 w_h\|_{L^2(x_d>0)} \leq C\varepsilon + C_\varepsilon h^{1/2}. \quad (46)$$

Lemma 3.7. *Let $g \in L^2(x_d > 0)$ be supported in $\mathbb{R}^{d-1} \times [0, 1]$. We assume $D_{x_d}g \in L^2(x_d > 0)$ then*

$$h|g|_{x_d=0}|_{L^2}^2 \leq 2\|hD_{x_d}g\|_{L^2(x_d>0)}\|g\|_{L^2(x_d>0)}. \quad (47)$$

Proof. Since

$$\begin{aligned} h \int_{\mathbb{R}^{d-1}} |g(x', 0)|^2 dx' &= -i \int_{\mathbb{R}^{d-1}} \int_0^\infty hD_{x_d} |g(x', x_d)|^2 dx' dx_d \\ &= -2i \operatorname{Re} \int_{\mathbb{R}^{d-1}} \int_0^\infty hD_{x_d} g(x', x_d) \overline{g(x', x_d)} dx' dx_d, \end{aligned}$$

we obtain the lemma by Cauchy-Schwarz inequality. \square

Lemma 3.8. *There exists $C > 0$, such that for all $g \in L^2(x_d > 0)$ be supported in $\mathbb{R}^{d-1} \times [0, 1]$. Moreover we assume $D_{x_d}g \in L^2(x_d > 0)$ and $D_{x_d}^2g \in L^2(x_d > 0)$, we have*

$$\|hD_{x_d}g\|_{L^2(x_d>0)}^2 \leq C\|h^2D_{x_d}^2g\|_{L^2(x_d>0)}\|g\|_{L^2(x_d>0)} \quad (48)$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \int_0^\infty |hD_{x_d}g(x', x_d)|^2 dx' dx_d &= \int_{\mathbb{R}^{d-1}} \int_0^\infty hD_{x_d} (hD_{x_d}g(x', x_d) \overline{g(x', x_d)}) dx' dx_d \\ &\quad - \int_{\mathbb{R}^{d-1}} \int_0^\infty h^2D_{x_d}^2g(x', x_d) \overline{g(x', x_d)} dx' dx_d, \end{aligned}$$

we yields

$$\|hD_{x_d}g\|_{L^2(x_d>0)}^2 \leq h\|hD_{x_d}g(x', 0)\|_{L^2}\|g(x', 0)\|_{L^2} + \|h^2D_{x_d}^2g\|_{L^2(x_d>0)}\|g\|_{L^2(x_d>0)}. \quad (49)$$

As $g \in H^2(x_d > 0)$, we can apply Lemma 3.7 to $hD_{x_d}g$ to obtain

$$h\|(hD_{x_d}g)|_{x_d=0}\|_{L^2}^2 \leq 2\|h^2D_{x_d}^2g\|_{L^2(x_d>0)}\|hD_{x_d}g\|_{L^2(x_d>0)}. \quad (50)$$

This estimate and (47) yield

$$h|g|_{x_d=0}|_{L^2}\|(hD_{x_d}g)|_{x_d=0}\|_{L^2} \leq 2\|h^2D_{x_d}^2g\|_{L^2(x_d>0)}^{1/2}\|hD_{x_d}g\|_{L^2(x_d>0)}\|g\|_{L^2(x_d>0)}^{1/2}.$$

From this estimate and (49) we obtain (48). \square

Lemma 3.9. *There exists $C > 0$ such that, for any $g \in H^2(x_d > 0)$ supported in $\mathbb{R}^{d-1} \times [0, 1]$, we have*

$$h|g|_{x_d=0}|_{L^2}^2 \leq C\|h^2D_{x_d}^2g\|_{L^2(x_d>0)}^{1/2}\|g\|_{L^2(x_d>0)}^{3/2} \quad (51)$$

$$h\|(hD_{x_d}g)|_{x_d=0}\|_{L^2}^2 \leq C\|h^2D_{x_d}^2g\|_{L^2(x_d>0)}^{3/2}\|g\|_{L^2(x_d>0)}^{1/2} \quad (52)$$

Proof. The first estimate is obtained from (47) and (48). The second estimate is obtained from (50) and (48). \square

Before applying the previous lemma to w_h , we have to estimate uniformly this function. As χ depend on ε the norm of $\operatorname{op}_{sc}(\chi)$ as an operator on L^2 depends on ε . Nevertheless applying (11) a consequence of Gårding inequality and as $|\chi| \leq 1$ we have

$$\|w_h\|_{L^2(x_d>0)} = \|\operatorname{op}_{sc}(\chi)v_h\|_{L^2(x_d>0)} \leq C\|v_h\|_{L^2(x_d>0)} + C_\varepsilon h^{1/2}\|v_h\|_{L^2(x_d>0)} \leq C + C_\varepsilon h^{1/2}.$$

From (46) and (51), we moreover have

$$h|(w_h)|_{x_d=0}|_{L^2}^2 \leq (C\varepsilon + C_\varepsilon h^{1/2})^{1/2}(C + C_\varepsilon h^{1/2})^{3/2} \leq C\varepsilon^{1/2} + C_\varepsilon h^{1/4},$$

which gives (42).

From (46) and (52), we also have

$$h\|(hD_{x_d}w_h)|_{x_d=0}\|_{L^2}^2 \leq (C\varepsilon + C_\varepsilon h^{1/2})^{3/2}(C + C_\varepsilon h^{1/2})^{1/2} \leq C\varepsilon^{3/2} + C_\varepsilon h^{1/4},$$

which gives (43) as

$$(hD_{x_d} \operatorname{op}_{sc}(\chi)v_h)|_{x_d=0} = \operatorname{op}_{sc}(\chi|_{x_d=0})(hD_{x_d}v_h)|_{x_d=0} + h \operatorname{op}_{sc}(hD_{x_d}\chi)|_{x_d=0}(v_h)|_{x_d=0},$$

and the last term can be estimated by (31). \square

3.1.4 Elliptic points

We start with the notation introduced in Proposition 2.2. Let $\chi = \chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{2d-1})$, be such that

$$\chi(x, \xi') = \begin{cases} 1 & \text{if } R(x, \xi') - 1 \geq 2\varepsilon \text{ and } x_d \leq \delta \\ 0 & \text{if } R(x, \xi') - 1 \leq \varepsilon \text{ or } x_d \geq 2\delta, \end{cases} \quad (53)$$

and $0 \leq \chi \leq 1$. We have $\chi \in S_{\text{tan}}^0$. In this region the support of χ is not bounded, we have to take care of the symbol classes we use.

Let $\rho(x, \xi') = (R(x, \xi') - 1)^{1/2}$ if (x, ξ') satisfies $R(x, \xi') - 1 > 0$. Let $\chi_1(x, \xi') \in S_{\text{tan}}^0$ be such that, $\chi_1 = 1$ on the support of χ , and $\text{supp } \chi_1 \subset \{R(x, \xi') - 1 \geq \varepsilon/2\} \cup \{x_d \leq 3\delta\}$. Observe that $\chi_1 \rho \in S_{\text{tan}}^1$. We have

$$\begin{aligned} (hD_{x_d} + i \text{op}_{\text{sc}}(\chi_1 \rho))(hD_{x_d} - i \text{op}_{\text{sc}}(\chi_1 \rho)) &= h^2 D_{x_d}^2 - ihD_{x_d} \text{op}_{\text{sc}}(\chi_1 \rho) + i \text{op}_{\text{sc}}(\chi_1 \rho) hD_{x_d} \\ &\quad + \text{op}_{\text{sc}}(\chi_1 \rho)^2 \\ &= h^2 D_{x_d}^2 - i[hD_{x_d}, \text{op}_{\text{sc}}(\chi_1 \rho)] + \text{op}_{\text{sc}}(\chi_1 \rho)^2, \end{aligned}$$

and

$$\begin{aligned} [hD_{x_d}, \text{op}_{\text{sc}}(\chi_1 \rho)] &= h \text{op}_{\text{sc}}(D_{x_d}(\chi_1 \rho)), \text{ where } D_{x_d}(\chi_1 \rho) \in S_{\text{tan}}^1, \\ \text{op}_{\text{sc}}(\chi_1 \rho)^2 &= \text{op}_{\text{sc}}(\chi_1^2(R(x, \xi') - 1)) + h \text{op}_{\text{sc}}(r_1), \text{ where } r_1 \in S_{\text{tan}}^1. \end{aligned}$$

Then

$$(hD_{x_d} + i \text{op}_{\text{sc}}(\chi_1 \rho))(hD_{x_d} - i \text{op}_{\text{sc}}(\chi_1 \rho)) = h^2 D_{x_d}^2 + \text{op}_{\text{sc}}(\chi_1^2(R(x, \xi') - 1)) + h \text{op}_{\text{sc}}(\tilde{r}_1),$$

where $\tilde{r}_1 \in S_{\text{tan}}^1$.

Applying Lemma 3.3 we obtain

$$(hD_{x_d} + i \text{op}_{\text{sc}}(\chi_1 \rho))(hD_{x_d} - i \text{op}_{\text{sc}}(\chi_1 \rho)) \text{op}_{\text{sc}}(\chi) v_h = h q_2^h, \quad (54)$$

where $\|q_2^h\|_{L^2(x_d > 0)} \leq C_\varepsilon$.

Let $z = (hD_{x_d} - i \text{op}_{\text{sc}}(\chi_1 \rho)) \text{op}_{\text{sc}}(\chi) v_h$; z depends on h but for the sake of simplicity we prefer to denote it z . From (54) we have

$$(hD_{x_d} + i \text{op}_{\text{sc}}(\chi_1 \rho))z = h q_2^h,$$

in $x_d > 0$. We then have

$$2 \text{Re}((hD_{x_d} + i \text{op}_{\text{sc}}(\chi_1 \rho))z | i \text{op}_{\text{sc}}(\chi_1 \rho) z)_{L^2(x_d > 0)} \lesssim h \|q_2^h\|_{L^2(x_d > 0)} \| \text{op}_{\text{sc}}(\chi_1 \rho) z \|_{L^2(x_d > 0)}. \quad (55)$$

We recall the integration by parts formula in semiclassical context,

$$(u | hD_{x_d} w)_{L^2(x_d > 0)} = (hD_{x_d} u | w)_{L^2(x_d > 0)} - ih(u|_{x_d=0} | w|_{x_d=0})_0, \quad (56)$$

for u and w sufficiently smooth. Taking $w = i \text{op}_{\text{sc}}(\chi_1 \rho) z$ and $u = z$ we have

$$(z | ihD_{x_d} \text{op}_{\text{sc}}(\chi_1 \rho) z)_{L^2(x_d > 0)} = (hD_{x_d} z | i \text{op}_{\text{sc}}(\chi_1 \rho) z)_{L^2(x_d > 0)} - ih(z|_{x_d=0} | i \text{op}_{\text{sc}}(\chi_1 \rho) z|_{x_d=0})_0.$$

As $ihD_{x_d} \text{op}_{\text{sc}}(\chi_1 \rho) = i \text{op}_{\text{sc}}(\chi_1 \rho) hD_{x_d} + h \text{op}_{\text{sc}}(\partial_{x_d}(\chi_1 \rho))$, and $\text{op}_{\text{sc}}(\chi_1 \rho) = \text{op}_{\text{sc}}(\chi_1 \rho)^* + h \text{op}_{\text{sc}}(r_0)$, where $r_0 \in S_{\text{tan}}^0$ we obtain

$$2 \text{Re}(hD_{x_d} z | i \text{op}_{\text{sc}}(\chi_1 \rho) z)_{L^2(x_d > 0)} = h \text{Re}(z|_{x_d=0} | \text{op}_{\text{sc}}(\chi_1 \rho) z|_{x_d=0})_0 + hK, \quad (57)$$

where

$$|K| \lesssim \|hDz\|_{L^2(x_d > 0)} \|z\|_{L^2(x_d > 0)} + \|z\|_{L^2(x_d > 0)}^2.$$

From equation satisfied by z , we have

$$\begin{aligned} \|hD_{x_d}z\|_{L^2(x_d>0)} &\lesssim \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} + h\|q_2^h\|_{L^2(x_d>0)} \\ &\lesssim \|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} + h\|q_2^h\|_{L^2(x_d>0)}. \end{aligned}$$

Then

$$|K| \lesssim \|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)}\|z\|_{L^2(x_d>0)} + h^2\|q_2^h\|_{L^2(x_d>0)}^2. \quad (58)$$

From (55), (57) and (58) we yield

$$\begin{aligned} \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)}^2 + h\text{Re}(z|_{x_d=0}|\text{op}_{sc}(\chi_1\rho)z|_{x_d=0})_0 \\ \lesssim h\|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)}\|z\|_{L^2(x_d>0)} + h^2\|q_2^h\|_{L^2(x_d>0)}^2, \end{aligned} \quad (59)$$

as we can estimate

$$h\|q_2^h\|_{L^2(x_d>0)}\|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} \leq \alpha\|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)}^2 + C_\alpha h^2\|q_2^h\|_{L^2(x_d>0)}^2,$$

and absorb the term $\|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)}^2$, by the left hand side of (59) if α is sufficiently small.

Lemma 3.10.

$$\|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} \lesssim \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} + h\|hDv_h\|_{L^2(x_d>0)} + h\|v_h\|_{L^2(x_d>0)}.$$

Proof. Let $\chi_2(x, \xi') \in S_{\text{tan}}^0$ be such that, $\chi_2 = 1$ on the support of χ_1 , and $\text{supp } \chi_1 \subset \{R(x, \xi') - 1 \geq \varepsilon/4\} \cup \{x_d \leq 4\delta\}$. We have by symbol calculus

$$\text{op}_{sc}(\chi_2\rho^{-1})\text{op}_{sc}(\chi_1\rho) = \text{op}_{sc}(\chi_1) + h\text{op}_{sc}(r_1),$$

where $r_1 \in S_{\text{tan}}^{-1}$. Thus we obtain

$$\|\text{op}_{sc}(\langle\xi'\rangle)\text{op}_{sc}(\chi_1)z\|_{L^2(x_d>0)} \lesssim \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} + h\|z\|_{L^2(x_d>0)}.$$

From this estimate we obtain

$$\begin{aligned} \|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} &\lesssim \|\text{op}_{sc}(\langle\xi'\rangle)\text{op}_{sc}(1 - \chi_1)z\|_{L^2(x_d>0)} \\ &\quad + \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} + h\|z\|_{L^2(x_d>0)}. \end{aligned} \quad (60)$$

From definition of z , we can write $z = \text{op}_{sc}(\chi)(hD_{x_d} + \text{op}_{sc}(r_1'))v_h$, where $r_1' \in S_{\text{tan}}^1$. From symbol calculus and support properties of χ and χ_1 the operator $\text{op}_{sc}(\langle\xi'\rangle)\text{op}_{sc}(1 - \chi_1)\text{op}_{sc}(\chi)$ is bounded on L^2 by Ch . From (60) we thus have

$$\|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} \lesssim h\|hDv_h\|_{L^2(x_d>0)} + h\|v_h\|_{L^2(x_d>0)} + \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)} + h\|z\|_{L^2(x_d>0)}.$$

We obtain the statement as we can absorb $h\|z\|_{L^2(x_d>0)}$ by the left hand side. \square

From (59) and Lemma 3.10 we deduce

$$\begin{aligned} \|\text{op}_{sc}(\chi_1\rho)z\|_{L^2(x_d>0)}^2 + h\text{Re}(z|_{x_d=0}|\text{op}_{sc}(\chi_1\rho)z|_{x_d=0})_0 \\ \lesssim h^2\|hDv_h\|_{L^2(x_d>0)}^2 + h^2\|v_h\|_{L^2(x_d>0)}^2 + h^2\|q_2^h\|_{L^2(x_d>0)}^2, \end{aligned}$$

as $\|z\|_{L^2(x_d>0)} \leq \|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)}$.

From Lemma 3.10 we obtain

$$\|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} \lesssim h\|hDv_h\|_{L^2(x_d>0)} + h\|v_h\|_{L^2(x_d>0)} + h\|q_2^h\|_{L^2(x_d>0)}. \quad (61)$$

From equation satisfied by z we have

$$\|hD_{x_d}z\|_{L^2(x_d>0)} \lesssim \|\text{op}_{sc}(\langle\xi'\rangle)z\|_{L^2(x_d>0)} + h\|q_2^h\|_{L^2(x_d>0)}.$$

From this estimate, (61) and trace formula (8), we deduce

$$|z|_{x_d=0}|_{H_{sc}^{1/2}} \leq C_\varepsilon h^{1/2}. \quad (62)$$

From definition of z , we have, for $x_d > 0$ and by symbol calculus,

$$z = \text{op}_{sc}(\chi)hD_{x_d}v_h - i \text{op}_{sc}(\chi\rho)v_h + hz_1,$$

where $z_1 = \text{op}_{sc}(r_0)v_h$ and $r_0 \in S_{\tan}^0$. In particular

$$\|z_1\|_{L^2(x_d>0)} + \|hDz_1\|_{L^2(x_d>0)} \leq C_\varepsilon.$$

Let $u_0 = h(D_{x_d}v_h)|_{x_d=0}$, $u_1 = (v_h)|_{x_d=0}$, $\chi_0 = \chi|_{x_d=0}$ and $\rho_0 = \rho|_{x_d=0}$. From (62), we have

$$\text{op}_{sc}(\chi_0)u_0 - i \text{op}_{sc}(\chi_0\rho_0)u_1 = h^{1/2}z_4, \text{ where } |z_4|_{L^2} \leq C_\varepsilon.$$

Let $\Phi \in \mathcal{C}^\infty(\partial\Omega)$, we have

$$\Phi \text{op}_{sc}(\chi_0) = \text{op}_{sc}(\chi_0)\Phi + h \text{op}_{sc}(r_0) \text{ and } \Phi \text{op}_{sc}(\chi_0\rho_0) = \text{op}_{sc}(\chi_0\rho_0)\Phi + h \text{op}_{sc}(\tilde{r}_0),$$

where $r_0, \tilde{r}_0 \in S_{\tan}^0$ by symbol calculus. From Proposition 2.6, we have

$$\text{op}_{sc}(\chi_0)\Phi u_0 - i \text{op}_{sc}(\chi_0\rho_0)\Phi u_1 = h^{1/2}z_5, \text{ where } |z_5|_{L^2} \leq C_\varepsilon. \quad (63)$$

With this equation we can obtain trace estimates into $\partial\Omega_D$ and $\partial\Omega_N$.

Proposition 3.11. *Let $\Phi \in \mathcal{C}^\infty(\partial\Omega)$.*

If Φ is supported on $\partial\Omega_D$. Then

$$|\text{op}_{sc}(\chi_0)\Phi u_0|_{L^2} \leq C_\varepsilon h^{1/2}.$$

If Φ is supported on $\partial\Omega_N$. Then

$$|\text{op}_{sc}(\chi_0)\Phi u_1|_{H_{sc}^1} \leq C_\varepsilon h^{1/2}.$$

Proof. If Φ is supported on $\partial\Omega_D$, then $\Phi u_1 = 0$, (63) gives the first result. If Φ is supported on $\partial\Omega_N$, then $\Phi u_0 = 0$. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ be such that $\chi_1 = 1$ on the support of χ_0 , $\text{supp } \chi_1 \subset \{R(x', 0, \xi') - 1 \geq \varepsilon/2\}$, $\chi_1 \in S_{\tan}^0$, in particular we have $\chi_0\chi_1 = \chi_0$ and $\rho_0 \neq 0$ on support of χ_1 . We have $\text{op}_{sc}(\xi') \text{op}_{sc}(\chi_1\rho_0^{-1}) \text{op}_{sc}(\chi_0\rho_0) = \text{op}_{sc}(\xi') \text{op}_{sc}(\chi_0) + h \text{op}_{sc}(r_0)$, where $r_0 \in S_{\tan}^0$, by symbol calculus. From Proposition 2.6 and (63), we have

$$\begin{aligned} |\text{op}_{sc}(\chi_0)\Phi u_1|_{H_{sc}^1} &\leq |\text{op}_{sc}(\xi') \text{op}_{sc}(\chi_1\rho_0^{-1}) \text{op}_{sc}(\chi_0\rho_0)\Phi u_1|_{L^2} + h|\text{op}_{sc}(r_0)\Phi u_1|_{L^2} \\ &\leq h^{1/2}|\text{op}_{sc}(\xi') \text{op}_{sc}(\chi_1\rho_0^{-1})z_5|_{L^2} + h|\text{op}_{sc}(r_0)\Phi u_1|_{L^2} \\ &\leq C_\varepsilon h^{1/2}. \end{aligned}$$

This gives the second estimates. \square

Proposition 3.12. *Let $\Phi \in \mathcal{C}^\infty(\partial\Omega)$. We assume that Φ is supported either on $\partial\Omega_D$ or on $\partial\Omega_N$.*

$$\begin{aligned} h^{1/2}|(v_h)|_{x_d=0}|_{H_{sc}^1} &\rightarrow 0 \text{ as } h \rightarrow 0, \\ h^{1/2}|(hD_{x_d}v_h)|_{x_d=0}|_{L^2} &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

Proof. Let $\varepsilon > 0$, we can find χ_H, χ_G and χ_E satisfying respectively the assumption of Propositions 3.2, 3.6, 3.11 and furthermore the relation $\chi_H + \chi_G + \chi_E = 1$. Applying the results of Propositions in each region, we deduce the proposition. \square

In what follows we shall prove estimates in a neighborhood of $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$. The result is more delicate and less classical than these obtained for Dirichlet or Neumann boundary condition.

Now we assume that Φ is supported in a neighborhood of a point of $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$. We can assume that locally this set is given by $x_1 = 0$ and the support of Φ is contained into a fixed domain in x' and into $\{|x_1| \leq \mu\varepsilon/2\}$ where $\mu > 0$ will be fixed below sufficiently small. Here and in what follows ε is the one used to define hyperbolic, glancing, elliptic regions (see respectively (35), (41) and (53)). We assume that $\text{supp } u_0 \subset \{x_1 \leq 0\}$ and $\text{supp } u_1 \subset \{x_1 \geq 0\}$. We can choose the local coordinates such that

$$R(x', 0, \xi') = \xi_1^2 + R_0(x'', \xi'') + x_1 r_2(x', \xi'), \text{ where } x' = (x_1, x'') \text{ and } \xi' = (\xi_1, \xi''), \quad (64)$$

$R_0 \in S(\langle \xi'' \rangle^2, (dx'')^2 + \langle \xi'' \rangle^{-2} (d\xi'')^2)$ and $r_2 \in S_{\text{tan}}^2$. Indeed in normal geodesic coordinates we have $R(x', 0, \xi') = \xi_1^2 + R_1(x', \xi'') = \xi_1^2 + R_0(x'', \xi'') + x_1 r_2(x', \xi')$ and in fact $r_2(x', \xi')$ does not depend on ξ_1 but we do not use this property in what follows.

Let $\alpha(x'', \xi') = (\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2}$, and let

$$\beta(x'', \xi'') = (R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2}, \quad (65)$$

be such that $\text{Im } \beta(x'', \xi'') > 0$, for all $(x'', \xi'') \in \mathbb{R}^{d-2} \times \mathbb{R}^{d-2}$. We have

$$\alpha(x'', \xi') = (\xi_1 + i\beta(x'', \xi''))^{1/2} (\xi_1 - i\beta(x'', \xi''))^{1/2},$$

Observe that $\text{Re } \beta(x'', \xi'') > 0$, we deduce that

$$\begin{aligned} \xi_1 &\mapsto (\xi_1 + i\beta(x'', \xi''))^{\pm 1/2} \text{ are holomorphic functions in } \{\text{Im } \xi_1 > 0\}, \\ \xi_1 &\mapsto (\xi_1 - i\beta(x'', \xi''))^{\pm 1/2} \text{ are holomorphic functions on } \{\text{Im } \xi_1 < 0\}. \end{aligned}$$

Let $v_1 = \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2} \Phi u_1$. The operator $\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2}$ is a convolution operator with respect x_1 and its kernel is supported in $x_1 \geq 0$. As u_1 is supported in $x_1 \geq 0$, this implies that v_1 is supported in $x_1 \geq 0$. Let $v_0 = \text{op}_{sc}(\xi_1 + i\beta(x'', \xi''))^{-1/2} \Phi u_0$. As u_0 is supported in $x_1 \leq 0$ and $(\xi_1 + i\beta(x'', \xi''))^{-1/2}$ is a holomorphic function in $\{\text{Im } \xi_1 > 0\}$, v_0 is supported in $x_1 \leq 0$. We first prove the following lemma.

Lemma 3.13. *There exists $C > 0$, such that for all $\varepsilon \in (0, 1)$ we have, for every $h \in (0, 1)$*

$$\begin{aligned} |v_1|_{L^2} &\leq C\varepsilon^{1/5} h^{-1/2} + C_\varepsilon h^{-3/8}, \\ |v_0|_{L^2} &\leq C\varepsilon^{1/5} h^{-1/2} + C_\varepsilon h^{-3/8}, \end{aligned}$$

where $C_\varepsilon > 0$ depends on ε .

Proof. Observe that $(\xi_1 - i\beta(x'', \xi''))^{\pm 1/2} \in S(\langle \xi' \rangle^{\pm 1/2}, (dx')^2 + (d\xi')^2)$ then

$$\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2} = Id + h \text{op}_{sc}(r_0),$$

where $r_0 \in S(1, (dx')^2 + (d\xi')^2)$. From the definition of v_1 we have

$$\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1 = \Phi u_1 + h \text{op}_{sc}(r_0) u_1. \quad (66)$$

As $\text{op}_{sc}(\chi_0 \rho_0) \text{op}_{sc}(r_0)$ has a symbol in $S(\langle \xi' \rangle, (dx')^2 + (d\xi')^2)$, we obtain

$$|\text{op}_{sc}(\chi_0 \rho_0) \text{op}_{sc}(r_0) v_1|_{H_{sc}^{-1/2}} \leq C_\varepsilon |u_1|_{H_{sc}^{1/2}} \leq C_\varepsilon h^{-1/2}$$

by Proposition 2.6. Then from (63) we obtain

$$\text{op}_{sc}(\chi_0) \Phi u_0 - i \text{op}_{sc}(\chi_0 \rho_0) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1 = h^{1/2} z_6, \text{ where } |z_6|_{H_{sc}^{-1/2}} \leq C_\varepsilon. \quad (67)$$

To determine u_0 and u_1 we have to use the support properties of these functions. To do that, we have to modify the operators acting on these functions, note that $\text{op}_{sc}(\chi_0\rho_0)$ does *not* preserve the support of u_0 .

We introduce three cutoff functions χ_H , χ_G and χ_E , be such that $\chi_H + \chi_G + \chi_E = 1$ and

$$\begin{aligned} \text{supp } \chi_H &\subset \{R(x', 0, \xi') - 1 \leq -\varepsilon/2\}, \quad 0 \leq \chi_H \leq 1 \\ \text{supp } \chi_G &\subset \{|R(x', 0, \xi') - 1| \leq \varepsilon\}, \quad 0 \leq \chi_G \leq 1 \\ \text{supp } \chi_E &\subset \{R(x', 0, \xi') - 1 \geq \varepsilon/2\}, \quad 0 \leq \chi_E \leq 1. \end{aligned}$$

We then have χ_H , χ_G and χ_E , the $\chi|_{x_d=0}$, defined respectively in the hyperbolic, glancing and elliptic regions (after multiplying ε by a fix factor). We have $\Phi u_0 = \text{op}_{sc}(\chi_H)\Phi u_0 + \text{op}_{sc}(\chi_G)\Phi u_0 + \text{op}_{sc}(\chi_E)\Phi u_0$. From hyperbolic estimate given in Proposition 3.2, Proposition 2.6 and symbol calculus, we have

$$|\text{op}_{sc}(\chi_H)(\Phi u_0)|_{L^2} \leq C_\varepsilon.$$

From glancing estimate given in Proposition 3.6, Proposition 2.6 and symbol calculus, we have

$$|\text{op}_{sc}(\chi_G)(\Phi u_0)|_{L^2} \leq C\varepsilon^{3/4}h^{-1/2} + C_\varepsilon h^{-3/8}.$$

We deduce that

$$|\Phi u_0 - \text{op}_{sc}(\chi_E)(\Phi u_0)|_{L^2} \leq C\varepsilon^{3/4}h^{-1/2} + C_\varepsilon h^{-3/8}. \quad (68)$$

Lemma 3.14. *With the previously defined notations, we have*

$$|\text{op}_{sc}(\chi_H\alpha) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \leq C_\varepsilon \quad (69)$$

$$|\text{op}_{sc}(\chi_G\alpha) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \leq C\varepsilon^{3/4}h^{-1/2} + C_\varepsilon h^{-3/8}, \quad (70)$$

where $\alpha(x'', \xi') = (\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2}$.

Proof. Let $\tilde{\chi}_H$ be such that $\tilde{\chi}_H = 1$ on the support of χ_H and $\tilde{\chi}_H = 0$ if $R_0(x', \xi') - 1 \geq -\varepsilon/4$. Let $\tilde{\chi}_G$ be such that $\tilde{\chi}_G = 1$ on the support of χ_G and $\tilde{\chi}_G = 0$ if $|R_0(x', \xi') - 1| \geq 2\varepsilon$. Let $J = G$ or H .

By symbol calculus, we have

$$\text{op}_{sc}(\alpha\chi_J) = \text{op}_{sc}(\alpha\chi_J) \text{op}_{sc}(\tilde{\chi}_J) + h \text{op}_{sc}(r_0),$$

where $r_0 \in S_{\text{tan}}^0$. From (66), we deduce

$$\begin{aligned} &|\text{op}_{sc}(\chi_J\alpha) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \\ &\leq |\text{op}_{sc}(\chi_J\alpha) \text{op}_{sc}(\tilde{\chi}_J) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} + h |\text{op}_{sc}(r_0) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \\ &\leq |\text{op}_{sc}(\chi_J\alpha) \text{op}_{sc}(\tilde{\chi}_J) u_1|_{H_{sc}^{-1/2}} + C_\varepsilon h |u_1|_{H_{sc}^{1/2}}. \end{aligned}$$

If $J = H$ we have

$$|\text{op}_{sc}(\chi_H\alpha) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \leq C_\varepsilon |\text{op}_{sc}(\tilde{\chi}_H) u_1|_{H_{sc}^{1/2}} + C_\varepsilon h |u_1|_{H_{sc}^{1/2}}.$$

we obtain (69) from Propositions 2.6 and 3.2.

If $J = G$, using that $|x_1| \leq \mu\varepsilon/2$, on the support of χ_G , we have $|\alpha| \leq C\varepsilon^{1/2}$. As

$$|\text{op}_{sc}(\chi_J\alpha) \text{op}_{sc}(\tilde{\chi}_J) u_1|_{H_{sc}^{-1/2}} \leq C |\text{op}_{sc}(\chi_J\alpha) \text{op}_{sc}(\tilde{\chi}_J) u_1|_{L^2},$$

we apply Gårding inequality (12) and we obtain

$$|\text{op}_{sc}(\chi_G\alpha) \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H_{sc}^{-1/2}} \leq C\varepsilon^{1/2} |\text{op}_{sc}(\tilde{\chi}_G) u_1|_{H_{sc}^{1/2}} + C_\varepsilon h |u_1|_{H_{sc}^{1/2}},$$

where, at the right hand side, we have estimated the L^2 -norm by the $H^{1/2}$ -norm. We obtain (70) from Propositions 2.6 and 3.6. \square

Following (67) (with $\chi_0 = \chi_E$), (68) and Lemma 3.14, we have

$$\begin{aligned} \tilde{\Phi}u_0 - i \operatorname{op}_{\text{sc}}(\alpha(\chi_H + \chi_G)) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2}v_1 \\ - i \operatorname{op}_{\text{sc}}(\chi_E\rho_0) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2}v_1 = z_7, \end{aligned}$$

where $|z_7|_{H_{\text{sc}}^{-1/2}} \leq C\varepsilon^{3/4}h^{-1/2} + C_\varepsilon h^{-3/8}$. Applying $\operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2}$ to this equation, we obtain

$$\begin{aligned} v_0 - i \operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2} \operatorname{op}_{\text{sc}}(\alpha(\chi_H + \chi_G)) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2}v_1 \\ - i \operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2} \operatorname{op}_{\text{sc}}(\chi_E\rho_0) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2}v_1 \\ = \operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2}z_7. \quad (71) \end{aligned}$$

We have to precisely estimate $z_8 = \operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2}z_7$. By the definition of $\beta(x'', \xi'')$ there exists $C_0 > 0$ such that $\operatorname{Re} \beta(x'', \xi'') \geq C_0\varepsilon$, as $R_0(x'', \xi'') - 1 \geq -1$ moreover if $|\xi''| \geq 2$, then $\operatorname{Re} \beta(x'', \xi'') \geq C_0\langle \xi'' \rangle$. We deduce that $|\xi_1 + i\beta(x'', \xi'')| \geq C_0\varepsilon\langle \xi'' \rangle$. This implies that $|(\xi_1 + i\beta(x'', \xi''))^{-1/2}| \leq C_0\varepsilon^{-1/2}\langle \xi'' \rangle^{-1/2}$, where $C_0 > 0$. We have by symbol calculus

$$|z_8|_{L^2} \leq |\operatorname{op}_{\text{sc}}((\xi_1 + i\beta(x'', \xi''))^{-1/2}\langle \xi'' \rangle^{1/2}) \operatorname{op}_{\text{sc}}(\langle \xi'' \rangle^{-1/2})z_7|_{L^2} + C_\varepsilon h |\operatorname{op}_{\text{sc}}(\langle \xi'' \rangle^{-1/2})z_7|_{L^2},$$

and by Gårding inequality (12) and the estimate on z_7 , we have

$$\begin{aligned} |z_8|_{L^2} &\leq C\varepsilon^{-1/2} |\operatorname{op}_{\text{sc}}(\langle \xi'' \rangle^{-1/2})z_7|_{L^2} + C_\varepsilon h |\operatorname{op}_{\text{sc}}(\langle \xi'' \rangle^{-1/2})z_7|_{L^2}, \\ &\leq C\varepsilon^{1/4}h^{-1/2} + C_\varepsilon h^{-3/8}. \end{aligned}$$

By symbol calculus, as $(\xi_1 + i\beta(x'', \xi''))^{-1/2}\alpha(\xi_1 - i\beta(x'', \xi''))^{-1/2} = 1$, we have

$$\begin{aligned} \operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2} \operatorname{op}_{\text{sc}}(\alpha(\chi_H + \chi_G)) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2} \\ = \operatorname{op}_{\text{sc}}(\chi_H + \chi_G) + h \operatorname{op}_{\text{sc}}(r_0), \quad (72) \end{aligned}$$

where $r_0 \in S(1, (dx')^2 + (d\xi')^2)$.

By the same argument, we have

$$\operatorname{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2} \operatorname{op}_{\text{sc}}(\chi_E\rho_0) \operatorname{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{-1/2} = \operatorname{op}_{\text{sc}}(\chi_E\rho_0\alpha^{-1}) + h \operatorname{op}_{\text{sc}}(r_0), \quad (73)$$

where $r_0 \in S(1, (dx')^2 + (d\xi')^2)$. Indeed $(\xi_1 \pm i\beta(x'', \xi''))^{-1/2} \in S(\langle \xi'' \rangle^{-1/2}, (dx')^2 + (d\xi')^2)$ and $\chi_E\rho_0 \in S(\langle \xi'' \rangle, (dx')^2 + (d\xi')^2)$.

The following lemma gives a precise estimate on $\chi_E(\rho_0\alpha^{-1} - 1)$. We shall exploit that α and ρ_0 are close.

Lemma 3.15. *We have*

$$|\operatorname{op}_{\text{sc}}(\chi_E\rho_0\alpha^{-1})v_1 - \operatorname{op}_{\text{sc}}(\chi_E)v_1|_{L^2} \leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}.$$

Proof. Let $\tilde{\Phi}(x_1)$ supported in $\{|x_1| \leq \mu\varepsilon\}$ and $\tilde{\Phi} = 1$ on the support of Φ . Let b be either the symbol $\chi_E\rho_0\alpha^{-1}$ or χ_E , we have $b \in S(1, (dx')^2 + (d\xi')^2)$.

By symbol calculus we have $\operatorname{op}_{\text{sc}}(b(1 - \tilde{\Phi}))v_1 = h \operatorname{op}(r)u_1$, where $r \in S(\langle \xi'' \rangle^{1/2}, (dx')^2 + (d\xi')^2)$. By Proposition 2.6 we have $|\operatorname{op}_{\text{sc}}(b(1 - \tilde{\Phi}))v_1|_{L^2} \leq C_\varepsilon h |u_1|_{H_{\text{sc}}^{1/2}} \leq C_\varepsilon h^{1/2}$. Then we can considerate $\operatorname{op}_{\text{sc}}(\tilde{\Phi}\chi_E\rho_0\alpha^{-1})$ and $\operatorname{op}_{\text{sc}}(\tilde{\Phi}\chi_E)$, instead of respectively $\operatorname{op}_{\text{sc}}(\chi_E\rho_0\alpha^{-1})$ and $\operatorname{op}_{\text{sc}}(\chi_E)$.

We introduce three cutoff functions $\chi_j \in \mathcal{C}^\infty(\mathbb{R}^{d-2} \times \mathbb{R}^{d-1})$ be such that

$$\begin{aligned} \chi_1 &= \begin{cases} 1 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \leq \varepsilon^{4/5}, \\ 0 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \geq 3\varepsilon^{4/5}, \end{cases} \\ \chi_2 &= \begin{cases} 1 & \text{if } 1 \geq \xi_1^2 + R_0(x'', \xi'') - 1 \geq 2\varepsilon^{4/5}, \\ 0 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \leq \varepsilon^{4/5} \text{ and } |\xi''|^2 - 1 \geq 3, \end{cases} \\ \chi_3 &= \begin{cases} 1 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \geq 2, \\ 0 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \leq 1, \end{cases} \end{aligned}$$

$\chi_1 + \chi_2 + \chi_3 = 1$ and $0 \leq \chi_j \leq 1$ for $j = 1, 2, 3$.

Estimation on the support of $\chi_E \chi_1$.

On the support of $\chi_E \chi_1$ we have $\xi_1^2 + R_0(x'', \xi'') - 1 \leq 3\varepsilon^{4/5}$ and $R(x', 0, \xi') - 1 \geq \varepsilon/2$, in particular $|\xi'|$ is bounded. We compute on this domain

$$\rho_0(x', \xi') - \alpha(x'', \xi'') = D(x', \xi')(x_1 r_2(x', \xi') - i\varepsilon), \quad (74)$$

where $D(x', \xi') = \left((R(x', 0, \xi') - 1)^{1/2} + (\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2} \right)^{-1}$. Observe that we have $\text{Re}(\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2} > 0$ and $(R(x', 0, \xi') - 1)^{1/2} \geq 2^{-1/2}\varepsilon^{1/2}$, then $|D(x', \xi')| \leq C_0\varepsilon^{-1/2}$. We deduce that $|\rho_0 - \alpha| \leq C_0\varepsilon^{1/2}$ if $|x_1| \leq \mu\varepsilon$ where μ was introduced in the definition of Φ . As

$$|\xi_1^2 + R_0(x'', \xi'') - 1 - (R(x', 0, \xi') - 1)| \leq C|x_1| \leq C\mu\varepsilon,$$

if μ is chosen sufficiently small, on the support of χ_E , we have $\xi_1^2 + R_0(x'', \xi'') - 1 \geq C_0\varepsilon$, for $C_0 > 0$. We deduce that $|\alpha(x'', \xi'')| \geq C_1\varepsilon^{1/2}$, for $C_1 > 0$.

This implies that

$$|(\rho_0 - \alpha)\alpha^{-1}| \leq C_2 \text{ on the support of } \chi_E \chi_1, \text{ for } C_2 > 0 \text{ and } |x_1| \leq \mu\varepsilon. \quad (75)$$

Let

$$\tilde{\chi}_1(x', \xi') = \begin{cases} 1 & \text{on the support of } \chi_1, \\ 0 & \text{if } \xi_1^2 + R_0(x'', \xi'') - 1 \geq 4\varepsilon^{4/5} \text{ or } R(x', 0, \xi') - 1 \leq \varepsilon/4. \end{cases}$$

By symbol calculus in classes $S(\langle \xi' \rangle^s, (dx')^2 + (d\xi')^2)$, we have

$$(\text{op}_{sc}(\tilde{\Phi}\chi_E\chi_1\rho_0\alpha^{-1}) - \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_1))v_1 = \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_1(\rho_0 - \alpha)\alpha^{-1})\text{op}_{sc}(\tilde{\chi}_1)v_1 + h\text{op}_{sc}(r_0)v_1,$$

where $r_0 \in S(1, (dx')^2 + (d\xi')^2)$. By Gårding inequality (12) and (75) we have

$$\left| (\text{op}_{sc}(\tilde{\Phi}\chi_E\chi_1\rho_0\alpha^{-1}) - \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_1))v_1 \right|_{L^2} \leq C|\text{op}_{sc}(\tilde{\chi}_1)v_1|_{L^2} + C_\varepsilon h|v_1|_{L^2}. \quad (76)$$

Observe that

$$\text{op}_{sc}(\tilde{\chi}_1)v_1 = \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2}\Phi\text{op}_{sc}(\tilde{\chi}_1)u_1 + h\text{op}_{sc}(r_{1/2})u_1,$$

by semiclassical symbol calculus in $S(\langle \xi' \rangle^s, (dx')^2 + (d\xi')^2)$, where $r_{1/2} \in S(\langle \xi' \rangle^{1/2}, (dx')^2 + (d\xi')^2)$. As $\tilde{\chi}_1$ is supported in $|R(x', 0, \xi') - 1| \leq C\varepsilon^{4/5}$, we can apply Proposition 3.6 with $\varepsilon^{4/5}$ instead of ε . From (76), Proposition 2.6 and as $|v_1|_{L^2} \leq C_\varepsilon|u_1|_{H_{sc}^{1/2}}$, we have

$$\begin{aligned} \left| (\text{op}_{sc}(\chi_E\chi_1\rho_0\alpha^{-1}) - \text{op}_{sc}(\chi_E\chi_1))v_1 \right|_{H_{sc}^{1/2}} &\leq C|\text{op}_{sc}(\tilde{\chi}_1)u_1|_{H_{sc}^1} + C_\varepsilon h|u_1|_{H_{sc}^{1/2}} \\ &\leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}. \end{aligned} \quad (77)$$

Estimation on the support of $\chi_E \chi_2$.

Equation (74) is valid in the supports of $\chi_E \chi_2$ and $\tilde{\Phi}$. As $\xi_1^2 + R_0(x'', \xi'') - 1 \geq \varepsilon^{4/5}$, on this domain, we have $\text{Re}(\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2} \geq C_0\varepsilon^{2/5}$, for $C_0 > 0$. We deduce that $|D(x', \xi')| \leq C\varepsilon^{-2/5}$ and $|\alpha^{-1}(x'', \xi'')| \leq C\varepsilon^{-2/5}$, then $|(\rho_0 - \alpha)\alpha^{-1}| \leq C\varepsilon^{1/5}$. We conclude by semiclassical symbol calculus in $S(\langle \xi' \rangle^s, (dx')^2 + (d\xi')^2)$, Gårding inequality (12) and Proposition 2.6 that

$$\begin{aligned} &\left| \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_2\rho_0\alpha^{-1}) - \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_2)v_1 \right|_{L^2} \\ &= \left| \text{op}_{sc}(\tilde{\Phi}\chi_E\chi_2(\rho_0 - \alpha)\alpha^{-1})\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2}\Phi u_1 \right|_{L^2} \leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon. \end{aligned} \quad (78)$$

Estimation on the support of $\chi_E \chi_3$.

On the supports of $\chi_E \chi_3$ and $\tilde{\Phi}$, we have $\text{Re}(\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon) \geq C_0\langle \xi' \rangle^2$ for $C_0 > 0$. We deduce that $\text{Re}(\xi_1^2 + R_0(x'', \xi'') - 1 + i\varepsilon)^{1/2} \geq C_0\langle \xi' \rangle$ for $C_0 > 0$ and $|(\rho_0 - \alpha)\alpha^{-1}| \leq C\varepsilon$. By

semiclassical symbol calculus in $S(\langle \xi' \rangle^s, (dx')^2 + (d\xi')^2)$, Gårding inequality and Proposition 2.6 we have

$$\begin{aligned} & \left| \text{op}_{\text{sc}}(\tilde{\Phi}\chi_E\chi_3\rho_0\alpha^{-1}) - \text{op}_{\text{sc}}(\tilde{\Phi}\chi_E\chi_3)v_1 \right|_{L^2} \\ &= \left| \text{op}_{\text{sc}}(\tilde{\Phi}\chi_E\chi_3(\rho_0 - \alpha)\alpha^{-1}) \text{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{1/2} \Phi u_1 \right|_{L^2} \leq C\varepsilon h^{-1/2} + C_\varepsilon. \end{aligned} \quad (79)$$

As $\chi_1 + \chi_2 + \chi_3 = 1$, Formulas (77), (78) and (79) give the conclusion of Lemma 3.15. \square

From (71)–(73) and Lemma 3.15 we have

$$v_0 - iv_1 = z_9, \text{ where } |z_9|_{L^2} \leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}. \quad (80)$$

As v_0 is supported in $x_1 \leq 0$ and v_1 is supported in $x_1 \geq 0$, if we restrict (80) on $x_1 > 0$, we obtain

$$|v_1|_{L^2(x_1 > 0)} = |v_1|_{L^2} \leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8},$$

and we deduce $|v_0|_{L^2} \leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}$. This proves Lemma 3.13. \square

Proposition 3.16. *With the previously defined notation, we have*

$$\begin{aligned} h^{1/2}|(v_h)|_{x_d=0}|_{H_{\text{sc}}^{1/2}} &= h^{1/2}|u_1|_{H_{\text{sc}}^{1/2}} \rightarrow 0 \text{ as } h \rightarrow 0, \\ h^{1/2}|(hD_{x_d}v_h)|_{x_d=0}|_{H_{\text{sc}}^{-1/2}} &= h^{1/2}|u_0|_{H_{\text{sc}}^{-1/2}} \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

Proof. We have to introduce another small parameter $\nu > 0$ chosen below such that $\nu \gg \varepsilon$. Let χ_H, χ_G and χ_E in $\mathcal{C}^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$ such that

$$\begin{aligned} \chi_H(x', \xi') &\text{ is supported in } R(x', 0, \xi') - 1 \leq -\nu, \\ \chi_G(x', \xi') &\text{ is supported in } |R(x', 0, \xi') - 1| \leq 2\nu, \\ \chi_E(x', \xi') &\text{ is supported in } R(x', 0, \xi') - 1 \geq \nu, \\ \chi_H + \chi_G + \chi_E &= 1. \end{aligned}$$

Let ψ_D, ψ_Z and ψ_N in $\mathcal{C}^\infty(\partial\Omega)$ such that

$$\begin{aligned} \psi_D &\text{ is supported in } x_1 \leq -\varepsilon\mu/4 \\ \psi_Z &\text{ is supported in } |x_1| \leq \varepsilon\mu/2 \\ \psi_N &\text{ is supported in } x_1 \geq \varepsilon\mu/4 \\ \psi_D + \psi_Z + \psi_N &= 1. \end{aligned}$$

To be clear, the ε is the one used in elliptic region. We recall the estimates obtained in previous sections. From Proposition 3.2 we have

$$\begin{aligned} |\text{op}_{\text{sc}}(\chi_H)u_1|_{H_{\text{sc}}^1} &\leq C_\nu \\ |\text{op}_{\text{sc}}(\chi_H)u_0|_{L^2} &\leq C_\nu. \end{aligned} \quad (81)$$

From Proposition 3.6 we have

$$\begin{aligned} |\text{op}_{\text{sc}}(\chi_G)u_1|_{H_{\text{sc}}^1} &\leq C\nu^{1/4}h^{-1/2} + C_\nu h^{-3/8} \\ |(\text{op}_{\text{sc}}(\chi_G)u_0)|_{x_d=0}|_{L_{\text{sc}}^2} &\leq C\nu^{3/4}h^{-1/2} + C_\nu h^{-3/8}. \end{aligned} \quad (82)$$

In the elliptic region, we estimate v_j and we have to estimate u_j for $j = 0, 1$. To be precise, $v_0 = \text{op}_{\text{sc}}(\xi_1 + i\beta(x'', \xi''))^{-1/2}\psi_Z u_0$ and $v_1 = \text{op}_{\text{sc}}(\xi_1 - i\beta(x'', \xi''))^{1/2}\psi_Z u_1$, where β is defined by formula (65) and $\Phi = \psi_Z$. By Lemma 3.13, we have

$$\begin{aligned} |v_1|_{L^2} &\leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}, \\ |v_0|_{L^2} &\leq C\varepsilon^{1/5}h^{-1/2} + C_\varepsilon h^{-3/8}. \end{aligned}$$

Thus $\text{op}_{\text{sc}}(\chi_E(\xi_1 - i\beta(x'', \xi''))^{-1/2})v_1 = \text{op}_{\text{sc}}(\chi_E)\psi_Z u_1 + h \text{op}_{\text{sc}}(r_0)u_1$, where $r_0 \in S(1, (dx')^2 + (d\xi')^2)$.

Lemma 3.17. *On the support of χ_E , we have $|(\xi_1 - i\beta(x'', \xi''))^{-1/2}| \leq C\nu^{-1/4}\langle \xi' \rangle^{-1/2} = C\nu\langle \xi' \rangle^{-1/2}$, where C_ν does not depend on ε .*

Proof. We have to consider different cases.

If $|\xi''| \geq C$, where C sufficiently large to have $R_0(x'', \xi'') \geq 2$, we have $\operatorname{Re} \beta(x'', \xi'') \geq C_1\langle \xi'' \rangle$, for $C_1 > 0$. Then $|\xi_1 - i\beta(x'', \xi'')| \geq C_2\langle \xi' \rangle$, for $C_2 > 0$.

If $|\xi''| \leq C$ and $|\xi_1|$ sufficiently large, we have $|\xi_1 - i\beta(x'', \xi'')| \geq C_3\langle \xi_1 \rangle \geq C_4\langle \xi' \rangle$.

If $|\xi'|$ bounded, on the support of χ_E , if ε is sufficiently small with respect ν , we have $\xi_1^2 + R_0(x'', \xi'') - 1 \geq \nu/2$. If $R_0(x'', \xi'') - 1 \geq \delta_1\nu$, for $\delta_1 > 0$, then $\operatorname{Re} \beta(x'', \xi'') \geq \delta_2\nu^{1/2}$, for $\delta_2 > 0$. If δ_1 is sufficiently small and $R_0(x'', \xi'') - 1 \leq \delta_1\nu$, then $\xi_1^2 \geq \nu/4$ and $|\xi_1 - i\beta(x'', \xi'')| \geq \delta_3\nu^{1/2}$.

In all cases, we get that $|\xi_1 - i\beta(x'', \xi'')| \geq \delta_3\nu^{1/2}\langle \xi' \rangle$. This implies the result. \square

By symbol calculus, we have

$$\begin{aligned} |\operatorname{op}_{\text{sc}}(\chi_E)\psi_Z u_1|_{H_{\text{sc}}^{1/2}} &\leq |\operatorname{op}_{\text{sc}}(\langle \xi' \rangle^{1/2}) \operatorname{op}_{\text{sc}}(\chi_E(\xi_1 - i\beta(x'', \xi''))^{-1/2})v_1|_{L^2} + C_{\nu,\varepsilon}h|u_1|_{H_{\text{sc}}^{1/2}} \\ &\leq |\operatorname{op}_{\text{sc}}(\langle \xi' \rangle^{1/2}\chi_E(\xi_1 - i\beta(x'', \xi''))^{-1/2})v_1|_{L^2} + C_{\nu,\varepsilon}h|u_1|_{H_{\text{sc}}^{1/2}}, \end{aligned}$$

and by Gårding inequality (12) and Lemma 3.13 we have

$$\begin{aligned} |\operatorname{op}_{\text{sc}}(\chi_E)\psi_Z u_1|_{H_{\text{sc}}^{1/2}} &\leq C_\nu|v_1|_{L^2} + C_{\nu,\varepsilon}h^{1/2} + C_{\nu,\varepsilon}h|u_1|_{H_{\text{sc}}^{1/2}} \\ &\leq C_\nu\varepsilon^{1/5}h^{-1/2} + C_{\nu,\varepsilon}h^{-3/8} + C_{\nu,\varepsilon}h|u_1|_{H_{\text{sc}}^{1/2}}. \end{aligned} \quad (83)$$

For v_0 we have

$$\operatorname{op}_{\text{sc}}(\chi_E(\xi_1 + i\beta(x'', \xi''))^{1/2})v_0 = \operatorname{op}_{\text{sc}}(\chi_E)\psi_Z u_0 + h \operatorname{op}_{\text{sc}}(r_0)u_0, \quad \text{where } r_0 \in S(1, (dx')^2 + (d\xi')^2).$$

A proof analogous to the one of Lemma 3.17 gives $|(\xi + i\beta(x'', \xi''))^{1/2}| \leq C_\nu\langle \xi' \rangle^{1/2}$, we have by Gårding inequality and symbol calculus

$$\begin{aligned} |\operatorname{op}_{\text{sc}}(\chi_E)\psi_Z u_0|_{H^{-1/2}} &\leq C_\nu|v_0|_{L^2} + C_{\nu,\varepsilon}h^{1/2} + C_{\nu,\varepsilon}h|u_0|_{H_{\text{sc}}^{-1/2}} \\ &\leq C_\nu\varepsilon^{1/5}h^{-1/2} + C_{\nu,\varepsilon}h^{-3/8} + C_{\nu,\varepsilon}h|u_0|_{H_{\text{sc}}^{-1/2}}. \end{aligned} \quad (84)$$

From Proposition 3.11, we have

$$\begin{aligned} |\operatorname{op}_{\text{sc}}(\chi_E)\psi_D u_0|_{L^2} &\leq C_{\nu,\varepsilon}h^{1/2} \text{ and } \psi_D u_1 = 0 \\ |\operatorname{op}_{\text{sc}}(\chi_E)\psi_N u_1|_{H_{\text{sc}}^1} &\leq C_{\nu,\varepsilon}h^{1/2} \text{ and } \psi_N u_0 = 0. \end{aligned} \quad (85)$$

As $u_j = \operatorname{op}_{\text{sc}}(\chi_H)u_j + \operatorname{op}_{\text{sc}}(\chi_G)u_j + \operatorname{op}_{\text{sc}}(\chi_E)\psi_N u_j + \operatorname{op}_{\text{sc}}(\chi_E)\psi_Z u_j + \operatorname{op}_{\text{sc}}(\chi_E)\psi_D u_j$, we have, by (81)–(85),

$$h^{1/2}|u_j|_{H_{\text{sc}}^{-1/2+j}} \leq C\nu^{(3-2j)/4} + C_\nu\varepsilon^{1/5} + C_{\nu,\varepsilon}h^{1/8} + C_{\nu,\varepsilon}h^{3/2}|u_j|_{H_{\text{sc}}^{-1/2+j}}.$$

Choosing first ν sufficiently small, second ε sufficiently small, we can absorb the right hand side term $C_{\nu,\varepsilon}h^{3/2}|u_j|_{H_{\text{sc}}^{-1/2+j}}$ by the left hand side term taking h sufficiently small. The limit superior with respect to h of the left hand side can be estimated by any positive number. This proves Proposition 3.16. \square

3.2 Support of semiclassical measure in a neighborhood of boundary

We can now prove Proposition 3.1, that is, the measure μ is supported on $p = 0$, in a neighborhood of $x_0 \in \partial\Omega$.

Proof. The proof is based on the results obtained by Proposition 2.2 and Proposition 3.16. We recall that in local coordinates $p(x, \xi) = \xi_d^2 + R(x, \xi') - 1$. We have with the notation (13)

$$[h^2 D_{x_d} + R(x, hD_{x'}) - 1]v_h = h\underline{q}_h - ih(hD_{x_d}v_h)|_{x_d=0} \otimes \delta_{x_d=0} - ih(v_h)|_{x_d=0} \otimes hD_{x_d}\delta_{x_d=0}. \quad (86)$$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_{\xi_d})$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}_{x_d})$ be such that χ is supported in a neighborhood of 0. Let $\ell \in \mathcal{C}_0^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1})$. By symbol calculus we have

$$\begin{aligned} & \text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi'))[h^2 D_{x_d} + R(x, hD_{x'}) - 1] \\ &= \text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi')(\xi_d^2 + R(x, \xi') - 1)) + h \text{Op}_{sc}(r_0), \end{aligned}$$

where $r_0 \in S^0$. Then

$$\begin{aligned} I &:= \left(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi'))[h^2 D_{x_d} + R(x, hD_{x'}) - 1] \underline{v}_h | \underline{v}_h \right) \\ &= \underbrace{\left(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi')(\xi_d^2 + R(x, \xi') - 1)) \underline{v}_h | \underline{v}_h \right)}_{=:A} + \underbrace{h(\text{Op}_{sc}(r_0) \underline{v}_h | \underline{v}_h)}_{=:B}. \end{aligned}$$

By definition of the semiclassical measure, the term A converges to $\langle \mu | \chi(x_d)\varphi(\xi_d)\ell(x', \xi')(\xi_d^2 + R(x, \xi') - 1) \rangle$ as h to 0. The term B is estimated by $Ch\|v_h\|_{L^2(\Omega)}^2$ and this converges to 0 as h to 0 by Proposition 2.2.

By (86) we also have

$$\begin{aligned} I &= h(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi')) \underline{q}_h | \underline{v}_h) \\ &\quad - ih(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi'))(hD_{x_d}v_h)|_{x_d=0} \otimes \delta_{x_d=0} | \underline{v}_h) \\ &\quad - ih(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi'))(v_h)|_{x_d=0} \otimes hD_{x_d}\delta_{x_d=0} | \underline{v}_h) = I_1 + I_2 + I_3. \end{aligned}$$

Obviously we have

$$|I_1| \leq Ch\|q_h\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)},$$

as $\chi(x_d)\varphi(\xi_d)\ell(x', \xi') \in S^0$. Then $I_1 \rightarrow 0$ as $h \rightarrow 0$ by (19). By exact calculus, we have

$$\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi')) = \chi(x_d) \text{op}_{sc}(\ell(x', \xi')) \text{Op}_{sc}(\varphi(\xi_d)).$$

Let $w_j = ((hD_{x_d})^{1-j}v_h)|_{x_d=0}$, we have for $j = 0, 1$

$$\begin{aligned} & h|(\text{Op}_{sc}(\chi(x_d)\varphi(\xi_d)\ell(x', \xi'))w_j \otimes (hD_{x_d})^j \delta_{x_d=0} | \underline{v}_h)| \\ &= h|(\chi(x_d) \text{op}_{sc}(\ell(x', \xi'))w_j \otimes \text{Op}_{sc}(\varphi(\xi_d))(hD_{x_d})^j \delta_{x_d=0} | \underline{v}_h)| \\ &\leq h|w_j|_{H^{j-1/2}}\|\text{Op}_{sc}(\varphi(\xi_d))(hD_{x_d})^j \delta_{x_d=0}\|_{L^2(\mathbb{R})}\|v_h\|_{L^2(\Omega)}, \end{aligned} \tag{87}$$

where we have used, to estimate w_0 , that $\langle \xi' \rangle^{1/2}\ell(x', \xi')$ is bounded on L^2 , as ℓ is compactly supported. A direct computation in Fourier variable gives that $\|\text{Op}_{sc}(\varphi(\xi_d))(hD_{x_d})^j \delta_{x_d=0}\|_{L^2(\mathbb{R})} \lesssim h^{-1/2}$. From (87), we obtain

$$|I_2| + |I_3| \leq C(|(v_h)|_{x_d=0}|_{H^{1/2}} + |(D_{x_d}v_h)|_{x_d=0}|_{H^{-1/2}})h^{1/2} \rightarrow 0 \text{ as } h \rightarrow 0,$$

by Proposition 3.16. We conclude that $\langle \mu | \chi(x_d)\varphi(\xi_d)\ell(x', \xi')(\xi_d^2 + R(x, \xi') - 1) \rangle = 0$ and by density of functions spanned by $\chi(x_d)\varphi(\xi_d)\ell(x', \xi')$ in $\mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have that $\langle p(x, \xi)\mu | \phi(x, \xi) \rangle = 0$, for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. This gives the conclusion of Proposition 3.1. \square

3.3 The semiclassical measure is not identically null

Proposition 3.18. *The measure μ constructed at the beginning of Section 3.1 for the sequence $(v_h)_h$ satisfying (19) is not identically 0, i.e. $\mu \neq 0$.*

Proof. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\phi = 1$ in a neighborhood of 0. Let $s \in (0, 1/2)$. Let $\phi_R(\xi) = \phi(|\xi|/R)$, we have

$$\begin{aligned} \|\text{Op}_{sc}(1 - \phi_R)\underline{v}_h\|_{L^2(\mathbb{R}^d)} &\leq CR^{-s}\|\text{Op}_{sc}(\langle \xi \rangle^s)\underline{v}_h\|_{L^2(\mathbb{R}^d)} \leq CR^{-s}\|v_h\|_{H_{sc}^s(\Omega)} \\ &\leq CR^{-s}\|v_h\|_{H_{sc}^1(\Omega)} \leq CR^{-s}, \end{aligned}$$

as $\|w\|_{H_{sc}^s(\mathbb{R}^d)}$ is equivalent to $\|w\|_{H_{sc}^s(\Omega)}$ (uniformly with respect to $h \in (0, 1)$) if w is supported in $\overline{\Omega}$. Then for R sufficiently large, $\|\text{Op}_{sc}(1 - \phi(|\xi|/R))\underline{v}_h\|_{L^2(\mathbb{R}^d)} \leq 1/2$. We thus have

$$\begin{aligned} (\text{Op}_{sc}(\phi_R)\underline{v}_h|\underline{v}_h)_{L^2(\mathbb{R}^d)} &= \|\underline{v}_h\|_{L^2(\mathbb{R}^d)}^2 - (\text{Op}_{sc}(1 - \phi_R)\underline{v}_h|\underline{v}_h)_{L^2(\mathbb{R}^d)} \\ &\geq 1 - \|\text{Op}_{sc}(1 - \phi(|\xi|/R))\underline{v}_h\|_{L^2(\mathbb{R}^d)} \geq 1/2. \end{aligned}$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be such that $\chi(x) = 1$ for $x \in \Omega$ and $\chi \geq 0$, we have

$$(\text{Op}_{sc}(\phi_R)\underline{v}_h|\underline{v}_h)_{L^2(\mathbb{R}^d)} = (\text{Op}_{sc}(\phi_R(\xi)\chi(x))\underline{v}_h|\underline{v}_h)_{L^2(\mathbb{R}^d)} \rightarrow \langle \mu, \chi(x)\phi_R(\xi) \rangle \text{ as } h \rightarrow 0,$$

we obtain $\langle \mu, \chi(x)\phi_R(\xi) \rangle \geq 1/2$. Then μ is not identically null. \square

3.4 The semiclassical measure is null on the support of a

Before proving the result we need to extend the space of test functions acting on μ . We have the following lemma.

Let $b(x, \xi') \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, we can give a sense to the expression $\langle \mu, b(x, \xi')\xi_d^j \rangle$ for all $j \in \mathbb{N}$.

Lemma 3.19. *Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\Phi(\sigma) = 1$, for $|\sigma| \leq 1$. Let $j \in \mathbb{N}$, then the quantity $\langle \mu, b(x, \xi')\xi_d^j\Phi(|\xi|/R) \rangle$ does not depend on R for sufficiently large R .*

By definition, we denote $\langle \mu, b(x, \xi')\xi_d^j \rangle = \lim_{R \rightarrow \infty} \langle \mu, b(x, \xi')\xi_d^j\Phi(|\xi|/R) \rangle$.

Proof. As $p\mu = 0$, μ is supported in $|\xi'|^2 + \xi_d^2 \leq C_0$, for $C_0 > 0$, sufficiently large. If R is sufficiently large and $R' > R$, $\Phi(|\xi|/R) - \Phi(|\xi|/R') = 0$, if $|\xi| \leq R$, in particular if $R^2 > C_0$. Then $b(x, \xi')(\Phi(|\xi|/R) - \Phi(|\xi|/R')) = 0$, on the support of μ . This proves that $\langle \mu, b(x, \xi')\xi_d^j\Phi(|\xi|/R) \rangle$, does not depend on R if R is sufficiently large. \square

Proposition 3.20. *We have $a\mu = 0$.*

Proof. From Proposition 2.2 we have $-h^2Pv_h + v_h - ihav_h = hq_h$. The inner product with v_h lead to $(-h^2Pv_h + v_h - ihav_h|v_h) = h(q_h|v_h)$. Taking the imaginary part of this equation, as $(Pv_h|v_h)$ is real, we have $-(av_h|v_h) = \text{Im}(q_h|v_h)$. As $|\text{Im}(q_h|v_h)| \leq \|q_h\|\|v_h\| \rightarrow 0$, as $h \rightarrow 0$, we have $(av_h|v_h) \rightarrow 0$, as $h \rightarrow 0$. Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\Phi = 1$ in a neighborhood of 0, and $0 \leq \Phi \leq 1$.

By Lemma 3.19, the limit when h goes to 0 of $(\text{Op}_{sc}(a(x)\Phi(|\xi|/R))v_h|v_h)$ does not depend on R for R large enough. By Gårding inequality (9), as $a \geq 0$, $(\text{Op}_{sc}(a(x)\Phi(|\xi|/R))v_h|v_h)$ and $(\text{Op}_{sc}(a(x)(1 - \Phi(|\xi|/R)))v_h|v_h)$ are non negative modulo $\mathcal{O}(h)$.

Consequently, we have $\lim_{h \rightarrow 0} (\text{Op}_{sc}(a(x)\Phi(|\xi|/R))v_h|v_h) \geq 0$, $\lim_{h \rightarrow 0} (av_h|v_h) = 0$ moreover $(\text{Op}_{sc}(a(x)(1 - \Phi(|\xi|/R)))v_h|v_h)$ has a limit and $\lim_{h \rightarrow 0} (\text{Op}_{sc}(a(x)(1 - \Phi(|\xi|/R)))v_h|v_h) \geq 0$. Since

$$\lim_{h \rightarrow 0} (\text{Op}_{sc}(a(x)\Phi(|\xi|/R))v_h|v_h) + \lim_{h \rightarrow 0} (\text{Op}_{sc}(a(x)(1 - \Phi(|\xi|/R)))v_h|v_h) = 0,$$

we deduce that $\lim_{h \rightarrow 0} (\text{Op}_{sc}(a(x)\Phi(|\xi|/R))v_h|v_h) = 0$. This implies that $\langle \mu, a \rangle = 0$, and as a and μ are non negative, we deduce $a\mu = 0$. \square

4 Measure properties

4.1 Action of Hamiltonian

We first recall the main results proved in previous sections. There exists a sequence $(v_h)_h$ satisfying the following properties

$$\begin{aligned} h^2Pv_h - v_h + ihav_h &= hq_h, \\ \|v_h\|_{L^2(\Omega)} &= 1 \text{ and } \|h\nabla v_h\|_{L^2(\Omega)} \leq 2, \\ \|q_h\|_{L^2(\Omega)} &\rightarrow 0 \text{ as } h \rightarrow 0, \\ h^{1/2}|(v_h)|_{x_d=0}|_{H_{sc}^{1/2}} &\rightarrow 0 \text{ as } h \rightarrow 0, \\ h^{1/2}|(hD_{x_d}v_h)|_{x_d=0}|_{H_{sc}^{-1/2}} &\rightarrow 0 \text{ as } h \rightarrow 0, \\ p\mu &= 0. \end{aligned} \tag{88}$$

We also proved that $a\mu = 0$ but in the following we do not systematically use this property. These results was stated in Proposition 2.2, Proposition 3.1 and Proposition 3.16.

4.1.1 Interior formula

We begin by stated that the measure μ is propagated along the H_p flow in interior of domain. This property is classical but the proof is simpler in this case than in a neighborhood of boundary, even if the main ideas are used.

Proposition 4.1. *Let $b \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^d)$. We have $\langle H_p \mu - 2a\mu, b \rangle = 0$.*

Proof. We consider the following quantity

$$\begin{aligned} A &= ih^{-1}(b(x, hD)(h^2P - 1 + iha)v_h|v_h)_{L^2(\Omega)} \\ &\quad - ih^{-1}(b(x, hD)v_h|(h^2P - 1 + iha)v_h)_{L^2(\Omega)} \\ &= ih^{-1}(b(x, hD)hq_h|v_h)_{L^2(\Omega)} - ih^{-1}(b(x, hD)v_h|hq_h)_{L^2(\Omega)}. \end{aligned}$$

We have $|A| \lesssim \|v_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}$, then A goes to 0 as h . As b is supported far away the boundary of Ω , we have $(b(x, hD)v_h|(h^2P - 1 + iha)v_h)_{L^2(\Omega)} = ((h^2P - 1 - iha)b(x, hD)v_h|v_h)_{L^2(\Omega)}$. As the principal symbol of $b(x, hD)a - ab(x, hD)$ is $\mathcal{O}(h)$, we have

$$A = ih^{-1}([b(x, hD), h^2P - 1]v_h|v_h)_{L^2(\Omega)} - 2(ab(x, hD)v_h|v_h)_{L^2(\Omega)} + \mathcal{O}(h)\|v_h\|_{L^2(\Omega)}^2,$$

and by symbol calculus the principal symbol of $[b(x, hD), h^2P - 1]$ is $-ih\{b, p\}$. Then

$$A = ((\text{Op}_{sc}(\{b, p\}) - 2a(x)b(x, hD))v_h|v_h)_{L^2(\Omega)} + \mathcal{O}(h)\|v_h\|_{L^2(\Omega)}^2,$$

then $A \rightarrow \langle \mu, \{b, p\} - 2ab \rangle$ as $h \rightarrow 0$, which gives the result. \square

4.1.2 Limit computations

In the following section quantities as $(\text{op}_{sc}(a)h^j D_{x_d}^j v_h|v_h)$ for $j = 0, 1, 2$ appear. We need to evaluate their limits in term of the measure μ . To do so we shall now state some technical results.

Proposition 4.2. *Let $b_0(x, \xi') \in S(1, (dx)^2 + (d\xi')^2)$, and $b_1(x, \xi'')$, $b_2(x, \xi'') \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-2})$. Let $b(x, \xi') = b_0(x, \xi') + b_1(x, \xi'')\xi_1 + b_2(x, \xi'')\xi_1^2$. We have $(\text{op}_{sc}(b)v_h|v_h)_{L^2(x_d > 0)} \rightarrow \langle \mu, b \rangle$ as $h \rightarrow 0$.*

For the proof we need the following lemma.

Lemma 4.3. *Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\Phi(\sigma) = 1$, for σ in a neighborhood of 0. For all $s \in (0, 1/2)$, there exists $C > 0$, such that*

$$\begin{aligned} \left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R))\right) 1_{x_d > 0} v \right\|_{L^2(\mathbb{R}^d)} &\leq CR^{-s} \|v\|_{H_{sc}^s(x_d > 0)} \\ \left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R)\Phi(|\xi'|/R))\right) 1_{x_d > 0} v \right\|_{L^2(\mathbb{R}^d)} &\leq CR^{-s} \|v\|_{H_{sc}^s(x_d > 0)}, \end{aligned}$$

for all $v \in H^s(x_d > 0)$, for all $h \in (0, 1)$ and all $R > 1$.

We recall that

$$H_{sc}^s(x_d > 0) = \{u \in \mathcal{D}'(x_d > 0) : \exists w \in H_{sc}^s(\mathbb{R}^d), w|_{x_d > 0} = u\},$$

and for $u \in H_{sc}^s(x_d > 0)$, we define $\|u\|_{H_{sc}^s(x_d > 0)} = \inf\{\|w\|_{H_{sc}^s(\mathbb{R}^d)}, w|_{x_d > 0} = u\}$. We recall that for $s \in [0, 1/2)$ and $u \in H_{sc}^s(x_d > 0)$, one has $1_{x_d > 0} u \in H_{sc}^s(\mathbb{R}^d)$ and $\|u\|_{H_{sc}^s(x_d > 0)}$ and $\|1_{x_d > 0} u\|_{H_{sc}^s(\mathbb{R}^d)}$ define two equivalent norms (uniformly with respect to $h \in (0, 1)$).

Proof. Let $w = 1_{x_d > 0}v$. We have

$$\begin{aligned} \left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R))\right) 1_{x_d > 0}v \right\|_{L^2(\mathbb{R}^d)}^2 &\lesssim \int (1 - \Phi(h\xi_d/R))^2 |\hat{w}(\xi', \xi_d)|^2 d\xi \\ &\lesssim \int \left((1 - \Phi(h\xi_d/R)) / (|h\xi_d|^s) \right)^2 \langle h\xi_d \rangle^{2s} |\hat{w}(\xi', \xi_d)|^2 d\xi. \end{aligned}$$

But

$$|1 - \Phi(h\xi_d/R)| / (|h\xi_d|^s) \leq R^{-s} |1 - \Phi(h\xi_d/R)| / (|h\xi_d/R|^s) \lesssim R^{-s},$$

then, as $s < 1/2$, we obtain that

$$\left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R))\right) 1_{x_d > 0}v \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim R^{-2s} \int \langle h\xi \rangle^{2s} |\hat{w}(\xi', \xi_d)|^2 d\xi \lesssim R^{-2s} \|v\|_{H^s(x_d > 0)}^2, \quad (89)$$

which is the first estimate of statement.

By the same method, we prove that (for any $s > 0$)

$$\left\| \left(1 - \text{Op}_{sc}(\Phi(\xi'/R))\right) 1_{x_d > 0}v \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim R^{-2s} \int \langle h\xi' \rangle^{2s} |\hat{w}(\xi', \xi_d)|^2 d\xi \lesssim R^{-2s} \|v\|_{H^s(x_d > 0)}^2, \quad (90)$$

As $1 - \Phi(\xi_d/R)\Phi(|\xi'|/R) = 1 - \Phi(\xi_d/R) + \Phi(\xi_d/R)(1 - \Phi(|\xi'|/R))$, we have

$$\begin{aligned} \left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R)\Phi(|\xi'|/R))\right) 1_{x_d > 0}v \right\|_{L^2(\mathbb{R}^d)} &\leq \left\| \left(1 - \text{Op}_{sc}(\Phi(\xi_d/R)) 1_{x_d > 0}v\right) \right\|_{L^2(\mathbb{R}^d)} \\ &\quad + \left\| \left(1 - \text{Op}_{sc}(\Phi(|\xi'|/R))\right) 1_{x_d > 0}v \right\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

as $\text{Op}_{sc}(\Phi(\xi_d/R))$ is bounded by 1 on L^2 . From (89) and (90), we obtain the second estimate of statement. \square

Proof of Proposition 4.2. Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\Phi(\sigma) = 1$, for σ in a neighborhood of 0. We treat the terme b_0 . We have as $\text{op}_{sc}(b_0)$ is a tangential operator

$$\begin{aligned} (\text{op}_{sc}(b_0)v_h|v_h)_{L^2(x_d > 0)} &= (\text{op}_{sc}(b_0)1_{x_d > 0}v_h|1_{x_d > 0}v_h)_{L^2(\mathbb{R}^d)} \\ &= (\text{op}_{sc}(b_0)\text{Op}_{sc}(\Phi(\xi_d/R)\Phi(|\xi'|/R))1_{x_d > 0}v_h|1_{x_d > 0}v_h)_{L^2(\mathbb{R}^d)} \\ &\quad + (\text{op}_{sc}(b_0)\text{Op}_{sc}((1 - \Phi(|\xi'|/R))\Phi(\xi_d/R))1_{x_d > 0}v_h|1_{x_d > 0}v_h)_{L^2(\mathbb{R}^d)} \\ &\quad + (\text{op}_{sc}(b_0)\text{Op}_{sc}(1 - \Phi(\xi_d/R))1_{x_d > 0}v_h|1_{x_d > 0}v_h)_{L^2(\mathbb{R}^d)} = A_1 + A_2 + A_3. \end{aligned}$$

By definition of semiclassical measure and from Lemma 3.19 we have

$$A_1 \rightarrow \langle \mu, b_0(x, \xi')\Phi(\xi_d/R)\Phi(|\xi'|/R) \rangle = \langle \mu, b_0 \rangle \text{ as } h \rightarrow 0.$$

We have

$$\begin{aligned} |A_2| &\lesssim \|1_{x_d > 0}\text{Op}_{sc}(1 - \Phi(|\xi'|/R))v_h\|_{L^2(\mathbb{R}^d)} \|1_{x_d > 0}v_h\|_{L^2(\mathbb{R}^d)} \\ &\lesssim R^{-1} \|v_h\|_{L^2(\mathbb{R}, H_{\frac{1}{2}c}^1(\mathbb{R}^{d-1}))} \|v_h\|_{L^2(x_d > 0)}, \end{aligned}$$

and, for $s \in (0, 1/2)$ we have

$$|A_3| \lesssim \|\text{Op}_{sc}((1 - \Phi(\xi_d/R))1_{x_d > 0}v_h)\|_{L^2(\mathbb{R}^d)} \|1_{x_d > 0}v_h\|_{L^2(\mathbb{R}^d)} \lesssim R^{-s} \|v\|_{H_{sc}^s(x_d > 0)} \|v\|_{L^2(x_d > 0)},$$

as $\text{op}_{sc}(b_0)$ is bounded on $L^2(\mathbb{R}^d)$ and from Lemma 4.3. As $A_1 + A_2 + A_3$ does not depend on R we obtain the result for b_0 .

In the following we only consider the term $b_2\xi_1^2$, the term $b_1\xi_1$ can be managed as the previous with some minor modifications.

$$\begin{aligned}
(\text{op}_{sc}(b_2\xi_1^2)v_h|v_h)_{L^2(x_d>0)} &= (\text{op}_{sc}(b_2\xi_1^2)1_{x_d>0}v_h|1_{x_d>0}v_h)_{L^2(\mathbb{R}^d)} \\
&= (\text{op}_{sc}(b_2\xi_1^2)\text{Op}_{sc}(\Phi(\xi_d/R)\Phi(|\xi_1|/R))1_{x_d>0}v_h|1_{x_d>0}v_h)_{L^2(\mathbb{R}^d)} \\
&\quad + (\text{op}_{sc}(b_2\xi_1^2)\text{Op}_{sc}((1-\Phi(\xi_d/R))\Phi(|\xi_1|/R))1_{x_d>0}v_h|1_{x_d>0}v_h)_{L^2(\mathbb{R}^d)} \\
&\quad + (\text{op}_{sc}(b_2\xi_1^2)\text{Op}_{sc}((1-\Phi(|\xi_1|/R))1_{x_d>0}v_h|1_{x_d>0}v_h)_{L^2(\mathbb{R}^d)} \\
&= B_1 + B_2 + B_3
\end{aligned}$$

As previously $B_1 \rightarrow \langle \mu, b_2\xi_1^2\Phi(\xi_d/R)\Phi(|\xi_1|/R) \rangle = \langle \mu, b_2\xi_1^2 \rangle$ as $h \rightarrow 0$. We need to prove a regularity result on v_h given by the following lemma which is proven in Appendix A.

Lemma 4.4 (Zaremba regularity result). *Let $s \in (0, 1/2)$. There exists $C > 0$, such that for any v_h satisfying (88), we have $\|v_h\|_{H_{sc}^{1+s}(\Omega)} \leq C$.*

We have after an integration by parts and symbol calculus, for $s \in (0, 1/2)$

$$\begin{aligned}
|B_2| &\lesssim \|\text{Op}_{sc}((1-\Phi(|\xi_d|/R))1_{x_d>0}hD_{x_1}v_h)\|_{L^2(x_d>0)}\|v_h\|_{L^2(x_d>0, H_{sc}^1(\mathbb{R}^{d-1}))} + h\|v_h\|_{H_{sc}^1(x_d>0)}^2 \\
&\lesssim R^{-s}\|hD_{x_1}v_h\|_{H^s(x_d>0)}\|v_h\|_{L^2(x_d>0, H_{sc}^1(\mathbb{R}^{d-1}))} + h\|v_h\|_{H_{sc}^1(x_d>0)}^2
\end{aligned}$$

and

$$\begin{aligned}
|B_3| &\lesssim \|((1-\Phi(|\xi_1|/R))v_h)\|_{L^2(x_d>0, H_{sc}^1(\mathbb{R}^{d-1}))}\|v_h\|_{L^2(x_d>0, H_{sc}^1(\mathbb{R}^{d-1}))} + h\|v_h\|_{H_{sc}^1(x_d>0)}^2 \\
&\lesssim R^{-s}\|v_h\|_{L^2(x_d>0, H_{sc}^{1+s}(\mathbb{R}^{d-1}))}\|v_h\|_{L^2(x_d>0, H_{sc}^1(\mathbb{R}^{d-1}))} + h\|v_h\|_{H_{sc}^1(x_d>0)}^2
\end{aligned}$$

where we have applied Lemma 4.4. Then we can conclude as for the term b_0 . \square

In the following lemma we consider the quantity $(\text{op}_{sc}(b)h^2D_{x_d}^2v_h|v_h)_{L^2(x_d>0)}$, which is not clearly well defined. But as v_h satisfied (88), we can prove that $h^2D_{x_d}^2v_h \in L^2(x_d > 0, H_{sc}^{-1}(\mathbb{R}^{d-1}))$. The inner product in tangential variables need to be interpreted as a duality product H^{-1}, H^1 .

Proposition 4.5. *Let $b(x, \xi') \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, we have*

$$(\text{op}_{sc}(b)h^2D_{x_d}^2v_h|v_h)_{L^2(x_d>0)} \rightarrow \langle \mu, b\xi_d^2 \rangle \text{ as } h \rightarrow 0.$$

Proof. We use the equation satisfied by v_h (see (88)).

$$\begin{aligned}
(\text{op}_{sc}(b)h^2D_{x_d}^2v_h|v_h)_{L^2(x_d>0)} &= (\text{op}_{sc}(b)h(q_h - iav_h)|v_h)_{L^2(x_d>0)} \\
&\quad - (\text{op}_{sc}(b)(R(x, hD_{x'}) - 1)v_h|v_h)_{L^2(x_d>0)} \\
&= A + B.
\end{aligned}$$

Clearly

$$|A| \lesssim h(\|q_h\|_{L^2(x_d>0)} + \|v_h\|_{L^2(x_d>0)})\|v_h\|_{L^2(x_d>0)} \rightarrow 0 \text{ as } h \rightarrow 0.$$

By symbol calculus, $\text{op}_{sc}(b)(R(x, hD_{x'}) - 1) = \text{op}_{sc}(b(x, \xi')(R(x, \xi') - 1)) + h\text{op}_{sc}(r_0)$, where $r_0 \in S_{\text{tan}}^0$. Then, by Proposition 4.2, we have $B \rightarrow -\langle \mu, b(R - 1) \rangle$. Let Φ be as given in Lemma 3.19, we have for λ sufficiently large

$$\begin{aligned}
-\langle \mu, b(x, \xi')(R(x, \xi') - 1) \rangle &= -\langle \mu, b(x, \xi')(R(x, \xi') - 1)\Phi(\xi_d/\lambda) \rangle \\
&= -\langle \mu, b(x, \xi')(\xi_d^2 + R(x, \xi') - 1)\Phi(\xi_d/\lambda) \rangle + \langle \mu, b(x, \xi')\xi_d^2\Phi(\xi_d/\lambda) \rangle \\
&= \langle \mu, b(x, \xi')\xi_d^2\Phi(\xi_d/\lambda) \rangle = \langle \mu, b(x, \xi')\xi_d^2 \rangle,
\end{aligned}$$

as $p\mu = 0$. Which gives the lemma. \square

Proposition 4.6. *Let $b(x, \xi') \in S(1, (dx)^2 + (d\xi')^2)$, we have*

$$\left(\text{op}_{\text{sc}}(b)hD_{x_d}v_h|v_h \right)_{L^2(x_d>0)} \rightarrow \langle \mu, b\xi_d \rangle \text{ as } h \rightarrow 0.$$

Proof. Let Φ be as given in Lemma 3.19, we have for $\lambda > 0$,

$$\begin{aligned} \left(\text{op}_{\text{sc}}(b)hD_{x_d}v_h|v_h \right)_{L^2(x_d>0)} &= \left(\text{op}_{\text{sc}}(b)1_{x_d>0}hD_{x_d}v_h|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \\ &= \left(\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))1_{x_d>0}hD_{x_d}v_h|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \\ &\quad + \left(\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(1 - \Phi(\xi_d/\lambda))1_{x_d>0}hD_{x_d}v_h|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \\ &= A + B. \end{aligned}$$

By symbol calculus in $S(1, (dx)^2 + (d\xi)^2)$, we have

$$\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(1 - \Phi(\xi_d/\lambda)) = \text{Op}_{\text{sc}}(1 - \Phi(\xi_d/\lambda))^* \text{op}_{\text{sc}}(b) + h \text{Op}_{\text{sc}}(r_0),$$

where $r_0 \in S(1, (dx)^2 + (d\xi)^2)$. Then

$$\begin{aligned} |B| &\lesssim \left| \left(\text{op}_{\text{sc}}(b)1_{x_d>0}hD_{x_d}v_h| \text{Op}_{\text{sc}}(1 - \Phi(\xi_d/\lambda))1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \right| + h \|hD_{x_d}v_h\|_{L^2(\mathbb{R})} \|v_h\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \lambda^{-s} + h, \end{aligned}$$

by Lemma 4.3 and a priori estimates (88).

Next we treat the term A . We have $1_{x_d>0}hD_{x_d}v_h = hD_{x_d}(1_{x_d>0}v_h) + ih(v_h)|_{x_d=0} \otimes \delta_{x_d=0}$. Then we have

$$\begin{aligned} A &= \left(\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))hD_{x_d}(1_{x_d>0}v_h)|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \\ &\quad + ih \left(\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))((v_h)|_{x_d=0} \otimes \delta_{x_d=0})|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} = A_1 + A_2. \end{aligned}$$

We have $\text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))hD_{x_d} = \text{Op}_{\text{sc}}(\xi_d\Phi(\xi_d/\lambda))$ and by symbol calculus in $S(1, (dx)^2 + (d\xi)^2)$, we have $\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(\xi_d\Phi(\xi_d/\lambda)) = \text{Op}_{\text{sc}}(\xi_d\Phi(\xi_d/\lambda)b) + h \text{Op}_{\text{sc}}(r_0)$, where $r_0 \in S(1, (dx)^2 + (d\xi)^2)$. Then $A_1 = \left(\text{op}_{\text{sc}}(b) \text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))hD_{x_d}(1_{x_d>0}v_h)|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \rightarrow \langle \mu, \xi_d\Phi(\xi_d/\lambda)b \rangle = \langle \mu, \xi_db \rangle$, if λ is sufficiently large.

Let $\tilde{\Phi}$ be such that $\mathcal{F}\tilde{\Phi} = \Phi$, where \mathcal{F} is the Fourier transform. We have $\text{Op}_{\text{sc}}(\Phi(\xi_d/\lambda))\delta_{x_d=0} = h^{-1}\lambda\tilde{\Phi}(\lambda x_d/h)$. Then

$$\begin{aligned} |A_2| &\lesssim \left| \left(\text{op}_{\text{sc}}(b)((v_h)|_{x_d=0}\lambda\tilde{\Phi}(\lambda x_d/h))|1_{x_d>0}v_h \right)_{L^2(\mathbb{R}^d)} \right| \\ &\lesssim \lambda \|(v_h)|_{x_d=0}\|_{L^2(\mathbb{R}^{d-1})} \|\tilde{\Phi}(\lambda x_d/h)\|_{L^2(\mathbb{R})} \|v_h\|_{L^2(x_d>0)} \\ &\lesssim \lambda^{1/2}h^{1/2} \|(v_h)|_{x_d=0}\|_{L^2(\mathbb{R}^{d-1})}, \end{aligned}$$

as $|\tilde{\Phi}(\lambda x_d/h)|_{L^2(\mathbb{R})} = Ch^{1/2}\lambda^{-1/2}$. From (88), we have $A_2 \rightarrow 0$, as $h \rightarrow 0$. From estimates on A_1 , A_2 and B we obtain the result. \square

4.1.3 Boundary formulas

The two next propositions are the analogous of Proposition 4.1 at the boundary.

Proposition 4.7. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ (resp. $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-2})$) be real valued functions. We have the following formula*

$$\begin{aligned} \langle H_p\mu - 2a\mu, b \rangle &= \langle \mu, \{b, p\} - 2ab \rangle = \lim_{h \rightarrow 0} 2 \text{Re} \left(b(x', 0, hD')(v_h)|_{x_d=0} |(hD_{x_d}v_h)|_{x_d=0} \right)_0 \\ &\quad \left(\text{resp. } \lim_{h \rightarrow 0} 2 \text{Re} \left(b(x', 0, hD'')(v_h)|_{x_d=0} |(hD_{x_d}v_h)|_{x_d=0} \right)_0 \right). \end{aligned} \quad (91)$$

In particular $(b(x', 0, hD')(v_h)|_{x_d=0} |(hD_{x_d}v_h)|_{x_d=0})_0$ (resp. $(b(x', 0, hD'')(v_h)|_{x_d=0} |(hD_{x_d}v_h)|_{x_d=0})_0$), have limits as $h \rightarrow 0$.

Remark 7. In what follow we write $b(x', 0, hD')$ even if b only depends on variables (x, ξ'') . With Proposition 3.16 we can only proof that $(b(x', 0, hD')(v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0$, is a $o(h^{-1})$. In the proof below we show that the quantity has a limit and converges to the left hand side of (91). We are not able to prove that $(v_h)|_{x_d=0}$ or $(hD_{x_d}v_h)|_{x_d=0}$ are bounded.

Proof. We recall (56) the integration by parts formula in semiclassical context.

$$(u|hD_{x_d}w)_{L^2(x_d>0)} = (hD_{x_d}u|w)_{L^2(x_d>0)} - ih(u|_{x_d=0}|w|_{x_d=0})_0,$$

for u and w sufficiently smooth.

To proof the lemma we compute the following quantity by two different manners.

$$\begin{aligned} A &= ih^{-1}(b(x, hD')(h^2D_{x_d}^2 + R(x, hD') - 1 + iha)v_h|v_h)_{L^2(x_d>0)} \\ &\quad - ih^{-1}(b(x, hD')v_h|(h^2D_{x_d}^2 + R(x, hD') - 1 + iha)v_h)_{L^2(x_d>0)} \\ &= ih^{-1}(b(x, hD')hq_h|v_h)_{L^2(x_d>0)} - ih^{-1}(b(x, hD')v_h|hq_h)_{L^2(x_d>0)}. \end{aligned}$$

Then $|A| \lesssim \|q_h\|_{L^2(x_d>0)}\|v_h\|_{L^2(x_d>0)} \rightarrow 0$ as $h \rightarrow 0$ from (88). To compute A we now integrate by parts. We have

$$\begin{aligned} (b(x, hD')v_h|(R(x, hD') - 1)v_h)_{L^2(x_d>0)} &= ((R(x, hD') - 1)b(x, hD')v_h|v_h)_{L^2(x_d>0)}, \\ (ihb(x, hD')av_h|v_h)_{L^2(x_d>0)} - (b(x, hD')v_h|ihav_h)_{L^2(x_d>0)} \\ &= 2ih(\text{op}_{sc}(b(x, \xi')a(x))v_h|v_h)_{L^2(x_d>0)} + \mathcal{O}(h^2), \end{aligned}$$

as first $R(x, hD')$ is self-adjoint and does not contain derivative with respect x_d and second by symbol calculus $b(x, hD')a = ab(x, hD') = \text{op}_{sc}(a(x)b(x, \xi'))$ up to $\mathcal{O}(h)$. We have

$$\begin{aligned} (b(x, hD')v_h|h^2D_{x_d}^2v_h)_{L^2(x_d>0)} &= (hD_{x_d}b(x, hD')v_h|hD_{x_d}v_h)_{L^2(x_d>0)} \\ &\quad - ih(b(x', 0, hD')(v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0 \\ &= (h^2D_{x_d}^2b(x, hD')v_h|v_h)_{L^2(x_d>0)} \\ &\quad - ih(b(x', 0, hD')(hD_{x_d}v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 \\ &\quad - ih(b(x', 0, hD')(v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0. \end{aligned}$$

Then we have

$$\begin{aligned} A &= ih^{-1}([b(x, hD'), (h^2D_{x_d}^2 + R(x, hD') - 1)]v_h|v_h)_{L^2(x_d>0)} - 2(\text{op}_{sc}(b(x, \xi')a(x))v_h|v_h)_{L^2(x_d>0)} \\ &\quad - (b(x', 0, hD')(hD_{x_d}v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 - (b(x', 0, hD')(v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0 + \mathcal{O}(h). \end{aligned}$$

From the structure of b we claim that

$$[b(x, hD'), (h^2D_{x_d}^2 + R(x, hD') - 1)] = -ih \text{Op}_{sc}(\{b, p\}) + h^2 \text{op}_{sc}(r_0)$$

where $r_0 \in S(1, (dx)^2 + (d\xi')^2)$. Indeed, the assumptions imply that $b(x, \xi') \in S(1, (dx)^2 + (d\xi')^2)$, and $h^2D_{x_d}^2 + R$ is a sum of terms $c(x)h^2D_{x_j}D_{x_k}$. By exact symbol calculus we have

$$\begin{aligned} [b(x, hD'), c(x)h^2D_{x_j}D_{x_k}] &= [b(x, hD'), c(x)]h^2D_{x_j}D_{x_k} + c(x)[b(x, hD'), hD_{x_j}]hD_{x_k} \\ &\quad + c(x)hD_{x_j}[b(x, hD'), hD_{x_k}] \\ &= [b(x, hD'), c(x)]h^2D_{x_j}D_{x_k} + c(x)[b(x, hD'), hD_{x_j}]hD_{x_k} \\ &\quad + c(x)[b(x, hD'), hD_{x_k}]hD_{x_j} + c(x)[hD_{x_j}, [b(x, hD'), hD_{x_k}]] \\ &= -ih \text{Op}_{sc}(\{b, c\xi_j\xi_k\}) + h^2 \text{op}_{sc}(\tilde{r}_0), \end{aligned}$$

where \tilde{r}_0 is in $S(1, (dx)^2 + (d\xi')^2)$.

First $\{b, p\} = -2\xi_d \partial_{x_d} b + \{b, R\}$, we can apply Propositions 4.2 and 4.6 as $\{b, R\}$ is the sum of terms in $S(1, (dx)^2 + (d\xi')^2)$ or of the form $q(x, \xi'')$, $q(x, \xi'')\xi_1$ and $q(x, \xi'')\xi_1^2$, where $q \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-2})$. Second $|(\text{op}_{sc}(r_0)v_h|v_h)| \lesssim \|v_h\|_{L^2(x_d>0)}^2$ we can conclude that

$$\begin{aligned} & ih^{-1}([b(x, hD'), (h^2 D_{x_d}^2 + R(x, hD') - 1)]v_h|v_h)_{L^2(x_d>0)} - 2(\text{op}_{sc}(b(x, \xi')a(x))v_h|v_h)_{L^2(x_d>0)} \\ & \rightarrow \langle \mu, \{b, p\} - 2ba \rangle \text{ as } h \rightarrow 0. \end{aligned}$$

By symbol calculus we have $\text{op}_{sc}(b(x', 0, \xi'))^* = \text{op}_{sc}(b(x', 0, \xi')) + h \text{op}_{sc}(r_0)$, where $r_0 \in S(1, (dx)^2 + (d\xi')^2)$. Then $(b(x', 0, hD')(hD_{x_d}v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 = ((hD_{x_d}v_h)|_{x_d=0}|b(x', 0, hD')(v_h)|_{x_d=0})_0 + B$, where $|B| \leq h|(v_h)|_{x_d=0}|_{H^{1/2}}|(hD_{x_d}v_h)|_{x_d=0}|_{H^{-1/2}} \rightarrow 0$ as $h \rightarrow 0$ by (88). This gives the conclusion of Lemma. \square

Proposition 4.8. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ be a real valued we have the following formula*

$$\begin{aligned} \langle -H_p \mu + 2a\mu, b\xi_d \rangle &= \langle \mu, \{p, \xi_d b\} + 2ab\xi_d \rangle \\ &= \lim_{h \rightarrow 0} \text{Re} \left((b(x', 0, hD')(R(x', 0, hD') - 1)(v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 \right. \\ & \quad \left. - (b(x', 0, hD')(hD_{x_d}v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0 \right). \end{aligned} \quad (92)$$

In particular this means that

$$\begin{aligned} & \text{Re} \left((b(x', 0, hD')(R(x', 0, hD') - 1)(v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 \right. \\ & \quad \left. - (b(x', 0, hD')(hD_{x_d}v_h)|_{x_d=0}|(hD_{x_d}v_h)|_{x_d=0})_0 \right), \end{aligned}$$

has a limit as $h \rightarrow 0$.

Remark 8. *As in Proposition 4.7 the right hand side of (92) does not have a priori limit and we do not know if each term of the sum has a limit.*

Proof. We begin by an observation on regularity of traces. From the definition of q_h (see (25)), the terms $\theta(h^2 P)g_j^h$ are in $\mathcal{D}(P)$ for $j = 1, 2$. The term $[\theta(h^2 P), a]u_h = \theta(h^2 P)(au_h) - a\theta(h^2 P)u_h$ is in $\mathcal{D}(P)$, it is clear for $\theta(h^2 P)(au_h)$ and $a\theta(h^2 P)u_h$ is in $H^1(\Omega)$, a direct computation shows that $P(a\theta(h^2 P)u_h)$ is in $L^2(\Omega)$ and $a\theta(h^2 P)u_h$ satisfies the Zaremba trace condition as $\theta(h^2 P)u_h$ satisfies it. This implies that $q_h \in H^1(\Omega)$. In particular we have $h^2 D_{x_d}^2 v_h = hq_h - (R(x, hD') - 1)v_h$, then $(\text{op}_{sc}(b)h^2 D_{x_d}^2 v_h)|_{x_d=0} = (h \text{op}_{sc}(b)q_h)|_{x_d=0} - (\text{op}_{sc}(b)(R(x, hD') - 1 + iha)v_h)|_{x_d=0} \in L^2(x_d = 0)$, for b compactly supported and using properties (88). In this analysis we do not estimate the size of the norm with respect h but this allows to give a sense to some terms appearing in what follows. We introduce the following quantity which is real

$$\begin{aligned} A &= ih^{-1} \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h|(h^2 D_{x_d}^2 + R(x, hD') - 1 + iha)v_h)_{L^2(x_d>0)} \right. \\ & \quad \left. - ((h^2 D_{x_d}^2 + R(x, hD') - 1 + iha)v_h|(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{L^2(x_d>0)} \right) \\ &= \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h|q_h)_{L^2(x_d>0)} \right. \\ & \quad \left. - (q_h|(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{L^2(x_d>0)} \right). \end{aligned}$$

As $hD_{x_d}b(x, hD')^* = b(x, hD')^*hD_{x_d} + \text{op}_{sc}(r_0)$ where $r_0 \in S_{\text{tan}}^0$, we obtain

$$|A| \lesssim \|q_h\|_{L^2(x_d>0)} (\|v_h\|_{L^2(x_d>0)} + \|hD_{x_d}v_h\|_{L^2(x_d>0)}) \rightarrow 0 \text{ as } h \rightarrow 0,$$

by (88).

Let

$$B = ((h^2 D_{x_d}^2 + R(x, hD') - 1 + iha)v_h|(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{L^2(x_d>0)}. \quad (93)$$

We have by integrations by parts

$$\begin{aligned}
B &= \left((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)(h^2D_{x_d}^2 + R(x, hD') - 1 + iha)v_h|v_h \right)_{L^2(x_d>0)} \\
&\quad + \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)ihav_h|v_h) \right)_{L^2(x_d>0)} \\
&\quad - ih \left(((b(x, hD') + b(x, hD')^*)(h^2D_{x_d}^2 + R(x, hD') - 1)v_h)_{|x_d=0} |(v_h)_{|x_d=0} \right)_0. \tag{94}
\end{aligned}$$

Let

$$C = \left((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h|h^2D_{x_d}^2v_h \right)_{L^2(x_d>0)}.$$

We have by integration by parts

$$\begin{aligned}
C &= \left(hD_{x_d}(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h|hD_{x_d}v_h \right)_{L^2(x_d>0)} \\
&\quad - ih \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{|x_d=0} |(hD_{x_d}v_h)_{|x_d=0} \right)_0 \\
&= \left(h^2D_{x_d}^2(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h|v_h \right)_{L^2(x_d>0)} \\
&\quad - ih \left([hD_{x_d}(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h]_{|x_d=0} |(v_h)_{|x_d=0} \right)_0 \\
&\quad - ih \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{|x_d=0} |(hD_{x_d}v_h)_{|x_d=0} \right)_0. \tag{95}
\end{aligned}$$

The terms with damping term a , coming from A and B , give a term

$$\begin{aligned}
&-4(iahb(x, hD')hD_{x_d}v_h|v_h)_{L^2(x_d>0)} + h^2(\text{op}_{sc}(r_0)v_h|v_h)_{L^2(x_d>0)} \\
&\quad + h^2(\text{op}_{sc}(\tilde{r}_0)hD_{x_d}v_h|v_h)_{L^2(x_d>0)},
\end{aligned}$$

where $r_0, \tilde{r}_0 \in S_{\text{tan}}^0$. From this, (94) and (95) we obtain

$$\begin{aligned}
A &= ih^{-1} \left([h^2D_{x_d}^2 + R(x, hD') - 1, b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*]v_h|v_h \right)_{L^2(x_d>0)} \\
&\quad + 4(ab(x, hD')hD_{x_d}v_h|v_h)_{L^2(x_d>0)} + h\mathcal{O}(\|v_h\|_{L^2(x_d>0)}^2 + \|hDv_h\|_{L^2(x_d>0)}^2) \\
&\quad - \left(((b(x, hD') + b(x, hD')^*)(h^2D_{x_d}^2 + R(x, hD') - 1)v_h)_{|x_d=0} |(v_h)_{|x_d=0} \right)_0 \\
&\quad + \left([hD_{x_d}(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h]_{|x_d=0} |(v_h)_{|x_d=0} \right)_0 \\
&\quad + \left(((b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*)v_h)_{|x_d=0} |(hD_{x_d}v_h)_{|x_d=0} \right)_0. \tag{96}
\end{aligned}$$

By symbol calculus we have

$$ih^{-1}[h^2D_{x_d}^2 + R(x, hD') - 1, b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*] \tag{97}$$

$$= \text{Op}_{sc}\{\xi_d^2 + R(x, \xi') - 1, 2b(x, \xi')\xi_d\} + h\text{op}_{sc}(r_0) + h\text{op}_{sc}(\tilde{r}_0)hD_{x_d}, \tag{98}$$

where $r_0, \tilde{r}_0 \in S_{\text{tan}}^0$. Propositions 4.2, 4.5 and 4.6 imply that

$$(\text{Op}_{sc}\{\xi_d^2 + R(x, \xi') - 1, 2b(x, \xi')\xi_d\}v_h|v_h)_{L^2(x_d>0)} \rightarrow \langle \mu, \{p, 2b\} \rangle \text{ as } h \rightarrow 0. \tag{99}$$

By Proposition 4.6 the term $4(ab(x, hD')hD_{x_d}v_h|v_h)_{L^2(x_d>0)} \rightarrow \langle \mu, 4ba\xi_d \rangle$ as $h \rightarrow 0$. And we have

$$|(h\text{op}_{sc}(r_0)v_h|v_h)_{L^2(x_d>0)}| \lesssim h\|v_h\|_{L^2(x_d>0)}^2 \rightarrow 0 \text{ as } h \rightarrow 0,$$

$$|(h\text{op}_{sc}(\tilde{r}_0)hD_{x_d}v_h|v_h)_{L^2(x_d>0)}| \lesssim h\|hD_{x_d}v_h\|_{L^2(x_d>0)}\|v_h\|_{L^2(x_d>0)} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then this and (99) imply that

$$\begin{aligned}
&ih^{-1} \left([h^2D_{x_d}^2 + R(x, hD') - 1, b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*]v_h|v_h \right)_{L^2(x_d>0)} \\
&\quad + 4(ab(x, hD')hD_{x_d}v_h|v_h)_{L^2(x_d>0)} \\
&\quad \rightarrow \langle \mu, \{p, 2b\} + 4ab\xi_d \rangle \text{ as } h \rightarrow 0. \tag{100}
\end{aligned}$$

We now treat the boundary terms coming from (96). We have by symbol calculus

$$\begin{aligned} b(x, hD') + b(x, hD')^* &= 2b(x, hD') + h \operatorname{op}_{sc}(r_0), \\ b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^* &= 2b(x, hD')hD_{x_d} + h \operatorname{op}_{sc}(r_0), \end{aligned}$$

$$\begin{aligned} hD_{x_d}(b(x, hD')hD_{x_d} + hD_{x_d}b(x, hD')^*) \\ = (b(x, hD') + b(x, hD')^*)h^2D_{x_d}^2 + h \operatorname{op}_{sc}(r_0) + h \operatorname{op}_{sc}(\tilde{r}_0)hD_{x_d}, \end{aligned}$$

where $r_0, \tilde{r}_0 \in S_{\tan}^0$. Then boundary terms of A can be written as

$$\begin{aligned} - \left(((2b(x, hD'))(R(x, hD') - 1)v_h)|_{x_d=0} |(v_h)|_{x_d=0} \right)_0 \\ + \left((2b(x, hD')hD_{x_d}v_h)|_{x_d=0} |(hD_{x_d}v_h)|_{x_d=0} \right)_0, \quad (101) \end{aligned}$$

up to terms estimated by $h|((\operatorname{op}(r_0)v_h)|_{x_d=0} |(v_h)|_{x_d=0})_0| + h|((\operatorname{op}(\tilde{r}_0)hD_{x_d}v_h)|_{x_d=0} |(v_h)|_{x_d=0})_0|$, and these terms converge to 0 from (88). Recalling that $A \rightarrow 0$ as $h \rightarrow 0$ and A real valued, we deduce Proposition 4.8 from (96), (100) and (101). \square

4.2 Properties and support of semiclassical measure

Here we decompose the measure into an interior measure and a measure supported at boundary. Moreover this last measure is supported on $\xi_d = 0$.

Lemma 4.9. *There exists a non negative Radon measure μ^∂ on $x_d = \xi_d = 0$ such that $\mu = 1_{x_d > 0}\mu + \mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$. Furthermore μ^∂ is supported on $R(x', 0, \xi') - 1 = 0$.*

Proof. We apply Proposition 4.8. We observe that $H_p = 2\xi_d \partial_{x_d} - (\partial_{x_d} R) \partial_{\xi_d} + H'_R$ where $H'_R = \sum_{j=1}^{d-1} ((\partial_{\xi_j} R) \partial_{x_j} - (\partial_{x_j} R) \partial_{\xi_j})$. We have $H_p(\xi_d b) = 2\xi_d^2 (\partial_{x_d} b) - (\partial_{x_d} R) b + \xi_d H'_R b$. Let $b = b^\varepsilon = \varepsilon \chi(x_d/\varepsilon) \ell(x, \xi')$, where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, such that $\chi(0) = 0$ and $\chi'(0) = 1$, $\ell \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$. We have

$$\begin{aligned} H_p(\xi_d b^\varepsilon) + 2ab^\varepsilon \xi_d &= 2\xi_d^2 \chi'(x_d/\varepsilon) \ell(x, \xi') + 2\varepsilon \chi(x_d/\varepsilon) \xi_d^2 \partial_{x_d} \ell - (\partial_{x_d} R) \varepsilon \chi(x_d/\varepsilon) \ell(x, \xi') \\ &\quad + \xi_d \varepsilon \chi(x_d/\varepsilon) H'_R \ell + 2\varepsilon a \chi(x_d/\varepsilon) \ell(x, \xi'). \end{aligned}$$

Clearly $H_p(\xi_d b^\varepsilon) + 2ab^\varepsilon \xi_d$ is uniformly bounded on the support of μ and $H_p(\xi_d b^\varepsilon) + 2ab^\varepsilon \xi_d \rightarrow 2\xi_d^2 \chi'(0) \ell 1_{x_d=0}$ everywhere as $\varepsilon \rightarrow 0$. Then by Lebesgue's dominated convergence theorem we have $\langle \mu, H_p(\xi_d b^\varepsilon) + 2ab^\varepsilon \xi_d \rangle \rightarrow \langle \mu, 2\xi_d^2 \ell 1_{x_d=0} \rangle$ as $\varepsilon \rightarrow 0$. As $\chi(0) = 0$, the right hand side of (92) is 0 for every h . Then $\langle \mu, 2\xi_d^2 \ell 1_{x_d=0} \rangle = 0$. This means that $1_{x_d=0}\mu$ is supported on $\xi_d = 0$. We denote μ^∂ the measure $1_{\xi_d=0} 1_{x_d=0} \mu$. As $\mu = 1_{x_d > 0} \mu + 1_{x_d=0} \mu$, we have $\mu = 1_{x_d > 0} \mu + \mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$. We have by Proposition 3.1, $(\xi_d^2 + R(x, \xi') - 1) 1_{x_d=0} \mu = 0$, then $(R(x', 0, \xi') - 1) \mu^\partial = 0$. This gives the conclusion of Lemma. \square

The Hamiltonian of the interior measure is a priori a distribution of order one supported on $x_d = 0$. The following lemma says that this quantity is a measure if the Hamiltonian vector field is transverse to the boundary.

Lemma 4.10. *We assume that $a\mu = 0$. There exists a distribution (of order 1) μ_0 defined on $x_d = 0$, such that $H_p(\mu 1_{x_d > 0}) = \delta_{x_d=0} \otimes \mu_0$. Moreover, in a neighborhood where $H_p x_d > 0$, μ_0 is a non negative Radon measure, and in a neighborhood where $H_p x_d < 0$, μ_0 is a non positive Radon measure.*

Proof. The support of $H_p(\mu 1_{x_d > 0})$ as a distribution is $x_d \geq 0$ and $H_p(\mu) = 0$ on $x_d > 0$. Then $H_p(\mu 1_{x_d > 0})$ is supported on $x_d = 0$. This implies that there exist $n \geq 0$ and μ_j distributions on

$x_d = 0$ such that $H_p(\mu 1_{x_d > 0}) = \sum_{j=0}^n \delta_{x_d=0}^{(j)} \otimes \mu_k$. Suppose that $n \geq 1$ and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\chi^{(k)}(0) = 0$ for $k = 0, \dots, n-1$ and $\chi^{(n)}(0) = 1$. Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, we have

$$\begin{aligned} -\langle \mu 1_{x_d > 0}, H_p(\varepsilon^n \chi(x_d/\varepsilon) b(x, \xi')) \rangle &= \langle H_p \mu, \varepsilon^n \chi(x_d/\varepsilon) b(x, \xi') \rangle \\ &= \left\langle \sum_{j=0}^n \delta_{x_d=0}^{(j)} \otimes \mu_k, \varepsilon^n \chi(x_d/\varepsilon) b(x, \xi') \right\rangle = \langle \mu_n, b(x', 0, \xi') \rangle. \end{aligned} \quad (102)$$

We also have

$$H_p(\varepsilon^n \chi(x_d/\varepsilon) b(x, \xi')) = 2\xi_d \varepsilon^{n-1} \chi'(x_d/\varepsilon) b(x, \xi') + \varepsilon^n \chi(x_d/\varepsilon) H_p b(x, \xi'),$$

which is bounded uniformly with respect ε and supported in a fixed compact set. Then if $n \geq 2$, $H_p(\varepsilon^n \chi(x_d/\varepsilon) b(x, \xi')) \rightarrow 0$ everywhere as $\varepsilon \rightarrow 0$ and (102), Lebesgue's dominated convergence theorem imply that $\mu_n = 0$ for $n \geq 2$. If $n = 1$, $H_p(\varepsilon \chi(x_d/\varepsilon) b(x, \xi')) \rightarrow 2\xi_d 1_{x_d=0} b(x, \xi')$ everywhere as $\varepsilon \rightarrow 0$. Lebesgue's dominated convergence theorem and (102) imply that

$$-\langle \mu 1_{x_d > 0}, 2\xi_d 1_{x_d=0} b(x, \xi') \rangle = \langle \mu_1, b(x', 0, \xi') \rangle,$$

as $\langle \mu 1_{x_d > 0}, 2\xi_d 1_{x_d=0} b(x, \xi') \rangle = 0$ we find that $\mu_1 = 0$. Then we have $H_p(\mu 1_{x_d > 0}) = \delta_{x_d=0} \otimes \mu_0$, where μ_0 is a distribution of order 1.

If $H_p x_d \neq 0$, let $(x(s; x', \xi), \xi(s; x', \xi))$ the solution to $(\dot{x}, \dot{\xi}) = H_p$ satisfying the initial condition $(x(0; x', \xi), \xi(0; x', \xi)) = (x', 0, \xi)$. We verify that the map $(s, x', \xi) \rightarrow (x(s; x', \xi), \xi(s; x', \xi))$ locally is one to one and transforms ∂_s in H_p . Moreover, $s = 0$ is transformed in $x_d = 0$ and if $H_p x_d > 0$, $s > 0$ is transformed in $x_d > 0$, if $H_p x_d < 0$, $s < 0$ is transformed in $x_d > 0$. In coordinates (s, x', ξ) the equation $H_p(\mu 1_{x_d > 0}) = \delta_{x_d=0} \otimes \mu_0$ is transformed in $\partial_s(\mu 1_{s > 0}) = \delta_{s=0} \otimes \mu_0$ if $H_p x_d > 0$ and $\partial_s(\mu 1_{s < 0}) = \delta_{s=0} \otimes \mu_0$ if $H_p x_d < 0$, where we keep the notations μ, μ_0 in variables (s, x', ξ) for the images of μ, μ_0 . If $H_p x_d > 0$, we have $\mu 1_{s > 0} = (1_{s > 0} ds) \otimes \mu_0$ and if $H_p x_d < 0$, we have $\mu 1_{s < 0} = -(1_{s < 0} ds) \otimes \mu_0$. As μ is non negative, we obtain that μ_0 is a measure and its sign. \square

At the hyperbolic region the measure μ_0 has a particular structure given by this lemma.

Lemma 4.11. *We assume that $a\mu = 0$. Let (x'_0, ξ'_0) be a hyperbolic point (i.e. $R(x'_0, 0, \xi'_0) < 1$). Locally in a neighborhood of (x'_0, ξ'_0) , there exist μ^+ and μ^- non negative measures on $\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1}$ such that $\mu_0 = \mu^+ \otimes \delta_{\xi_d = \sqrt{1-R(x', 0, \xi')}} - \mu^- \otimes \delta_{\xi_d = -\sqrt{1-R(x', 0, \xi')}}$. Moreover, if in a neighborhood of $x'_0, (v_h)_h$ satisfies the Dirichlet boundary condition or the Neumann boundary condition then $\mu^+ = \mu^-$.*

In a neighborhood of Γ we have the following property, if $\mu^+ = 0$ (resp $\mu^- = 0$) then $\mu^- = 0$ (resp $\mu^+ = 0$).

Proof. As μ is supported on $\xi_d^2 + R(x, \xi') - 1 = 0$, this implies that $\delta_{x_d=0} \otimes \mu_0$ is supported on $\xi_d^2 + R(x, \xi') - 1 = 0$, then μ_0 is supported on $\xi_d = \pm \sqrt{1 - R(x', 0, \xi')}$. Moreover $H_p = 2\xi_d \partial_{x_d} + X$ where X is a vector field tangent to $x_d = 0$, in particular $H_p x_d = 2\xi_d$. This implies that $H_p x_d > 0$ if $\xi_d = \sqrt{1 - R(x', 0, \xi')}$ and $H_p x_d < 0$ if $\xi_d = -\sqrt{1 - R(x', 0, \xi')}$. From Lemma 4.10, we obtain $\mu_0 = \mu^+ \otimes \delta_{\xi_d = \sqrt{1-R(x', 0, \xi')}} - \mu^- \otimes \delta_{\xi_d = -\sqrt{1-R(x', 0, \xi')}}$. From Proposition 4.7 if $b|_{x_d=0}$ is supported on a part of the boundary such that $(v_h)_h$ satisfies the Dirichlet boundary condition (resp. Neumann boundary condition), then the right hand side of (91) is 0. Then we have $\langle H_p \mu, b \rangle = 0$ which implies $\langle \mu^+ - \mu^-, b|_{x_d=0} \rangle = 0$ as $b|_{x_d=0}$ describes every \mathcal{C}_0^∞ function supported in a neighborhood of (x'_0, ξ'_0) , this implies that $\mu^+ = \mu^-$ in a neighborhood of (x'_0, ξ'_0) .

Now to obtain the result in a neighborhood of Γ , it suffices to take $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-2})$ positive. Here we use notation defined above Formula (64). As $\mathcal{C}_0^\infty(x_1 > 0)$ is dense in $H^{1/2}(x_1 \geq 0)$ (see [33, Theorem 11.1]), we can approach in $H_{sc}^{1/2}$ (for h fixed) $(v_h)|_{x_d=0}$ by a sequence $(w_n)_n$ of smooth functions supported on $x_1 > 0$. We have $(b(x', 0, hD'') w_n | (hD_{x_d} v_h)|_{x_d=0})_0 = 0$, by support properties and passing to the limit we have $(b(x', 0, hD'') (v_h)|_{x_d=0} | (hD_{x_d} v_h)|_{x_d=0})_0 = 0$, for each h . As above we obtain $\langle \mu^+ - \mu^-, b|_{x_d=0} \rangle = 0$. If $\mu^- = 0$ we have $\langle \mu^+, b|_{x_d=0} \rangle = 0$ for all b independent of ξ_1 supported on a neighborhood of a point ρ of $\Gamma \times \mathbb{R}^{d-1}$, as μ^+ is a non negative measure we obtain $\mu^+ = 0$ in a neighborhood of ρ . \square

We recall that $\mathcal{G}_d = \{(x', \xi'), R(x', 0, \xi') = 1 \text{ and } \partial_{x_d} R(x', 0, \xi') < 0\}$ is the set of diffractive points.

The following lemma states that the diffractive points whose projection belong to $\partial\Omega_D$ are not in the support of the measure.

Lemma 4.12. *We assume $a\mu = 0$. We have $1_{\mathcal{G}_d \cap (\partial\Omega_{DN} \times \mathbb{R}^{d-1})} \mu^\partial = 0$, where $\partial\Omega_{DN} = \partial\Omega_D \cup \partial\Omega_N$.*

Proof. We apply Proposition 4.8. We have to choose an adapted function b . Let $\chi \in \mathcal{C}^\infty$ be such that $\chi(\sigma) = 0$ if $|\sigma| \geq 2$, $\chi(\sigma) = 1$ if $|\sigma| \leq 1$. We apply Proposition 4.8 with $b(x, \xi') = \chi((1 - R(x, \xi'))/\varepsilon) \ell(x, \xi') \chi(x_d/\varepsilon)$, where $\varepsilon > 0$ will be chosen in what follows and ℓ is supported in a neighborhood of a point of $\partial\Omega_{DN} \times \mathbb{R} \times \mathbb{R}^{d-1}$. We recall that $H_p = 2\xi_d \partial_{x_d} - (\partial_{x_d} R) \partial_{\xi_d} + H'_R$ (see the proof of Lemma 4.9). We have

$$\begin{aligned} H_p(\xi_d b) &= 2\xi_d^2 \left(-(\partial_{x_d} R) \chi'((1 - R(x, \xi'))/\varepsilon) \ell(x, \xi') \chi(x_d/\varepsilon)/\varepsilon \right. \\ &\quad + \chi((1 - R(x, \xi'))/\varepsilon) (\partial_{x_d} \ell(x, \xi')) \chi(x_d/\varepsilon) \\ &\quad + \chi((1 - R(x, \xi'))/\varepsilon) \ell(x, \xi') \chi'(x_d/\varepsilon)/\varepsilon \left. \right) \\ &\quad - (\partial_{x_d} R) \chi((1 - R(x, \xi'))/\varepsilon) \ell(x, \xi') \chi(x_d/\varepsilon) \\ &\quad + \chi((1 - R(x, \xi'))/\varepsilon) \chi(x_d/\varepsilon) \xi_d H'_R \ell(x, \xi'). \end{aligned} \quad (103)$$

We claim

$$\begin{aligned} H_p(b\xi_d) &\text{ is uniformly bounded on } \xi_d^2 + R(x, \xi') = 1, \\ H_p(b\xi_d) &\rightarrow -(\partial_{x_d} R(x', 0, \xi')) 1_{R(x', 0, \xi')=1} 1_{x_d=0} \ell(x', 0, \xi') \text{ as } \varepsilon \rightarrow 0 \\ &\text{ for all } (x, \xi), \text{ such that } \xi_d^2 + R(x, \xi') = 1. \end{aligned} \quad (104)$$

As μ is supported on $\xi_d^2 + R(x, \xi') - 1 = 0$, then $\xi_d^2/\varepsilon = (1 - R(x, \xi'))/\varepsilon$ on the support of μ , this implies that the three first terms in (103) are bounded. It is easy to prove that they converge to 0 as ε to 0. The fourth term is bounded and converges to $-(\partial_{x_d} R(x', 0, \xi')) 1_{R(x', 0, \xi')=1} 1_{x_d=0} \ell(x', 0, \xi')$. In the last term as $|R(x, \xi') - 1|/\varepsilon$ is bounded, thus $|\xi_d|$ is bounded by $C\sqrt{\varepsilon}$ and then this term converges to 0 as $\varepsilon \rightarrow 0$. This proves (104). From that we can conclude that $\langle \mu, H_p(\ell\xi_d) \rangle$ converges to

$$\begin{aligned} \langle \mu, -(\partial_{x_d} R(x', 0, \xi')) 1_{R(x', 0, \xi')=1} 1_{x_d=0} \ell(x', 0, \xi') \rangle \\ = \langle \mu^\partial, -(\partial_{x_d} R(x', 0, \xi')) 1_{R(x', 0, \xi')=1} \ell(x', 0, \xi') \rangle, \end{aligned} \quad (105)$$

as $\varepsilon \rightarrow 0$, which is a non negative term if l is non negative and supported in a neighborhood of a point of \mathcal{G}_d .

We now assume ℓ supported on a neighborhood of a point of $\partial\Omega_D$. Let

$$\begin{aligned} A_\varepsilon &= \lim_{h \rightarrow 0} \left((b(x', 0, hD') (R(x', 0, hD') - 1) (v_h)|_{x_d=0} | (v_h)|_{x_d=0})_0 \right. \\ &\quad \left. - (b(x', 0, hD') (hD_{x_d} v_h)|_{x_d=0} | (hD_{x_d} v_h)|_{x_d=0})_0 \right) \\ &= - \lim_{h \rightarrow 0} (b(x', 0, hD') (hD_{x_d} v_h)|_{x_d=0} | (hD_{x_d} v_h)|_{x_d=0})_0, \end{aligned} \quad (106)$$

as v_h satisfies the Dirichlet boundary condition. We want to prove that $A_\varepsilon \leq 0$. By Gårding inequality (9) (in fact used for \mathbb{R}^{d-1} instead \mathbb{R}^d) we have

$$-(b(x', 0, hD') (hD_{x_d} v_h)|_{x_d=0} | (hD_{x_d} v_h)|_{x_d=0})_0 \leq C_\varepsilon h |(hD_{x_d} v_h)|_{x_d=0}|_{L^2}^2.$$

Taking the limit as $h \rightarrow 0$, Proposition 3.12 implies that $A_\varepsilon \leq 0$. As (105) is non negative, this prove that

$$\langle \mu^\partial, -(\partial_{x_d} R(x', 0, \xi')) 1_{R(x', 0, \xi')=1} \ell(x', 0, \xi') \rangle = 0. \quad (107)$$

Then $1_{\mathcal{G}_d}\mu^\partial = 0$ on $\partial\Omega_D$ as ℓ can be arbitrary chosen.

We now assume ℓ supported on a neighborhood of a point of $\partial\Omega_N$. With A_ε defined in (106) we have, as $(hD_{x_d}v_h)|_{x_d=0} = 0$ on the support of ℓ

$$\begin{aligned} A_\varepsilon &= \lim_{h \rightarrow 0} (b(x', 0, hD')(R(x', 0, hD') - 1)(v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0 \\ &= \lim_{h \rightarrow 0} (\text{op}_{sc}(\chi((1 - R(x', 0, \xi'))/\varepsilon)\ell(x', 0, \xi')(R(x', 0, h\xi'') - 1))(v_h)|_{x_d=0}|(v_h)|_{x_d=0})_0. \end{aligned}$$

From Proposition B.7, A_ε goes to 0 as $\varepsilon \rightarrow 0$. Then from (105) and Proposition 4.8 we obtain (107) in this case and $1_{\mathcal{G}_d}\mu^\partial = 0$ on $\partial\Omega_N$. \square

This lemma describes how the support of the boundary measure propagates along the boundary.

Lemma 4.13. *Let $(x'_0, \xi'_0) \in T^*\partial\Omega$ be such that $x'_0 \in \partial\Omega_D \cup \partial\Omega_N$. Let $(x, \xi) \in T^*\Omega$, we denote $\gamma_{(x, \xi)}$ the integral curve of H_p starting from (x, ξ) . We assume that μ is locally supported in a neighborhood of $(x'_0, 0, \xi'_0, 0)$, in the set $\{(x, \xi), \xi_d^2 + R(x, \xi') = 1, \text{ such that } \gamma_{(x, \xi)} \text{ hits } x_d = 0, \xi_d = 0\}$. In particular μ_0 is supported on $\xi_d = 0$ and $\mu_0 = \tilde{\mu}_0 \otimes \delta_{\xi_d=0} + \tilde{\mu}_1 \otimes \delta'_{\xi_d=0}$, where $\tilde{\mu}_0$ is a distribution and $\tilde{\mu}_1$ is a Radon measure. Then $H'_R\mu^\partial - 2a\mu^\partial + \tilde{\mu}_0 = 0$, $\tilde{\mu}_1 = 0$ and $H_p\mu = -\partial_{x_d}R(x', 0, \xi')\mu^\partial \otimes \delta_{x_d=0} \otimes \delta'_{\xi_d=0} + 2a\mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$, in a neighborhood of $(x'_0, 0, \xi'_0, 0)$.*

Proof. We have $\mu = 1_{x_d>0}\mu + \mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$. Then we have

$$\begin{aligned} H_p\mu &= \mu_0 \otimes \delta_{x_d=0} + 2\xi_d\mu^\partial \otimes \delta'_{x_d=0} \otimes \delta_{\xi_d=0} - (\partial_{x_d}R(x', 0, \xi'))\mu^\partial \otimes \delta_{x_d=0} \otimes \delta'_{\xi_d=0} \\ &\quad + H'_R\mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0} \\ &= \mu_0 \otimes \delta_{x_d=0} - (\partial_{x_d}R(x', 0, \xi'))\mu^\partial \otimes \delta_{x_d=0} \otimes \delta'_{\xi_d=0} + H'_R\mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}, \end{aligned} \quad (108)$$

as $\xi_d\mu^\partial \delta'_{x_d=0} \otimes \delta_{\xi_d=0} = 0$. As μ_0 is of order 1 and supported on $\xi_d = 0$, we have $\mu_0 = \tilde{\mu}_0 \otimes \delta_{\xi_d=0} + \tilde{\mu}_1 \otimes \delta'_{\xi_d=0}$, where $\tilde{\mu}_j$ are distributions.

To prove that $\tilde{\mu}_1$ is a Radon measure, we test μ_0 on $\varphi = \xi_d\psi(x', \xi')\chi(\xi_d/\varepsilon)$. The first derivative of φ are estimated by supremum of $\psi(x', \xi')\chi(\xi_d/\varepsilon)$, $\psi(x', \xi')\chi'(\xi_d/\varepsilon)\xi_d/\varepsilon$ and $\xi_d\partial\psi(x', \xi')\chi(\xi_d/\varepsilon)$. When $\varepsilon \rightarrow 0$ the supremum on φ is estimated by supremum of $\psi(x', \xi')$. We also have $\langle \mu_0, \varphi \rangle$ converging to $\langle \tilde{\mu}_1, \psi(x', \xi') \rangle$ as $\varepsilon \rightarrow 0$. We deduce that $\langle \tilde{\mu}_1, \psi(x', \xi') \rangle$ is estimated by the supremum of $\psi(x', \xi')$, this implies that $\tilde{\mu}_1$ is a Radon measure.

By Proposition 4.7 and if $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ is supported in $\partial\Omega_D \cup \partial\Omega_N$ and in a neighborhood of $(x'_0, 0, \xi'_0, 0)$, we have $\langle H_p\mu - 2a\mu, b(x, \xi') \rangle = 0$.

Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, be such that $b|_{x_d=0} = \ell \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$, and $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, be such that $\chi(\sigma) = 1$ in a neighborhood of 0. From (108) we obtain

$$\langle H_p\mu - 2a\mu, b\chi(x_d/\varepsilon) \rangle = \langle \tilde{\mu}_0 + H'_R\mu^\partial - 2a\mu^\partial, \ell \rangle - 2\langle a1_{x_d>0}\mu, b\chi(x_d/\varepsilon) \rangle.$$

As $b\chi(x_d/\varepsilon)$ is uniformly bounded and $b\chi(x_d/\varepsilon)$ converges to $b1_{x_d=0}$ everywhere, we obtain

$$\langle a1_{x_d>0}\mu, b\chi(x_d/\varepsilon) \rangle \rightarrow \langle a1_{x_d>0}\mu, b1_{x_d=0} \rangle = 0 \text{ as } \varepsilon \rightarrow 0.$$

We deduce that $H'_R\mu^\partial - 2a\mu^\partial + \tilde{\mu}_0 = 0$, which gives the first conclusion of Lemma. We deduce from (108)

$$H_p\mu = (\tilde{\mu}_1 - (\partial_{x_d}R(x', 0, \xi'))\mu^\partial) \otimes \delta_{x_d=0} \otimes \delta'_{\xi_d=0} + 2a\mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}. \quad (109)$$

We then can write

$$\langle \mu, H_p b(x, \xi) \rangle = \langle \tilde{\mu}_1 - (\partial_{x_d}R(x', 0, \xi'))\mu^\partial, \partial_{\xi_d}b(x', 0, \xi', 0) \rangle - \langle 2a\mu^\partial, b(x', 0, \xi', 0) \rangle, \quad (110)$$

for $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

Now we choose an adapted b to apply (110). Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\chi(\sigma) = 1$ for σ in a neighborhood of 0. Let $\ell \in \mathcal{C}_0^\infty(x', \xi')$ be supported in a neighborhood of (x'_0, ξ'_0) . We set $b(x, \xi) = \xi_d\ell(x', \xi')\chi(\xi_d/\varepsilon)\chi(x_d/\varepsilon)$, where $\varepsilon > 0$. We have

$$\begin{aligned} H_p b(x, \xi) &= 2\ell(x', \xi')\chi(\xi_d/\varepsilon)\chi'(x_d/\varepsilon)\xi_d^2/\varepsilon - \ell(x', \xi')(\partial_{x_d}R(x, \xi'))\chi(x_d/\varepsilon)(\chi(\xi_d/\varepsilon) + \chi'(\xi_d/\varepsilon)\xi_d/\varepsilon) \\ &\quad + \xi_d H'_R(\ell(x', \xi'))\chi(\xi_d/\varepsilon)\chi(x_d/\varepsilon). \end{aligned}$$

As $\chi'(x_d, \varepsilon)\chi(\xi_d/\varepsilon)\xi_d^2/\varepsilon$, $\chi(x_d/\varepsilon)\chi(\xi_d/\varepsilon)\xi_d/\varepsilon$ and $\xi_d\chi(\xi_d/\varepsilon)\chi(x_d/\varepsilon)$ are uniformly bounded and goes to 0 as ε goes to 0, and $\chi(\xi_d/\varepsilon)\chi(x_d/\varepsilon)$ goes to $1_{x_d=0, \xi_d=0}$ and is uniformly bounded. We have $\langle \mu, H_p b \rangle$ goes to $\langle \mu, -1_{x_d=0, \xi_d=0}(\partial_{x_d} R(x', 0, \xi'))\ell(x', \xi') \rangle$ as ε goes to 0 and we have

$$\langle \mu, -1_{x_d=0, \xi_d=0}\partial_{x_d} R(x', 0, \xi')\ell(x', \xi') \rangle = -\langle \mu^\partial, (\partial_{x_d} R(x', 0, \xi'))\ell(x', \xi') \rangle. \quad (111)$$

The term $\langle 2a\mu^\partial, b(x', 0, \xi', 0) \rangle$ goes to 0 as ε , as b goes to 0 uniformly as ε goes to 0. Now we compute the limite as $\varepsilon \rightarrow 0$ of $\langle \tilde{\mu}_1, \partial_{\xi_d} b(x', 0, \xi', 0) \rangle$. We have

$$\partial_{\xi_d} b(x', x_d, \xi', \xi_d) = \ell(x', \xi')\chi(\xi_d/\varepsilon)\chi(x_d/\varepsilon) + \ell(x', \xi')\chi'(\xi_d/\varepsilon)\chi(x_d/\varepsilon)\xi_d/\varepsilon.$$

Then $\partial_{\xi_d} b(x', 0, \xi', 0) = \ell(x', \xi')$ and

$$\begin{aligned} \langle \tilde{\mu}_1 - (\partial_{x_d} R(x', 0, \xi'))\mu^\partial, \partial_{\xi_d} b(x', 0, \xi', 0) \rangle &= \langle \tilde{\mu}_1 - (\partial_{x_d} R(x', 0, \xi'))\mu^\partial, \ell(x', \xi') \rangle \\ &= -\langle \mu^\partial, (\partial_{x_d} R(x', 0, \xi'))\ell(x', \xi') \rangle \end{aligned}$$

from (110) and (111). We deduce that $\tilde{\mu}_1 = 0$ and the last result of the lemma from (109). \square

We have an analogous result to Lemma 4.13 in a neighborhood of Zaremba condition. To be precise we recall the notations defined in Formula (64), we have $R(x', 0, \xi') = \xi_1^2 + R_0(x'', \xi'') + x_1 r_2(x, \xi')$, where $x' = (x_1, x'')$ and $\xi' = (\xi_1, \xi'')$, $R_0 \in S(\langle \xi'' \rangle^2, \langle dx'' \rangle^2 + \langle \xi'' \rangle^{-2} \langle d\xi'' \rangle^2)$ and $r_2 \in S_{\tan}^2$.

Lemma 4.14. *We assume that $(v_h)_h$ satisfies the boundary Zaremba condition neighborhood of $(0, x''_0)$. Let $\ell \in \mathcal{C}_0^\infty(\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi''}^{d-2})$, where $\xi' = (\xi_1, \xi'')$. Then $\langle H'_R \mu^\partial - 2a\mu^\partial + \mu_0, \ell \rangle = 0$. In particular if μ^∂ is a measure supported on $x_1 = \xi_1 = 0$, this means that $\mu^\partial = \tilde{\mu}^\partial \otimes \delta_{x_1=\xi_1=0}$ and if μ_0 is supported on $x_1 = \xi_1 = \xi_d = 0$, this means that there exists $\tilde{\mu}_0$, and $\tilde{\mu}_{\alpha, \beta}$ distributions of order 1 on $\mathbb{R}_{x''}^{d-1} \times \mathbb{R}_{\xi''}^{d-1}$, for $\alpha = (0, 0, 0)$ or $(1, 0, 0)$ and $\beta \in \{(j, 0, k), j, k = 0 \text{ or } 1\}$ such that $\mu_0 = \tilde{\mu}_0 \otimes \delta_{x_1=\xi_1=\xi_d=0} + \sum_{|\alpha|+|\beta|=1} \tilde{\mu}_{\alpha, \beta} \otimes \partial_x^\alpha \partial_{\xi'}^\beta \delta_{x_1=\xi_1=\xi_d=0}$. Then $H''_{R_0} \tilde{\mu}^\partial - 2a\tilde{\mu}^\partial + \tilde{\mu}_0 = 0$, where $H''_{R_0} = \sum_{2 \leq j \leq d-1} (\partial_{\xi_j} R_0(x'', \xi'') \partial_{x_j} - \partial_{x_j} R_0(x'', \xi'') \partial_{\xi_j})$.*

Proof. Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-2})$ be such that $b|_{x_d=0} = \ell$. We apply Proposition 4.7 to $b(x, \xi'')$. As in the end of the proof of Lemma 4.11 we have $(\ell(x', hD'')(v_h)|_{x_d=0} | (hD_{x_d} v_h)|_{x_d=0})_0 = 0$. Then we have $\langle H_p \mu - 2a\mu, b(x, \xi'') \rangle = 0$. We follow the same ideas of the proof of Lemma 4.13. From Formula (108) we have

$$\langle H'_R \mu^\partial - 2a\mu^\partial + \mu_0, \ell \rangle - 2\langle a1_{x_d>0} \mu, b \rangle = 0.$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\chi(\sigma) = 1$ in a neighborhood of 0. Taking $b_\varepsilon(x, \xi'') = \chi(x_d/\varepsilon)\ell(x', \xi'')$ and letting ε goes to 0, we obtain $\langle H'_R \mu^\partial - 2a\mu^\partial + \mu_0, \ell \rangle = 0$. We have $H'_R = 2\xi_1 \partial_{x_1} + H''_{R_0} + H'_{x_1 r_2}$, and $H'_{x_1 r_2} \ell(x', \xi'') = x_1 H'_{r_2} \ell(x', \xi'')$, as ℓ independent of ξ_1 . We deduce from the form of μ^∂ that $H'_R \mu^\partial = H''_{R_0} \tilde{\mu}^\partial \otimes \delta_{x_1=\xi_1=0}$. Taking $\ell(x', \xi'') = \chi(x_1) \tilde{\ell}(x'', \xi'')$, we deduce from the form of μ_0 $\langle H'_R \mu^\partial - 2a\mu^\partial + \mu_0, \ell \rangle = \langle H''_{R_0} \tilde{\mu}^\partial - 2a\tilde{\mu}^\partial + \tilde{\mu}_0, \ell \rangle = 0$. This implies the result. \square

5 Support propagation results

In this section we prove propagation of support of semiclassical measure under the assumption mGCC, see Definition 1.2.

Proposition 5.1. *We assume that P , $a(x)$ and Ω satisfy mGCC. Let μ the semiclassical measure constructed from $(v_h)_h$ and satisfying (33), we have $\mu = 0$.*

We prove the propagation result, first in interior which is a classical result, second in a neighborhood of a point on the boundary with Dirichlet or Neumann conditions and third in a neighborhood of a point on Γ .

5.1 Propagation in interior domain Ω

Here we use the fact that $a\mu = 0$. From Proposition 4.1, we have $H_p\mu = 0$. It is then classical that μ is invariant by the flow of H_p . More precisely, let $\rho_0 \in T^*\Omega$ and we assume that $\gamma(s, \rho_0) \in T^*\Omega$ for $s \in [0, t]$. Let $b \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^d)$ be such that $b(\gamma(-s, \cdot))$ is supported in $T^*\Omega$ for $s \in [0, t]$, we have $\langle \gamma_*(t, \cdot)\mu, b \rangle = \langle \mu, b(\gamma(-t, \cdot)) \rangle = \langle \mu, b \rangle$.

5.2 Propagation at boundary: hyperbolic points

The propagation results given in this section are classical for Dirichlet boundary condition. We prove a propagation result for Neumann boundary and in a neighborhood of Γ which is new in context of semiclassical measure. We use the geometry context defined in section 1.2, in particular j and the definition of the different flows.

We prove that the support of measure is locally empty in the future assuming that in the past the support of measure is locally empty but by symmetry we can deduce that the support of measure is locally empty in the past if we assume that the support of measure is locally empty in the future.

Recall that we choose coordinates such that $p(x, \xi) = \xi_d^2 + R(x, \xi') - 1$ and locally $\Omega = \{x_d > 0\}$. In this section we use that $a\mu = 0$.

We recall that a point $(x'_0, \xi'_0) \in T^*\partial\Omega$ is in \mathcal{H} , if $R(x'_0, 0, \xi'_0) - 1 < 0$. We apply Lemmas 4.9 and 4.11. We have $\mu = 1_{x_d > 0}\mu$ and $H_p\mu = \mu^+ \otimes \delta_{\xi_d = \sqrt{1 - R(x'_0, 0, \xi'_0)}} - \mu^- \otimes \delta_{\xi_d = -\sqrt{1 - R(x'_0, 0, \xi'_0)}}$ with $\mu^+ = \mu^-$.

We call γ^\pm the integral curve of H_p starting from $(x'_0, x_d, \xi'_0, \pm\sqrt{1 - R(x'_0, 0, \xi'_0)})$. If we assume that the support of $\mu = 1_{x_d > 0}\mu$ is empty in a neighborhood of $\gamma^-(s)$ for $s < 0$ and $|s|$ sufficiently small, then $\mu^- = 0$. This implies $\mu^+ = 0$ and $H_p\mu = 0$. As $\mu|_{x_d < 0} = 0$ and $\gamma^+(s)$ is in $x_d < 0$ for $s < 0$ and $|s|$ sufficiently small, this implies that $\mu = 0$ in a neighborhood of $\gamma^+(0)$.

Remark 9. *In a neighborhood of points in $\partial\Omega_D \cup \partial\Omega_N$ we can prove a propagation of measure because we have proved $\mu_+ = \mu_-$ but we do not know if this property is true for points in Γ .*

5.3 Propagation at $\partial\Omega_D \cup \partial\Omega_N$

5.3.1 Propagation at gliding points

We recall that a point $(x'_0, \xi'_0) \in T^*\partial\Omega$ is in \mathcal{G}_g , if $R(x'_0, 0, \xi'_0) - 1 = 0$ and $\partial_{x_d}R(x'_0, 0, \xi'_0) > 0$. Let γ be the integral curve of H_p starting from $(x'_0, 0, \xi'_0, 0)$. Then $\gamma(s)$ into $\{x_d < 0\}$ for $s \neq 0$ and $|s|$ sufficiently small. In a neighborhood of (x'_0, ξ'_0) in $T^*\partial\Omega$ all the point are either hyperbolic, or gliding. We assume that $j^{-1}(\gamma_g(s_0; x'_0, \xi'_0)) \cap \text{supp } \mu = \emptyset$ for $s_0 < 0$ where $|s_0|$ is sufficiently small. Here $\gamma_g(s; x'_0, \xi'_0) = \Gamma(s; x'_0, \xi'_0)$, then all the point ρ in a neighborhood of $j^{-1}(\gamma_g(s_0; x'_0, \xi'_0))$ are not in the support of μ . By continuity of Γ the curve $\Gamma(s; \rho)$ hit the boundary at ρ' in a neighborhood of (x'_0, ξ'_0) . If ρ' is an hyperbolic point, by the previous result the point $j^{-1}(\Gamma(s; \rho))$ are not in the support of μ . If ρ' is a gliding point, all the points $\Gamma(s; \rho)$ are strictly gliding. In particular this implies that μ is supported on $x_d = 0$, then $1_{x_d > 0}\mu = 0$ and $\mu_0 = 0$. We can apply Lemma 4.13 and μ^∂ satisfied $H'_R\mu^\partial = 0$. Let γ_g be the integral curve of H'_R starting from (x'_0, ξ'_0) . As by assumption $\mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0} = \mu$ is 0 in a neighborhood of $j^{-1}(\gamma_g(s_0; x'_0, \xi'_0))$, we have $\mu^\partial = 0$ in a neighborhood of $\gamma_g(s_0)$ and $H'_R\mu^\partial = 0$, this implies that $\gamma_g(s)$ is not in the support of μ^∂ in a neighborhood of $s = 0$. As $\mu = \mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$ we have $\mu = 0$ in a neighborhood of $j^{-1}(x'_0, \xi'_0)$.

5.3.2 Propagation at diffractive points

We recall that a point $(x'_0, \xi'_0) \in T^*\partial\Omega$ is in \mathcal{G}_g , if $R(x'_0, 0, \xi'_0) - 1 = 0$ and $\partial_{x_d}R(x'_0, 0, \xi'_0) < 0$. We keep the previous notation for γ . For a point ρ in a neighborhood of $(x'_0, \xi'_0) \in T^*\partial\Omega \cup T^*\Omega$, there are three cases, first the integral curve passing through ρ hits $x_d = 0$ at an hyperbolic point and by previous result the integral curve is not in support of μ , second it does not hit $x_d = 0$ and the integral curve is not in the support of μ by propagation result in interior, third the integral curve hits $x_d = 0$ at a diffractive point. Then the support of μ^∂ is in \mathcal{G}_g and the support of $1_{x_d > 0}\mu$ is into $\{(x, \xi), \xi_d^2 + R(x, \xi') = 1, \text{ such that } \gamma_{(x, \xi)} \text{ hits } x_d = 0, \xi_d = 0\}$.

If $(x'_0, 0) \in \partial\Omega_D \cup \partial\Omega_N$ we can apply Lemma 4.12, then $\mu^\partial = 0$ in a neighborhood of $(x'_0, 0, \xi'_0)$. As the integral curves hitting $x_d = 0$ at an hyperbolic points are not in the support of $1_{x_d > 0}\mu$, then μ_0 is supported on $\xi_d = 0$. We can apply Lemma 4.13 to obtain $\mu_0 = 0$. Then $H_p\mu = H_p(1_{x_d > 0}\mu) = 0$ and as, by assumption, $\gamma(s)$ is not in the support of μ for $s < 0$, $|s|$ sufficiently small, we deduce that $\gamma(0)$ is not in the support of μ .

5.3.3 Propagation at boundary: integral curves with high contact order

We recall that if (x'_0, ξ'_0) is such that $R(x'_0, 0, \xi'_0, 0) = 1$, $\partial_{x_d}R(x'_0, 0, \xi'_0, 0) = 0$ and if we denote by $\gamma(s) = (x'(s), x_d(s), \xi'(s), \xi_d(s))$ the integral curve of H_p starting from $(x'_0, 0, \xi'_0, 0)$. By the assumption made (see Definition 1.1) there exist $k \in \mathbb{N}$, $k \geq 3$ and $\alpha \neq 0$ such that $x_d(s) = \alpha s^k + \mathcal{O}(s^{k+1})$. We denote $\gamma_g(s) = (x'_g(s), \xi'_g(s))$ the integral curve of H'_R , starting from (x'_0, ξ'_0) . For each k we assume that we have already proved that the integral curves hitting $x_d = 0$ at a point in \mathcal{G}^j for $j < k$ or \mathcal{H} are not in the support of μ .

Case k even, $\alpha < 0$ The integral curve of H_p starting from a point belonging to $T^*\Omega$ in a neighborhood of (x'_0, ξ'_0) in $T^*\partial\Omega \cup T^*\Omega$ eventually hits $x_d = 0$ at a point ρ' , in \mathcal{H} or \mathcal{G}^j for $j \leq k$ (see Section 1.2 for definition of \mathcal{G}^j). By assumptions and by induction this integral curve is not in the support of μ except if ρ' is in \mathcal{G}^k , but in this case this integral curve is in $x_d \leq 0$. This implies that $1_{x_d > 0}\mu = 0$, then $\mu_0 = 0$. By Lemma 4.13, we have $H'_R\mu^\partial = 0$ and as, by assumption, $\gamma_g(s)$ is not in the support of μ^∂ for $s < 0$, $|s|$ sufficiently small, we deduce that $\gamma_g(0)$ is not in the support of μ^∂ .

Case k odd, $\alpha < 0$ By the same argument as in previous case, the integral curve of H_p starting from a point belonging to $T^*\Omega$ in a neighborhood of (x'_0, ξ'_0) in $T^*\partial\Omega \cup T^*\Omega$ hits \mathcal{G}^k or is not in the support of μ . Denote by ρ' the point of this integral curve hitting $x_d = 0$. The generalized bicharacteristic starting from ρ' is on $x_d = 0$ for $s > 0$ and in $x_d > 0$ for $s < 0$, and for $s > 0$ all the points on the integral curve of H'_R are in \mathcal{G}_g , if $|s|$ is sufficiently small. As, by assumption the generalized bicharacteristic is not in support of μ in the past, this means that $1_{x_d > 0}\mu = 0$ then $\mu_0 = 0$. We can apply Lemma 4.13 then $H'_R\mu^\partial = 0$. But $\gamma_g(s)$ is not in support of $\mu = \mu^\partial$ for $s < 0$ and $|s|$ sufficiently small as $\gamma_g(s) \in \mathcal{G}_d$, then $\gamma_g(0)$ is not in the support of μ^∂ .

Case k even, $\alpha > 0$ By induction, only the generalized bicharacteristics with the same order of contact k and the same sign condition $\alpha > 0$ can be in the support of μ . Applying Lemma 4.13 we have $-\partial_{x_d}R(x', 0, \xi')\mu^\partial \otimes \delta_{x_d=0} \otimes \delta'_{\xi_d=0} = 0$ as by induction, $\mu^\partial = 0$ when $\partial_{x_d}R(x', 0, \xi') \neq 0$. We deduce $H_p\mu = 0$. Then the propagation the support of μ is invariant by the flow of H_p .

Case k odd, $\alpha > 0$ By induction, only the generalized bicharacteristics with the same order of contact k and the same sign condition $\alpha > 0$ can be in the support of μ . We can apply Lemma 4.13, and by the same argument used in the previous case, we have $H_p\mu = 0$ and as $\gamma(s)$ is in $x_d < 0$ for $|s| < 0$ sufficiently small, $\gamma(s)$ is not in the support of μ for $s < 0$ and by propagation $\gamma(0)$ is not in support of μ .

Proof of Proposition 5.1, first case. By assumption for a point $\rho \in T_b\Omega = T^*\Omega \cup T^*\partial\Omega$ with $\pi(\rho) \in \Omega \cup \partial\Omega_D \cup \partial\Omega_N$, we have assumed that $\pi(\Gamma(s_0, \rho)) \in \{x \in \overline{\Omega}, a(x) > 0\}$ for some $s_0 \in \mathbb{R}$ and for every $s \in [0, s_0]$, if $\pi\Gamma(s, \rho) \in \Gamma$ then $\Gamma(s, \rho) \in \mathcal{H}$. As $\text{supp } \mu$ is a closed set, if $\rho \in \text{supp } \mu$ there exist $s_1 \in [0, s_0]$ such that $j^{-1}\Gamma(s_1, \rho) \cap \text{supp } \mu \neq \emptyset$ and $j^{-1}\Gamma(s, \rho) \cap \text{supp } \mu = \emptyset$ for $s \in [s_0, s_1)$. At $\Gamma(s_1, \rho)$ we can apply the results obtained in this section to prove that $j^{-1}\Gamma(s_1, \rho) \cap \text{supp } \mu = \emptyset$, and reach a contradiction. \square

5.4 Propagation on Γ

Now we prove Proposition 5.1 in the second case, i.e. $\pi(\rho) \in \Gamma$. We recall that we can change the coordinates such that locally in a neighborhood of Γ we have $\Omega = \{x_d > 0\}$ and $\Gamma = \{x_d = x_1 =$

0}, moreover, in the coordinates $(x_1, x'', x_d) = (x', x_d)$, we have $R(x', 0, \xi') = \xi_1^2 + R_0(x'', \xi'') + x_1 r_2(x', \xi')$. By the result obtained from the previous section, $\text{supp } \mu \subset \{x_1 = 0, x_d = 0, \xi_d = 0\}$, in particular $1_{x_d > 0} \mu = 0$ and $\mu_0 = 0$. Let $\rho = (x'_0, \xi'_0) \in T^* \partial \Omega$ such that $x'_0 \in \Gamma$. Then the measure verifies $\mu = \mu^\partial \otimes \delta_{x_d=0} \otimes \delta_{\xi_d=0}$ and μ^∂ is supported on $x_1 = 0$.

We have to distinguish two cases, first if $R_0(x'', \xi'') - 1 < 0$ and second $R_0(x'', \xi'') = 1$ even if the result is the same in both cases.

5.4.1 Case $R_0(x'', \xi'') - 1 < 0$

Let $\xi_1(x'', \xi'')$ the positive solution in ξ_1 of $\xi_1^2 + R_0(x'', \xi'') - 1 = 0$. There exist $\mu^\pm(x'', \xi'')$ measures such that $\mu^\partial = \mu^+(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=\xi_1(x'', \xi'')} + \mu^-(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=-\xi_1(x'', \xi'')}$. Lemma 4.14 implies that $\langle H'_R \mu^\partial, \ell(x', \xi') \rangle = 0$.

Let $S(x', \xi') = R_0(x'', \xi'') + x_1 r_2(x', \xi')$, we have on $x_d = \xi_d = 0$,

$$H'_R = 2\xi_1 \partial_{x_1} - (\partial_{x_1} S(x', \xi')) \partial_{\xi_1} + H''_R, \quad (112)$$

where $H''_R = \sum_{j=2}^{d-1} (\partial_{\xi_j} R(x', 0, \xi') \partial_{x_j} - \partial_{x_j} R(x', 0, \xi') \partial_{\xi_j})$. Observe that on $x_1 = 0$, $H''_R = H''_{R_0}$.

$$\begin{aligned} H'_R \mu^\partial &= 2\xi_1(x'', \xi'') \mu^+(x'', \xi'') \otimes \delta'_{x_1=0} \otimes \delta_{\xi_1=\xi_1(x'', \xi'')} \\ &\quad - 2\xi_1(x'', \xi'') \mu^-(x'', \xi'') \otimes \delta'_{x_1=0} \otimes \delta_{\xi_1=-\xi_1(x'', \xi'')} \\ &\quad - (\partial_{\xi_1} S(x', \xi')) \mu^+(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta'_{\xi_1=\xi_1(x'', \xi'')} \\ &\quad - (\partial_{\xi_1} S(x', \xi')) \mu^-(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta'_{\xi_1=-\xi_1(x'', \xi'')} \\ &\quad + H''_{R_0} \mu^+(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=\xi_1(x'', \xi'')} + H''_{R_0} \mu^-(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=-\xi_1(x'', \xi'')}. \end{aligned}$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\chi(\sigma) = 1$ for σ in a neighborhood of 0. Let $\ell(x', \xi') = x_1 \chi(x_1) b(x', \xi'')$ where $b \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-2})$. We have

$$\langle H'_R \mu^\partial, \ell(x', \xi') \rangle = -2 \langle \xi_1(x'', \xi'') \mu^+(x'', \xi''), b(0, x'', \xi'') \rangle + 2 \langle \xi_1(x'', \xi'') \mu^-(x'', \xi''), b(0, x'', \xi'') \rangle = 0.$$

Then $\mu^+ = \mu^-$ as $\xi_1(x'', \xi'') \neq 0$ in a neighborhood of (x''_0, ξ''_0) .

Now we take $\ell(x', \xi') = b(x'', \xi'') \chi(x_1)$. Observe that $(\partial_{\xi_1} S(x', \xi')) = x_1 \partial_{\xi_1} r_2(x', \xi')$ is null on $x_1 = 0$, we deduce that

$$\langle H'_R \mu^\partial, \ell(x', \xi') \rangle = \langle H''_{R_0} (\mu^+(x'', \xi'') + \mu^-(x'', \xi'')), b(x'', \xi'') \rangle = 0,$$

then, as $\mu^+ = \mu^-$, $H''_{R_0} \mu^+ = H''_{R_0} \mu^- = 0$ with the previous equation. Then the support of μ^+ , μ^- and μ^∂ propagate along the integral curves of H''_{R_0} . By assumption mGCC (see Definition 1.2) all these curves hit the set $a \geq \delta > 0$, we obtain that $\mu^\partial = 0$ in a neighborhood of such a point (x'_0, ξ'_0) .

5.4.2 Case $R_0(x'', \xi'') = 1$

By the result obtained in previous section the measure μ^∂ is supported on $x_1 = \xi_1 = 0$, then we have $\mu^\partial = \tilde{\mu}(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=0}$, where $\tilde{\mu}$ is a non negative Radon measure. We deduce from (112), as $\partial_{\xi_1} S(x', \xi') = 0$ on $x_1 = 0$,

$$H'_R \mu^\partial = H''_{R_0} \tilde{\mu}(x'', \xi'') \otimes \delta_{x_1=0} \otimes \delta_{\xi_1=0}.$$

Taking $\ell(x', \xi') = b(x'', \xi'') \chi(x_1)$, where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\chi(s) = 1$ for s in a neighborhood of 0, by Lemma 4.14, and arguing as in the previous case, we have $H''_{R_0} \tilde{\mu} = 0$, then the supports of $\tilde{\mu}$ and μ^∂ propagate along the integral curves of H''_{R_0} . As in the previous case we obtain $\mu^\partial = 0$ in a neighborhood of such a point (x'_0, ξ'_0) .

A Proof of Lemma 4.4, Zaremba regularity result

It is well known that the solution of elliptic equation of second order with Zaremba boundary condition is in H^s with $s < 3/2$ for a data in L^2 , see for instance Shamir [42], Savaré [41]. Here we have to prove that the solution is in semiclassical Sobolev spaces.

We start from (88) and equation $h^2 P v_h - v_h + i h a v_h = h q_h$. We have $h^2 P v_h + v_h = r_h$ where $r_h = 2v_h - i h a v_h + h q_h$ and we deduce $\|r_h\|_{L^2(\Omega)} \leq C$. We observe that $h^2 P + 1$ is a semiclassical elliptic operator. To prove the result we follow the method used in Section 3.1.4 with the advantage that the operator is globally elliptic and we keep more or less the same notations introduced in this section. In particular we do not have to use microlocal cutoff.

We work in a neighborhood of the boundary in coordinate (x', x_d) and Ω is given by $x_d > 0$ and Γ by $x_1 = 0$. The symbol of the operator is given by $\xi_d^2 + R(x, \xi') + 1$ and $R(x', 0, \xi') = \xi_1^2 + R_0(x'', \xi'') + x_1 r_2(x', \xi')$.

Let $\tilde{\chi}$ a cutoff in a neighborhood of a point of Γ , it will fix to 0 in what follows. We set $w_h = \tilde{\chi}_\delta v_h$ where $\tilde{\chi}_\delta(x) = \tilde{\chi}(x/\delta)$. Let χ another cutoff function such that $\chi(x) = 1$ if x is contained in a neighborhood of $\text{supp } \tilde{\chi}$ and we set $\chi_\delta(x) = \chi(x/\delta)$. As v_h is uniformly in $H_{sc}^1(x_d > 0)$ and r_2 is a differential operator, we have

$$h^2 D_{x_d}^2 w_h + h^2 D_{x_1}^2 w_h + \text{op}_{sc}(R_0(0, x'', x_d, \xi'')) w_h + x_1 \chi_\delta(x) \text{op}_{sc}(r_2(x, \xi')) w_h = r_h^0,$$

where $\|r_h^0\|_{L^2(\Omega)} \leq C$. Let $\rho(x, \xi') = (\xi_d^2 + \xi_1^2 + R_0(0, x'', x_d, \xi'') + x_1 \chi_\delta(x) r_2(x, \xi') + 1)^{1/2}$. By symbol calculus, we have in $x_d > 0$

$$(hD_{x_d} + i \text{op}_{sc}(\rho))(hD_{x_d} + i \text{op}_{sc}(\rho))w_h = r_h^1 \text{ where } \|r_h^1\|_{L^2(\Omega)} \leq C. \quad (113)$$

Let $z = (hD_{x_d} + i \text{op}_{sc}(\rho))w_h$, we then have

$$(hD_{x_d} + i \text{op}_{sc}(\rho))z = r_h^1.$$

$$2 \text{Re}((hD_{x_d} + i \text{op}_{sc}(\rho))z | i \text{op}_{sc}(\rho)z) \leq 2\|r_h^1\|_{L^2(\Omega)} \| \text{op}_{sc}(\rho)z \|_{L_{sc}^2(x_d > 0)}. \quad (114)$$

Integrating by parts (see (56)) we have

$$(i \text{op}_{sc}(\rho)z | hD_{x_d}z) = (ihD_{x_d} \text{op}_{sc}(\rho)z | z) - ih(i \text{op}_{sc}(\rho)z |_{x_d=0} | z |_{x_d=0})_0.$$

We deduce

$$2 \text{Re}(hD_{x_d}z | i \text{op}_{sc}(\rho)z) = (i[hD_{x_d}, \text{op}_{sc}(\rho)]z | z) + h(\text{op}_{sc}(\rho)z |_{x_d=0} | z |_{x_d=0})_0.$$

As $|(i[hD_{x_d}, \text{op}_{sc}(\rho)]z | z)| \lesssim h\|z\|_{L^2(0, +\infty, H_{sc}^1)}$, we deduce from (114)

$$\| \text{op}_{sc}(\rho)z \|_{L^2(x_d > 0)}^2 + h(\text{op}_{sc}(\rho)z |_{x_d=0} | z |_{x_d=0})_0 \lesssim \|r_h^1\|_{L^2(\Omega)} \| \text{op}_{sc}(\rho)z \|_{L^2(x_d > 0)} + h\|z\|_{L^2(0, +\infty, H_{sc}^1)}^2.$$

As $\text{op}_{sc}(\rho^{-1}) \text{op}_{sc}(\rho) = Id + h \text{op}_{sc}(r_1)$, where $r_1 \in S_{\text{tan}}^{-1}$ we have

$$\|z\|_{L^2(0, +\infty, H_{sc}^1)}^2 \lesssim \| \text{op}_{sc}(\rho)z \|_{L^2(x_d > 0)}^2 + h\|z\|_{L^2}^2.$$

We deduce

$$h(\text{op}_{sc}(\rho)z |_{x_d=0} | z |_{x_d=0})_0 \lesssim \|r_h^1\|_{L^2(\Omega)}^2.$$

And by tangential Gårding inequality (9) (in \mathbb{R}^{d-1} instead of \mathbb{R}^d) applied to

$$(\text{op}_{sc}(\langle \xi' \rangle^{-1/2}) \text{op}_{sc}(\rho) \text{op}_{sc}(\langle \xi' \rangle^{-1/2}) \text{op}_{sc}(\langle \xi' \rangle^{1/2})z |_{x_d=0} | \text{op}_{sc}(\langle \xi' \rangle^{1/2})z |_{x_d=0})_0$$

we obtain, as $\langle \xi' \rangle^{-1} \rho \geq C > 0$, $h|z|_{x_d=0}|_{H_{sc}^{1/2}} \lesssim \|r_h^1\|_{L^2(\Omega)}^2$. By definition of z , we have

$$(hD_{x_d} w_h) |_{x_d=0} + i \text{op}_{sc}(\rho_0)(w_h) |_{x_d=0} = h^{-1/2} r_h^2, \text{ where } |r_h^2|_{H_{sc}^{1/2}} \leq C, \quad (115)$$

where $\rho_0(x', \xi') = (R(x', 0, \xi') + 1)^{1/2}$.

Let $\beta(x'', \xi'') = (R_0(x'', \xi'') + 1)^{1/2}$. We have $\rho_0(x', \xi') = (\xi_1^2 + \beta^2(x'', \xi'') + x_1 \chi_{\delta r_2}(x', \xi'))^{1/2}$. Let $u_0 = (hD_{x_d} w_h)|_{x_d=0}$ and $u_1 = (w_h)|_{x_d=0}$, we recall that $\text{supp } u_0 \subset \{x_1 \leq 0\}$ and $\text{supp } u_1 \subset \{x_1 \geq 0\}$. As in Section 3.1.4, we have

$$\begin{aligned} \xi_1 &\mapsto (\xi_1 + i\beta(x'', \xi''))^{\pm 1/2} \text{ are holomorphic functions on } \text{Im } \xi_1 > 0, \\ \xi_1 &\mapsto (\xi_1 - i\beta(x'', \xi''))^{\pm 1/2} \text{ are holomorphic functions on } \text{Im } \xi_1 < 0. \end{aligned}$$

Let $v_0 = \text{op}_{sc}(\xi_1 + i\beta)^{-1/2} u_0$. As u_0 is supported in $x_1 \leq 0$ and $(\xi_1 + i\beta)^{-1/2}$ is a holomorphic function on $\text{Im } \xi_1 > 0$, v_0 is supported in $x_1 \leq 0$.

We have $\xi_1 - i\beta(x'', \xi'') \in S(\langle \xi' \rangle, (dx')^2 + (d\xi')^2)$, then $(\xi_1 - i\beta(x'', \xi''))^{\pm 1/2} \in S(\langle \xi' \rangle^{\pm 1/2}, (dx')^2 + (d\xi')^2)$. This implies by symbol calculus that

$$\text{op}_{sc}(\xi_1 - i\beta)^{-1/2} \text{op}_{sc}(\xi_1 - i\beta)^{1/2} = Id + h \text{op}_{sc}(s_0), \quad (116)$$

where $s_0 \in S(1, (dx')^2 + (d\xi')^2)$. If h is sufficiently small, $Id + h \text{op}_{sc}(s_0)$ is invertible on H_{sc}^s for every s . Let $v_1 = \text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{1/2} (Id + h \text{op}_{sc}(s_0))^{-1} u_1$, thus we have $\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1 = u_1$. Moreover from (116), $\text{op}_{sc}(s_0)$ map distribution supported on $x_1 \geq 0$ to distribution supported on $x_1 \geq 0$, then by Neumann series, $(Id + h \text{op}_{sc}(s_0))^{-1}$ also satisfies this property. This implies that v_1 is supported on $x_1 \geq 0$. From (115), we obtain

$$\text{op}_{sc}(\xi_1 + i\beta)^{-1/2} v_0 + i \text{op}_{sc}(\xi_1 + i\beta)^{-1/2} \text{op}_{sc}(\rho_0) \text{op}_{sc}(\xi_1 - i\beta)^{-1/2} v_1 = h^{-1/2} r_h^3, \quad (117)$$

where $|r_h^3|_{H_{sc}^1} \leq C$. The principal symbol of $\text{op}_{sc}(\xi_1 + i\beta)^{-1/2} \text{op}_{sc}(\rho_0) \text{op}_{sc}(\xi_1 - i\beta)^{-1/2}$ is by a simple computation

$$(\xi_1 + i\beta)^{-1/2} \rho_0 (\xi_1 - i\beta)^{-1/2} = 1 + \frac{x_1 \chi_{\delta r_2}}{(\xi_1^2 + \beta^2)^{1/2} (\rho_0 + (\xi_1^2 + \beta^2)^{1/2})} = 1 + x_1 \chi_{\delta r_3},$$

where $r_3 \in S(1, (dx')^2 + (d\xi')^2)$. Formula (117) reads

$$\text{op}_{sc}(\xi_1 + i\beta)^{-1/2} v_0 + i v_1 + i x_1 \chi_{\delta} \text{op}_{sc}(r_3) v_1 = h^{-1/2} r_h^3.$$

We restrict this equation on $x_1 > 0$, as $\text{op}_{sc}(\xi_1 + i\beta)^{-1/2} v_0$ is supported on $x_1 \leq 0$, we obtain

$$i(v_1)|_{x_1>0} + i x_1 \chi_{\delta} (\text{op}_{sc}(r_3) v_1)|_{x_1>0} = h^{-1/2} (r_h^3)|_{x_1>0}.$$

As $|w|_{x_1>0}|_{H_{sc}^s} \leq |w|_{H_{sc}^s}$, we obtain that

$$|(v_1)|_{x_1>0}|_{H_{sc}^s(x_1>0)} \leq |x_1 \chi_{\delta} \text{op}_{sc}(r_3) v_1|_{H_{sc}^s} + h^{-1/2} |r_h^3|_{H_{sc}^s}. \quad (118)$$

Lemma A.1. *Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\chi_\delta(x) = \chi(x/\delta)$, where $\delta > 0$. Then there exist $C > 0$ such that for every $\delta > 0$, $\|x_1 \chi_\delta\|_{C^\alpha} \leq C \delta^{1-\alpha}$, for $\alpha \in (0, 1)$ and $\|x_1 \chi_\delta\|_{L^\infty} \leq C \delta$.*

Proof. First, we have $|x_1 \chi(x/\delta)| \leq C \delta$. Second, $|(x_1 + y_1) \chi((x+y)/\delta) - x_1 \chi(x/\delta)| \leq 2C \delta$ and $|\partial_x(x_1 \chi(x/\delta))| \leq C$, then we have $|(x_1 + y_1) \chi((x+y)/\delta) - x_1 \chi(x/\delta)| \leq C |y|$. For $\alpha \in (0, 1)$, interpolating both estimates, we have $|(x_1 + y_1) \chi((x+y)/\delta) - x_1 \chi(x/\delta)| \leq C \delta^{1-\alpha} |y|^\alpha$. Which gives the result. \square

Lemma A.2. *Let $s \in (0, 1)$ and $0 < s < \alpha < 1$, there exists $C > 0$, such that for every $f \in C_{sc}^\alpha(\mathbb{R}^d)$, and $g \in H_{sc}^s(\mathbb{R}^d)$, $fg \in H_{sc}^s(\mathbb{R}^d)$ and we have*

$$\|fg\|_{H_{sc}^s(\mathbb{R}^d)} \leq C \|f\|_{C_{sc}^\alpha(\mathbb{R}^d)} \|g\|_{H_{sc}^s(\mathbb{R}^d)}.$$

Here we say that $f \in C_{sc}^\alpha(\mathbb{R}^d)$ if f is bounded and $h^\alpha |f(x+y) - f(x)| \leq C |y|^\alpha$. The norm on $C_{sc}^\alpha(\mathbb{R}^d)$ is $\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x,y \in \mathbb{R}^d} h^\alpha |f(x+y) - f(x)| |y|^{-\alpha}$.

Proof. We can follow the classical proof that the multiplication by C^α functions are bounded operators on H^s , using Littlewood-Paley theory and para-product in spirit of Bony [9].

We recall the Littlewood-Paley decomposition (see (20) for notations), we have $w = \sum_{k \geq -1} \Delta_k w$ where $\mathcal{F}(\Delta_k(w))(\xi) = \phi(2^{-k}h\xi)\mathcal{F}(w)(\xi)$ for $k \geq 0$ and $\mathcal{F}(\Delta_{-1}(w))(\xi) = \psi(h\xi)\mathcal{F}(w)(\xi)$. Let $S_k = \sum_{-1 \leq j \leq k} \Delta_j$.

By assumptions on f , we have $\|S_k f\|_{L^\infty(\mathbb{R}^d)} \leq C\|f\|_{L^\infty(\mathbb{R}^d)}$ and by the usual proof of characterization of C^α functions with Littlewood-Paley decomposition we have $\|\Delta_k f\|_{L^\infty(\mathbb{R}^d)} \leq C2^{-k\alpha}\|f\|_{C_{sc}^\alpha(\mathbb{R}^d)}$ for $k \geq 0$. By assumptions on g , we have $\|\Delta_k g\|_{L^2(\mathbb{R}^d)} \leq c_k 2^{-sk}$ for $k \geq 0$, where $\|(c_k)\|_{\ell^2(\mathbb{N})} \leq C\|g\|_{H_{sc}^s(\mathbb{R}^d)}$, $\|\Delta_{-1}g\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$ and $\|S_k g\|_{L^2(\mathbb{R}^d)} \leq C\|g\|_{L^2(\mathbb{R}^d)}$.

The product $fg = \sum_{k \geq -1} S_k(f)\Delta_k(g) + \sum_{k \geq 0} S_{k-1}(g)\Delta_k(f)$. We estimate each term in previous formula. For $j \geq 2$ we have

$$\begin{aligned} \|\Delta_j(\sum_{k \geq -1} S_k(f)\Delta_k(g))\|_{L^2(\mathbb{R}^d)} &\lesssim \|\Delta_j(\sum_{k \geq j-2} S_k(f)\Delta_k(g))\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \sum_{k \geq j-2} \|S_k(f)\|_{L^\infty(\mathbb{R}^d)} \|\Delta_k(g)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim 2^{-sj} \|f\|_{L^\infty(\mathbb{R}^d)} \sum_{k \geq j-2} c_k 2^{-(k-j)s}, \end{aligned}$$

and $d_j = \sum_{k \geq j-2} c_k 2^{-(k-j)s} \in \ell^2 * \ell^1 \subset \ell^2$, where $\|(d_j)\|_{\ell^2} \lesssim \|g\|_{H_{sc}^s(\mathbb{R}^d)}$.

$$\begin{aligned} \|\Delta_j(\sum_{k \geq 0} S_{k-1}(g)\Delta_k(f))\|_{L^2(\mathbb{R}^d)} &\leq \|\Delta_j(\sum_{k \geq j-2} S_{k-1}(g)\Delta_k(f))\|_{L^2(\mathbb{R}^d)} \\ &\leq \sum_{k \geq j-2} \|S_{k-1}(g)\|_{L^2(\mathbb{R}^d)} \|\Delta_k(f)\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \|g\|_{L^2(\mathbb{R}^d)} \|f\|_{C_{sc}^\alpha(\mathbb{R}^d)} \sum_{k \geq j-2} 2^{-k\alpha} \\ &\lesssim \|g\|_{L^2(\mathbb{R}^d)} \|f\|_{C_{sc}^\alpha(\mathbb{R}^d)} 2^{-js} 2^{-j(\alpha-s)}. \end{aligned}$$

As $(2^{-j(\alpha-s)})$ is in ℓ^2 , and as the result is obvious for $j \leq 1$ we obtain the result. \square

Observe that $\|f\|_{C_{sc}^\alpha(\mathbb{R}^d)} \lesssim \|f\|_{C^\alpha(\mathbb{R}^d)}$. Then by Lemmas A.1 and A.2 we have $|x_1 \chi_\delta w|_{H_{sc}^s} \leq C\delta^{1-\alpha} |w|_{H_{sc}^s}$.

From (118) we have

$$|(v_1)|_{x_1 > 0}|_{H_{sc}^s(x_1 > 0)} \leq C\delta^{1-\alpha} |v_1|_{H_{sc}^s} + h^{-1/2} |r_h^3|_{H_{sc}^s},$$

for $\leq s < \alpha < 1/2$. As $|v_1|_{H_{sc}^s} = |(v_1)|_{x_1 > 0}|_{H_{sc}^s(x_1 > 0)}$ for $s \in [0, 1/2)$, we have

$$|(v_1)|_{x_1 > 0}|_{H_{sc}^s(x_1 > 0)} \leq Ch^{-1/2} |r_h^3|_{H_{sc}^s} \leq Ch^{-1/2},$$

for δ sufficiently small. We obtain

$$|u_1|_{H^{s+1/2}(x_1 > 0)} \leq |\text{op}_{sc}(\xi_1 - i\beta(x'', \xi''))^{-1/2} v_1|_{H^{s+1/2}} \leq C|v_1|_{H^s} \leq Ch^{-1/2}.$$

From (115) we deduce that

$$|u_0|_{H^{s-1/2}} \leq Ch^{-1/2}.$$

The solution w_h of semiclassical elliptic problem with boundary condition satisfying

$$|(w_h)|_{x_d=0}|_{H_{sc}^{s+1/2}} \leq Ch^{-1/2} \text{ and } |(hD_{x_d} w_h)|_{x_d=0}|_{H^{s-1/2}} \leq Ch^{-1/2},$$

is in $H_{sc}^{1+s}(x_d > 0)$ and $\|w_h\|_{H^{1+s}(x_d > 0)} \leq C$. This result is well-known and it is a consequence of Formula (60) in [39]. This achieves the proof of Lemma 4.4.

B A priori estimate for the trace of solution for Neumann boundary condition

We begin this section by recall some result on semiclassical Fourier Integral Operator.

Lemma B.1. *Let (x'_0, ξ'_0) be such that $R(x'_0, 0, \xi'_0) - 1 = 0$. For all x_d in a neighborhood of 0, there exist a smooth symplectic transformation $\kappa : U_0 \rightarrow U_1$ where U_0 and U_1 are some open set respectively of $\mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{\xi'}^{d-1}$ and of $\mathbb{R}_{y'}^{d-1} \times \mathbb{R}_{\eta'}^{d-1}$, satisfying $(x'_0, \xi'_0) \in U_0$, $(0, 0) \in U_1$, $\kappa(x'_0, \xi'_0) = (0, 0)$, and $\kappa^*(\eta_1) = R - 1$. Moreover x_d acts as a parameter and κ is smooth with respect $x_d = y_d$.*

This lemma is classical. We can find a proof in Hörmander [26, Theorem 21.1.6]. This means we can complete the coordinate $R - 1$ in a symplectic manner.

To avoid ambiguity even if $x_d = y_d$ we denote x_d when we work in (x, ξ') variables and y_d otherwise.

We call a symbol of order 0 a symbol $a \in S(1, |dx|^2 + |d\xi'|^2)$ or in $S(1, |dy|^2 + |d\eta'|^2)$. In this section we only use tangential symbol, but as in what follows we have to use different classes of symbols, we prefer use everywhere the same kind of notation.

Lemma B.2. *Associated with κ , there exists F a semiclassical Fourier Integral Operator satisfying the following properties,*

- i) F is a unitary operator uniformly with respect x_d .
- ii) For all $\tilde{a} \in \mathcal{C}_0^\infty(U_0)$, $F^{-1} \text{op}_{\text{sc}}(\tilde{a})F = \text{op}_{\text{sc}}(a)$, where $a = \kappa^*\tilde{a} + hb$ where b is a symbol of order 0. In particular we have $F^{-1} \text{op}_{\text{sc}}(\eta_1 \tilde{\chi}^2(y, \eta'))F = \text{op}_{\text{sc}}(\chi^2(R-1)) + h \text{op}_{\text{sc}}(b)$, where $\tilde{\chi} \in \mathcal{C}_0^\infty(U_0)$, $\tilde{\chi} \geq 0$ and b a symbol of order 0.
- iii) there exist θ a symbol of order 0, B a bounded operator on L^2 such that $\text{op}_{\text{sc}}(\theta)^* = \text{op}_{\text{sc}}(\theta)$, $\kappa^*\tilde{\chi} = \chi$ and $(\partial_{x_d}F)F^{-1} = ih^{-1} \text{op}_{\text{sc}}(\theta) + hB$.
- iv) If the operators A and \tilde{A} are such that $A = F^{-1}\tilde{A}F$ then

$$\partial_{x_d}A = F^{-1}(\partial_{y_d}\tilde{A} + ih^{-1}[\tilde{A}, \text{op}_{\text{sc}}(\theta)] + h[\tilde{A}, B])F$$

where B is the operator defined previously.

- v) In particular we have $\kappa^*\{\eta_1, \theta\} = \partial_{x_d}R$ in a neighborhood of $(x'_0, 0, \xi'_0)$.

Remark 10. Zworski states the result for Weyl quantification. It is clear that we can deduce the result for classical quantification. In the proof of Lemma B.2 we use Weyl quantification but in the rest of this section we shall use classical quantification to be coherent with notation used in this article.

A proof of Lemma B.2 is given in Section C.

Here we adapt, in the framework of semiclassical analysis, the results obtained by Tataru [44] especially Lemma 4.3, Propositions 4.5 and 4.7. We essentially keep the notation used in that paper.

From now we shall use two semiclassical quantifications of symbol, one with parameter h and the other with parameter $h^{1/3}$. To avoid ambiguity or confusion between both, we do not use the notation op_{sc} but we use classical quantification. For instance, for a a symbol of order 0 we have $\text{op}_{\text{sc}}(a) = \text{op}(a(x, h\xi'))$ that is we keep the h or $h^{1/3}$ in the notation.

Let $g = |dy|^2 + h^{2/3}\langle h^{1/3}\eta_1 \rangle^{-2}|d\eta'|^2$, this metric gives symbol classes essentially as semiclassical symbol classes with $h^{1/3}$ for semiclassical parameter. We let to the reader to check that g is slowly varying and σ temperate. The "h" defined by Hörmander associated with g is $h^{1/3}\langle h^{1/3}\eta_1 \rangle^{-1}$. It is the quantity we gain in the asymptotic expansion for the symbol calculus. In particular the function $\langle h^{1/3}\eta_1 \rangle^\nu$ is a g continuous and σ, g temperate for every $\nu \in \mathbb{R}$. We refer to [26, Chapter 18, Sections 4 and 5] for definitions used freely here.

From (88) we have $-h^2\partial_{x_d}^2 v_h + \text{op}(R(x, h\xi') - 1)v_h = hq_h$. Let $v_h = \text{op}(\chi_1(x, h\xi'))v_h$, where χ_1 is supported where $\chi = 1$ and χ is the cutoff function define in Lemma B.2. By symbol calculus we have

$$-h^2\partial_{x_d}^2 v_h + \text{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))v_h = hq_h \quad (119)$$

where

$$\begin{aligned} q_h &= \text{op}(\chi_1(x, h\xi'))q_h - h^{-1}[h^2D_{x_d}^2, \text{op}(\chi_1(x, h\xi'))]v_h \\ &\quad + h^{-1}(-\text{op}(\chi_1(x, h\xi'))\text{op}(R(x, h\xi') - 1) + \text{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))\text{op}(\chi_1(x, h\xi'))) \end{aligned}$$

is bounded on $L^2(x_d > 0)$ from symbol calculus and support properties of χ and χ_1 . Moreover we have $|v_h|_{L^2(\mathbb{R}^{d-1})} \lesssim |v_h|_{L^2(\mathbb{R}^{d-1})}$.

We recall some properties of the Airy function which is denoted by Ai . It verifies the equation $\text{Ai}''(z) - z\text{Ai}(z) = 0$ for $z \in \mathbb{C}$, Ai is real on the real axis and $\overline{\text{Ai}(z)} = \text{Ai}(\bar{z})$. Let $\omega = e^{2i\varepsilon\pi/3}$ for $\varepsilon = \pm 1$.

For $x \in \mathbb{R}$, let $\alpha(x) = -\omega \text{Ai}'(\omega x) / \text{Ai}(\omega x) \in \mathcal{C}^\infty(\mathbb{R})$. As the zero of Ai are on the negative real axis, the function α is well defined for $x \in \mathbb{R}$ and smooth. The function α satisfies the following properties.

Lemma B.3. *We have*

- i) $\alpha(x) = -\sqrt{x} + \frac{1}{4x} + b_1(x)$ for $x > 0$
- ii) $\alpha(x) = \varepsilon i\sqrt{-x} + \frac{1}{4x} + b_2(x)$ for $x < 0$
- iii) $\text{Re}\alpha(x) < 0$ for all $x \in \mathbb{R}$,
- iv) α satisfies the differential equation $\alpha'(x) = \alpha^2(x) - x$.

where $b_j \in S(\langle x \rangle^{-5/2}, |dx|^2)$ for $j = 1, 2$.

The proof of lemma is given in Section C.

Let \tilde{r}_d be such that $\kappa^* \tilde{r}_d = -\partial_{x_d} R$. We assume that locally $\partial_{x_d} R < 0$, this implies that $\tilde{r}_d > 0$ in a neighborhood of $(0, 0)$.

Let $\tilde{a}(y, \eta') = h^{1/3}\tilde{\chi}(y, h\eta')\tilde{r}_d^{1/3}(y, h\eta')\alpha(\zeta)$ where $\zeta = h^{1/3}\eta_1\tilde{r}_d^{-2/3}(y, h\eta')$. We assume that on the support of $\tilde{\chi}(y, h\eta')$, $\tilde{r}_d(y, h\eta') > 0$. In what follows we denote $\tilde{\rho} = (y, h\eta')$. We define $\tilde{A} = \text{op}(\tilde{a})$, $\tilde{\Psi} = h^{-1/3}\text{op}(\langle h^{1/3}\eta_1 \rangle^{-1/2})$, and let $A = F^{-1}\tilde{A}F$, $\Psi = F^{-1}\tilde{\Psi}F$.

We have $\tilde{a} \in S(h^{1/3}\langle h^{1/3}\eta_1 \rangle^{1/2}, g)$ as $h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2} \in S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g)$ and from Lemma B.3

Proposition B.4. *Let v_h satisfying properties (119). There exist $C_0 > 0$ such that*

$$|(h\partial_{x_d} v_h - Av_h)|_{x_d=0}|_{L^2(\mathbb{R}^{d-1})}^2 + \|\Psi(h\partial_{x_d} v_h - Av_h)\|_{L^2(x_d>0)}^2 \leq C_0\|v_h\|_{H_{sc}^1(x_d>0)}^2 + C_0\|q_h\|_{L^2(x_d>0)}^2.$$

This result is equivalent to the following. Let $w_h = F(h\partial_{x_d} v_h - Av_h)$, there exist $C_0 > 0$ such that

$$|(w_h)|_{y_d=0}|_{L^2(\mathbb{R}^{d-1})}^2 + \|\tilde{\Psi}w_h\|_{L^2(y_d>0)}^2 \leq C_0\|v_h\|_{H_{sc}^1(x_d>0)}^2 + C_0\|q_h\|_{L^2(x_d>0)}^2.$$

Here and in this section we denote $\|u\|_{H_{sc}^1(x_d>0)} = \|u\|_{L^2(x_d>0)} + \|h\nabla u\|_{L^2(x_d>0)}$.

Proof. To ease notation we write $|u|$ instead of $|u|_{L^2(x_d)}$ when there is no ambiguity on the fact that the L^2 norm is taken on variables x' or y' at point x_d or y_d . By the same abuse of notation we write the inner product (\cdot, \cdot) instead of $(\cdot, \cdot)_{L^2(\mathbb{R}^{d-1})(x_d)}$. We compute

$$\begin{aligned} \frac{1}{2}\partial_{x_d}|h\partial_{x_d} v_h - Av_h|^2 &= \text{Re}(\partial_{x_d}(h\partial_{x_d} v_h - Av_h)|h\partial_{x_d} v_h - Av_h) \\ &= \text{Re}(h\partial_{x_d}^2 v_h - (\partial_{x_d} A)v_h - A\partial_{x_d} v_h |h\partial_{x_d} v_h - Av_h) \\ &= \text{Re}(h^{-1}\text{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))v_h - q_h - (\partial_{x_d} A)v_h - A\partial_{x_d} v_h |h\partial_{x_d} v_h - Av_h), \end{aligned}$$

from (119). Then we have

$$\frac{1}{2}\partial_{x_d}|h\partial_{x_d}v_h - Av_h|^2 = I_1 + I_2 + I_3 + I_4, \quad (120)$$

where

$$\begin{aligned} I_1 &= \operatorname{Re}(-h^{-1}A(h\partial_{x_d}v_h - Av_h)|h\partial_{x_d}v_h - Av_h) \\ I_2 &= \operatorname{Re}(h^{-1}(\operatorname{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1)) - A^2)v_h|h\partial_{x_d}v_h - Av_h) \\ I_3 &= \operatorname{Re}(-(\partial_{x_d}A)v_h|h\partial_{x_d}v_h - Av_h) \\ I_4 &= -\operatorname{Re}(q_h|h\partial_{x_d}v_h - Av_h). \end{aligned}$$

Using $w_h = F(h\partial_{x_d}v_h - Av_h)$, we have

$$I_1 = \operatorname{Re}(-h^{-1}FAF^{-1}w_h|w_h) = \operatorname{Re}(-h^{-1}\tilde{A}w_h|w_h).$$

From Lemma B.3 we have

$$-h^{-1}\operatorname{Re}\tilde{a}(y, \eta) \geq \delta h^{-2/3}\tilde{\chi}^2(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1},$$

and $h^{-1}\operatorname{Re}\tilde{a}(y, \eta) \in S(h^{-2/3}\langle h^{1/3}\eta_1 \rangle^{1/2}, g)$. Then from Fefferman-Phong inequality (see [26, Theorem 18.6.8]) and as the real part of symbol of $\operatorname{op}(h^{-1/3}\tilde{\chi}(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1/2})^* \operatorname{op}(h^{-1/3}\tilde{\chi}(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1/2})$ is $h^{-2/3}\tilde{\chi}^2(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1}$ modulo an operator bounded on L^2 , we have

$$I_1 \geq \delta|h^{-1/3}\operatorname{op}(\tilde{\chi}(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1/2})w_h|^2 - C|w_h|^2,$$

for $C > 0$.

From Lemma C.1 we obtain

$$I_1 \geq \delta|\tilde{\Psi}w_h|^2 - C(|w_h|^2 + |v_h|^2 + |h\partial_{x_d}v_h|^2). \quad (121)$$

We have

$$\begin{aligned} I_2 &= \operatorname{Re}(h^{-1}F(\operatorname{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1) - A^2)v_h|w_h) \\ &= \operatorname{Re}(h^{-1}(\operatorname{op}(h\eta_1\tilde{\chi}^2(\tilde{\rho})) - \tilde{A}^2)Fv_h|w_h) + \operatorname{Re}(B_0v_h|w_h), \end{aligned}$$

where $B_0 = \operatorname{op}_{\text{sc}}(b)$ is bounded on L^2 (see Lemma B.2).

The symbol of \tilde{A}^2 is $\tilde{a}^2 \in S(h^{2/3}\langle h^{1/3}\eta_1 \rangle, g)$ modulo a term in $S(h, g)$. From definition of \tilde{a} and Lemma B.3 we have

$$\begin{aligned} \tilde{a}^2 &= h^{2/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})\alpha^2(\zeta) \\ &= h^{2/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})(\zeta + \alpha'(\zeta)) \\ &= h\eta_1\tilde{\chi}^2(\tilde{\rho}) + h^{2/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})\alpha'(\zeta). \end{aligned}$$

We have

$$h^{-1/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})\alpha'(\zeta) \in S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g),$$

we then obtain

$$|I_2| \lesssim |v_h|(|\tilde{\Psi}w_h| + |w_h|) \quad (122)$$

We have

$$\begin{aligned} I_3 &= -\operatorname{Re}(F(\partial_{x_d}A)F^{-1}Fv_h|w_h) \\ &= -\operatorname{Re}((\operatorname{op}(\partial_{y_d}\tilde{a}(\tilde{\rho})) + ih^{-1}[\operatorname{op}(\tilde{a}(\tilde{\rho}))], \operatorname{op}(\theta(\tilde{\rho}))])Fv_h|w_h) \\ &\quad - \operatorname{Re}(h[\operatorname{op}(\tilde{a}(\tilde{\rho}))], B]Fv_h|w_h), \end{aligned}$$

from Lemma B.2.

Observe that $\tilde{a}(\tilde{\rho}) \in S(1, g)$ as $h^{1/3}\langle h^{1/3}\eta_1 \rangle^{1/2}$ is bounded on support of $\tilde{\chi}(\tilde{\rho})$. Then $\text{op}(\partial_{y_d}\tilde{a}(\tilde{\rho}))$ and $h[\text{op}(\tilde{a}(\tilde{\rho})), B]$ are bounded operator on L^2 .

From properties of α and symbol calculus, the symbol of

$$h^{-1}[\text{op}(\tilde{a}(\tilde{\rho})), \text{op}(\theta(\tilde{\rho}))] \text{ is in } S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g).$$

Then we obtain

$$|I_3| \lesssim |v_h|(|\tilde{\Psi}w_h| + |w_h|). \quad (123)$$

We have

$$|I_4| = |\text{Re}(q_h, F^{-1}w_h)| \lesssim |q_h||w_h| \quad (124)$$

From (120), (121), (122), (123) and (124) we have

$$\begin{aligned} \frac{1}{2}\partial_{x_d}|h\partial_{x_d}v_h - Av_h|^2 &\geq \delta|\tilde{\Psi}w_h|^2 - C|w_h|^2 - C|\tilde{\Psi}w_h||v_h| - C|v_h|^2 - C|h\partial_{x_d}v_h|^2 - C|q_h|^2 \\ &\geq \delta'|\tilde{\Psi}w_h|^2 - C'(|w_h|^2 + |v_h|^2 + |h\partial_{x_d}v_h|^2 + |q_h|^2). \end{aligned} \quad (125)$$

Observe that $h^{1/3}\langle h^{1/3}\eta_1 \rangle^{1/2} \lesssim \langle h\eta' \rangle^{1/2}$, then $h^{1/3}\tilde{\chi}(\tilde{\rho})^2\tilde{r}_d^{1/3}\alpha(\zeta)$ is bounded.

We then have

$$|Av_h| \lesssim |\tilde{A}Fv_h| \lesssim |v_h|. \quad (126)$$

We also have $|h\partial_{x_d}v_h| \lesssim |h\partial_{x_d}v_h| + |v_h|$. Then

$$|w_h| \lesssim |h\partial_{x_d}v_h| + |v_h|. \quad (127)$$

We deduce from (125) that

$$\frac{1}{2}\partial_{x_d}|h\partial_{x_d}v_h - Av_h|^2 \geq \delta'|\tilde{\Psi}w_h|^2 - C|q_h|^2 - C|v_h|^2 - C|h\partial_{x_d}v_h|^2.$$

Integrating this inequality between 0 and $\sigma > 0$ we have

$$|h\partial_{x_d}v_h - Av_h|^2(0) + \delta' \int_0^\sigma |\tilde{\Psi}w_h|^2(x_d)dx_d \lesssim \|q_h\|^2 + \|v_h\|_{H_{sc}^1(x_d>0)}^2 + |h\partial_{x_d}v_h|^2(\sigma) + |v_h|^2(\sigma)$$

from (127). Integrating this inequality between two positive values of σ and as $|\tilde{\Psi}w_h|(y_d) = |\Psi(h\partial_{x_d}v_h - Av_h)|(x_d)$ we obtain the result. \square

To state the next result we have to introduce another operator.

Lemma B.5. *There exist a function $\beta \in \mathcal{C}^\infty(\mathbb{R})$ satisfying the following properties*

- i) $-\beta' - \beta \text{Re } \alpha \geq 0$
- ii) $\beta \in S(\langle x \rangle^{-1/4}, |dx|^2)$
- iii) $\beta \gtrsim \langle x \rangle^{-1/4}$.

A proof is given in Section C.

We recall the notation $\tilde{a}(y, \eta') = h^{1/3}\tilde{\chi}(\tilde{\rho})\tilde{r}_d^{1/3}(\tilde{\rho})\alpha(\zeta)$ where $\zeta = h^{1/3}\eta_1\tilde{r}_d^{-2/3}(\tilde{\rho})$ and by assumption, $\tilde{r}_d(\tilde{\rho}) > 0$ on the support of $\tilde{\chi}(\tilde{\rho})$.

Let $\tilde{c}(y, \eta') = h^{-1/6}\tilde{\chi}_2(\tilde{\rho})\beta(\zeta)$, where $\tilde{\chi}_2$ is supported on $\{\tilde{\chi} = 1\}$ and $\tilde{\chi}_2 = 1$ on a neighborhood of $(0, 0)$. We have $\tilde{a} \in S(h^{1/3}\langle h^{1/3}\eta_1 \rangle^{1/2}, g)$ and $\tilde{c} \in S(h^{-1/6}\langle h^{1/3}\eta_1 \rangle^{-1/4}, g)$.

We define $\tilde{C} = \text{op}(\tilde{c})$ and $C = F^{-1}\tilde{C}F$.

Proposition B.6. *Let v_h satisfying properties (119). There exists $C_0 > 0$ such that*

$$|C(h\partial_{x_d}v_h - Av_h)|_{x_d=0}|_{L^2(\mathbb{R}^{d-1})} \leq C_0\|v_h\|_{H_{sc}^1(x_d>0)}^2 + C_0\|q_h\|_{L^2(x_d>0)}^2.$$

Proof. We have

$$\begin{aligned}
\frac{1}{2}\partial_{x_d}|C(h\partial_{x_d}v_h - Av_h)|^2 &= \operatorname{Re}(\partial_{x_d}(C(h\partial_{x_d}v_h - Av_h))|C(h\partial_{x_d}v_h - Av_h)) \\
&= \operatorname{Re}((\partial_{x_d}C)(h\partial_{x_d}v_h - Av_h) + C(h\partial_{x_d}^2v_h - (\partial_{x_d}A)v_h - A\partial_{x_d}v_h)|C(h\partial_{x_d}v_h - Av_h)) \\
&= \operatorname{Re}((\partial_{x_d}C)(h\partial_{x_d}v_h - Av_h) + C(h^{-1}\operatorname{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))v_h - q_h \\
&\quad - (\partial_{x_d}A)v_h - A\partial_{x_d}v_h)|C(h\partial_{x_d}v_h - Av_h)),
\end{aligned}$$

from (119). Then we have

$$\frac{1}{2}\partial_{x_d}|C(h\partial_{x_d}v_h - Av_h)|^2 = J_1 + J_2 + J_3 + J_4 + J_5 = K_1 + K_2 + K_3 + K_4 + K_5, \quad (128)$$

where

$$\begin{aligned}
J_1 &= \operatorname{Re}(C^*(\partial_{x_d}C)(h\partial_{x_d}v_h - Av_h)|h\partial_{x_d}v_h - Av_h) \\
J_2 &= -\operatorname{Re}((\partial_{x_d}A)v_h|C^*C(h\partial_{x_d}v_h - Av_h)) \\
J_3 &= -\operatorname{Re}(q_h|C^*C(h\partial_{x_d}v_h - Av_h)) \\
J_4 &= \operatorname{Re}((-C^*C)A\partial_{x_d}v_h|h\partial_{x_d}v_h - Av_h) \\
J_5 &= \operatorname{Re}(h^{-1}\operatorname{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))v_h|C^*C(h\partial_{x_d}v_h - Av_h)).
\end{aligned}$$

Taking as in the proof of Proposition B.4, $w_h = F(h\partial_{x_d}v_h - Av_h)$ and from Lemma B.2 we write

$$\begin{aligned}
K_1 &= \operatorname{Re}(\tilde{C}^*(\partial_{y_d}\tilde{C})w_h|w_h) \\
K_2 &= -\operatorname{Re}((\partial_{y_d}\tilde{A})Fv_h|\tilde{C}^*\tilde{C}w_h) - \operatorname{Re}(h[\tilde{A}, B]Fv_h|\tilde{C}^*\tilde{C}w_h) - \operatorname{Re}(F\operatorname{op}(b(x, h\xi'))v_h|\tilde{C}^*\tilde{C}w_h) \\
K_3 &= \operatorname{Re}(Fq_h|\tilde{C}^*\tilde{C}w_h) \\
K_4 &= \operatorname{Re}((-h^{-1}\tilde{C}^*\tilde{C}\tilde{A} + ih^{-1}\tilde{C}^*[\tilde{C}, \operatorname{op}(\theta(\tilde{\rho}))])w_h|w_h) + \operatorname{Re}(h\tilde{C}^*[\tilde{C}, B]w_h|w_h) \\
K_5 &= -\operatorname{Re}((h^{-1}\tilde{A}^2 + ih^{-1}[\tilde{A}, \operatorname{op}(\theta(\tilde{\rho}))]) - h^{-1}\operatorname{op}(h\eta_1\tilde{\chi}^2(\tilde{\rho})))Fv_h|\tilde{C}^*\tilde{C}w_h).
\end{aligned}$$

To estimate K_1 , observe that the symbol of $\tilde{C}^*(\partial_{y_d}\tilde{C})$ is in $\in S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g)$, then

$$|K_1| \lesssim |\tilde{\Psi}w_h||w_h|. \quad (129)$$

From symbol calculus the symbol of $(\partial_{y_d}\tilde{A})^*\tilde{C}^*\tilde{C}$ is in $S(1, g)$, thus this operator is bounded on L^2 . Clearly the terms $[\tilde{A}, B]$ coming from remainder term of $\partial_{x_d}A$ (see **iv**) Lemma B.2 and $\operatorname{op}(b(x, h\xi'))$ coming from remainder term of $F\operatorname{op}(\chi^2(x, h\xi')(R(x, h\xi') - 1))F^{-1}$ (see **ii**) Lemma B.2) are bounded on L^2 . As $\tilde{C}^*\tilde{C}$ has a symbol in $S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g)$, we obtain

$$|K_2| \lesssim |w_h||v_h| + |v_h||\tilde{\Psi}w_h|. \quad (130)$$

For the same argument we have

$$|K_3| \lesssim |q_h||\tilde{\Psi}w_h|. \quad (131)$$

To estimate the last term of K_4 we write $\tilde{C}^*[\tilde{C}, B] = \tilde{C}^*\tilde{C}B - \tilde{C}^*B\tilde{C}$, the first term gives a term estimated by $|\tilde{\Psi}w_h||w_h|$ and the second is estimated $|\tilde{C}w_h|^2 \lesssim |\tilde{\Psi}w_h||w_h|$.

To estimate the other terms of K_4 observe that the symbol of $-h^{-1}\tilde{C}^*\tilde{C}\tilde{A} + ih^{-1}\tilde{C}^*[\tilde{C}, \operatorname{op}(\theta)]$ is $-h^{-1}\tilde{c}^2\tilde{a} + h^{-1}\tilde{c}\{\tilde{c}, \theta(\tilde{\rho})\}$ modulo a symbol in $S(h^{-2/3}\langle h^{1/3}\eta_1 \rangle^{-1}, g)$ and this term can be estimate by $|\tilde{\Psi}w_h|^2$. We compute

$$\begin{aligned}
h^{1/6}\{\tilde{c}, \theta(\tilde{\rho})\} &= \{\tilde{\chi}_2(\tilde{\rho}), \theta(\tilde{\rho})\}\beta(\zeta) + \{\beta(\zeta), \theta(\tilde{\rho})\}\chi_2(\tilde{\rho}) \\
&= \{\tilde{\chi}_2(\tilde{\rho}), \theta(\tilde{\rho})\}\beta(\zeta) + \{\zeta, \theta(\tilde{\rho})\}\chi_2(\tilde{\rho})\beta'(\zeta).
\end{aligned}$$

The term $h^{-1}\tilde{c}h^{-1/6}\{\tilde{\chi}_2(\tilde{\rho}), \theta(\tilde{\rho})\}\beta(\zeta) \in S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g)$, and the term of K_4 coming from this term can be estimate by $|\tilde{\Psi}w_h||w_h|$. For the other term we have

$$h^{-1/3}\{\zeta, \theta(\tilde{\rho})\} = \{\tilde{r}_d^{-2/3}(\tilde{\rho}), \theta(\tilde{\rho})\}\eta_1 + \{\eta_1, \theta(\tilde{\rho})\}\tilde{r}_d^{-2/3}(\tilde{\rho}).$$

The term $h^{-1}\tilde{c}h^{-1/6}\chi_2(\tilde{\rho})\beta'(\zeta)h^{1/3}\{\tilde{r}_d^{-2/3}(\tilde{\rho}),\theta(\tilde{\rho})\}\eta_1 \in S(h^{-1/3}\langle h^{1/3}\eta_1\rangle^{-1/2},g)$ and the term of K_4 coming from this term can be estimate by $|\tilde{\Psi}w_h||w_h|$.

Thus, modulo remainder terms, the symbol of $-h^{-1}\tilde{C}^*\tilde{C}\tilde{A} + ih^{-1}\tilde{C}^*[\tilde{C},\text{op}(\theta)]$ is given by

$$\begin{aligned} L &= -h^{-1}\tilde{c}^2\tilde{a} + h^{-1}\tilde{c}h^{-1/6}\tilde{\chi}_2(\tilde{\rho})\beta'(\zeta)h^{1/3}\{\eta_1,\theta(\tilde{\rho})\}\tilde{r}_d^{-2/3}(\tilde{\rho}) \\ &= -h^{-1}h^{-1/3}\tilde{\chi}_2^2(\tilde{\rho})\beta^2(\zeta)h^{1/3}\tilde{\chi}(\tilde{\rho})\tilde{r}_d^{1/3}(\tilde{\rho})\alpha(\zeta) \\ &\quad - h^{-1}h^{-1/6}\tilde{\chi}_2(\tilde{\rho})\beta(\zeta)h^{-1/6}\tilde{\chi}_2(\tilde{\rho})\beta'(\zeta)h^{1/3}\tilde{r}_d\tilde{r}_d^{-2/3}(\tilde{\rho}) \end{aligned}$$

from **v**) of Lemma **B.2**. We thus obtain

$$L = h^{-1}\tilde{\chi}_2^2(\tilde{\rho})\beta(\zeta)\tilde{r}_d^{1/3}(\tilde{\rho})(-\beta(\zeta)\tilde{\chi}(\tilde{\rho})\alpha(\zeta) - \beta'(\zeta)) \in S(h^{-1},g).$$

As $\tilde{\chi}$ is equal 1 on the support of $\tilde{\chi}_2^2$, $\beta \geq 0$ and $-\beta(\zeta)\text{Re}\alpha(\zeta) - \beta'(\zeta) \geq 0$ we have $\text{Re}L \geq 0$. We can apply sharp Gårding inequality (see [26, Theorem 18.6.7]), we yield, taking account remainder terms

$$K_4 \geq -C(|\tilde{\Psi}w_h|^2 + |\tilde{\Psi}w_h||w_h|). \quad (132)$$

The last term is K_5 . The symbol of $h^{-1}\tilde{A}^2 + ih^{-1}[\tilde{A},\text{op}(\theta(\tilde{\rho}))] - h^{-1}\text{op}(h\eta_1\tilde{\chi}^2(\tilde{\rho}))$ is $h^{-1}\tilde{a}^2 + h^{-1}\{\tilde{a},\theta(\tilde{\rho})\} - \eta_1\tilde{\chi}^2(\tilde{\rho})$ modulo a symbol in $S(1,g)$. We have

$$\begin{aligned} \{\tilde{a},\theta(\tilde{\rho})\} &= h^{1/3}\alpha(\zeta)\{(\tilde{\chi}\tilde{r}_d^{1/3})(\tilde{\rho}),\theta(\tilde{\rho})\} + h^{2/3}(\tilde{\chi}\tilde{r}_d^{1/3})(\tilde{\rho})\alpha'(\zeta)\eta_1\{\tilde{r}_d^{-2/3}(\tilde{\rho}),\theta(\tilde{\rho})\} \\ &\quad + h^{2/3}(\tilde{\chi}\tilde{r}_d^{-1/3})(\tilde{\rho})\alpha'(\zeta)\{\eta_1,\theta(\tilde{\rho})\}. \end{aligned}$$

The first two terms give a term estimated by $h^{1/3}\langle h^{1/3}\eta_1\rangle^{1/2} \lesssim 1$ as on the support of $\tilde{\chi}$ we have $|\eta_1| \lesssim h^{-1}$. Then both terms give associated operators bounded on L^2 . Modulo a bounded operator on L^2 we have to consider the symbol, taking account **v**) of Lemma **B.2**

$$\begin{aligned} &h^{-1/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})\alpha^2(\zeta) - h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})\alpha'(\zeta) - \eta_1\tilde{\chi}^2(\tilde{\rho}) \\ &= h^{-1/3}\tilde{\chi}^2(\tilde{\rho})\tilde{r}_d^{2/3}(\tilde{\rho})(\alpha^2(\zeta) - \alpha'(\zeta) - h^{1/3}\tilde{r}_d^{-2/3}(\tilde{\rho})\eta_1) \\ &\quad - h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})(1 - \tilde{\chi}(\tilde{\rho}))\alpha'(\zeta). \end{aligned}$$

The first term is null from differential equation satisfying by α and the value of ζ . We claim that

$$|\text{op}(h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})(1 - \tilde{\chi}(\tilde{\rho}))\alpha'(\zeta))Fv_h| \lesssim |v_h|. \quad (133)$$

The proof of the claim is given below. With this claim, (126) and what we do above, the operator $h^{-1}\tilde{A}^2 + ih^{-1}[\tilde{A},\text{op}(\theta(\tilde{\rho}))] - h^{-1}\text{op}(h\eta_1\tilde{\chi}^2(\tilde{\rho}))$ gives a term bounded by $|v_h|$. As the symbol of $\tilde{C}^*\tilde{C}$ is in $S(h^{-1/3}\langle h^{1/3}\eta_1\rangle^{-1/2},g)$, we obtain that

$$|K_5| \lesssim |v_h||\tilde{\Psi}w_h|. \quad (134)$$

From (127), (128), (129), (130), (131), (132) and (134) we obtain

$$\frac{1}{2}\partial_{x_d}|C(h\partial_{x_d}v_h - Av_h)|^2 \gtrsim -(|q_h|^2 + |\tilde{\Psi}w_h|^2 + |v_h|^2 + |h\partial_{x_d}v_h|^2).$$

Integrating this inequality between 0 and $\sigma > 0$, we have, estimating the term coming from $|\tilde{\Psi}w_h|$ by Proposition **B.4**,

$$|C(h\partial_{x_d}v_h - Av_h)|^2(0) \lesssim \|q_h\|^2 + \|v_h\|_{H_{sc}^1(x_d>0)}^2 + |C(h\partial_{x_d}v_h - Av_h)|^2(\sigma). \quad (135)$$

As

$$|C(h\partial_{x_d}v_h - Av_h)|^2(\sigma) = (\tilde{C}^*\tilde{C}w_h, w_h),$$

we have from (127)

$$|C(h\partial_{x_d}v_h - Av_h)|^2(\sigma) \lesssim |\tilde{\Psi}w_h|^2(\sigma) + |v_h|^2(\sigma) + |h\partial_{x_d}v_h|^2(\sigma)$$

Integrating estimate (135) between two positive values of σ and estimating as above the term $|\tilde{\Psi}w_h|^2(\sigma)$, we obtain the conclusion of Proposition **B.6**. \square

Proof of Claim (133). As $\kappa^* \tilde{\chi}_1 = \chi_1$, from Lemma B.2 we thus have

$$F^{-1} \text{op}(\tilde{\chi}_1(\tilde{\rho}))F = \text{op}(\chi_1(x, h\xi')) + hK,$$

where K is bounded on L^2 . We then have $Fv_h = \text{op}(\tilde{\chi}_1(\tilde{\rho}))Fv_h + hK'v_h$, where K' is bounded on L^2 . Then

$$\begin{aligned} & \text{op}(h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})(1 - \tilde{\chi}(\tilde{\rho}))\alpha'(\zeta))Fv_h \\ &= \text{op}(h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})(1 - \tilde{\chi}(\tilde{\rho}))\alpha'(\zeta))(\text{op}(\tilde{\chi}_1(\tilde{\rho}))Fv_h + hK'v_h). \end{aligned}$$

The first term coming from $\text{op}(\tilde{\chi}_1(\tilde{\rho}))$ gives an operator with null symbol. As

$$h^{-1/3}(\tilde{\chi}\tilde{r}_d^{2/3})(\tilde{\rho})(1 - \tilde{\chi}(\tilde{\rho}))\alpha'(\zeta) \in S(h^{-1/3}, g),$$

then the second term is also bounded by $|v_h|$. This proves the claim. \square

Proposition B.7. *Let $(x'_0, 0) \in \partial\Omega_N$. Let $\chi_4 \in \mathcal{C}_0^\infty(\mathbb{R})$ be supported on a neighborhood of 0 and $\chi_4 = 1$ in a neighborhood of 0. Let $(x'_0, 0, \xi'_0)$ and U_0 be as in the statement of Lemma B.1. Let $\ell \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ supported on $\{\chi_2 = 1\}$ for every x_d , where $\kappa^* \tilde{\chi}_2 = \chi_2$. We moreover assume $\partial_{x_d} R(x, \xi') < 0$ on support of ℓ . We have*

$$\lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} \left| \left(\text{op}(\ell(x', 0, h\xi')\chi_4((R(x', 0, h\xi') - 1)/\varepsilon)(R(x', 0, h\xi') - 1))(v_h)|_{x_d=0} | (v_h)|_{x_d=0} \right)_0 \right| = 0$$

Proof. We can assume that the support of ℓ is contained in $\{\chi_1 = 1\}$. We then have from symbol calculus

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} \left(\text{op}(\ell(x', 0, h\xi')\chi_4((R(x', 0, h\xi') - 1)/\varepsilon)(R(x', 0, h\xi') - 1))(v_h)|_{x_d=0} | (v_h)|_{x_d=0} \right)_0 \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} \left(\text{op}(\ell(x', 0, h\xi')\chi_4((R(x', 0, h\xi') - 1)/\varepsilon)(R(x', 0, h\xi') - 1))(v_h)|_{x_d=0} | (v_h)|_{x_d=0} \right)_0 \end{aligned}$$

Let $z_h = (Fv_h)|_{x_d=0}$. From Proposition B.6 we obtain $|\tilde{C}\tilde{A}z_h|$ is bounded. By symbol calculus and the support properties of $\tilde{\chi}$ and $\tilde{\chi}_2$

$$\tilde{C}\tilde{A} = h^{1/6} \text{op}(\tilde{\chi}_2(\tilde{\rho})\beta(\zeta)\tilde{r}_d^{1/3}(\tilde{\rho})\alpha(\zeta))$$

modulo an operator with symbol in $S(h^{1/2}\langle h^{1/3}\eta_1 \rangle^{-3/4}, g)$. From properties of traces, see (88), this remainder term goes to 0 as h to 0.

Let $\tilde{\ell}$ be such that $\kappa^* \tilde{\ell} = \ell$ and $\kappa^* \chi_4(\eta_1/\varepsilon) = \chi_4((R(x', 0, \xi') - 1)/\varepsilon)$. In what follow, to be coherent with our notation we define $\tilde{\chi}_4 = \chi_4$ and we use the notation $\tilde{\chi}_4$ when the function is defined in (y, η) variables.

From Lemma B.2 we have

$$\begin{aligned} & F^{-1} \text{op}(h\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon))F \\ &= \text{op}(\ell(x', 0, h\xi')\chi_4((R(x', 0, h\xi') - 1)/\varepsilon)(R(x', 0, h\xi') - 1)) + h \text{op}(r_0(x, h\xi')), \end{aligned}$$

where r_0 is of order 0. The term coming from r_0 goes to 0 as h to 0, from properties of traces. Then it is sufficient to prove that $\lim_{\varepsilon \rightarrow 0} \lim_{h \rightarrow 0} \left(\text{op}(h\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon))z_h|z_h \right)_0 = 0$. Considering the symbol $h\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon) \in S(h^{2/3}\langle h^{1/3}\eta_1 \rangle, g)$ and from support properties of $\tilde{\ell}$ and $\tilde{\chi}_2$, we have

$$\text{op}(h\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon)) = \tilde{\gamma}^* \text{op}(h^{2/3}\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon)\beta^{-2}(\zeta)\tilde{r}_d^{-2/3}(\tilde{\rho})|\alpha(\zeta)|^{-2})\tilde{\gamma}$$

where $\tilde{\gamma} = \text{op}(h^{1/6}\tilde{\chi}_2(\tilde{\rho})\beta(\zeta)\tilde{r}_d^{1/3}(\tilde{\rho})\alpha(\zeta))$, modulo an operator with symbol in $S(h, g)$ then this last term involves a term going to 0 as h to 0. Then we obtain an estimation, modulo a term going to 0 as h to 0,

$$\begin{aligned} & \left| \left(\text{op}(h\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon))z_h|z_h \right)_0 \right| \\ & \lesssim \left| \text{op}(h^{2/3}\eta_1 \tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon)\beta^{-2}(\zeta)\tilde{r}_d^{-2/3}(\tilde{\rho})|\alpha(\zeta)|^{-2})z_h \right| |z_h|, \quad (136) \end{aligned}$$

where $z_h = \text{op}(h^{1/6}\tilde{\chi}_2(\tilde{\rho})\beta(\zeta)\tilde{r}_d^{1/3}(\tilde{\rho})\alpha(\zeta))z_h$. Observe from Proposition B.6, $|z_h|$ is bounded.

We claim that $h^{2/3}\eta_1\tilde{\chi}_4(h\eta_1/\varepsilon) \in S(\langle h^{1/3}\eta_1 \rangle^{1/2}, g)$. Indeed $h^{2/3}|\eta_1| \lesssim \langle h\eta_1 \rangle^{1/2} \langle h^{1/3}\eta_1 \rangle^{1/2}$, and this gives the sought estimate. We have for $k \geq 1$,

$$\partial_{\eta_1}^k (h^{2/3}\eta_1\tilde{\chi}_4(h\eta_1/\varepsilon)) = h^{2/3}(h/\varepsilon)^{k-1}\tilde{\chi}_4^{(k-1)}(h\eta_1/\varepsilon) + h^{2/3}\eta_1(h/\varepsilon)^k\tilde{\chi}_4^{(k)}(h\eta_1/\varepsilon).$$

As $h\eta_1$ is bounded on the support of $\tilde{\chi}_4^{(k)}(h\eta_1/\varepsilon)$ both terms are bounded by $h^{k-1/3}$. From estimate $h^{2/3}\langle h^{1/3}\eta_1 \rangle \lesssim \langle h\eta_1 \rangle$ we have

$$h^{k-1/3} \lesssim h^{k/3} (\langle h\eta_1 \rangle / \langle h^{1/3}\eta_1 \rangle)^{k-1/2},$$

which proves the claim. Observe that the constant of estimation are not uniform with respect ε .

With the previous claim and as $\beta^{-2}(\zeta)|\alpha(\zeta)|^{-2} \lesssim \langle h^{1/3}\eta_1 \rangle^{-1/2}$, we have

$$L(y', \eta') = h^{2/3}\eta_1\tilde{\ell}(\tilde{\rho})\tilde{\chi}_4(h\eta_1/\varepsilon)\beta^{-2}(\zeta)\tilde{r}_d^{-2/3}(\tilde{\rho})|\alpha(\zeta)|^{-2} \in S(1, g).$$

As $h^{2/3}|\eta_1| \lesssim |h\eta_1|^{1/2} \langle h^{1/3}\eta_1 \rangle^{1/2} \lesssim \sqrt{\varepsilon} \langle h^{1/3}\eta_1 \rangle^{1/2}$, on the support of $\tilde{\chi}_4(h\eta_1/\varepsilon)$, we deduce from Gårding inequality (see [26, Theorem 18.6.7]) that the operator norm from L^2 to L^2 of $\text{op}(L)$ is bounded by $C\sqrt{\varepsilon} + C_\varepsilon h^{1/3}$ where C is independent of ε and C_ε may depend on ε . From that and (136) we deduce the result. \square

C Proof of Lemmas

Lemma C.1. *Let $\tilde{\chi}_3 \in \mathcal{C}_0^\infty(U_1)$ be such that $(1 - \tilde{\chi}_3)\tilde{\chi}_1 = 0$ where $\kappa^*\tilde{\chi}_1 = \chi_1$. We have*

$$|\tilde{\Psi}w_h| \lesssim |h^{-1/3} \text{op}(\tilde{\chi}_3(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1/2})w_h| + |v_h| + |h\partial_{x_d}v_h|.$$

Proof. We write $\tilde{\Psi} = h^{-1/3} \text{op}(\tilde{\chi}_3(\tilde{\rho})\langle h^{1/3}\eta_1 \rangle^{-1/2}) + \text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})$, it suffices to estimates terms coming from the second operator. Recall that $w_h = F(h\partial_{x_d}v_h - Av_h)$, we have

$$\text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})FA = \text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})\tilde{A}F,$$

as $h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2} \in S(h^{-1/3}\langle h^{1/3}\eta_1 \rangle^{-1/2}, g)$ and $\tilde{a} \in S(h^{1/3}\langle h^{1/3}\eta_1 \rangle^{1/2}, g)$, we have

$$|\text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})FAv_h| \lesssim |v_h|.$$

We have

$$Fh\partial_{x_d} \text{op}(\chi_1(x, h\xi')) = F \text{op}(\chi_1(x, h\xi'))h\partial_{x_d} + Fh \text{op}(\partial_{x_d}\chi_1(x, h\xi')). \quad (137)$$

The second term gives

$$|\text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})Fh \text{op}(\partial_{x_d}\chi_1(x, h\xi'))v_h| \lesssim h^{1/3}|v_h|.$$

As $\kappa^*\tilde{\chi}_1 = \chi_1$, from Lemma B.2 we thus have $F^{-1} \text{op}(\tilde{\chi}_1(\tilde{\rho}))F = \text{op}(\chi_1(x, h\xi')) + hK$, where K is bounded on L^2 . Then the first term at the right hand side of (137) gives

$$\begin{aligned} & |\text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})F \text{op}(\chi_1(x, h\xi'))h\partial_{x_d}v_h| \\ & \lesssim |\text{op}(h^{-1/3}(1 - \tilde{\chi}_3(\tilde{\rho}))\langle h^{1/3}\eta_1 \rangle^{-1/2})(\text{op}(\tilde{\chi}_1(\tilde{\rho})) - hFKF^{-1})Fh\partial_{x_d}v_h| \leq |h\partial_{x_d}v_h|, \end{aligned}$$

as the both terms are bounded on L^2 indeed the asymptotic expansion of the first symbol is null and the second is bounded by $h^{2/3}$ times an operator bounded on L^2 . This concludes the proof of the lemma. \square

Proof of Lemma B.2. We recall that Zworski use Weyl quantification. We give the proof in this context. From that, it is easy to obtain the result for classical quantification. We denote by $\text{op}_{sc}^w(a)$ the operator associated with symbol a by the Weyl quantification. Items **i)** and **ii)** come for Zworski [46, Theorem 11.5]. To prove the others Items we have to use the construction of F given by Zworski [46, Section 11.1 and 11.2]. Let κ_t a smooth family of symplectic transformations, $t \in [0, 1]$, $\kappa_0 = Id$ and $\kappa_1 = \kappa$ and $q_t \in \mathcal{C}_0^\infty(U_0)$ be real valued, such that $\partial_t \kappa_t = (\kappa_t)_* H_{q_t}$ (see [46, Theorem 11.4]). Let $Q(t) = \text{op}_{sc}^w(q_t)$ (here as we use the Weyl quantification, $Q(t)$ is selfadjoint). Let $F(t)$ the solution of

$$\begin{cases} hD_t F(t) + F(t)Q(t) = 0 \\ F(0) = Id. \end{cases}$$

The Fourier Integral Operator we search, is $F = F(1)$. We then have for $G(t) = \partial_{x_d} F(t)$

$$\begin{cases} hD_t G(t) + G(t)Q(t) = -F(t)\partial_{x_d} Q(t) \\ G(0) = 0. \end{cases}$$

The Duhamel formula yields

$$G(t)F^{-1}(t) = -ih^{-1} \int_0^t F(\sigma)\partial_{x_d} Q(\sigma)F^{-1}(\sigma)d\sigma.$$

Taking $\tilde{q}_0(t)$ such that $\kappa_t^* \tilde{q}_0(t) = \partial_{x_d} q(t)$ which it is possible as κ_t is a diffeomorphism, we have from Item **ii)**, $F^{-1}(\sigma) \text{op}_{sc}^w(\tilde{q}_0(\sigma))F(\sigma) = \partial_{x_d} Q(\sigma) + h \text{op}_{sc}^w(q_1)$. We can repeat the construction taking $\kappa_t^* \tilde{q}_1(t) = q_1(t)$ and we have $F^{-1}(\sigma) \text{op}_{sc}^w(\tilde{q}_0(\sigma) - h\tilde{q}_1(\sigma))F(\sigma) = \partial_{x_d} Q(\sigma) + h^2 \text{op}_{sc}^w(q_2(\sigma))$. This implies that $F(\sigma)\partial_{x_d} Q(\sigma)F^{-1}(\sigma) = \text{op}_{sc}^w(\tilde{q}_0(\sigma) - h\tilde{q}_1(\sigma)) + h^2 B(\sigma)$, where $B(\sigma)$ is bounded on L^2 uniformly with respect σ . From that and taking $t = 1$ we deduce Item **iii)**.

To prove Item **iv)** we have $\partial_{x_d}(F^{-1}) = -F^{-1}\partial_{x_d}(F)F^{-1}$, we deduce

$$\begin{aligned} \partial_{x_d} A &= -F^{-1}(\partial_{x_d} F)F^{-1}\tilde{A}F + F^{-1}(\partial_{x_d} \tilde{A})F + F^{-1}\tilde{A}(\partial_{x_d} F) \\ &= -F^{-1}(ih^{-1} \text{op}_{sc}^w(\theta) + hB)\tilde{A}F + F^{-1}(\partial_{x_d} \tilde{A})F + F^{-1}\tilde{A}(ih^{-1} \text{op}_{sc}^w(\theta) + hB)F \\ &= F^{-1}((\partial_{x_d} \tilde{A}) + ih^{-1}[\tilde{A}, \text{op}_{sc}^w(\theta)] + h[\tilde{A}, B])F, \end{aligned}$$

Which gives Item **iv)** as there exist a symbol θ_1 such that $\text{op}_{sc}(\theta_1) = \text{op}_{sc}^w(\theta)$.

Let χ_1 and χ_2 be \mathcal{C}_0^∞ functions such that $\chi_1 = 1$ and $\chi_2 = 1$ in a neighborhood of $(x'_0, 0, \xi'_0)$, and we assume χ_2 supported on $\{\chi_1 = 1\}$. Applying **iv)** to $A = (R-1)\chi_1$, taking account **ii)**, we obtain $\kappa^*(\{\eta_1, \theta\}\tilde{\chi}_2(y, \eta')) = \chi_2 \partial_{x_d} R$ where $\kappa^* \tilde{\chi}_2 = \chi_2$. We deduce Item **v)** taking the previous formula on $\{\chi_2 = 1\}$. \square

Proof of Lemma B.3. The asymptotic expansion of Ai is well-known (see [1, Formula 10.4.59]). We recall that for $z \in \mathbb{C}$ with $|\arg z| < \pi$ we have

$$\begin{aligned} \text{Ai}(z) &\sim 2^{-1}\pi^{-1/2}z^{-1/4}e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n c_n \zeta^{-n}, \quad \text{with } \zeta = \frac{2}{3}z^{2/3} \\ \text{Ai}'(z) &\sim -2^{-1}\pi^{-1/2}z^{1/4}e^{-\zeta} \sum_{n=0}^{\infty} (-1)^n d_n \zeta^{-n}, \end{aligned} \tag{138}$$

where $c_0 = d_0 = 1$, $c_1 = \frac{5}{2332}$ and $d_1 = -\frac{7c_1}{5}$. From that we obtain

$$\frac{\text{Ai}'(z)}{\text{Ai}(z)} \sim -z^{1/2} \sum_{n=0}^{\infty} f_n \zeta^{-n} \sim -z^{1/2} \sum_{n=0}^{\infty} \ell_n z^{-3n/2},$$

where $f_0 = \ell_0 = 1$, $f_1 = 1/6$ and $\ell_1 = 1/4$.

For $x > 0$, as $\omega^{3/2} = -1$, we have,

$$\alpha(x) \sim \omega \omega^{1/2} x^{1/2} \sum_{n=0}^{\infty} \ell_n \omega^{-3n/2} x^{-3n/2} \sim -x^{1/2} \sum_{n=0}^{\infty} (-1)^n \ell_n x^{-3n/2}.$$

As this asymptotic expansion is also valid for derivative with respect x we deduce Item **i**).

For $x < 0$, we have $\omega x = -e^{2i\epsilon\pi/3}|x| = e^{-i\epsilon\pi/3}|x|$ to have $|\arg(\omega x)| < \pi$. As $-\omega e^{-i\epsilon\pi/6} = -\epsilon i$ we obtain

$$\alpha(x) \sim \epsilon i |x|^{1/2} \sum_{n=0}^{\infty} \ell_n (e^{-i\epsilon\pi/3})^{-3n/2} |x|^{-3n/2} \sim \epsilon i |x|^{1/2} \sum_{n=0}^{\infty} (\epsilon i)^n \ell_n |x|^{-3n/2}.$$

This gives Item **ii**) from properties of this asymptotic expansion.

Let $F(z) = \text{Ai}'(z)/\text{Ai}(z)$ we have $F'(z) = z - A^2(z)$ for z different of a zero of Ai which are on the negative real axis. As $\alpha(x) = -\omega F(\omega x)$ we have

$$\alpha'(x) = -\omega^2 F'(\omega x) = -\omega^2 (\omega x - F^2(\omega x)) = \alpha^2(x) - x.$$

This gives Item **iv**).

Item **iii**) is probably classical but we do not find this property in literature. Here we give a proof of that. Let $\alpha_1(x)$ and $\alpha_2(x)$ real valued be such that $\alpha(x) = \alpha_1(x) + i\alpha_2(x)$. We have

$$\begin{cases} \alpha_1'(x) = \alpha_1^2(x) - \alpha_2^2(x) - x \\ \alpha_2'(x) = 2\alpha_1(x)\alpha_2(x). \end{cases} \quad (139)$$

We also use a nice formula given in [45, Section 3]

$$F(z) = C_1 + \sum_{j=1}^{\infty} ((z - \nu_j)^{-1} + \nu_j^{-1}),$$

where ν_j 's are the zeros of Ai (observe that $\nu_j < 0$) and $C_1 \in \mathbb{R}$ is an explicit negative constant. We deduce

$$F'(z) = -\sum_{j=1}^{\infty} (z - \nu_j)^{-2}$$

It is easy to prove that both series converge.

We fix $\epsilon = 1$, observe that $\overline{\alpha(x)}$ is the $\alpha(x)$ defined with $\epsilon = -1$. Observe that $\alpha_2(x) > 0$, indeed if $\alpha_2(x_0) = 0$ for some $x_0 \in \mathbb{R}$, α_2 is identically null by uniqueness of System (139). Then $\alpha_2(x) > 0$ as it is true for $x \ll 0$ from Item **ii**).

We have

$$\begin{aligned} \alpha'(x) &= -\omega^2 F'(\omega x) = \omega^2 \sum_{j=1}^{\infty} (\omega x - \nu_j)^{-2} \\ &= \sum_{j=1}^{\infty} |\omega x - \nu_j|^{-4} (x^2 - 2\omega \nu_j x + \omega^2 \nu_j^2). \end{aligned}$$

We deduce that

$$\alpha_2'(x) = \sum_{j=1}^{\infty} |\omega x - \nu_j|^{-4} \left(-\sqrt{3}\nu_j x - \frac{\sqrt{3}}{2}\nu_j^2 \right)$$

Assuming $x \leq 0$ we have $\alpha_2'(x) < 0$ and from (139), $\alpha_1(x) < 0$.

Now for $x \geq 0$ we compute

$$(\alpha_1/\alpha_2)'(x) = \frac{-\alpha_1^2(x)\alpha_2(x) - \alpha_2^3(x) - x\alpha_2(x)}{\alpha_2^2(x)} < 0.$$

This implies $(\alpha_1/\alpha_2)(x) \leq (\alpha_1/\alpha_2)(0) < 0$ then $\alpha_1(x) < 0$. This concludes Item **iii**). \square

Proof of Lemma B.5. Let $\gamma_1(x) = |\text{Ai}(\omega x)|$ and $\gamma_2(x) = C_0 \langle x \rangle^{-1/4}$ where $C_0 > 0$ will be fixed below. We shall begin to prove $\gamma_1' + \gamma_1 \text{Re } \alpha = 0$. Writing $\gamma_1(x) = \sqrt{\text{Ai}(\omega x) \text{Ai}(\bar{\omega} x)}$ we obtain

$$\gamma_1'(x) = \frac{\omega \text{Ai}'(\omega x) \text{Ai}(\bar{\omega} x) + \bar{\omega} \text{Ai}(\omega x) \text{Ai}'(\bar{\omega} x)}{2|\text{Ai}(\omega x)|},$$

and

$$\text{Re } \alpha(x) = -\text{Re} \frac{\omega \text{Ai}'(\omega x) \text{Ai}(\bar{\omega} x)}{|\text{Ai}(\omega x)|^2} = -\frac{\gamma_1'(x)}{\gamma_1(x)}.$$

From definition of γ_2 we have for $x > 0$, $\gamma_2 > 0$, $\gamma_2' < 0$ and as $\text{Re } \alpha < 0$ we have $\gamma_2' + \gamma_2 \text{Re } \alpha < 0$.

Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\chi_0(x) = 0$ if $x \leq 0$, $\chi_0(x) = 1$ if $x \geq 1$ and we assume $\chi_0' \geq 0$ on \mathbb{R} . Let $\beta = \chi_0 \gamma_2 + (1 - \chi_0) \gamma_1$. Clearly β is a smooth function. We have $\beta' = \chi_0 \gamma_2' + (1 - \chi_0) \gamma_1' + \chi_0'(\gamma_2 - \gamma_1)$. As $\gamma_1 > 0$, if C_0 is chosen sufficiently small, $\gamma_2 - \gamma_1 < 0$ on the support of χ_0' . As $\chi_0' \geq 0$, $\chi_0'(\gamma_2 - \gamma_1) \leq 0$. This and above properties imply **i)**. We only have to prove **ii)** for $x < 0$. That is a consequence of asymptotic expansion of Airy function (138). Indeed for $x < 0$, $\omega x = |x|e^{-i\pi/3}$, then $(\omega x)^{3/2} = -\varepsilon i|x|^{3/2}$. We then have $\beta(x) = |\text{Ai}(\omega x)| \sim 2^{-1}\pi^{-1/2}|x|^{-1/4} \left| \sum_{n=0}^{\infty} (-1)^n c_n \zeta^{-n} \right|$, with $\zeta = \frac{2}{3}(\omega x)^{2/3}$. Clearly the asymptotic expansion satisfies symbol estimates. This asymptotic expansion also gives estimates **iii)** for $x < 0$ with $|x|$ sufficiently large. For $x > 0$ **iii)** is obvious, and by construction $\beta > 0$. This achieves the proof. \square

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