



HAL
open science

ON RADIUS OF CONVERGENCE OF q -DEFORMED REAL NUMBERS

Ludivine Leclere, Sophie Morier-Genoud, Valentin Ovsienko, Alexander
Veselov

► **To cite this version:**

Ludivine Leclere, Sophie Morier-Genoud, Valentin Ovsienko, Alexander Veselov. ON RADIUS OF CONVERGENCE OF q -DEFORMED REAL NUMBERS. Moscow Mathematical Journal, In press. hal-03363862v1

HAL Id: hal-03363862

<https://hal.science/hal-03363862v1>

Submitted on 4 Oct 2021 (v1), last revised 28 Sep 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON RADIUS OF CONVERGENCE OF q -DEFORMED REAL NUMBERS

LUDIVINE LECLERE, SOPHIE MORIER-GENOUD, VALENTIN OVSIENKO, AND ALEXANDER VESELOV

ABSTRACT. We study analytic properties of “ q -deformed real numbers”, a notion recently introduced by two of us. A q -deformed positive real number is a power series with integer coefficients in one formal variable q . We study the radius of convergence of these power series assuming that $q \in \mathbb{C}$. Our main conjecture, which can be viewed as a q -analogue of Hurwitz’s Irrational Number Theorem, provides a lower bound for these radii, given by the radius of convergence of the q -deformed golden ratio. The conjecture is proved in several particular cases and confirmed by a number of computer experiments. For an interesting sequence of “Pell polynomials”, we obtain stronger bounds.

1. INTRODUCTION

There is a famous result due to Hurwitz [6], which roughly claims that the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is the most irrational number. More precisely, for any real number $x \in \mathbb{R}$ one can define as the measure of irrationality the Markov constant $\mu(x)$, which is the infimum of c , for which the inequality $|x - \frac{p}{q}| < \frac{c}{q^2}$ holds for infinitely many integer p, q . Hurwitz’s Irrational Number Theorem claims that for every $x \in \mathbb{R}$

$$(1.1) \quad \mu(x) \leq \mu(\varphi) = \frac{1}{\sqrt{5}}$$

with equality holding only for x which are $\text{PSL}(2, \mathbb{Z})$ -equivalent to the golden ratio.

In this paper we discuss a possible q -analogue of this classical result. Namely, we consider the q -deformations (or “ q -analogues”) of real numbers, which have been recently introduced in [12, 13] and studied further in [10, 14, 8]. They have several nice properties and connections, including theory of Conway-Coxeter friezes and knot invariants.

For a rational $x = \frac{r}{s} > 0$ the q -deformation is a rational function

$$(1.2) \quad \left[\frac{r}{s} \right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)},$$

where $\mathcal{R}(q)$ and $\mathcal{S}(q)$ are polynomials with positive integer coefficients both depending on r and s . When $x \geq 1$ is an irrational, the q -deformation of x is defined as a power series in q :

$$(1.3) \quad [x]_q = 1 + \varkappa_1 q + \varkappa_2 q^2 + \varkappa_3 q^3 + \dots$$

with coefficients $\varkappa_k \in \mathbb{Z}$. To obtain the series (1.3), one chooses an arbitrary sequence of rationals $(x_i)_{i \in \mathbb{Z}}$ converging to x . It turns out that the Taylor series of the rational functions $[x_i]_q$ stabilize, as i grows; see [13]. Moreover, the stabilized series depends only on x (and not on the approximating sequence of rationals). The series $[x]_q$ is defined as the stabilization of the Taylor series of $[x_i]_q$. It is unknown, in general, how to characterize the class of power series that represent q -deformed real numbers. One of the goals of this paper is to show that series arising in this context are not arbitrary. In particular, they must have non-zero radius of convergence.

We study analytic properties of q -deformed real numbers. Considering the parameter of deformation q as a *complex variable*, $q \in \mathbb{C}$, we study the radius of convergence of the series (1.3).

The following conjecture can be considered as a possible q -analogue of the Hurwitz claim.

Conjecture 1.1. *For every real $x \geq 1$, the radius of convergence $R(x)$ of the series $[x]_q$ satisfies*

$$(1.4) \quad R(x) \geq R_* := R(\varphi) = \frac{3 - \sqrt{5}}{2}$$

and the equality holding only for x which are $\text{PSL}(2, \mathbb{Z})$ -equivalent to $\varphi = \frac{1+\sqrt{5}}{2}$.

Recall that the inverse $\rho = R^{-1}$ of the radius of convergence of power series (1.3) can be given by the standard formula (see e.g. [3])

$$(1.5) \quad \rho(x) = \limsup_{n \rightarrow \infty} |\varkappa_n|^{\frac{1}{n}}$$

and thus describes the growth of the coefficients \varkappa_n . Conjecture 1.1 can be reformulated in the form, more similar to Hurwitz, as the inequality

$$(1.6) \quad \rho(x) \leq \rho(\varphi) = \frac{3 + \sqrt{5}}{2}$$

holding for every real x , and thus means that $[\varphi]_q$ has the fastest growth of the coefficients among all q -deformed real numbers. In other words, the series $[x]_q$ always “converges better” than $[\varphi]_q$, whenever x is not equivalent to φ . In the particular case where $x = \frac{r}{s}$ is rational, Conjecture 1.1 means that the polynomial $\mathcal{S}(q)$ in (1.2) has no roots q with $|q| < R_* \approx 0.381966$.

We should mention that the analogy with the Diophantine analysis here cannot be extended to the celebrated Markov theorem [9], claiming that the set of all possible Markov constants $\mu = \mu(x) > \frac{1}{3}$ is discrete (see e.g. [1]). The situation might be more similar to the Lyapunov spectrum of Markov and Euclid trees, which fills the whole segment $[0, \ln \varphi]$, see [16]. In particular, for every R such that $R_* < R < 1$, there exists a number x , such that $R(x) \leq R$.

Conjecture 1.1 was checked by a long series of computer experimentation, and will be proved in several particular cases. Provided it is correct, this gives a restriction for the series (1.3) that can appear as a q -deformed real number. However, it is quite clear that the radius of convergence is not the only condition. It would be interesting to obtain more information about this class of power series.

Let us give here an idea about the approach we use to prove the main statements. Our main analytical tool is the classical Rouché theorem (see [3, 17]) that states the following. Let f and g be two functions of one complex variable q , holomorphic inside a disc D and continuous on the bound ∂D . Suppose that $f(q) > g(q)$ on ∂D , then f and $f + g$ have the same number of zeros inside D . We will usually suppose that f has no zeros inside D , and conclude that the same is true for $f + g$.

The paper is organized as follows.

In Section 2, we briefly recall the notion of q -rationals and q -irrationals. Following [8], we emphasize the role of the modular group $\text{PSL}(2, \mathbb{Z})$. We also give explicit formulas in terms of continued fractions.

In Section 3, we consider the important example of q -deformed golden ratio. Its approximation by the quotients of consecutive Fibonacci numbers leads to interesting “Fibonacci polynomials”. We check the statement of Conjecture 1.1 in this particular case.

In Section 4, we consider the q -number $[\sqrt{2}]_q$, and its approximation by the quotients of consecutive “Pell polynomials”. We strengthen the statement of Conjecture 1.1 in this case, replacing R_* by another, greater value.

Finally, in Section 5 we prove the statement of Conjecture 1.1 for a large class of irrational numbers with a certain restriction (that, as we believe, is technical and can perhaps be removed). We give two more interesting examples.

2. DEFINITION AND EXPLICIT FORMULAS FOR q -REALS

We start with recalling an axiomatic definition of q -deformed rational numbers. For more equivalent definitions; see [12, 14].

2.1. An axiomatic definition. Recall that every rational can be obtained from 0, by applying a sequence of the operations $x \mapsto x + 1$ and $x \mapsto -\frac{1}{x}$.

The following two recurrences (see [8], and also [14]) suffice for calculation of q -rationals.

Definition 2.1. The q -deformation sends every rational number $x = \frac{r}{s}$ to a rational function in q

$$x \mapsto [x]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)},$$

in such a way that the following two recurrent formulas are satisfied

$$(2.1) \quad [x+1]_q = q[x]_q + 1, \quad \left[-\frac{1}{x}\right]_q = -\frac{1}{q[x]_q}.$$

Recurrences (2.1) determine the q -deformation of a rational in a unique way, starting from the “initial” trivial deformation $[0]_q = 0$.

Example 2.2. The following examples are easy to obtain applying (2.1).

(a) The q -deformation of integers corresponds to the standard formulas of Euler and Gauss. For $n \in \mathbb{N}$, one has

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}, \quad [-n]_q = -q^{-n} - q^{1-n} - q^{2-n} - \cdots - q^{-1}.$$

Both cases can be written as $[x]_q = \frac{1-q^x}{1-q}$, where x is integer.

(b) Already the simplest example of $\frac{1}{2}$ shows that the q -deformation is not the quotient of the q -deformed integers appearing in numerator and denominator:

$$\left[\frac{1}{2}\right]_q = \frac{q}{1+q}, \quad \left[-\frac{1}{2}\right]_q = -\frac{1}{q(1+q)}.$$

The above expression $[x]_q = \frac{1-q^x}{1-q}$ is no longer true.

(c) The next examples

$$\left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}, \quad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

illustrate the fact that the numerator and denominator in (1.2) depend simultaneously on r and s . Indeed, the “quantized 5” in the numerator depends on the denominator.

Remark 2.3. In the case of q -rationals, one also has the formula for the negation and inverse:

$$(2.2) \quad [-x]_q = -\frac{[x]_{q^{-1}}}{q}, \quad \left[\frac{1}{x}\right]_q = \frac{1}{[x]_{q^{-1}}}.$$

2.2. The action and a central extension of $\mathrm{PSL}(2, \mathbb{Z})$. We briefly mention here a more conceptual way of understanding Definition 2.1. This observation will not be used in the sequel, and will be part of a separate work.

Recurrences (2.1) can be reformulated as a statement that “ q -deformation commutes with $\mathrm{PSL}(2, \mathbb{Z})$ -action”. Indeed, recurrences (2.1) define an action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ on q -deformed rationals. This action is given by fractional-linear transformations and is generated by the matrices

$$(2.3) \quad T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$$

which are q -deformations of the standard generators, T, S , of $\mathrm{PSL}(2, \mathbb{Z})$ corresponding to $q = 1$ in (2.3).

The relations $S^2 = \mathrm{Id}$ and $(TS)^3 = \mathrm{Id}$, defining $\mathrm{PSL}(2, \mathbb{Z})$, become

$$S_q^2 = q \mathrm{Id}, \quad (T_q S_q)^3 = q^3 \mathrm{Id},$$

The matrices T_q, S_q generate an interesting non-trivial *central extension* of $\mathrm{PSL}(2, \mathbb{Z})$, which is different from the braid group B_3 , yet contains it as a subgroup. The centre, $\{q^n \mathrm{Id} \mid n \in \mathbb{Z}\}$, of the extended modular group acts trivially on q -deformed rationals, so that one still has the well-defined action of $\mathrm{PSL}(2, \mathbb{Z})$.

Remark 2.4. (a) Let us stress on the fact that emergence of a central extension of the symmetry group is a usual situation in geometry; see [7]. However, the centre of the extended group usually acts trivially on quantized objects.

(b) The matrix S_q arose in the context of quantum groups; see [2], while T_q can be viewed as a “standard” matrix connected to quantum integers. However, we did not find in the literature simultaneous appearance of T_q and S_q .

2.3. q -deformed continued fractions. Every rational number $\frac{r}{s} > 0$, where $r, s \in \mathbb{Z}_{>0}$ are coprime, has a standard finite continued fraction expansion

$$\frac{r}{s} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2m}}}},$$

where $a_i \geq 1$ (except for $a_1 \geq 0$). It is usually denoted by $\frac{r}{s} = [a_1, \dots, a_{2m}]$. Note that, choosing an even number of coefficients, one removes the ambiguity $[a_1, \dots, a_n, 1] = [a_1, \dots, a_n + 1]$ and makes the expansion unique.

There is also a unique continued fraction expansion with minus signs, often called the Hirzebruch-Jung continued fraction:

$$\frac{r}{s} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_k}}},$$

where $c_j \geq 2$ (except for $c_1 \geq 1$). The notation used by Hirzebruch is $\frac{r}{s} = \llbracket c_1, \dots, c_k \rrbracket$. The coefficients a_i and c_j of the above expansions are connected by the Hirzebruch formula; see, e.g., [5, 11] and Section 5.1.

Definition 2.5. (a) The q -deformed regular continued fraction is defined by

$$(2.4) \quad [a_1, \dots, a_{2m}]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_{q^{-1}} + \frac{q^{-a_2}}{[a_3]_q + \frac{q^{a_3}}{[a_4]_{q^{-1}} + \frac{q^{-a_4}}{\ddots}}}}}$$

where $[a]_q$ is the Euler q -integer.

(b) The q -deformed Hirzebruch-Jung continued fraction is

$$(2.5) \quad \llbracket c_1, \dots, c_k \rrbracket_q := [c_1]_q - \frac{q^{c_1-1}}{[c_2]_q - \frac{q^{c_2-1}}{\ddots}}}$$

For every rational $\frac{r}{s}$ written in two different ways:

$$\frac{r}{s} = [a_1, \dots, a_{2m}] = \llbracket c_1, \dots, c_k \rrbracket,$$

the rational functions (2.4) and (2.5) coincide, and also coincide with the rational function $\left[\frac{r}{s}\right]_q$ provided by Definition 2.1; see [12].

2.4. Stabilization phenomenon, q -irrationals. The notion of q -deformed rational was extended to irrational numbers in [13]. Let $x \geq 1$ be an irrational number, and choose a sequence of rationals $(x_n)_{n \geq 1}$, converging to x . Consider the corresponding sequence of q -rationals $[x_1]_q, [x_2]_q, \dots$. It turns out that this sequence of rational functions also converge, but in the sense of formal power series.

Consider the Taylor expansions at $q = 0$ of the rational functions $[x_n]_q$, that, abusing notation, we also denote by $[x_n]_q$:

$$[x_n]_q = \sum_{k \geq 0} \varkappa_{n,k} q^k.$$

One has the following stabilization property.

Theorem 2.1 ([13]). (i) For every $k \geq 0$, the coefficients $\varkappa_{n,k}$ of the Taylor series of the functions $[x_n]_q$ stabilize, as n grows.

(ii) The limit coefficients, $\varkappa_k := \lim_{n \rightarrow \infty} \varkappa_{n,k}$, do not depend on the choice of the sequence of rationals, but only on the irrational number x .

The q -deformation, $[x]_q$, is the limit power series in q , which has the form (1.3) (i.e., the coefficient of degree 0 is equal to 1). Furthermore, the recurrence (2.1) also holds for the series $[x]_q$, and allows one to extend the q -deformation to the case of $x < 0$. The resulting series in q is a *Laurent series* (with integer coefficients):

$$[x]_q = -q^{-N} + \varkappa_{1-N} q^{1-N} + \varkappa_{2-N} q^{2-N} + \dots$$

where $N \in \mathbb{Z}_{>0}$ such that $-N \leq x < 1 - N$.

Remark 2.6. Let us mention that the stabilization phenomenon fails when a sequence of rationals $(x_n)_{n \geq 1}$, converges to another rational, cf. [13].

3. CONVERGENCE RADIUS OF q -GOLDEN RATIO AND ROOTS OF THE FIBONACCI POLYNOMIALS

The simplest example of q -irrational is the q -deformation of the celebrated golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. The series $[\varphi]_q$ is obtained as stabilized Taylor series of the q -deformed quotients of the consecutive Fibonacci numbers $\left[\frac{F_{n+1}}{F_n}\right]_q$ (see [12, 13]). We call the polynomials in the numerator and the denominator of these rational functions the *Fibonacci polynomials*.

We prove that the radius of convergence of $[\varphi]_q$ is $R_* = \frac{3-\sqrt{5}}{2}$, and that all roots of the Fibonacci polynomials belong to the annulus bounded by the circles with the radius $R_* = \frac{3-\sqrt{5}}{2}$ and $R_*^{-1} = \frac{3+\sqrt{5}}{2}$. This result is in an accordance with Conjecture 1.1.

3.1. The q -deformed golden ratio. The q -deformation of the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$, was considered in [13]. The series $[\varphi]_q$ can be written as an infinite continued fraction:

$$(3.1) \quad [\varphi]_q = 1 + \frac{q^2}{q + \frac{1}{1 + \frac{q^2}{q + \frac{1}{\ddots}}}} = 1 + \frac{1}{q^{-1} + \frac{1}{q^2 + \frac{1}{q^{-3} + \frac{1}{\ddots}}}}$$

The series starts as follows

$$\begin{aligned} [\varphi]_q &= 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 + 185q^{10} \\ &\quad - 423q^{11} + 978q^{12} - 2283q^{13} + 5373q^{14} - 12735q^{15} + 30372q^{16} \\ &\quad - 72832q^{17} + 175502q^{18} - 424748q^{19} + 1032004q^{20} \dots \end{aligned}$$

The sequence of coefficients in $[\varphi]_q$ coincides (up to the alternating sign) with the remarkable sequence A004148 of [15] called the “generalized Catalan numbers”.

The series $[\varphi]_q$ is a solution of the functional equation

$$(3.2) \quad qX^2 = (q^2 + q - 1)X + 1,$$

which can be deduced from (3.1).

Proposition 3.1. The radius of convergence of the series $[\varphi]_q$ is equal to R_* .

Proof. It follows from (5.2), that the generating function of $[\varphi]_q$ can therefore be written in radicals:

$$(3.3) \quad \text{GF}_{[\varphi]_q} = \frac{q^2 + q - 1 + \sqrt{(q^2 + 3q + 1)(q^2 - q + 1)}}{2q},$$

and the series $[\varphi]_q$ is the Taylor expansion of $\text{GF}_{[\varphi]_q}$ at $q = 0$. The number R_* is the modulus of the smallest (i.e., closest to 0) root of the polynomials under the radical in (3.3). Indeed,

$$q^2 + 3q + 1 = (q + R_*)(q + R_*^{-1}),$$

and therefore the Taylor series of (3.3) converges for $|q| < R_*$. \square

Remark 3.2. Observe that formula (3.1) has a certain similarity with the celebrated Rogers-Ramanujan continued fraction

$$R(q) = 1 + \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{\ddots}}}}$$

but the deformation (3.1) has very different properties.

3.2. The Fibonacci polynomials. The most natural choice of a sequence of rationals converging to φ is related to a quite remarkable and well-known sequence of polynomials.

Let F_n be the n^{th} Fibonacci number, the sequence of rationals $\frac{F_{n+1}}{F_n}$ converges to φ . Quantizing this sequence, one obtains a sequence of rational functions

$$(3.4) \quad \left[\frac{F_{n+1}}{F_n} \right]_q =: \frac{\tilde{\mathcal{F}}_{n+1}(q)}{\mathcal{F}_n(q)}.$$

The polynomials $\tilde{\mathcal{F}}_{n+1}(q)$ and $\mathcal{F}_n(q)$ in the numerator and denominator of (3.4) are q -deformations of the Fibonacci numbers, considered in [12].

Both sequences of polynomials $\mathcal{F}_n(q)$ and $\tilde{\mathcal{F}}_n(q)$ are of degree $n - 2$ (for $n \geq 2$) and are mirror of each other:

$$\tilde{\mathcal{F}}_n(q) = q^{n-2} \mathcal{F}_n(q^{-1}).$$

The polynomials $\mathcal{F}_n(q)$ can be calculated recursively. It will be convenient to separate the sequence of polynomials $\mathcal{F}_n(q)$ into two subsequences, with even n and odd n . Both of these sequences satisfy the same recurrence, which is a q -analogue of the classical recurrence

$$F_{n+2} = 3F_n - F_{n-2}$$

for the Fibonacci numbers. We have the following.

Proposition 3.3. *The polynomials $\mathcal{F}_n(q)$ in the denominator of (3.4) are determined by the recurrence*

$$(3.5) \quad \mathcal{F}_{n+2}(q) = [3]_q \mathcal{F}_n(q) - q^2 \mathcal{F}_{n-2}(q),$$

where $[3]_q = 1 + q + q^2$, and the initial conditions

$$(\mathcal{F}_0(q) = 0, \mathcal{F}_2(q) = 1) \quad \text{and} \quad (\mathcal{F}_1(q) = 1, \mathcal{F}_3(q) = 1 + q).$$

Proof. It was shown in [12] that

$$\begin{aligned} \mathcal{F}_{2\ell+1} &= q\mathcal{F}_{2\ell} + \mathcal{F}_{2\ell-1}, \\ \mathcal{F}_{2\ell+2} &= \mathcal{F}_{2\ell+1} + q^2\mathcal{F}_{2\ell}, \end{aligned}$$

Recurrence (3.5) follows readily. \square

Lemma 3.6 is a standard exercise of complex analysis. The minimum and maximum values of the function

$$|q^2 + q + 1| = \left| (q + e^{i\pi/3})(q + e^{-i\pi/3}) \right|$$

on the circle C_0 are attained at the points $q = -R_*$ and $q = R_*$, respectively. Therefore, we have

$$R_*^2 - R_* + 1 \leq |q^2 + q + 1|_{C_0} \leq R_*^2 + R_* + 1.$$

Lemma 3.6 then follows from the relation $R_*^2 = 3R_* - 1$ satisfied by the radius $R_* = \frac{3-\sqrt{5}}{2}$. \square

Lemma 3.7 follows directly from (3.6) and Lemma 3.6. \square

It remains to prove that the polynomials $\tilde{\mathcal{F}}_n(q)$ have no roots inside D_0 . The proof is similar, since $\tilde{\mathcal{F}}_n(q)$ also satisfy (3.5). This follows from the recurrent formulas

$$\begin{aligned} \tilde{\mathcal{F}}_{2\ell+1} &= \tilde{\mathcal{F}}_{2\ell} + q^2 \tilde{\mathcal{F}}_{2\ell-1}, \\ \tilde{\mathcal{F}}_{2\ell+2} &= q \tilde{\mathcal{F}}_{2\ell+1} + \tilde{\mathcal{F}}_{2\ell}, \end{aligned}$$

proved in [12]. Theorem 3.1 is proved. \square

It is easy to prove that the bounds R_* and R_*^{-1} of Theorem 3.1 are tight. This follows from the fact that R_* is the radius of convergence of $[\varphi]_q$, and the stabilization phenomenon.

4. RADIUS OF $[\sqrt{2}]_q$ AND ROOTS OF THE PELL POLYNOMIALS

The second example we consider is $\sqrt{2}$. It is important in Markov theory of Diophantine approximations. The slight modification, $\sqrt{2} + 1$, has some tiny advantages, so we will often work with it; note that $\sqrt{2} + 1$ is approximated by the quotients of consecutive *Pell numbers* $\frac{P_{n+1}}{P_n} = \underbrace{[2, 2, \dots, 2]}_n$.

We show that the convergence radius of the series $[\sqrt{2}]_q$ and $[\sqrt{2} + 1]_q$ is equal to $R_{\sqrt{2}} := R(\sqrt{2})$,

$$(4.1) \quad R_{\sqrt{2}} = \frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2} \approx 0.53101,$$

and that the roots of the polynomials in the numerators and denominators of $\left[\frac{P_{n+1}}{P_n}\right]_q$ belong to the annulus bounded by the circles with radius $R_{\sqrt{2}}$ and $R_{\sqrt{2}}^{-1}$.

4.1. The series $[\sqrt{2}]_q$ and $[\sqrt{2} + 1]_q$. These series are related by $[\sqrt{2} + 1]_q = q[\sqrt{2}]_q + 1$ (cf. (2.1)) and obviously have the same convergence radius; we prefer to perform the calculations for $[\sqrt{2} + 1]_q$.

The q -deformation $[\sqrt{2} + 1]_q$ is given by the infinite 2-periodic continued fraction

$$(4.2) \quad [\sqrt{2} + 1]_q = 1 + q + \frac{q^4}{q + q^2 + \frac{1}{1 + q + \frac{q^4}{q + q^2 + \frac{1}{\ddots}}}}$$

see [13]. This is the q -deformed classical continued fraction expansion $\sqrt{2} + 1 = [2, 2, 2, 2, \dots]$.

The series $[\sqrt{2} + 1]_q$ satisfies the following functional equation:

$$(4.3) \quad qX^2 - (q^3 + 2q - 1)X - 1 = 0,$$

readily obtained from (4.2), and can be calculated from it recursively:

$$\begin{aligned} [\sqrt{2} + 1]_q &= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} - 55q^{13} + 18q^{14} \\ &\quad + 146q^{15} - 155q^{16} - 322q^{17} + 692q^{18} + 476q^{19} - 2446q^{20} + 307q^{21} \\ &\quad + 7322q^{22} - 6276q^{23} - 18277q^{24} + 33061q^{25} + 33376q^{26} - 129238q^{27} - 10899q^{28} \dots \end{aligned}$$

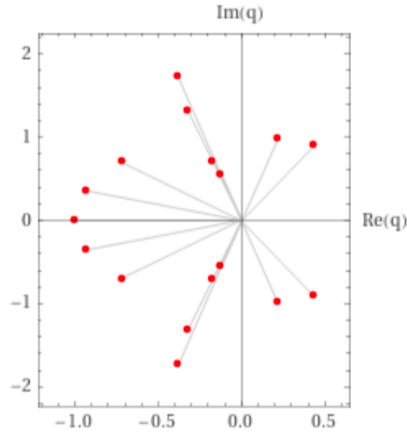


FIGURE 1. Roots of $\mathcal{P}_{10}(q)$: the smallest root modulus is $0.5668\dots$, and the largest $1.8832\dots$

Example 4.3. One has

$$\begin{aligned} \left[\frac{5}{2}\right]_q &= \frac{1 + 2q + q^2 + q^3}{1 + q + q^2}, \\ \left[\frac{12}{5}\right]_q &= \frac{1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5}{1 + q + 2q^2 + q^3}, \\ \left[\frac{29}{12}\right]_q &= \frac{1 + 3q + 5q^2 + 6q^3 + 6q^4 + 5q^5 + 2q^6 + q^7}{1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5}, \\ \left[\frac{70}{29}\right]_q &= \frac{1 + 3q + 7q^2 + 11q^3 + 13q^4 + 13q^5 + 11q^6 + 7q^7 + 3q^8 + q^9}{1 + 2q + 5q^2 + 6q^3 + 6q^4 + 5q^5 + 3q^6 + q^7}, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

4.3. Roots of the Pell polynomials. We prove that the roots of the Pell polynomials always belong to the annulus bounded by $R_{\sqrt{2}}$ and $R_{\sqrt{2}}^{-1}$; see Figure 4.3.

Theorem 4.1. *The roots of the polynomials, $\mathcal{P}_n(q)$ and $\tilde{\mathcal{P}}_n(q)$ belong to the annulus bounded by $R_{\sqrt{2}}$ and $R_{\sqrt{2}}^{-1}$:*

$$R_{\sqrt{2}} < |q_r| < R_{\sqrt{2}}^{-1},$$

for every q_r , such that $\mathcal{P}_n(q_r) = 0$, or $\tilde{\mathcal{P}}_n(q_r) = 0$.

Proof. Recurrence (4.4) lads to

$$(4.5) \quad \frac{\mathcal{P}_{n+2}(q)}{\mathcal{P}_n(q)} = \binom{4}{2}_q - q^4 \frac{\mathcal{P}_{n-2}(q)}{\mathcal{P}_n(q)}.$$

Once again, we use the Rouché theorem and want to prove that, for every n , the absolute value of the polynomial $\binom{4}{2}_q$ satisfies

$$(4.6) \quad \left| \binom{4}{2}_q \right|_{C_1} > R_{\sqrt{2}}^4 \left| \frac{\mathcal{P}_{n-2}(q)}{\mathcal{P}_n(q)} \right|_{C_1},$$

on the circle $C_1 = \{|q| = R_{\sqrt{2}}\}$. Indeed, all the roots of the polynomial $\binom{4}{2}_q = (q^2 + 1)(q^2 + q + 1)$ belong to the unit circle, so that (4.6) will guarantee, by induction on n , that $\mathcal{P}_{n+2}(q)$ has no roots inside D_1 .

The inequality (4.6) follows from the next two lemmas.

Lemma 4.4. *On the circle C_1 , one has*

$$R_{\sqrt{2}} \left(R_{\sqrt{2}}^2 + R_{\sqrt{2}} + 1 \right) \leq \left| \binom{4}{2}_q \right|_{C_1}.$$

Proof of Lemma 4.4. The minimum of the function

$$\left| \binom{4}{2}_q \right| = |(q^2 + 1)(q^2 + q + 1)|$$

on the circle C_1 is attained at the point $q = -R_{\sqrt{2}}$. Therefore, we have

$$(R_{\sqrt{2}}^2 + 1)(R_{\sqrt{2}}^2 - R_{\sqrt{2}} + 1) = R_{\sqrt{2}}^4 - R_{\sqrt{2}}^3 + 2R_{\sqrt{2}}^2 - R_{\sqrt{2}} + 1 \leq \left| \binom{4}{2}_q \right|_{C_1}.$$

Since the radius $R_{\sqrt{2}}$ satisfies the equation $R_{\sqrt{2}}^4 - 2R_{\sqrt{2}}^3 + R_{\sqrt{2}}^2 - 2R_{\sqrt{2}} + 1 = 0$, the above inequality becomes

$$R_{\sqrt{2}}^3 + R_{\sqrt{2}}^2 + R_{\sqrt{2}} \leq \left| \binom{4}{2}_q \right|_{C_1}.$$

Hence the lemma. \square

Lemma 4.5. *Assume that $\left| \frac{\mathcal{P}_{n-2}(q)}{\mathcal{P}_n(q)} \right|_{C_1} < \frac{1}{R_{\sqrt{2}}}$, then $\left| \frac{\mathcal{P}_n(q)}{\mathcal{P}_{n+2}(q)} \right|_{C_1} < \frac{1}{R_{\sqrt{2}}}$.*

Proof of Lemma 4.5. It follows from (4.5) that

$$\left| \frac{\mathcal{P}_{n+2}(q)}{\mathcal{P}_n(q)} \right|_{C_1} \geq \left| \binom{4}{2}_q \right|_{C_1} - R_{\sqrt{2}}^4 \left| \frac{\mathcal{P}_{n-2}(q)}{\mathcal{P}_n(q)} \right|_{C_1}.$$

Lemma 4.4 and assumption $\left| \frac{\mathcal{P}_{n-2}(q)}{\mathcal{P}_n(q)} \right|_{C_1} < \frac{1}{R_{\sqrt{2}}}$ then imply

$$\left| \frac{\mathcal{P}_{n+2}(q)}{\mathcal{P}_n(q)} \right|_{C_1} \geq R_{\sqrt{2}}^2 + R_{\sqrt{2}}.$$

Hence the lemma. \square

The proof that the polynomials $\tilde{\mathcal{P}}_n(q)$ have no roots inside D_1 is analogous since $\tilde{\mathcal{P}}_n(q)$ also satisfy (4.4). Theorem 4.1 is proved. \square

5. A GENERAL RESULT

In this section, we prove that Conjecture 1.1 holds for $x \in \mathbb{R}$, satisfying under some technical restrictions on the coefficients of the continued fraction.

5.1. Statement of the theorem. Given an irrational $x \geq 0$, consider the Hirzebruch-Jung continued fraction expansion $x = \llbracket c_1, c_2, c_3, \dots \rrbracket$, see Section 2.3.

Theorem 5.1. *If every coefficient of the Hirzebruch-Jung continued fraction expansion $x = \llbracket c_1, c_2, c_3, \dots \rrbracket$ of an irrational $x > 0$ satisfies*

$$(5.1) \quad c_i \geq 4,$$

for all $i \geq N$, and some fixed N , then the radius of convergence of $[x]_q$ is greater or equal to $R_ = \frac{3-\sqrt{5}}{2}$.*

Let us reformulate the inequality (5.1) in terms of the coefficients of the regular continued fraction expansion $x = [a_1, a_2, a_3, \dots]$. Recall the formula [5] expressing the coefficients of the Hirzebruch-Jung continued fraction:

$$x = \llbracket a_1 + 1, \underbrace{2, \dots, 2}_{a_2-1}, a_3 + 2, \underbrace{2, \dots, 2}_{a_4-1}, a_5 + 2, \dots, a_{2n-1} + 2, \underbrace{2, \dots, 2}_{a_{2n}-1}, \dots \rrbracket$$

In other words, the coefficients with odd indices, a_{2n-1} become $a_{2n-1} + 2$ (except for a_1 that becomes $a_1 + 1$), and coefficients with even indices, a_{2n} produce an $(a_{2n} - 1)$ -tuple of 2's. Inequality (5.1) becomes:

$$\begin{cases} a_{2M-1} & \geq 2, \\ a_{2M} & = 1, \end{cases}$$

starting from some M .

5.2. Proof of Theorem 5.1. Using the $\text{PSL}(2, \mathbb{Z})$ -invariance of the q -deformation, we can assume that condition (5.1) is satisfied by the coefficients c_i , for all $i \geq 1$.

Let us choose the rational approximation of x by the finite convergents of the Hirzebruch-Jung continued fraction: $\frac{r_n}{s_n} = \llbracket c_1, c_2, c_3, \dots, c_n \rrbracket$, and consider the q -deformation

$$\frac{\mathcal{R}_n(q)}{\mathcal{S}_n(q)} := \left[\frac{r_n}{s_n} \right]_q.$$

It suffices to prove that, for every n , the polynomial $\mathcal{S}_n(q)$ has no roots in the disc D_0 .

Formula (2.5) implies that the polynomial $\mathcal{S}_n(q)$ satisfy the recurrence

$$\mathcal{S}_{n+1}(q) = [c_{n+1}]_q \mathcal{S}_n(q) - q^{c_n-1} \mathcal{S}_{n-1}(q),$$

with the initial values $\mathcal{S}_0(q) = 0$ and $\mathcal{S}_1(q) = 1$, that we rewrite as follows

$$(5.2) \quad \frac{\mathcal{S}_{n+1}(q)}{\mathcal{S}_n(q)} = [c_{n+1}]_q - q^{c_n-1} \frac{\mathcal{S}_{n-1}(q)}{\mathcal{S}_n(q)}.$$

By the Rouché theorem, we need to prove that, for every n , the polynomial $[c_{n+1}]_q = \frac{1-q^{c_{n+1}}}{1-q}$ dominates the second summand of the right-hand-side of (5.2), when restricted on the circle C_0 :

$$(5.3) \quad \left| [c_{n+1}]_q \right|_{C_0} > R_*^{c_n-1} \left| \frac{\mathcal{S}_{n-1}(q)}{\mathcal{S}_n(q)} \right|_{C_0}.$$

Since for every positive integer c the polynomial $[c]_q$ has no roots in D_0 , we will then argue by induction that $\mathcal{S}_{n+1}(q)$ also has no roots in D_0 .

For every positive integer c , one obviously has

$$(5.4) \quad \frac{1 - R^c}{1 + R} \leq \left| [c]_q \right|,$$

on any circle of radius R .

We will assume, by induction, that $\left| \frac{\mathcal{S}_{n-1}(q)}{\mathcal{S}_n(q)} \right|_{C_0} < \frac{1}{R_*}$, and therefore

$$R_*^{c_n-1} \left| \frac{\mathcal{S}_{n-1}(q)}{\mathcal{S}_n(q)} \right|_{C_0} < R_*^{c_n-2}.$$

We then have from (5.2) and (5.4) that

$$\left| \frac{\mathcal{S}_{n+1}(q)}{\mathcal{S}_n(q)} \right|_{C_0} > \frac{1 - R_*^{c_{n+1}}}{1 + R_*} - R_*^{c_n-2}.$$

To prove the induction step, we need the following.

Lemma 5.1. *If $c_i \geq 4$ for all i , then $\left| \frac{\mathcal{S}_{n+1}(q)}{\mathcal{S}_n(q)} \right|_{C_0} > R_*$.*

Proof. Indeed,

$$\frac{1 - R_*^{c_{n+1}}}{1 + R_*} - R_*^{c_n-2} - R_* = \frac{1 - R_*^{c_{n+1}} - R_*^{c_n-2} - R_*^{c_n-1} - R_* - R_*^2}{1 + R_*}.$$

Since $c_i \geq 4$ for all i , we have

$$\frac{1 - R_*^{c_{n+1}} - R_*^{c_n-2} - R_*^{c_n-1} - R_* - R_*^2}{1 + R_*} \geq \frac{1 - R_*^4 - R_*^3 - 2R_*^2 - R_*}{1 + R_*} = \frac{14 - 36R_*}{1 + R_*},$$

because $R_*^2 = 3R_* - 1$. One checks that $7 - 18R_* > 0$.

Hence the lemma. \square

We proved that (5.3) holds for all n . Theorem 5.1 is proved.

Remark 5.2. Let us stress on the fact that, using the adopted approach, we are unable to improve the assumption (5.1). Indeed, assuming $c_i \geq 3$ in Lemma 5.1, leads in the proof to the quantity $3 - 8R_*$, which is negative. However, the statement of the theorem is true in the case $c_i = 3$ for all $i \geq 1$, that corresponds to the Fibonacci polynomials, see Section 3.

5.3. Miscellaneous experiments. Multiple computer experiments show that the situation in the q -deformed case is very different from classical Markov theory [1].

Example 5.3. For $\sqrt{3}$, one has: $\sqrt{3} = [1, \overline{1, 2}] = [\overline{2, 4}]$. The generating function of the series $[\sqrt{3}]_q$ is

$$\text{GF}_{[\sqrt{3}]_q} = \frac{q^3 + q^2 - q - 1 + \sqrt{q^6 + 2q^5 + 3q^4 + 3q^2 + 2q + 1}}{2q^2},$$

see [13]. The absolute value of the minimal root of the polynomial under the radical is

$$R_{\sqrt{3}} := R(\sqrt{3}) \approx 0.527756 \dots$$

which is between R_* and $R_{\sqrt{2}}$:

$$R_* < R_{\sqrt{3}} < R_{\sqrt{2}}.$$

This example demonstrates, that, unlike classical Markov theory [1], the series corresponding to $\sqrt{2}$ “converges better” than that of $\sqrt{3}$.

Let us also give another interesting example, which is the third “badly approximated” number in Markov theory, after φ and the “silver ratio” $\sqrt{2}$.

Example 5.4. The number $\alpha = \frac{9 + \sqrt{221}}{10} = [\overline{2, 2, 1, 1}]$, sometimes called “bronze ratio”, is the third most irrational number [1]. In this case, the radius of convergence can be calculated explicitly in radicals. More precisely, the radius of convergence of $[\frac{9 + \sqrt{221}}{10}]_q$ is

$$R_{\text{bronze}} := R(\alpha) = \frac{1 + \sqrt{13} - \sqrt{2(\sqrt{13} - 1)}}{4} \approx 0.58069 \dots$$

Indeed, a direct computation gives

$$\left[\frac{9 + \sqrt{221}}{10}\right]_q = \frac{q^6 + 2q^5 + 3q^4 + 3q^3 + q^2 - 1 + \sqrt{(q^4 + 3q^3 + 5q^2 + 3q + 1)(q^6 + 2q^5 + 3q^4 + 5q^3 + 3q^2 + 2q + 1)(q^2 - q + 1)}}{2q(q^3 + 2q^2 + q + 1)}$$

Note that $221 = 13 \cdot 17$, the factors under the radical are q -versions of these numbers. The radius of convergence is equal to the absolute value of the minimal root of the polynomial under the radical, which can be found explicitly.

We wonder if for other quadratic irrationals the radius of convergence is an algebraic number of degree 2^n (like in the theory of ruler-and-compass construction) as it was for φ , $\sqrt{2}$, and for the above example, but we have no explicit formulas in general.

Acknowledgements. We are very grateful to the Mathematisches Forschungsinstitut Oberwolfach for the hospitality during our RiP stay in summer 2020, when this project was started. We are pleased to thank Jenya Ferapontov and Sergei Tabachnikov for fruitful discussions. This paper was partially supported by the ANR project ANR-19-CE40-0021.

REFERENCES

- [1] M. Aigner, *Markov's Theorem and 100 Years of the Uniqueness Conjecture: A Mathematical Journey from Irrational Numbers to Perfect Matchings*. Springer, 2013.
- [2] J. Bernstein, T. Khovanova, *On the quantum group $SL_q(2)$* , Comm. Math. Phys. 177 (1996), no. 3, 691–708.
- [3] J.B. Conway, *Functions of One Complex Variable I*. Springer-Verlag, 1978.
- [4] G. Cotti, A. Varchenko, *The $*$ -Markov equation for Laurent polynomials*, arXiv:2006.11753.
- [5] F.E.P. Hirzebruch, *Hilbert modular surfaces*, Enseign. Math. (2) 19 (1973), 183–281.
- [6] A. Hurwitz *Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche*. Math. Annalen, **39** (1891), 279–284.
- [7] A.A. Kirillov, *Elements of the theory of representations*. Springer-Verlag, Berlin-New York, 1976.
- [8] L. Leclere, S. Morier-Genoud, *The q -deformed modular group and quadratic irrationals*, arXiv:2101.02953.
- [9] A.A. Markov *Sur les formes quadratiques binaires indéfinies*. Math. Annalen, **15** (1879), 381-406; **17** (1880), 379-399.
- [10] T. McConville, B.E. Sagan, C. Smyth, *On a rank-unimodality conjecture of Morier-Genoud and Ovsienko*, arXiv:2008.13232.
- [11] S. Morier-Genoud, V. Ovsienko, *Farey boat: continued fractions and triangulations, modular group and polygon dissections*, Jahresber. Dtsch. Math.-Ver. **121** (2019), no. 2, 91–136.
- [12] S. Morier-Genoud, V. Ovsienko, *q -deformed rationals and q -continued fractions*. Forum Math. Sigma 8 (2020), e13, 55 pp.
- [13] S. Morier-Genoud, V. Ovsienko, *On q -Deformed Real Numbers*, arXiv:1908.04365, Experimental Mathematics, DOI: 10.1080/10586458.2019.1671922.
- [14] S. Morier-Genoud, V. Ovsienko, *Quantum numbers and q -deformed Conway-Coxeter friezes*, arXiv:2011.10809.
- [15] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org>.
- [16] K. Spalding, A.P. Veselov, *Lyapunov spectrum of Markov and Euclid trees*. Nonlinearity **30** (2017), 4428-53.
- [17] E. C. Titchmarsh, *The theory of functions*. Oxford University Press, Oxford, 1958. x+454 pp.

LUDIVINE LECLERE, SOPHIE MORIER-GENOUD, SORBONNE UNIVERSITÉ, UNIVERSITÉ PARIS DIDEROT, CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, F-75005, PARIS, FRANCE

VALENTIN OVSIENKO, CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE, LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE REIMS, U.F.R. SCIENCES EXACTES ET NATURELLES, MOULIN DE LA HOUSSE - BP 1039, 51687 REIMS CEDEX 2, FRANCE

ALEXANDER VESELOV, DEPARTMENT OF MATHEMATICAL SCIENCES, LOUGHBOROUGH UNIVERSITY, LOUGHBOROUGH LE11 3TU, UK, MOSCOW STATE UNIVERSITY AND STEKLOV MATHEMATICAL INSTITUTE, MOSCOW, RUSSIA

E-mail address: ludivine.leclere1@gmail.com, sophie.morier-genoud@imj-prg.fr, valentin.ovsienko@univ-reims.fr, A.P.Veselov@lboro.ac.uk