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On a Competitive Selection Problem ^{*†}

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Abstract

We consider the problem in which n items arrive to a market sequentially over time, where two agents compete to choose the best possible item. When an agent selects an item, he leaves the market and obtains a payoff given by the value of the item, which is represented by a random variable following a known distribution with support contained in $[0, 1]$. We consider two different settings for this problem. In the first one, namely *competitive selection problem with no recall*, agents observe the value of each item upon its arrival and decide whether to accept or reject it, in which case they will not select it in future. In the second setting, called *competitive selection problem with recall*, agents are allowed to select any of the available items arrived so far. For each of these problems, we describe the game induced by the selection problem as a sequential game with imperfect information and study the set of subgame-perfect Nash equilibrium payoffs. We also study the efficiency of the game equilibria. More specifically, we address the question of how much better is to have the power of getting any available item against the take-it-or-leave-it fashion. To this end, we define and study the price of anarchy and price of stability of a game instance as the ratio between the maximal sum of payoffs obtained by players under any feasible strategy and the sum of payoffs for the worst and best subgame-perfect Nash equilibrium, respectively. For the no recall case, we prove that if there are two agents and two items arriving sequentially over time, both the price of anarchy and price of stability are upper bounded by the constant $4/3$ for any value distribution. Even more, we show that this bound is tight.

Keywords: Optimal stopping, Competing agents, Recall, Prophet inequalities, Price of anarchy, Price of stability, Game theory.

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1 Introduction

The theory of optimal stopping has a vast history and is concerned with the problem of a decision-maker who observes a sequence of random variables arriving over time and has to decide when to stop optimizing a particular objective. Probably, the two best-known problems in optimal stopping are the Secretary Problem and the Prophet Inequality. In classical model of the former, that was introduced in the '60s, a decision-maker observes a sequence of values arriving over time and has to pick one in a take-it-or-leave-it fashion maximizing the probability of picking the highest one. In other words, after observing an arrival, he has to decide whether to pick this value (and gets a reward equals to the value picked) or to pass and continue observing the sequence. Once a value was picked, the game ends and the goal of the decision-maker is to maximize the probability of getting the highest value. Lindley [13] proves that an optimal stopping rule for this problem consists on reject a particular amount of items first and then accept the first following item with value higher than the maximum observed so far. When the number of arrivals goes to infinity, the probability of picking the best value approaches to $1/e$. Since then, the secretary problem with a lot of model variants has been studied in the literature (see, e.g., [5, 7, 10]).

Related to this secretary problem is the optimal selection problem where the decision-maker knows not only the total number of arrivals but also the distribution behind them, and he has to decide when to stop maximizing the expected value of what he gets. Instead of looking at this problem as an optimal stopping problem, in the '70s researches starting to answer the question of how good can a decision-maker plays compared to what a prophet can do, where a prophet is someone who knows all the realization of the random variables in advance and pick the maximum. This inequalities are called Prophet Inequalities and it was in the '70s when Krengel and Sucheston, and Garling [12] proved that the decision maker can get at least $1/2$ of what a prophet gets, and that in fact this bound is tight. Later, in 1984 Samuel-Cahn [15] proved that in fact instead of looking at all feasible stopping rules, it is enough to look at a single threshold strategy to get the $1/2$ bound. These results are for a general setting where the random variables are independent but not necessarily identically distributed, and then one natural question that arose was if this bound could be improved assuming iid random variables. Kertz [11] answers this question on the positive given a lower bound of roughly 0.7451. Quite recently, Correa et al. [3] proved that this bound is in fact tight. A lot of work has been appeared considering different model variants (infinity many arrivals, feasibility constraints, multi-selection) but it was since the last decades that this problem gained a particular attention due to its surprising connection with online mechanisms (see, e.g., [2, 4, 9]).

Motivated by the real applications captured by this set of problems (a person who wants to sell a house, a company hiring an employee, the assignment of students to schools, among others), in this paper we consider two model variants of the classical setting. More specifically, we consider the classical iid setting of the optimal stopping problem but with two decision-makers competing to get the best possible item, and we call it *competitive selection problem with no recall* or *no recall case* for short. In addition to competition, we also consider another setting, namely *competitive selection problem with recall* or *full recall case* for short, where not only there are two decision-makers, but also is allowed to select any available item and not only the one just arrived.

The variant of considering two decision-makers leads to a two-player game and it is the goal of this paper the study of such games. More explicitly, our contribution can be divided into two main lines.

Subgame Perfect Equilibrium Payoffs. One of the main contributions of the papers is the study the set of subgame perfect equilibrium (SPE) payoffs for both the full recall and no recall cases.

In this context, for the former setting, we prove the surprising result that the set of SPE payoffs is symmetric and a subset of the two-dimensional identity line. Moreover, we fully characterize this set. Regarding the no recall case, the game behind it does not satisfy some properties we prove for the full recall case, and then computing the set of SPE payoffs seems more challenging. However, we compute the SPE payoffs corresponding to the equilibrium maximizing the sum of payoffs of players, namely best SPE, and the one that minimize the sum of payoffs, which is called worst SPE.

Efficiency of Subgame Perfect Equilibrium. After defining the games and studying the set of SPE payoffs, we turn to study the efficiency of such equilibrium. In other words, we answer the question of how good is to play the best or worst SPE comparing to the sum of payoffs obtained if the best possible feasible strategy is played. To address this question, we define the price of anarchy (PoA) and price of stability (PoS) of the games. For the full recall case, we have closed formulas for an instance of the game. For the no recall case, we not only reach closed formulas, but also prove that both PoS and PoA are upper bounded by $4/3$, and we show that this bound is tight. In addition to the PoS and PoA ratios, we analyse how socially good can be play the best equilibrium compared to what a prophet— someone who observes the realizations of the random variables in advance— is able to obtain. By definition, for the full recall case, this notion is equivalent to the PoS. For the no recall case, we have a closed formula to compute this prophet ratio (PR) for each game instance. We also tackle, numerically, the question of how many arrivals give the worst possible ratios for a sequence of items with uniformly distributed values in the interval $[0, 1]$ in the no recall case. Whereas for the PoA and PoS we obtain the intuitive result that this worst case is reached for two arrivals, for the PR we obtain that the number the arrivals giving the worst case is 5, which is surprising.

1.1 Related literature

As it was aforementioned, the literature in optimal stopping theory is extensive and it is mainly focused on finding optimal or near-optimal policies for the different model variants, as well as on studying the guarantees of some simple strategies, such as single threshold strategies, even when they are not optimal. However, this paper introduces a game-theoretic approach for a model with competition and where recall is allowed. In what follows, we revisit some of the existing literature regarding optimal stopping problems with some of these two particular features.

Optimal Stopping with competition. The paper by Abdelaziz and Krichen [1] surveys the literature on optimal stopping problems with more than one decision maker until 2000s. More recently, Immorlica et al. [10] and Ezra et al. [5] study the secretary problem with competition. The former considers a classical setting where decision are made in a take-it-or-leave-it fashion and ties are broken uniformly at random, and they show that as the number of competitors grows, the time at which is played accept at the first time in an equilibrium decreases. The latter incorporates the recalling option, and studies the structure and performance of equilibria in this game when the ties are broken uniformly at random or according to a global ranking.

Our paper considers a different model, since the problem is more related to the prophet inequality setting. In this sense, probably the work closest to ours is the recent paper by Ezra et al. [6], who introduce, independently of our paper, the no recall case, again with the two variants for tie breaking. However, the novelty of our paper is the incorporation of recalling in addition to the competition between players. Furthermore, instead of study the reward guarantees under single-threshold strategies, our paper is more focused on the study of equilibria of the game.

Optimal Stopping with recall. Allowing decision makers to choose between any of the items that have already arrived is a variant of the classic problem that may have interesting applications. If this extension is considered without competition, it is easily to see that the optimum is just to wait until the end and pick the best item. However, adding competition the game is not trivial anymore and the study of the game behind it as well as how bad is to play an equilibria are challenging questions we address in this paper. However, this notion of recalling is not new for some optimal stopping problems. For example, Yang [17] considers a variant of the secretary problem, where the interviewer is allowed to make an offer to any applicant already interviewed. In his model, the applicant reject the offer with some probability that depends on when the offer is made and he studies the optimal stopping rules in this context. Thereafter, different authors have been studying other variants of the secretary problem with recall (see, e.g., [5, 14, 16]). Our work differs from most of them papers not only in the model we consider but also in the focus of the research since we are more interested in a game-theoretic approach of the problem.

2 Model

Consider a sequence X_1, \dots, X_n of i.i.d. random variables distributed according to a continuous c.d.f. F with support included in $[0, 1]$. There are two players, or decision-makers, competing to pick the best item among X_1, \dots, X_n . More explicitly, at each time period $t = 1, \dots, n$, the decision-makers observe the realization X_t and simultaneously decide whether to select one item in the current *feasible set* \mathcal{F}_t or not. Once a decision maker gets an item l , he leaves the market obtaining a payoff of X_l and the corresponding item is not longer available. Decision makers must decide when to stop maximizing the expected value of their payoff. If at time t both agents want to take the same item, we break the tie uniformly at random. That is, each of them get the item with probability $1/2$, and the decision maker who gets the item leaves the market, whereas the other passes to the next period.

It remains to specify what are the sets of feasible items, distinguishing the two cases we will consider.

In the first one, namely *full recall case*, the set \mathcal{F}_t of feasible items at time t consists on all the items arrived so far that have not yet been selected by a decision maker, and a *feasible action* represents a probability distribution over the set $\mathcal{F}_t \cup \{\emptyset\}$, where \emptyset represents the action of not selecting anything. In this case, the game ends when all decision makers get an item, and then it could be later than n if both decision makers are still present at period N . If this is the case, we assume that the last present player takes at time $n + 1$ the best item still available.

The second variant we consider is the one we call *no recall case*. In this case, a take-or-leave-it decision is faced by the decision makers at each time period. That is, after observing the realization of the random variable just arrived they should decide whether to take it or not. Whether it was chosen by someone or not, the item cannot be chosen later. Thus, the set of feasible items at time t is just given by $\mathcal{F}_t = \{t\}$ and a feasible action is a probability distribution on $\{t, \emptyset\}$, where t represents the item arriving at time t and \emptyset the action of rejecting the item.

Notice that in the 1-player case (decision problem), the optimal strategy with full recall is simply to wait until the end and to pick the maximum of $\{X_1, \dots, X_n\}$. And in the 1-player case with no recall we have a standard prophet problem, with value smaller than the expectation of the maximum of $\{X_1, \dots, X_n\}$. In games setting, we cannot say *a priori* that the full recall case is beneficial to the players, as there are examples where having more information or more actions decreases the sum of the payoffs of the players at equilibrium. One question we want to address here is the following: how important is to have the power to be able to choose an item that came to the market

earlier? For each variant, we will study the set of SPE payoffs and we bound the Price of Anarchy (PoA) and Price of Stability (PoS).

2.1 Description of the induced games

Full recall case. Game Γ_n^{FR}

For each $t = 0, 1, \dots, n + 1$ we denote by \mathcal{H}_t the set of possible histories up to stage t . \mathcal{H}_0 only contains the empty history. \mathcal{H}_1 contains what happens at stage 1, i.e. the realization of X_1 , who tried to pick X_1 and who got X_1 (possibly nobody). \mathcal{H}_2 contains everything that happened at stage 1 and 2, etc...

As usual a strategy for player $j = 1, 2$ is an element $\sigma_j = (\sigma_{j,t})_{t=0,\dots,n}$ where $\sigma_{j,t}$ is a map which associates to every history in \mathcal{H}_t an available action, that is a probability distribution over $\mathcal{F}_t \cup \{\emptyset\}$. A strategy profile (σ_1, σ_2) induces a probability distribution over the set of possible plays \mathcal{H}_{n+1} , and the payoff (or utility) of each player is defined as the expectation of the value of the item he gets, with the convention that getting no item yields a payoff of 0.

No recall case. Game Γ_n^{NR}

Here, the set of available items at stage t is $\mathcal{F}_t = \{t\}$, and we only need to consider histories \mathcal{H}'_t for $t = 0, \dots, n$ where \mathcal{H}'_t contains everything that happened up to stage t under the no recall assumption. A strategy for player $j = 1, 2$ is an element $\sigma_j = (\sigma_{j,t})_{t=0,\dots,n-1}$ where $\sigma_{j,t}$ is a map which associates to every history in \mathcal{H}_t an available action, that is a probability distribution over $\{t, \emptyset\}$. A strategy profile (σ_1, σ_2) induces a probability distribution over the set of possible plays \mathcal{H}'_n , and payoffs are defined as in the full recall case.

Equilibrium notions. We recall here the usual notions of Nash equilibrium (NE) and subgame perfect equilibrium (SPE) (see, e.g., [8]). The following definitions apply to all games Γ_n^{FR} or Γ_n^{NR} .

Definition 1. Given σ_1, σ_2 strategies for player 1 and 2, respectively, we say that $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium (NE) of the game if for every agent i and every strategy σ'_i , player's i utility when (σ'_i, σ_{-i}) is played is not greater than the one obtained if σ is played.

Given a stage number t and a finite history h_t mentioning everything that happened up to stage t , we can define the continuation game after h_t .

Definition 2. Given σ_1, σ_2 strategies for player 1 and 2, respectively, we say that $\sigma = (\sigma_1, \sigma_2)$ is a subgame perfect equilibrium (SPE) if it induces a NE for every proper subgame of the game (i.e. for any continuation game after a finite history).

In the next section, we study the SPE payoffs of the games induced by both the full recall and no recall cases.

3 Results

3.1 Study of SPE payoffs

One of the goals of the paper is to study the set of SPE payoffs of the games described in Section 2.1. In this first part we present the results in this context, characterizing the entire set for the full recall case, and providing recursive formulas to compute the sum of SPE payoffs of players when they play the best or the worst equilibriums.

3.1.1 Full recall case.

Let us first study the game under the full recall case, that is, at time t , any of the available items from $\{1, \dots, t\}$ that has no been selected before can still be selected.

We introduce the two-player game $\Gamma_n^{FR}(a, b)$, where for each natural number n and $1 \geq a \geq b \geq 0$, $\Gamma_n^{FR}(a, b)$ is defined as Γ_n^{FR} with two initial active players, n items to arrive and a and b two available items present at the beginning of the game. That is, we have a time period “zero” where players choose between getting a, b or pass, before the sequential arrival of the items. We denote by $E_n(a, b) \subset \mathbb{R}_+^2$ the set of the SPE payoffs of the game $\Gamma_n^{FR}(a, b)$. Note that the set of the SPE payoffs of the game Γ_n^{FR} is just the set $E_n(0, 0)$,

Next theorem states that the set of SPE payoffs for the defined auxiliary game $\Gamma_n^{FR}(a, b)$ is contained in the 2-dimensional identity line, which we see as a surprising result.

Theorem 1. Consider an instance of the game $\Gamma_n^{FR}(a, b)$, for a, b real numbers such that $0 \leq b \leq a \leq 1$ and $n \in \mathbb{N}$.

The set of SPE payoffs is symmetric and contained in the 2-dimensional identity line. That is, $E_n(a, b) \subset \{(u, u), u \in [0, 1]\}$.

Furthermore, if we define $PE_n(a, b)$ the projection of $E_n(a, b)$ to \mathbb{R} , we have $\min PE_n(a, b) = l_n(a, b)$ and $\max PE_n(a, b) = h_n(a, b)$, where l_n and h_n are defined recursively as follows:

$$i) \quad l_0(a, b) = h_0(a, b) = \frac{a+b}{2};$$

ii) for $n \geq 1$:

$$l_n(a, b) = L(a, \mathbb{E}(X_{(n)} \vee b), d_n) \quad \text{with } d_n = \mathbb{E}_X(l_{n-1}(a \vee X, \text{med}[a, b, X])),$$

$$h_n(a, b) = H(a, \mathbb{E}(X_{(n)} \vee b), d_n) \quad \text{with } d_n = \mathbb{E}_X(h_{n-1}(a \vee X, \text{med}[a, b, X])),$$

with $L : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $H : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by:

$$L(x, y, z) = \begin{cases} z & \text{if } x \leq y, \\ \frac{1}{2}(x+y) & \text{if } x > y, \end{cases} \quad \text{and} \quad H(x, y, z) = \begin{cases} \frac{1}{2}(x+y) & \text{if } x > y \vee z, \\ z & \text{if } x \leq y \vee z. \end{cases}$$

Using the theorem above, we can prove the following result which fully characterizes the set of SPE payoffs of the games Γ_n^{FR} .

Theorem 2. For $n \geq 1$, the set E_n^{FR} of SPE payoffs of the game Γ_n^{FR} is:

$$E_n^{FR} := \{(u, u) : l_n \leq u \leq h_n\}, \quad \text{where } l_n = l_n(0, 0) \text{ and } h_n = h_n(0, 0).$$

In particular E_n^{FR} is convex, contained in the 2-dimensional identity line, and its extremepoints are given by $(l_n(0, 0), l_n(0, 0))$ and $(h_n(0, 0), h_n(0, 0))$, where $l_n(0, 0)$ and $h_n(0, 0)$ are defined in the statement of Theorem 1.

Remark 1. The proof shows that the above theorem extends to the cases where the c.d.f. F is not assumed to be continuous, e.g. for discrete distributions.

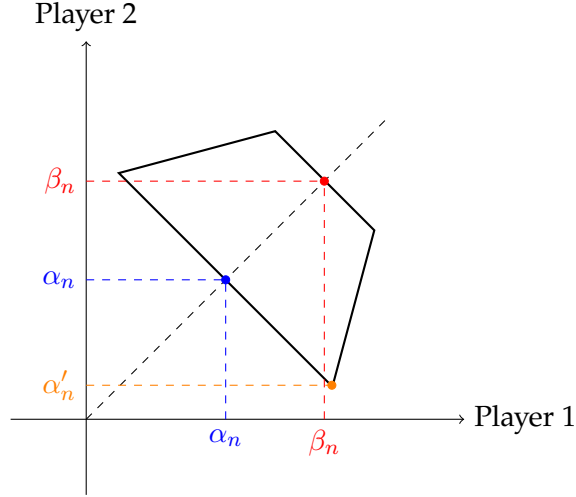


Figure 1: Representation of α_n , β_n and α'_n if the set E_n is given by the trapezoid.

3.1.2 No recall case.

We now consider the no recall variant, where players only can play on a take-it-or-leave-it fashion, without being able to select an item arrived in the past.

We introduce here the two-player game $\Gamma_n^{NR}(a)$, where for each natural number n and $a \in [0, 1]$, $\Gamma_n^{NR}(a)$ is defined as Γ^{NR} with two initial active players, n items to arrive, but with a the value of an available item already present. That is, we have a time period “zero” where players choose between getting a or pass, before the sequential arrival of the items.

Calling $E_n(a) \subset \mathbb{R}^2$ the set of SPE payoffs of the game $\Gamma_n^{NR}(a)$, we have that the set of the SPE payoffs of the game Γ_n^{NR} is just $E_n^{NR} := E_n(0) = \mathbb{E}_{a \sim F}(E_{n-1}(a))$.

E_n^{NR} will be convex and symmetric with respect to the identity line, but it will not be true here that E_n^{NR} is a subset of the identity line. We will give recursive formulas to compute the sum of the SPE payoffs for the best and worst equilibria under the no recall setting. Here, by best (resp. worst) equilibrium we mean a SPE which maximizes (minimizes) the sum of the payoffs of the 2 players.

We now introduce notations. We first define by induction (with $X \sim F$):

$$c_1 = \mathbb{E}(X), \text{ and } \forall n > 1, c_n = \mathbb{E}(X \vee c_{n-1}).$$

c_n is the value of the decision problem in a standard prophet setting.

We also denote by α_n (β_n) the smallest (highest) coordinate value of a point on E_n belonging to the identity line, and by α'_n the smallest coordinate of a point belonging to E_n^{NR} . That is, $\alpha_n := \min\{x : (x, x) \in E_n^{NR}\}$, $\beta_n := \max\{x : (x, x) \in E_n^{NR}\}$ and $\alpha'_n = \min\{\min\{x, y\} : (x, y) \in E_n^{NR}\}$ (see Figure 1). It is easy to see that:

$$\alpha'_1 = \alpha_1 = \beta_1 = \frac{1}{2}\mathbb{E}(X).$$

Theorem 3. *In the game Γ_n^{NR} ,*

a) *the worst payoff a player can get at equilibrium is α'_n , where for $n \geq 1$:*

$$\alpha'_{n+1} = \frac{(c_n + 1)}{2} - \int_{\alpha'_n}^{c_n} F(a)da - \frac{1}{2} \int_{c_n}^1 F(a)da.$$

b) the sum of payoffs for the **best SPE** is $2\beta_n$, where for $n > 1$:

$$2\beta_n = \int_0^{\alpha'_{n-1}} 2\beta_{n-1} dF(a) + \int_{\alpha'_{n-1}}^{\beta_{n-1}} \max\{a + c_{n-1}, 2\beta_{n-1}\} dF(a) + \int_{\beta_{n-1}}^1 (a + c_{n-1}) dF(a).$$

c) the sum of payoffs for the **worst SPE** is $2\alpha_n$, where for $n > 1$:

$$2\alpha_n = \int_{c_{n-1}}^1 (a + c_{n-1}) dF(a) + \int_{\beta_{n-1}}^{c_{n-1}} \psi_n(a) dF(a) + \int_{\alpha_{n-1}}^{\beta_{n-1}} 2a dF(a) + \int_{\alpha'_{n-1}}^{\alpha_{n-1}} \xi_n(a) dF(a) + \int_0^{\alpha'_{n-1}} 2\alpha'_{n-1} dF(a),$$

$$\text{where } \psi_n(a) = \frac{4ac_{n-1} - 2\beta_{n-1}(a + c_{n-1})}{c_{n-1} + a - 2\beta_{n-1}} \text{ and } \xi_n(a) = \min\{2\alpha_{n-1}, a + c_{n-1}\}.$$

3.2 Example: Unif[0, 1] values

3.2.1 Full recall case.

We now compute the set of SPE payoffs of the game Γ_3^{FR} , when we have three items arriving and each of them has a value following uniform $[0, 1]$ distribution. That is, we will compute what we called $E_3(0, 0)$. Notice that this is equivalent to compute $\mathbb{E}_a(E_2(a, 0))$, where a represents the value of the first arrived item and then $a \sim \text{Unif}[0, 1]$.

Given $1 \geq a \geq b \geq 0$, by Theorem 1 it holds that $\mathbb{E}_0(a, b) = \left\{ \left(\frac{a+b}{2}, \frac{a+b}{2} \right) \right\}$. Let us now compute a closed expression for $E_1(a, b)$.

Note that in this case, if both players pass at stage 0, each player will get a (conditionally expected) payoff of $(a+b)/2$ if $X_1 \leq b$, and of $(a + X)/2$ if $X_1 \geq b$. This is

$$d_1 = b(a + b)/2 + (1 - b)(a/2 + (1 + b)/4) = \frac{1}{2}a + \frac{1}{4} + \frac{1}{4}b^2.$$

On the other hand, the expected value of the maximum between a Unif[0, 1] random variable and a constant k can be computed as:

$$\begin{aligned} \mathbb{E}(X \vee k) &= k\mathbb{P}(X \leq k) + \mathbb{E}(X|X > k)\mathbb{P}(X > k) \\ &= \frac{1 + k^2}{2}. \end{aligned}$$

Then, applying Theorem 1 we obtain:

$$E_1(a, b) = \left\{ \left(\frac{a}{2} + \frac{1 + b^2}{4}, \frac{a}{2} + \frac{1 + b^2}{4} \right) \right\}.$$

To compute $E_2(a, b)$, we need to compute $\mathbb{E}(X_{(2)} \vee b)$. But, the expected value of the maximum between n Unif[0, 1] random variables and a constant k , is given by:

$$\begin{aligned} \mathbb{E}(X_{(n)} \vee k) &= k\mathbb{P}(X_{(n)} \leq k) + \mathbb{E}(X_{(n)}|X_{(n)} > k)\mathbb{P}(X_{(n)} > k) \\ &= k^{n+1} + \frac{n}{n+1}(1 - k^{n+1}) \\ &= \frac{n + k^{n+1}}{n+1}, \end{aligned}$$

and therefore $\mathbb{E}(X_{(2)} \vee b) = (2 + b^3)/3$.

On the other hand, since $E_1(a, b)$ is a singleton for all a and b , the expected payoff of players if both play “pass” (i.e. d) does not depend on which equilibrium is played in the continuation game, and thus it is given by

$$\mathbb{E}_X \left(\frac{a \vee X}{2} + \frac{1 + ((a \wedge X) \vee b)^2}{4} \right) = \frac{1 + a^2}{2} + \frac{b^3 - a^3}{6}.$$

Again, applying Theorem 1 and using the expressions we obtained above, we conclude that:

$$E_2(a, b) = \begin{cases} \{(x, x)\} & \text{if } a \leq \frac{2+b^3}{3} \\ \{(x, x), (y, y), (z, z)\} & \text{if } a \in \left(\frac{2+b^3}{3}, \frac{1+a^2}{2} + \frac{b^3-a^3}{6} \right) \\ \{(z, z)\} & \text{if } a \geq \frac{1+a^2}{2} + \frac{b^3-a^3}{6}, \end{cases}$$

where $x = \frac{1+a^2}{2} + \frac{b^3-a^3}{6}$, $y = \frac{dc-2ac+ad}{2d-a-c}$ and $z = \frac{a}{2} + \frac{2+b^3}{6}$.

Now, we compute $E_3 = \mathbb{E}_a(E_2(a, 0))$, where a represents the value of the first arrived item and then $a \sim \text{Unif}[0, 1]$.

Using the analysis above and defining $a^* = 1 - \sqrt[3]{\frac{2}{\sqrt{5}-1}} + \sqrt[3]{\frac{1}{2}(\sqrt{5}-1)} \approx 0.677$, we obtain that

$$E_2(a, 0) = \begin{cases} \left\{ \left(\frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6}, \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} \right) \right\} & \text{if } a \leq 2/3, \\ \left\{ (u, u) : u \in \left\{ \frac{1}{3} + \frac{1}{2}a, \frac{6-15a+6a^2+7a^3-3a^4}{6+18a^2-6a^3-18a}, \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} \right\} \right\} & \text{if } a \in (2/3, a^*) \\ \left\{ \left(\frac{1}{3} + \frac{1}{2}a, \frac{1}{3} + \frac{1}{2}a \right) \right\} & \text{if } a \geq a^*. \end{cases}$$

Then, $(u, u) \in [0, 1]^2$ belongs to $E_3(0, 0)$ if and only if

$$u = \int_0^{2/3} \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} da + \int_{a^*}^1 \frac{1}{3} + \frac{1}{2}ada + \int_{2/3}^{a^*} f(a)da,$$

where for each a , $f(a) \in \left\{ \frac{1}{3} + \frac{1}{2}a, \frac{6-15a+6a^2+7a^3-3a^4}{6+18a^2-6a^3-18a}, \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} \right\}$.

Calling

$$h_3(0, 0) := \int_0^{a^*} \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} da + \int_{a^*}^1 \frac{1}{3} + \frac{1}{2}ada \approx 0.6245$$

and

$$l_3(0, 0) := \int_0^{2/3} \frac{1}{2} + \frac{a^2}{2} - \frac{a^3}{6} da + \int_{2/3}^1 \frac{1}{3} + \frac{1}{2}ada = \frac{607}{972} \approx 0.6244,$$

we conclude that

$$E_3(0, 0) = \{(u, u) : u \in [l_3(0, 0), h_3(0, 0)]\}.$$

What is played at equilibrium? In the best equilibrium, both players pick X_1 if and only if $X_1 \geq a^*$, whereas in the worst equilibrium, both players pick X_1 if and only if $X_1 \geq 2/3$. Competition induces the players to pick X_1 with relatively low values in the worst equilibrium, and this decreases the sum of expected payoffs.

3.2.2 No recall case.

Now, we turn to reduce the formulas obtained in Theorem 3 for the particular case where the items' values are uniformly distributed in the interval $[0, 1]$. Recall that $c_1 = \mathbb{E}(X)$, $c_n = \mathbb{E}(X \vee c_{n-1})$ for $n > 1$ and $X \sim F$. To obtain the expressions of β_n and α_n for the uniform case, we use the following technical result.

Lemma 1. *If $\alpha'_n < a < \beta_n$, then $a + c_n \geq 2\beta_n$.*

Proof. It is enough to show that $\alpha'_n + c_n \geq 2\beta_n$. We prove that by induction on n .

Notice that $E_1 = \{(1/4, 1/4)\}$ and thus $\beta_1 = \alpha'_1 = 1/4$. On the other hand, $c_1 = \mathbb{E}(X) = 1/2$, and putting all together we have that

$$\alpha'_1 + c_1 \geq 2\beta_1,$$

and then the Lemma holds for $n = 1$.

Let us assume now that the inequality holds for n and we prove that it also holds for $n + 1$, that is:

$$\alpha'_{n+1} + c_{n+1} \geq 2\beta_{n+1}.$$

By Theorem 3, we have that

$$\begin{aligned} 2\beta_{n+1} &= 2\alpha'_n\beta_n + \int_{\alpha'_n}^{\beta_n} \max\{a + c_n, 2\beta_n\} da + (1 - \beta_n)c_n + \frac{1}{2}(1 - \beta_n^2) \\ &= 2\alpha'_n\beta_n + (\beta_n - \alpha'_n)c_n + \frac{1}{2}(\beta_n^2 - \alpha_n'^2) + (1 - \beta_n)c_n + \frac{1}{2}(1 - \beta_n^2) \\ &= 2\alpha'_n\beta_n + c_n(1 - \alpha'_n) + \frac{1}{2}(1 - \alpha_n'^2) \\ &\leq (\alpha'_n + c_n)\alpha'_n + c_n(1 - \alpha'_n) + \frac{1}{2}(1 - \alpha_n'^2) \\ &= c_n + \frac{1}{2}\alpha_n'^2 + \frac{1}{2}, \end{aligned}$$

where the second equality and the inequality follow by the induction hypothesis.

On the other hand, from the proof of Theorem 3, we obtain

$$\alpha'_{n+1} = \frac{1}{2}\alpha_n'^2 + \frac{1}{2}c_n - \frac{1}{4}c_n^2 + \frac{1}{4}.$$

Note that $c_n + \frac{1}{2}\alpha_n'^2 + \frac{1}{2} \leq \frac{1}{2}\alpha_n'^2 + \frac{1}{2}c_n - \frac{1}{4}c_n^2 + \frac{1}{4} + c_{n+1}$ if and only if

$$\frac{1}{4}(c_n + 1)^2 \leq c_{n+1},$$

and the last holds due to $c_{n+1} = \mathbb{E}(X \vee c_n) = \frac{1+c_n^2}{2}$.

Therefore, we have that $2\beta_{n+1} \leq \alpha'_{n+1} + c_{n+1}$, and the proof is completed. \square

Using Theorem 3 together with Lemma 1, we obtain that if $F = \text{Unif}[0, 1]$ the sum of payoffs for the best SPE for $n > 1$ is

$$\begin{aligned} 2\beta_n &= \int_0^{\alpha'_{n-1}} 2\beta_{n-1} da + \int_{\alpha'_{n-1}}^1 (a + c_{n-1}) da \\ &= 2\beta_{n-1}\alpha'_{n-1} + c_{n-1}(1 - \alpha'_{n-1}) + \frac{1}{2}(1 - \alpha'_{n-1}^2). \end{aligned}$$

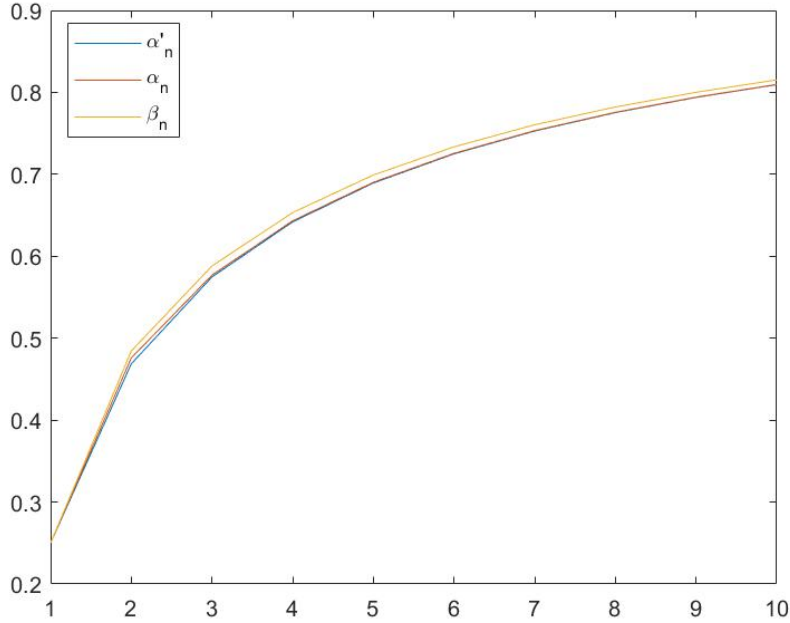


Figure 2: Values of α'_n , α_n and β_n for up to 10 arrivals and values uniformly distributed in $[0, 1]$.

Notice that by definition, $\beta_n \geq \alpha_n$ for all n , and therefore, by Lemma 1 we conclude that if $\alpha'_n < a < \alpha_n$ then $a + c_n \geq 2\alpha_n$. Thus, applying Theorem 3 we have that if $F = \text{Unif}[0, 1]$ the sum of payoffs for the worst SPE for $n > 1$ is

$$2\alpha_n = c_{n-1}^2 + 2\beta_{n-1}^2 + c_{n-1} - 3c_{n-1}\beta_{n-1} + \alpha_{n-1} + 2\alpha'_{n-1} - 2\alpha_{n-1}\alpha'_{n-1} - 2(c_{n-1} - \beta_{n-1})^2 \ln(2).$$

Therefore, if the value of the items are uniformly distributed in the interval $[0, 1]$ and we are under the no recall assumption, recursive formulas that are very easy to compute numerically, can be obtained for the sum of both the best and worst SPE payoffs.

In Table 1 we expose the values of α'_n , α_n and β_n for n from 1 to 4 and $F = \text{Unif}[0, 1]$, whereas in Figure 2 we compare this three values for n up to 10.

	α'_n	α_n	β_n
n=1	0.25	0.25	0.25
n=2	0.4688	0.4759	0.4844
n=3	0.5747	0.577	0.5881
n=4	0.6419	0.6432	0.6533

Table 1: Values of α'_n , α_n and β_n for one, two, three and four arrivals and values uniformly distributed in $[0, 1]$.

3.3 Efficiency of SPEs

The goal of this section is to study how efficient are the SPE in terms of the payoffs obtained by players. To this end, we define the Price of Anarchy, Price of Stability and Prophet Ratio of the

game. Given an instance of a game, the first two notions refers to the ratio between the maximal sum of payoffs obtained by players under any feasible strategy and the sum of payoffs for the worst and best SPEs, respectively. On the other hand, we define the Prophet Ratio of an instance of the problem as the ratio between the optimal Prophet value of the problem (that is, the expected value of the two best items) and the sum of payoffs for the best SPE. We call this quantity Prophet Ratio because we are comparing the best sum of payoffs obtained by playing a SPE strategy with what a prophet would do if he knows all the values of the items in advance.

Next, we formally introduce these definitions.

Definition 3. Consider an instance Γ_n^{FR} or Γ_n^{NR} , where n items with values distributed according to F , arrive sequentially over time. Then, we call

a) Price of Anarchy of this game instance– and we denote it by $PoA_n(F)$ – the following ratio

$$PoA_n(F) := \frac{\max_{\sigma \in \Sigma} \gamma^1(\sigma) + \gamma^2(\sigma)}{\gamma^1(\sigma_W) + \gamma^2(\sigma_W)},$$

where Σ represents the set of all feasible strategy pairs and σ_W represents the worst SPE of this instance of the game.

b) Price of Stability of this game instance– and we denote it by $PoS_n(F)$ – the following ratio

$$PoS_n(F) := \frac{\max_{\sigma \in \Sigma} \gamma^1(\sigma) + \gamma^2(\sigma)}{\gamma^1(\sigma_B) + \gamma^2(\sigma_B)},$$

where Σ represents the set of all feasible strategy pairs and σ_B represents the best SPE strategy of this instance of the game.

c) Prophet Ratio of this game instance– and we denote it by $PR_n(F)$ – the following ratio

$$PR_n(F) := \frac{\mathbb{E}(X_{(1:n)} + X_{(2:n)})}{\gamma^1(\sigma_B) + \gamma^2(\sigma_B)},$$

where $X_{(1:n)}$ and $X_{(2:n)}$ represent the first and second order statistics from the sequence $\{X_i\}_{i \in [n]}$ and σ_B represents the best SPE strategy of this instance of the game.

Clearly, by definition it holds that for each n and F

$$PoA_n(F) \geq PoS_n(F) \geq PR_n(F) \geq 1.$$

For each n , we define the Price of Anarchy, Price of Stability and Prophet Ratio of the game as the worst case ratio over all possible value distributions F . That is:

$$PoA_n := \max_F PoA_n(F), \quad PoS_n := \max_F PoS_n(F), \quad PR_n := \max_F PR_n(F).$$

3.3.1 Full recall case.

Note that in this case, the maximal feasible sum of payoffs obtained by the players is simply the expected value of the two best items (they can wait until the end of the horizon and pick the best and second best items). Thus, here, the notions of Price of Stability and Prophet Ratio are equivalent.

On the other hand, by Theorem 2, the sum of payoffs for the worst SPE is given by $2l_n$ and for the best SPE is $2h_n$, and therefore given an instance of the game we have that in this setting:

$$\text{PoA}_n^{FR}(F) = \frac{\mathbb{E}(X_{(1:n)} + X_{(2:n)})}{2l_n}$$

and

$$\text{PoS}_n^{FR}(F) = \text{PR}_n^{FR}(F) = \frac{\mathbb{E}(X_{(1:n)} + X_{(2:n)})}{2h_n}.$$

Notice that if $n = 2$, that is, we have only two arrivals, then under the full recall assumptions we have that $\text{PoA}_2^{FR}(F) = \mathbb{P}^{FR}S_2(F) = \text{PR}_2^{FR}(F) = 1$ for every value distribution F .

On the other hand, if n goes to infinity, then we also have that both the Price of Anarchy and Price of Stability goes to 1. Then, the interesting question is what happen with these ratios when n is finite and greater than 2.

Although we have a general characterization of the ratios for any value of n and distribution F , this quantities are sometimes difficult to compute for any n even if we fix the distribution F . However, if we fix $n = 3$ and $F = \text{Unif}[0, 1]$ it is easy to do the computations and we obtain that $\text{PoA}_3(F) = 1215/1214 \approx 1.000823$ and $\text{PoS}_3(F) \approx 1.0008$.

3.3.2 No recall case.

Under the no recall assumption, we no longer have that picking the two best items is a feasible strategy and then Price of Stability and Prophet Ratio are not equivalent anymore.

Note that in this case, the maximal feasible strategy is the same as the strategy of one player selecting two items among n in the classical online selection problem. The following Lemma gives us a recursive formula for the expected sum of payoffs of the maximal feasible strategy.

Lemma 2. *Assume we are in the no recall case with n arrivals following a distribution F with mean m . Let X denote a random variable with law F . Then, the expected maximal feasible sum of payoffs is given by*

- a) $s_1 = m$ if $n = 1$,
- b) $s_2 = 2m$ if $n = 2$,
- c) $s_n = \mathbb{P}(X \geq x_{n-1} - c_{n-1})\mathbb{E}(X + c_n | X \geq x_{n-1} - c_{n-1}) + s_{n-1}\mathbb{P}(X < x_{n-1} - c_{n-1})$ if $n > 2$.

Using Theorem 3 together with Lemma 2 we have that given the game Γ with n arrivals with value distribution F :

$$\text{PoA}_n^{NR}(F) = \frac{s_n}{2\alpha_n}, \quad \text{PoS}_n^{NR}(F) = \frac{s_n}{2\beta_n} \quad \text{and} \quad \text{PR}_n^{NR}(F) = \frac{\mathbb{E}(X_{(1:n)} + X_{(2:n)})}{2\beta_n}.$$

Recall that for each n and F

$$\text{PoA}_n^{NR}(F) \geq \text{PoS}_n^{NR}(F) \geq \text{PR}_n^{NR}(F) \geq 1.$$

Moreover, if $n = 1$ the ratios are equal to 1 and if we take n going to infinity, we also obtain that the ratios goes to 1. Then, as in the full recall case, the interesting cases are the one in the middle.

In particular, we fix $F = \text{Unif}[0, 1]$ and we compute $\text{PoA}_n(F)$ and $\text{PoS}_n(F)$ for $n = 2$ and $n = 3$ to compare the values with the ones obtained with full recall. The results are summarized in Table 2.

From the Table, we note that when n is small, both the Price of Anarchy and Price of Stability are closed to one, but the power of recalling gives a significant advantage: the ratios for the full recall cases are much more closer to 1.

	PoA _n (F)		PoS _n (F)	
	Full Recall	No Recall	Full Recall	No Recall
n=2	1	1.0507	1	1.0323
n=3	1.000823	1.0358	1.0008	1.0161

Table 2: Price of Anarchy and Price of Stability for full recall case vs. no recall case for two and three arrivals and values uniformly distributed in $[0, 1]$.

Another question we address here is the number of arrivals that gives the worst gaps, fixing the distribution F . In particular, we answer this question for the case $F = \text{Unif}[0, 1]$, obtaining that, for both the Price of Anarchy and Price of Stability, the ratios reached their maximum when $n = 2$. This result is intuitive due to as we are in the no recall case, there exists a positive probability of getting nothing and then the smaller the number of arrivals, the more likely this seems to happen. In Figures 3a and 3b are represented $\text{PoA}_n(F)$ and $\text{PoS}_n(F)$ as a function of n when $F = \text{Unif}[0, 1]$, where we can see that the maximum— in both cases— is reached at $n = 2$.

Regarding the Prophet Ratio, we also compute it as a function of n fixing the distribution $F = \text{Unif}[0, 1]$ (see Figure 3c), and we obtained that the maximum is reached when $n = 5$. Here, the result is more surprising.

To finish this section, we turn to study the efficiency of SPE when we fixed the number of arrivals at two and we look at the worst case ratios over F . In other words, we consider the game Γ with no recall and two arrivals and we want to study how bad may be to play the best or worst SPE in terms of the sum of payoffs obtained compared with the optimal sum of payoffs. Note that as we are considering only two arrivals, the maximal feasible sum of payoffs is equivalent to the sum of the expected value of the items.

Remark 2. Note that for the full recall case, if $n = 2$, both the price of anarchy and price of stability are equal to 1 for every distribution F so this question is trivial.

We obtain the following result, which states that both the Price of Anarchy and Price of Stability are upper bounded by $4/3$, and moreover, that this bound is in fact tight.

Proposition 1. If $n = 2$, under the no recall case it holds that for every distribution F ,

$$\text{PoS}_2^{\text{NR}}(F) \leq 4/3 \quad \text{and} \quad \text{PoA}_2^{\text{NR}}(F) \leq 4/3.$$

Furthermore, this bound is tight for both the price of stability and price of anarchy.

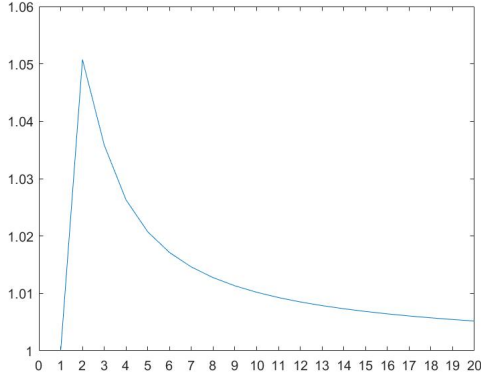
4 Proofs

In this section we provide the proofs omitted in Section 3.

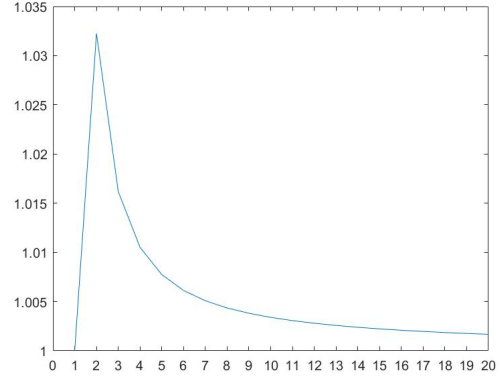
4.1 Omitted proofs from Section 3.1

Full recall case. In this section, we prove Theorems 1 and 2. Before that, we need some preliminary results stating properties of the game $\Gamma_n(a, b)$ defined in Section 3.1. Recall that this game is defined as Γ^{FR} with n arrivals but with two initial values $a > b$ present in the market.

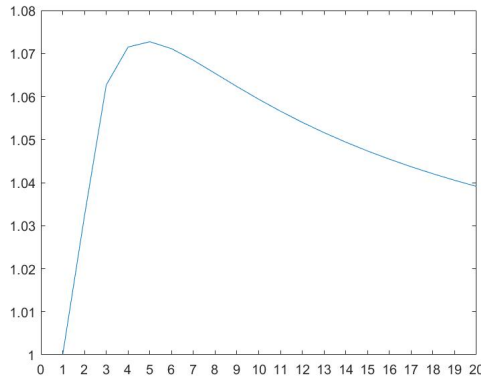
The first result states that the pure strategy consisting of choosing the item with value b is strictly dominated.



(a) $\text{PoA}_n^{NR}(F)$.



(b) $\text{PoS}_n^{NR}(F)$.



(c) $\text{PR}_n^{NR}(F)$.

Figure 3: Price of anarchy, price of stability and prophet ratio as function of n when $F = \text{Unif}[0, 1]$ in the no recall case.

Lemma 3. *Given $a > b$ in the support of F and n a positive natural number, let us consider the game $\Gamma_n(a, b)$. For both players, the strategy b is strictly dominated by mixed strategy consisting on playing a with probability $1/2$ and pass with probability $1/2$.*

Proof. Let us first compute the payoffs associated to each possible pair of pure strategies for player 1 and 2. Note that the pure strategies for players are to pick item a, b or pass. If both choose to pick a , then each player will get it with probability $1/2$ and will pass with probability $1/2$, staying along in the game with n items to see and the value b available. Then, each player will obtain in expectation $1/2a + 1/2\mathbb{E}(X_{(n)} \vee b)$. If one player chooses a and the other b , then payoffs will be a and b respectively. If one selects a (b) and the other passes, then they will get in expectation a (b) and $\mathbb{E}(X_{(n)} \vee b)$ ($\mathbb{E}(X_{(n)} \vee a)$), respectively. If both pass, they obtained a payoff belonging to $E_{n-1}(a, b)$, namely $(d_n(a, b), e_n(a, b))$. To not overload notation we just denote d_n , omitting its dependence on a and b . We summarize the expected payoffs in the 3×3 matrix represented in Table 3.

Due to the symmetry of the payoffs' matrix, it is enough to show that for one player (let us say player 1) the expected payoff if he plays a with probability $1/2$ and passes with probability $1/2$ is strictly higher that the expected payoff he obtains if he plays b .

Note that because $a > b$ we have that:

		Player 2		
		a	b	\emptyset
Player 1	a	$\left(\frac{a+\mathbb{E}(X_{(n)} \vee b)}{2}, \frac{a+\mathbb{E}(X_{(n)} \vee b)}{2}\right)$	(a, b)	$(a, \mathbb{E}(X_{(n)} \vee b))$
	b	(b, a)	$\left(\frac{b+\mathbb{E}(X_{(n)} \vee a)}{2}, \frac{b+\mathbb{E}(X_{(n)} \vee a)}{2}\right)$	$(b, \mathbb{E}(X_{(n)} \vee a))$
	\emptyset	$(\mathbb{E}(X_{(n)} \vee b), a)$	$(\mathbb{E}(X_{(n)} \vee a), b)$	(d_n, e_n)

Table 3: Expected payoffs matrix for $\Gamma_n^{FR}(a, b)$.

		Player 2	
		a	\emptyset
Player 1	a	$\left(\frac{a+\mathbb{E}(X_{(n)} \vee b)}{2}, \frac{a+\mathbb{E}(X_{(n)} \vee b)}{2}\right)$	$(a, \mathbb{E}(X_{(n)} \vee b))$
	\emptyset	$(\mathbb{E}(X_{(n)} \vee b), a)$	(d_n, e_n)

Table 4: Reduced expected payoffs matrix for $\Gamma_n^{FR}(a, b)$.

- i- $\frac{1}{2} \frac{a+\mathbb{E}(X_{(n)} \vee b)}{2} + \frac{1}{2} \mathbb{E}(X_{(n)} \vee b) = \frac{a}{4} + \frac{3}{4} \mathbb{E}(X_{(n)} \vee b) > b$,
- ii- $\frac{1}{2} a + \frac{1}{2} \mathbb{E}(X_{(n)} \vee a) > \frac{b+\mathbb{E}(X_{(n)} \vee a)}{2}$, and
- iii- $\frac{1}{2} a + \frac{1}{2} d_n > \frac{1}{2} b + \frac{1}{2} b = b$,

where the last inequality follows because $(d_n, e_n) \in \mathbb{E}_{n-1}(a, b)$ and then $d_n \geq b$.

We conclude that b is a strictly dominated strategy for both players (due to the symmetry of the game). \square

By Lemma 3, the payoffs matrix of the game $\Gamma_n^{FR}(a, b)$ is given by the 2×2 matrix in Table 4.

Moreover, denoting $c_n = \mathbb{E}(X_{(n)} \vee b)$, the payoff matrix has the particular form exposed in Table 5. Thus, it is enough to compute the NE of the game represented by Table 5, where (d_n, e_n) are values to determinate and represent the continuation equilibrium payoffs for players.

In the following proposition, we prove that if a player prefers to pass instead of take a given that the other player takes a , then he prefers to pass instead of take a if the other player passes.

Proposition 2. *If $c_n > a$ then $d_n > a$ and $e_n > a$.*

By the symmetry of the game, it is enough to prove that $d_n > a$, and then the same arguments will hold to prove $e_n > a$.

To prove the Proposition, we need the following two lemmas.

Lemma 4. *If X, Y are random variables and a a positive real number such that $\mathbb{E}(X \vee Y) \geq a$, then it holds that $\frac{1}{2} \mathbb{E}(X \vee a) + \frac{1}{2} \mathbb{E}((X \wedge a) \vee Y) \geq a$.*

Proof. Let us first prove that if x, y are real numbers, then

$$x \vee y \leq x^+ + (y \vee (x \wedge 0)). \quad (1)$$

Indeed, we show that for the four possible cases:

		Player 2	
		a	\emptyset
Player 1	a	$(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n))$	(a, c_n)
	\emptyset	(c_n, a)	(d_n, e_n)

Table 5: General expected payoffs matrix for $\Gamma_n^{FR}(a, b)$.

- (a) If $x \vee y = x \geq 0$, the left hand side in (1) is x and the right hand side is $x + \max\{y, 0\}$ which is at least x and the inequality holds.
- (b) If $x \vee y = x < 0$, the the left hand side in (1) is x and the right hand side is $0 + \max\{y, x\} = x$ and (1) follows.
- (c) If $x \vee y = y \geq 0$, the left hand side in (1) is y and the right hand side is $x^+ + y$ which is at least y and the inequality holds.
- (d) If $x \vee y = y < 0$, the left hand side in (1) is y and the right hand side is $0 + y$ and (1) follows.

Then, defining the radon variable $Z = X \vee Y - X^+ - (Y \vee (X \wedge 0))$ and using the inequality above, we have that $\mathbb{P}(Z \leq 0 | X = x, Y = y) = 1$. Therefore if we call g to the joint distribution for X and Y , we obtain

$$\begin{aligned}
 \mathbb{P}(X \vee Y \leq X^+ + (Y \vee (X \wedge 0))) &= \mathbb{P}(Z \leq 0) \\
 &= \int \int \mathbb{P}(Z \leq 0 | X = x, Y = y) g(x, y) dx dy \\
 &= \int \int g(x, y) dx dy = 1,
 \end{aligned}$$

meaning that for any X, Y random variables,

$$X \vee Y \leq X^+ + (Y \vee (X \wedge 0)) \quad \text{a.s.} \tag{2}$$

Take $X' = X - a$ and $Y' = Y - a$, then by (2) we have that

$$X' \vee Y' \leq X'^+ + (Y' \vee (X' \wedge 0)) \quad \text{a.s.} \tag{3}$$

But

- i) $X' \vee Y' = (X - a) \vee (Y - a) = (X \vee Y) - a$,
- ii) $X'^+ = (X - a)^+ = (X \vee a) - a$, and
- iii) $Y' \vee (X' \wedge 0) = (Y - a) \vee (X - a \wedge 0) = Y \vee (X \wedge a) - a$,

and thus (3) means that

$$(X \vee Y) - a \leq (X \vee a) - a + Y \vee (X \wedge a) - a \quad \text{a.s.},$$

which is equivalent to

$$(X \vee Y) + a \leq (X \vee a) + Y \vee (X \wedge a) \quad \text{a.s.}$$

In particular, the later implies that

$$\mathbb{E}(X \vee Y) + a \leq \mathbb{E}(X \vee a) + \mathbb{E}(Y \vee (X \wedge a)),$$

and due to $\mathbb{E}(X \vee Y) \geq a$, we conclude that

$$2a \leq \mathbb{E}(X \vee a) + \mathbb{E}(Y \vee (X \wedge a)),$$

and the result follows. \square

Lemma 5. For every $n \in \mathbb{N}$ and $a \geq b$, $d_n(a, b) \geq \frac{1}{2}\mathbb{E}(\alpha) + \frac{1}{2}\mathbb{E}(\beta)$, where $\alpha = X \vee a$ and $\beta = (X \wedge a) \vee X_{(n-1)} \vee b$

Proof. By definition, d_n represents the expected payoff of player 1 if both play \emptyset in $\Gamma_n^{FR}(a, b)$. Then we can lower bound d_n by the player 1 expected value obtained playing a feasible strategy, independently of the strategy of the other.

To this end, suppose that Player 1 plays the following strategy:

1. Observe the realization of X , namely x .
2. If $x \vee a \geq \mathbb{E}(\beta|X = x)$, bid for $x \vee a$.
3. If $x \vee a < \mathbb{E}(\beta|X = x)$, wait until the end and get the second best.

Then, defining $\Omega_1 := \{x : x \vee a \geq \mathbb{E}(\beta|X = x)\}$, $\Omega_2 := \{x : x \vee a < \mathbb{E}(\beta|X = x)\}$ and $\Omega = \Omega_1 \cup \Omega_2$, in expectation Player 1 gets

$$\int_{\Omega_1} \frac{1}{2}(x \vee a) + \frac{1}{2}\mathbb{E}(\beta|X = x)dF(x) + \int_{\Omega_2} \mathbb{E}(\beta|X = x)dF(x).$$

Note that the second integral is at least $\int_{\Omega_2} \frac{1}{2}(x \vee a) + \frac{1}{2}\mathbb{E}(\beta|X = x)dF(x)$ because $x \vee a < \mathbb{E}(\beta|X = x)$ under Ω_2 .

Therefore, we conclude that

$$\begin{aligned} d_n &\geq \int_{\Omega_1} \frac{1}{2}(x \vee a) + \frac{1}{2}\mathbb{E}(\beta|X = x)dF(x) + \int_{\Omega_2} \frac{1}{2}(x \vee a) + \frac{1}{2}\mathbb{E}(\beta|X = x)dF(x) \\ &= \int_{\Omega} \frac{1}{2}(x \vee a) + \frac{1}{2}\mathbb{E}(\beta|X = x)dF(x) \\ &= \frac{1}{2} \int_{\Omega} (x \vee a)dF(x) + \frac{1}{2} \int_{\Omega} \mathbb{E}(\beta|X = x)dF(x) = \frac{1}{2}\mathbb{E}(\alpha) + \frac{1}{2}\mathbb{E}(\beta), \end{aligned}$$

and the result follows. \square

Proof of Proposition 2. Take $X = X_1$ and $Y = X_{(n-1)} \vee b$. Then $\mathbb{E}(X \vee Y) = \mathbb{E}(X_{(n)} \vee b) \geq a$, and applying Lemma 4 we have

$$\frac{1}{2}\mathbb{E}(X_1 \vee a) + \frac{1}{2}\mathbb{E}((X_1 \wedge a) \vee X_{(n-1)} \vee b) \geq a.$$

On the other hand, by Lemma 5 $d_n \geq \frac{1}{2}\mathbb{E}(X_1 \vee a) + \frac{1}{2}\mathbb{E}((X_1 \wedge a) \vee X_{(n-1)} \vee b)$. Putting all together we obtain $d_n \geq a$ and the proof is completed. \square

		Player 2	
		a	\emptyset
Player 1	a	$(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n))$	(a, c_n)
	\emptyset	(c_n, a)	(d_n, d_n)

Table 6: Expected payoffs matrix for $\Gamma_n(a, b)$.

Corollary 1. Consider the game $\Gamma_n^{FR}(a, b)$. Then, the continuation equilibrium payoffs is the same for both players. That is, $d_n = e_n$.

Proof. The result follows directly from Proposition 2, since it implies that neither (a, \emptyset) nor (\emptyset, a) nor any convex combination of these strategies can be an equilibrium of the game. \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. From the Corollary 1 we conclude that the matrix represented in Table 6 corresponds to the expected payoffs matrix of the game $\Gamma_n^{FR}(a, b)$.

By Proposition 2 and using Table 6 we are able to compute the SPEs of the game, depending on the value of a . More specifically, we have that:

- (a) If $a > c_n \vee d_n$, then (a, a) is the unique NE with payoff $(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n))$.
- (b) If $d_n > a > c_n$, there are two pure NEs (a, a) and (\emptyset, \emptyset) and a symmetric mixed equilibrium in which both agents play a with probability $\frac{2(d_n - a)}{2d_n - a - c_n}$ and pass with probability $\frac{a - c_n}{2d_n - a - c_n}$. Furthermore, the equilibrium payoffs are:

$$\left(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n)\right), (d_n, d_n) \text{ and } \left(\frac{d_n c_n - 2ac_n + ad_n}{2d_n - a - c_n}, \frac{d_n c_n - 2ac_n + ad_n}{2d_n - a - c_n}\right), \text{ respectively.}$$

- (c) If $a \leq c_n$, then (\emptyset, \emptyset) is the unique NE, and (d_n, d_n) is the expected payoff.

From the analysis above we conclude that the set of SPE payoffs of the game is contained in the 2-dimensional identity line. That is:

$$E_n(a, b) \subset \{(u, u) : u \in \mathbb{R}_+\}.$$

and the first statement of the theorem is proved.

Let us now show that $\min PE_n(a, b) = l_n(a, b)$ and $\max PE_n(a, b) = h_n(a, b)$, where $PE_n(a, b)$ is the projection of $E_n(a, b)$ to \mathbb{R} .

Note that from the analysis done above, we have that if $a > c_n \vee d_n$ or $a \leq c_n$, there is only one NE but when $d_n > a > c_n$ we have multiple equilibrium and then the expected payoff of the players when both pass depends on which equilibrium is played in the following stages of the game. It is known that, in general, we cannot assume that if “the worst” or “the best” equilibrium is played at each stage, that results in the worst or best equilibrium of the game. However, below we show that this is indeed true for the game we are considering and then the result will follow just computing the expected payoff corresponding to the best and worst equilibrium.

To this end, it is enough to prove that the expected payoff of a player is increasing in d_n .

Define $\Omega = \{(x, y, z) \in \mathbb{R}^3 : \text{if } y > x \text{ then } z > x\}$ and consider the multivalued function $\psi : \Omega \rightrightarrows \mathbb{R}^3$ defined by

$$\psi(x, y, z) = \begin{cases} z & \text{if } y > x \\ \left\{ \frac{x+y}{2}, z \right\} & \text{if } y < x < z \\ \frac{x+y}{2} & \text{if } y \vee z < x. \end{cases}$$

We say that the multivalued function is increasing in z if for each x, y, z_1, z_2 such that $z_1 < z_2$ and $(x, y, z_i) \in \Omega$ for $i = 1, 2$, holds that $\min \psi(x, y, z_1) \leq \min \psi(x, y, z_2)$ and $\max \psi(x, y, z_1) \leq \max \psi(x, y, z_2)$. Let us see that ψ is increasing in z . To show that, fix x, y and take z_1, z_2 such that $z_1 < z_2$ and $(x, y, z_1) \in \Omega, (x, y, z_2) \in \Omega$. We have the following cases:

- (a) If $y > x$, then $\psi(x, y, z_1) = z_1 < z_2 = \psi(x, y, z_2)$.
- (b) If $y < x < z_1$, then also $y < x < z_2$ and $\frac{x+y}{2} < z_1 < z_2$. Thus, $\min \psi(x, y, z_1) = \frac{x+y}{2} = \min \psi(x, y, z_2)$ and $\max \psi(x, y, z_1) = z_1 \leq z_2 = \max \psi(x, y, z_2)$.
- (c) If $x > y \vee z_2$, then $\psi(x, y, z_1) = \frac{x+y}{2} = \psi(x, y, z_2)$.
- (d) If $y \vee z_1 < x < z_2$, then $\psi(x, y, z_1) = \frac{x+y}{2}$ and $\psi(x, y, z_2) = \left\{ \frac{x+y}{2}, z_2 \right\}$, obtaining $\psi(x, y, z_1) \leq \min \psi(x, y, z_2) \leq \max \psi(x, y, z_2)$.

Thus, we conclude that ψ is increasing in z .

Notice that if we set $x = a, y = c_n$ and $z = d_n$, $\psi(x, y, z)$ represents the NE payoffs of the game represented by Table 6, and thus the notion of monotonicity we proved for ψ means that playing a "better" equilibrium in the continuation game gives a "better" equilibrium for $\Gamma_n^{FR}(a, b)$ and playing a "worse" equilibrium one gives a "worse" equilibrium for $\Gamma_n^{FR}(a, b)$.

Furthermore, if $d_n > a > c_n$ holds that

$$d_n > \frac{d_n c_n - 2ac_n + ad_n}{2d_n - a - c_n} > \frac{1}{2}(a + c_n),$$

and thus when $d_n > a > c_n$, (\emptyset, \emptyset) is the *best equilibrium*, (a, a) is the *worst equilibrium* and the mixed one is the *middle equilibrium*.

Therefore, defining $L : \Omega \rightarrow \mathbb{R}$ and $H : \Omega \rightarrow \mathbb{R}$ by:

$$L(x, y, z) = \begin{cases} z & \text{if } x \leq y, \\ \frac{1}{2}(x + y) & \text{if } x > y \end{cases} \quad \text{and} \quad H(x, y, z) = \begin{cases} \frac{1}{2}(x + y) & \text{if } x > y \vee z, \\ z & \text{if } x \leq y \vee z \end{cases}$$

we have that taking $x = a, y = c_n$ and $z = d_n$, $L(x, y, z)$ and $H(x, y, z)$ represent the equilibrium expected payoffs if when $d_n > a > c_n$ the worst and the best equilibrium is played, respectively.

As we aforementioned, we are interested in computing the expected payoff corresponding to the best and the worst NE of the game, which represents the extremes values of the set $\mathbb{E}_n(a, b)$, and then it only remains to compute d_n . By the monotonicity of ψ , if we are interested in the best and the worst SPE, we have to consider the best and the worst equilibrium at each step of the continuation game, respectively. In what follows, we compute recursively d_n when the best or the worst equilibrium are played in the continuation game.

If the worst equilibrium is always played in the continuation game, we have that for each $n \geq 1$:

$$d_n = \mathbb{E}_X (L(a \vee X, \mathbb{E}(X_{(n-1)} \vee \text{med}[a, b, X]), d_{n-1})),$$

with $d_0 = \frac{a+b}{2}$. On the other hand, if the best equilibrium is always played in the continuation game, we have that for each $n \geq 1$:

$$d_n = \mathbb{E}_X (H(a \vee X, \mathbb{E}(X_{(n-1)} \vee \text{med}[a, b, X]), d_{n-1})).$$

		Player 2	
		a	\emptyset
Player 1	a	$(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n))$	(a, c_n)
	\emptyset	(c_n, a)	(d_n, e_n)

Table 7: Expected payoffs matrix for $\Gamma_n^{NR}(a)$.

Putting all together, we conclude that if (u, u) belongs to $E_n(a, b)$, then $u \in [l_n(a, b), h_n(a, b)]$, with $l_n(a, b) = L(a, c_n, d_n)$ and $h_n(a, b) = H(a, c_n, d_n)$, where L, H and d_n follows the above definitions. \square

Remark 3. Note that if $a = b$, then “pass” is a dominant strategy for both players and the unique equilibrium is given by (\emptyset, \emptyset) . Furthermore, in that case $c_n = \mathbb{E}(X_{(n)} \vee a) > a$ and then the same analysis as before holds for this case. Therefore the results so far also holds considering a discrete distribution.

Using Theorem 1, we now prove Theorem 2.

Proof of Theorem 2. By Theorem 1, we know that $E_n^{FR} \subset \{(u, u) : l_n \leq u \leq h_n\}$.

Furthermore, $E_n^{FR} = \left\{ \int_0^1 f(x) dF(x) : f(x) \in \mathbb{E}_{n-1}(x, 0) \right\}$, and then (l_n, l_n) and (h_n, h_n) belongs to E_n^{FR} (we obtain them just taking $f(x) = (l_{n-1}(x, 0), l_{n-1}(x, 0))$ and $f(x) = (h_{n-1}(x, 0), h_{n-1}(x, 0))$, respectively).

To obtain the result is then enough to prove that the set E_n is convex. To this end, let us define the function $\mu : [0, 1] \rightarrow \mathbb{R}$ by

$$\mu(\alpha) = \int_0^\alpha l_{n-1}(x, 0) dF(x) + \int_\alpha^1 h_{n-1}(x, 0) dF(x).$$

Notice that μ is continuous, $\mu(0) = h_n$ and $\mu(1) = l_n$, then for the intermediate value theorem all values between l_n and h_n are taken by μ . But $(\mu(\alpha), \mu(\alpha))$ belongs to E_n for every α and therefore E_n is convex. \square

No recall case. We pass now to the no recall case and the goal is to prove Theorem 3. To this end, we consider the game $\Gamma_n^{FR}(a)$ and the set $E_n(a)$ of its SPE payoffs. Recall that the set of SPE payoffs of the game Γ_n^{NR} is $E_n^{NR} = E_n(0) = \mathbb{E}_{a \sim F}(E_{n-1}(a))$.

Before proving the theorem, we present a preliminary result, stating that the set of SPE payoffs for the no recall case is convex and symmetric with respect to the identity line.

Proposition 3. For each natural number n , the set of the SPE payoffs E_n^{NR} is convex and symmetric with respect to the identity line.

Below, we prove the theorem.

Proof of Theorem 3. Given $a \in [0, 1]$ and n a natural number, we consider the game $\Gamma_n^{NR}(a)$ defined in Section 3.1. Note that if we call (d_n, e_n) to the continuation equilibrium payoffs for players and c_n the value of the decision problem in a standard prophet setting, matrix in Table 7 represents the expected payoffs matrix of the game $\Gamma_n^{NR}(a)$.

One first remark to do regarding this game, is that what a player gets if he stays alone in the game, that is c_n , is at least what he gets if both stay. In other words, $c_n \geq d_n$ and $c_n \geq e_n$.

Now, we use Table 7 and the remark above to study the SNEs of the game, depending on the relation between the parameters a, c_n, d_n and e_n . It is easy to check that the following holds:

- (a) If $a > c_n$, (a, a) is the unique NE with payoff $(\frac{1}{2}(a + c_n), \frac{1}{2}(a + c_n))$.
- (b) If $c_n > a > d_n \vee e_n$ there are two pure NE (a, \emptyset) and (\emptyset, a) , and a mixed equilibrium; with payoffs (a, c_n) , (c_n, a) and (γ_n^1, γ_n^2) , respectively, with $\gamma_n^1 = \frac{2ac_n - d_n(c_n + a)}{c_n + a - 2d_n}$, $\gamma_n^2 = \frac{2ac_n - e_n(c_n + a)}{c_n + a - 2e_n}$.
- (c) If $c_n > d_n \wedge e_n > a$, (\emptyset, \emptyset) is the unique NE with payoff (d_n, e_n) .
- (d) If $c_n > d_n > a > e_n$, (\emptyset, a) is the unique NE with payoff (c_n, a) .
- (e) If $c_n > e_n > a > d_n$, (a, \emptyset) is the unique NE with payoff (a, c_n) .

In particular, we are interested on computing the sum of the expected payoff corresponding to the best and the worst SPEs that is, $\max\{x + y : (x, y) \in E_{n+1}^{NR}\}$ and $\min\{x + y : (x, y) \in E_{n+1}^{NR}\}$, respectively; as well as the worst payoff a player can get at equilibrium, that is $\min\{\min\{x, y\} : (x, y) \in E_n^{NR}\}$.

Due to Proposition 3 we have that

$$\max\{x + y : (x, y) \in E_{n+1}^{NR}\} = 2 \max\{x : (x, x) \in E_{n+1}^{NR}\},$$

and

$$\min\{x + y : (x, y) \in E_{n+1}^{NR}\} = 2 \min\{x : (x, x) \in E_{n+1}^{NR}\}.$$

Therefore, defining $\alpha_n := \min\{x : (x, x) \in E_n^{NR}\}$, $\beta_n := \max\{x : (x, x) \in E_n^{NR}\}$ and $\alpha'_n = \min\{\min\{x, y\} : (x, y) \in E_n^{NR}\}$, it follows that it is enough to compute $2\alpha_{n+1}$, $2\beta_{n+1}$ and α'_{n+1} .

To prove the part a) of the theorem, we compute a recursive formula for α'_{n+1} . To this end, notice the following:

- i- if $\alpha'_n > a$, then we have that α'_{n+1} given a is α'_n ;
- ii- if $c_n > a > \alpha'_n$, it holds that α'_{n+1} given a is a ; and
- iii- if $a > c_n$, we obtain that α'_{n+1} given a is $\frac{a + c_n}{2}$.

Therefore, it follows that if $n \geq 1$

$$\begin{aligned} \alpha'_{n+1} &= \int_0^{\alpha'_n} \alpha'_n dF(a) + \int_{\alpha'_n}^{c_n} a dF(a) + \int_{c_n}^1 \frac{a + c_n}{2} dF(a) \\ &= \alpha'_n (F(\alpha'_n) - F(0)) + (c_n F(c_n) - \alpha'_n F(\alpha'_n)) - \int_{\alpha'_n}^{c_n} F(a) da \\ &+ \frac{c_n}{2} (F(1) - F(c_n)) + \frac{1}{2} (1F(1) - c_n F(c_n)) - \frac{1}{2} \int_{c_n}^1 F(a) da. \\ &= \frac{(c_n + 1)}{2} - \int_{\alpha'_n}^{c_n} F(a) da - \frac{1}{2} \int_{c_n}^1 F(a) da, \end{aligned} \tag{4}$$

and the first statement of Theorem 3 is proved.

Next, we compute $2\beta_{n+1}$ as a function of β_n , α'_n and c_n . Notice that if $c_n > a > d_n \vee e_n$, the mixed equilibrium gives a worse sum of payoffs than the pure, independently of the value of d_n and e_n . Thus, we do not need to consider the mixed equilibrium, but only the pure ones.

The sum of the expected payoffs it is just $d_n + e_n$ if $a < \min\{d_n, e_n\}$ and $a + c_n$, otherwise. More specifically:

- i- If $a < \alpha'_n$, then the sum of the payoffs for the best SPE is given by $2\beta_n$;

- ii- if $a > \beta_n$, the sum of the payoffs for the best SPE is given by $a + c_n$, and
- iii- if $\alpha'_n < a < \beta_n$, the sum of the payoffs for the best SPE is given by $\max\{a + c_n, 2\beta_n\}$.

Thus, we have that for $n \geq 1$

$$2\beta_{n+1} = \int_0^{\alpha'_n} 2\beta_n dF(a) + \int_{\alpha'_n}^{\beta_n} \max\{a + c_n, 2\beta_n\} dF(a) + \int_{\beta_n}^1 (a + c_n) dF(a), \quad (5)$$

and the second statement of the theorem is obtained.

It remains to compute sum of the expected payoff corresponding to the worst SPE, that is, $2\alpha_{n+1}$.

Note that if $c_n > a > d_n \vee e_n$, the mixed equilibrium gives a worse sum of payoffs than the pure, independently of the value of d_n and e_n . Thus, if $c_n > a > d_n \vee e_n$, only the mixed equilibrium will be considered.

Conditioned on the value of a , we have that:

- i- if $a > c_n$, the minimum payoff corresponding to the worst SPE is $a + c_n$,
- ii- if $c_n > a > \beta_n$, it is given by $\min_{\{(d_n, e_n) \in \mathbb{E}_n\}} \gamma_n^1 + \gamma_n^1 = 2 \frac{2ac_n - \beta_n(a + c_n)}{c_n + a - 2\beta_n}$,
- iii- if $\beta_n > a > \alpha_n$, we obtain $2a$,
- iv- if $\alpha_n > a > \alpha'_n$, the minimum payoff corresponding to the worst SPE is $\min\{2\alpha_n, a + c_n\}$, and
- v- if $\alpha'_n > a$, we have $2\alpha'_n$.

Therefore we obtain that for $n \geq 1$ holds that

$$\alpha_{n+1} = \int_{c_n}^1 (a + c_n) dF(a) + \int_{\beta_n}^{c_n} \psi_n(a) dF(a) + \int_{\alpha_n}^{\beta_n} 2a dF(a) + \int_{\alpha'_n}^{\alpha_n} \xi_n(a) dF(a) + \int_0^{\alpha'_n} 2\alpha'_n dF(a), \quad (6)$$

where $\psi_n(a) = \frac{4ac_n - 2\beta_n(a + c_n)}{c_n + a - 2\beta_n}$ and $\xi_n(a) = \min\{2\alpha_n, a + c_n\}$, which is the third statement of the theorem.

Putting together all the foregoing analysis, we obtain the desired result, concluding the proof. \square

4.2 Omitted proofs from Section 3.3

The main result in Section 3.3 we prove here is Proposition 1 and then we work on the competitive selection problem with no recall and two arrivals.

Before going to the proof, we obtain an expression for the price of anarchy and price of stability if the random variables are distributed according to F , namely $\text{PoA}_2^{NR}(F)$ and $\text{PoS}_2^{NR}(F)$ respectively.

Regarding the price of stability, we need to compute

$$\frac{\mathbb{E}(X_{(1:2)} + X_{(2:2)})}{2\beta_2},$$

where $2\beta_2$ is obtained from equation (5) by taking $n = 1$. That is:

$$2\beta_2 = \int_0^{\alpha'_1} 2\beta_1 dF(a) + \int_{\alpha'_1}^{\beta_1} \max\{a + c_1, 2\beta_1\} dF(a) + \int_{\beta_1}^1 (a + c_1) dF(a).$$

Now, due to $E_1^{NR} = \left\{ \left(\frac{\mathbb{E}(X)}{2}, \frac{\mathbb{E}(X)}{2} \right) \right\}$, we have that $\alpha'_1 = \beta_1 = \frac{\mathbb{E}(X)}{2}$. Noting that $c_1 = \mathbb{E}(X)$ and putting all together we have

$$\begin{aligned} 2\beta_2 &= \int_0^{\mathbb{E}(X)/2} \mathbb{E}(X) dF(a) + \int_{\mathbb{E}(X)/2}^{\mathbb{E}(X)/2} \max\{a + \mathbb{E}(X), \mathbb{E}(X)\} dF(a) + \int_{\mathbb{E}(X)/2}^1 (a + \mathbb{E}(X)) dF(a) \\ &= \int_0^1 \mathbb{E}(X) dF(a) + \int_{\mathbb{E}(X)/2}^1 a dF(a) \\ &= \mathbb{E}(X) + \mathbb{P}(X \geq \mathbb{E}(X)/2) \mathbb{E}(X|X \geq \mathbb{E}(X)/2). \end{aligned}$$

On the other hand, $\mathbb{E}(X_{(1:2)} + X_{(2:2)}) = 2\mathbb{E}(X)$, and therefore

$$\frac{1}{\text{PoS}_2^{NR}(F)} = \frac{\mathbb{E}(X) + \mathbb{P}(X \geq \mathbb{E}(X)/2) \mathbb{E}(X|X \geq \mathbb{E}(X)/2)}{2\mathbb{E}(X)} = \frac{1}{2} + \frac{\mathbb{P}(X \geq \mathbb{E}(X)/2) \mathbb{E}(X|X \geq \mathbb{E}(X)/2)}{2\mathbb{E}(X)}. \quad (7)$$

Regarding the price of anarchy, we need to compute

$$\frac{\mathbb{E}(X_{(1:n)} + X_{(2:n)})}{2\alpha_2},$$

where $2\alpha_2$ follows from equation (6) by taking $n = 1$. After some algebra, we obtain

$$\begin{aligned} 2\alpha_2 &= \mathbb{E}(X) + \int_{\mathbb{E}(X)/2}^{\mathbb{E}(X)} 2\mathbb{E}(X) - \frac{\mathbb{E}(X)^2}{a} dF(a) + \int_{\mathbb{E}(X)}^1 a dF(a) \\ &= 2\mathbb{E}(X) - \int_0^{\mathbb{E}(X)/2} a dF(a) - \int_{\mathbb{E}(X)/2}^{\mathbb{E}(X)} a - 2\mathbb{E}(X) + \frac{\mathbb{E}(X)^2}{a} dF(a), \end{aligned} \quad (8)$$

and thus

$$\frac{1}{\text{PoA}_2^{NR}(F)} = 1 - \frac{1}{2\mathbb{E}(X)} \int_0^{\mathbb{E}(X)/2} a dF(a) - \frac{1}{2\mathbb{E}(X)} \int_{\mathbb{E}(X)/2}^{\mathbb{E}(X)} a - 2\mathbb{E}(X) + \frac{\mathbb{E}(X)^2}{a} dF(a). \quad (9)$$

Also, we can write the inverse of price of anarchy as follows:

$$\frac{1}{\text{PoA}_2^{NR}(F)} = \frac{1}{\text{PoS}_2^{NR}(F)} - \frac{1}{2\mathbb{E}(X)} \int_{\mathbb{E}(X)/2}^{\mathbb{E}(X)} (a - 2\mathbb{E}(X) + \mathbb{E}(X)^2/2) dF(a). \quad (10)$$

We now prove Proposition 1 using the formulas obtained above.

Proof of Proposition 1. To prove that $\text{PoS}_2^{NR}(F) \leq 4/3$, let us consider the second term in the rhs of (7) and notice that

$$\begin{aligned} \mathbb{P}(X \geq \mathbb{E}(X)/2) \mathbb{E}(X|X \geq \mathbb{E}(X)/2) &= \mathbb{E}(X) - \mathbb{P}(X < \mathbb{E}(X)/2) \mathbb{E}(X|X < \mathbb{E}(X)/2) \\ &> \mathbb{E}(X) - \mathbb{P}(X < \mathbb{E}(X)/2) \frac{\mathbb{E}(X)}{2} \\ &\geq \mathbb{E}(X) - \frac{\mathbb{E}(X)}{2} = \frac{\mathbb{E}(X)}{2}. \end{aligned}$$

Then,

$$\frac{1}{\text{PoS}_2^{NR}(F)} \geq \frac{1}{2} + \frac{\mathbb{E}(X)}{2} \frac{1}{2\mathbb{E}(X)} = \frac{3}{4}.$$

Regarding the price of anarchy, by equation (9) and using that $a - 2\mathbb{E}(X) + \frac{\mathbb{E}(X)^2}{a} \leq \frac{\mathbb{E}(X)}{2}$ if $a \in [\mathbb{E}(X)/2, \mathbb{E}(X)]$, we have

$$\frac{1}{\text{PoA}_2^{NR}(F)} \geq 1 - \frac{\mathbb{P}(X < \mathbb{E}(X)/2)\mathbb{E}(X|X < \mathbb{E}(X)/2)}{2\mathbb{E}(X)} - \frac{1}{4}\mathbb{P}(\mathbb{E}(X)/2 \leq X \leq \mathbb{E}(X)).$$

But, $\mathbb{E}(X|X < \mathbb{E}(X)/2) \leq \mathbb{E}(X)/2$ and thus

$$\begin{aligned} \frac{1}{\text{PoA}_2^{NR}(F)} &\geq 1 - \frac{1}{4}\mathbb{P}(X < \mathbb{E}(X)/2) - \frac{1}{4}\mathbb{P}(\mathbb{E}(X)/2 \leq X \leq \mathbb{E}(X)) \\ &= 1 - \frac{1}{4}\mathbb{P}(X \leq \mathbb{E}(X)) \geq \frac{3}{4}, \end{aligned}$$

obtaining the desired inequalities.

To prove the tightness of the bound, let us take $\varepsilon > 0$ and $\eta > 0$ two small positive real numbers and consider the random variable $X_{\varepsilon,\eta} = (1 - \eta)X + \eta \text{Unif}[0, 1]$, where

$$X = \begin{cases} \varepsilon - \varepsilon^2 & \text{with probability } 1 - \varepsilon, \\ 1 & \text{with probability } \varepsilon. \end{cases}$$

Note that when η goes to 0,

$$\frac{1}{\text{PoS}_2^{NR}(F_{\varepsilon,\eta})} \rightarrow \frac{1}{2} + \frac{\mathbb{P}(X \geq \mathbb{E}(X)/2)\mathbb{E}(X|X \geq \mathbb{E}(X)/2)}{2\mathbb{E}(X)},$$

where $F_{\varepsilon,\eta}$ represents the c.d.f. of $X_{\varepsilon,\eta}$, and therefore it is enough to prove that

$$\frac{1}{2} + \frac{\mathbb{P}(X \geq \mathbb{E}(X)/2)\mathbb{E}(X|X \geq \mathbb{E}(X)/2)}{2\mathbb{E}(X)} \rightarrow 3/4$$

when ε goes to 0.

The c.d.f. of X is given by

$$F_{\varepsilon}(x) = \begin{cases} 0 & \text{if } x < \varepsilon - \varepsilon^2, \\ 1 - \varepsilon & \text{if } x \in [\varepsilon - \varepsilon^2, 1), \\ 1 & \text{if } x \geq 1; \end{cases}$$

and the expected value is $\mathbb{E}(X) = (\varepsilon - \varepsilon^2)(1 - \varepsilon) + \varepsilon = \varepsilon(1 + (1 - \varepsilon)^2)$.

Therefore, after some algebra, it follows that

$$\frac{1}{2} + \frac{\mathbb{P}(X \geq \mathbb{E}(X)/2)\mathbb{E}(X|X \geq \mathbb{E}(X)/2)}{2\mathbb{E}(X)} = \frac{1}{2} + \frac{\varepsilon}{2\varepsilon(1 + (1 - \varepsilon)^2)} = \frac{1}{2} + \frac{1}{2(1 + (1 - \varepsilon)^2)},$$

which converges to 3/4 when $\varepsilon \rightarrow 0$, and therefore we obtain price of stability 4/3.

On the other hand, note that by definition, for each distribution F , holds that $\text{PoS}_2(F) \leq \text{PoA}_2(F)$, and then $\text{PoA}_2(F_{\varepsilon,\eta}) \geq 4/3$, where 4/3 is the upper bound we already prove. Therefore, the bound is tight also for the price of anarchy. \square

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