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# HYDROSTATIC LIMIT OF THE NAVIER-STOKES-ALPHA MODEL

GLANGETAS LÉO, NGO VAN-SANG, SAID EL MEHDI

ABSTRACT. In this paper we study the hydrostatic limit of the Navier-Stokes-alpha model in a very thin striped domain. We derive some Prandtl-type limit equations for this model and we prove the global well-posedness of the limit system for small initial conditions in an appropriate analytic function space.

## 1. INTRODUCTION

**1.1. Motivation.** The characteristic feature of a turbulent fluid according to Kolmogorov's theory is that the energy cascades from large scales to small scales until it reaches the dissipation scale and then turns into heat. This feature leads to important costs of calculations in numerical simulation because the grid resolutions cannot keep up to the dissipation scale, which is extremely small when the Reynolds number is large (which corresponds to turbulent flows). So the idea is to consider the effects of smaller scales on larger scales instead of capturing all scales and as a consequence, one can achieve a balance between computational costs and precision.

One way to model turbulent flows is the so-called Large Eddy Simulation method. It consists in filtering the small scales and directly calculating the large scales of the turbulent cascade (see [13] for instance). Another approach is the Reynolds Averaged Navier-Stokes, based on Reynolds decomposition, which provides mean quantities of turbulent flow while fluctuations will be modeled. This second method is used in the industry due to its small computational costs. However, both approaches meet a common problem which is the closure of the model systems where there are more unknowns than equations.

In order to overcome this difficulty, the Navier-Stokes-alpha model was introduced, where an energy "penalty" inhibits the creation of small excitations below a certain length scale  $\alpha$  (also called the viscous Camassa-Holm equations, see [5, 9, 14, 15, 18, 20] and the references therein for a survey of Camassa-Holm equations). This "alpha-modification" leads to a change in the convection term of the Navier-Stokes equations. More precisely, this is the following system

$$(1.1) \quad \begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + v_1 \nabla u_1 + v_2 \nabla u_2 = \nu \Delta \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

where

$$\mathbf{v} = (v_1, v_2) = (1 - \alpha^2 \Delta) \mathbf{u} = ((1 - \alpha^2 \Delta) u_1, (1 - \alpha^2 \Delta) u_2)$$

and  $\mathbf{u}$  is the velocity of the fluid and  $q$  the modified pressure. In [11], Foias, Holm and Titi prove the global existence and uniqueness of the solution of (1.1) in a periodic domain for  $H^1$ -initial data and the convergence of the solution of 3D Camassa-Holm equation (NS- $\alpha$ ) towards a weak solution of 3D NS equations when  $\alpha$  tends to zero and in [4], Busuioc gives a simple proof of the global existence and uniqueness of (1.1) for  $H_0^1$ -initial data in bounded domains with Dirichlet boundary conditions.

In this paper, we consider the system (1.1) in the thin strip  $\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\}$ , where the width  $\varepsilon$  is supposed to be very small. This consideration is relevant for planetary-scale oceanic and atmospheric flows (see [22]), for which, the vertical scale (a few kilometers for oceans, 10-20 kilometers for the atmosphere) is much smaller than the horizontal scales (thousands of kilometers). In this framework, the fluid behaviors are approximated by the so-called hydrostatic model, in which the conservation

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momentum in the vertical direction is replaced by a simple hydrostatic equation. In the case of Navier-Stokes equations for a viscous fluid in a thin strip, the hydrostatic limit leads to the following rescaled system in domain  $\mathbb{R} \times ]0, 1[$

$$(1.2) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_z u = \partial_y^2 u - \partial_x p \\ \partial_y p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v)|_{y=0} = (u, v)|_{y=1} = 0 \\ u|_{t=0} = u_0. \end{cases}$$

This model and its three-dimensional counterpart are very important in oceanography and meteorology (see [3, 17, 22]). It is known that without any structural assumption on the initial data, real-analyticity is both necessary [23] and sufficient [16] for the local well-posedness of the system (1.2). In [12] the authors proved that for convex initial data, the local well-posedness holds under simple Gevrey regularity. In [21], Paicu *et al.* proved the existence and uniqueness of global analytic solution with small analytic initial data with respect to variable  $x$ . We want to emphasize the similarity of the system (1.2) and the classical Prandtl system. For the classical Prandtl, the well-posedness also requires analyticity frameworks (see [24]) or monotonicity hypotheses to be able to work in classical Sobolev frameworks (see for instance [1, 19]).

In this paper, we will study the hydrostatic limit of the Navier-Stokes-alpha equations as the strip width  $\varepsilon \rightarrow 0$ . Using the techniques of [21], we prove the global well-posedness of the limit system in appropriate analytic function spaces. We will justify this limit by proving the convergence of the Navier-Stokes-alpha system in a forthcoming paper. In the next subsection, we will give brief derivation of the hydrostatic limit of the system (1.1) as  $\varepsilon \rightarrow 0$ .

**1.2. Hydrostatic limit of the Navier-Stokes-alpha model.** We consider the system (1.1) in a thin strip

$$\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2, 0 < y < \varepsilon\},$$

where the width  $\varepsilon > 0$  is supposed to be very small. Equipped with no-slip boundary conditions, we rewrite (1.1) as

$$(1.3) \quad \begin{cases} \partial_t v_1^\varepsilon + u_1^\varepsilon \partial_x v_1^\varepsilon + u_2^\varepsilon \partial_y v_1^\varepsilon + v_1^\varepsilon \partial_x u_1^\varepsilon + v_2^\varepsilon \partial_x u_2^\varepsilon = \nu \Delta v_1^\varepsilon - \partial_x q^\varepsilon \\ \partial_t v_2^\varepsilon + u_1^\varepsilon \partial_x v_2^\varepsilon + u_2^\varepsilon \partial_y v_2^\varepsilon + v_1^\varepsilon \partial_y u_1^\varepsilon + v_2^\varepsilon \partial_y u_2^\varepsilon = \nu \Delta v_2^\varepsilon - \partial_y q^\varepsilon \\ \partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0 \\ u_1^\varepsilon = u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ \partial_z u_1^\varepsilon = \partial_z u_2^\varepsilon = 0 \quad \text{on} \quad \{y = 0\} \cup \{y = \varepsilon\} \\ (u_1^\varepsilon, u_2^\varepsilon)|_{t=0} = (u_{1,0}, u_{2,0})(x, y), \end{cases}$$

where, for  $i = 1, 2$ ,

$$v_i^\varepsilon = u_i^\varepsilon - \alpha^2 \Delta u_i^\varepsilon.$$

We consider the following rescaling

$$\begin{aligned} u_1^\varepsilon(t, x, y) &= u^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right), \\ u_2^\varepsilon(t, x, y) &= \varepsilon v^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right), \\ q^\varepsilon(t, x, y) &= \tilde{q}^\varepsilon \left( t, x, \frac{y}{\varepsilon} \right), \end{aligned}$$

we perform the change of variable  $z = \frac{y}{\varepsilon}$  and we set

$$\mathcal{S} = \{(x, z) \in \mathbb{R}^2, 0 < z < 1\}.$$

We remark that for any function  $\varphi(t, x, y) = \psi \left( t, x, \frac{y}{\varepsilon} \right)$ , we have

$$\partial_y^{(k)} (\varphi(t, x, y)) = \varepsilon^{-k} \partial_z^{(k)} \psi(t, x, z).$$

Then the following calculations are immediate

$$\begin{cases} v_1^\varepsilon(t, x, y) = u^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 u^\varepsilon(t, x, z) - \varepsilon^2 \alpha_1^2 \partial_x^2 u^\varepsilon(t, x, z) \\ v_2^\varepsilon(t, x, y) = \varepsilon(v^\varepsilon(t, x, z) - \alpha_1^2 \partial_z^2 v^\varepsilon(t, x, z)) - \varepsilon^3 \alpha_1^2 \partial_x^2 v^\varepsilon(t, x, z), \end{cases}$$

where  $\alpha_1 = \frac{\alpha}{\varepsilon}$ . The system (1.3) becomes

$$\begin{cases} \partial_t \omega^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_t \partial_x^2 u^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon - \alpha_1^2 \varepsilon^2 u^\varepsilon \partial_x^3 u^\varepsilon + v^\varepsilon \partial_z \omega^\varepsilon - \alpha_1^2 \varepsilon^2 v^\varepsilon \partial_z \partial_x^2 u^\varepsilon + \omega^\varepsilon \partial_x u^\varepsilon - \alpha_1^2 \varepsilon^2 \partial_x u^\varepsilon \partial_x^2 u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_x v^\varepsilon - \varepsilon^4 \alpha_1^2 \partial_x^2 v^\varepsilon \partial_x v^\varepsilon = \partial_z^2 \omega^\varepsilon + \varepsilon^2 \partial_x^2 \omega^\varepsilon - \varepsilon^2 \alpha_1^2 \partial_z^2 \partial_x^2 u^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^4 u^\varepsilon - \partial_x \tilde{q}^\varepsilon \\ \varepsilon^2 \partial_t \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_t \partial_x^2 v^\varepsilon + \varepsilon^2 u^\varepsilon \partial_x \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 u^\varepsilon \partial_x^3 v^\varepsilon + \varepsilon^2 v^\varepsilon \partial_z \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 v^\varepsilon \partial_z \partial_x^2 v^\varepsilon + \omega \partial_z u^\varepsilon - \alpha_1^2 \varepsilon^3 \partial_x^2 u^\varepsilon \partial_z u^\varepsilon \\ \quad + \varepsilon^2 \gamma^\varepsilon \partial_z v^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_x^2 v^\varepsilon \partial_z v^\varepsilon = \varepsilon^2 \partial_z^2 \gamma^\varepsilon + \varepsilon^4 \partial_x^2 \gamma^\varepsilon - \alpha_1^2 \varepsilon^4 \partial_z^2 \partial_x^2 v^\varepsilon - \alpha_1^2 \varepsilon^6 \partial_x^4 v^\varepsilon - \partial_z \tilde{q}^\varepsilon \\ \partial_x u^\varepsilon + \partial_z v^\varepsilon = 0, \end{cases}$$

where we denote  $\omega^\varepsilon = u^\varepsilon - \alpha_1^2 \partial_z^2 u^\varepsilon$  and  $\gamma^\varepsilon = v^\varepsilon - \alpha_1^2 \partial_z^2 v^\varepsilon$  to lighten the notations.

Formally taking  $\varepsilon \rightarrow 0$  we obtain

$$(1.4) \quad \begin{cases} \partial_t(u - \alpha_1^2 \partial_z^2 u) + u \partial_x(u - \alpha_1^2 \partial_z^2 u) + v \partial_z(u - \alpha_1^2 \partial_z^2 u) + (u - \alpha_1^2 \partial_z^2 u) \partial_x u = \partial_z^2(u - \alpha_1^2 \partial_z^2 u) - \partial_x \tilde{q} \\ (u - \alpha_1^2 \partial_z^2 u) \partial_z u = -\partial_z \tilde{q} \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 1 \\ u|_{t=0} = u_0. \end{cases}$$

We consider the modified pressure in the strip

$$p = \tilde{q} + \frac{1}{2} u^2 - \frac{1}{2} \alpha_1^2 (\partial_z u)^2$$

and we set

$$\omega = u - \alpha_1^2 \partial_z^2 u.$$

Then we can rewrite the system (1.4) as

$$(1.5) \quad \begin{cases} \partial_t \omega + u \partial_x \omega + v \partial_z \omega + \alpha_1^2 (\partial_z u \partial_x \partial_z u - \partial_z^2 u \partial_x u) = \partial_z^2 \omega - \partial_x p \\ \partial_z p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 0 \\ (u, v) = 0, \quad \partial_z u = 0 \quad \text{on } z = 1 \\ u|_{t=0} = u_0. \end{cases}$$

**Remark 1.1.** (1) *The no-slip boundary conditions  $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$  and the incompressibility  $\partial_x u + \partial_y v = 0$  imply*

$$v(t, x, y) = \int_0^y \partial_{\tilde{y}} v(t, x, \tilde{y}) d\tilde{y} = - \int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y}.$$

(2) *From the incompressibility condition we deduce that*

$$\partial_x \int_0^1 u(t, x, y) dy = - \int_0^1 \partial_y v(t, x, y) dy = v(t, x, 1) - v(t, x, 0) = 0,$$

*which together with the fact that  $u(t, x, y) \rightarrow 0$  as  $|x| \rightarrow +\infty$ , ensures that*

$$(1.6) \quad \int_0^1 u(t, x, y) dy = 0.$$

*Since  $\partial_z p = 0$ , integrating the first equation of system (1.5) and using the boundary conditions, we get*

$$\partial_x p(t, x) = \alpha_1^2 \partial_z^3 u|_{z=0} - \alpha_1^2 \partial_z^3 u|_{z=1} - \partial_x \int_0^1 u^2(t, x, y) dz - \alpha_1^2 \partial_x \int_0^1 (\partial_z u)^2(t, x, y) dz.$$

We emphasize that, similar to Prandtl equation, the nonlinear term  $v\partial_z\omega$  in the system (1.5) creates the loss of one derivative in  $x$  variable in energy-type estimates. So, in order to overcome this difficulty, it is natural to work in analytic function frameworks. In the next paragraph, we will introduce some elements of Littlewood-Paley theory and functional spaces that we are going to use in this paper.

**1.3. Littlewood-Paley theory and functional spaces.** We consider an even smooth function  $\chi$  in  $C_0^\infty(\mathbb{R})$  such that the support is contained in the ball  $B_{\mathbb{R}}(0, \frac{4}{3})$  and  $\psi$  is equal to 1 on a neighborhood of the ball  $B_{\mathbb{R}}(0, \frac{3}{4})$ . We set  $\psi(z) = \chi(\frac{z}{2}) - \chi(z)$  then the support of  $\psi$  is contained in the ring  $\{z \in \mathbb{R} : \frac{3}{4} \leq |z| \leq \frac{3}{2}\}$ , and  $\psi$  is identically equal to 1 on the ring  $\{z \in \mathbb{R} : \frac{4}{3} \leq |z| \leq \frac{3}{2}\}$ . Moreover, the functions  $\chi$  and  $\psi$  verify the following important properties

$$\sum_q \psi(2^{-q}z) = 1, \quad \forall z \in \mathbb{R}^*,$$

$$\chi(z) + \sum_{q \geq 0} \psi(2^{-q}z) = 1, \quad \forall z \in \mathbb{R}$$

and

$$\forall j, j' \in \mathbb{N}, |j - j'| \geq 2, \quad \text{supp } \psi(2^{-j}\cdot) \cap \text{supp } \psi(2^{-j'}\cdot) = \emptyset.$$

Let  $\mathcal{F}_h$  and  $\mathcal{F}_h^{-1}$  be the Fourier transform and the inverse Fourier transform respectively in the horizontal direction. We will also use the notation  $\widehat{f} = \mathcal{F}_h f$ . We introduce the following definitions of the homogeneous dyadic cut-off operators.

**Definition 1.2.** For all tempered distributions in the horizontal direction and for all  $q \in \mathbb{Z}$  we set

$$\Delta_q^h u(x, y) = \mathcal{F}_h^{-1}(\psi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)),$$

$$S_q^h u(x, y) = \mathcal{F}_h^{-1}(\chi(2^{-q}|\xi|)\mathcal{F}_h u(\xi, y)) = \sum_{q' \leq q-1} \Delta_{q'}^h u.$$

In our paper, we will use the same functional spaces as in [21], the definition of which is given in what follows.

**Definition 1.3.** Let  $s \in \mathbb{R}$  and  $\mathcal{S} = \mathbb{R} \times ]0, 1[$ . For any  $u \in S'_h(\mathcal{S})$ , i.e.,

$$u \in S'(\mathcal{S}) \quad \text{with} \quad \lim_{q \rightarrow -\infty} \|S_q^h u\|_{L^\infty} = 0,$$

we set

$$\|u\|_{B^s} = \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h u\|_{L^2}.$$

- For  $s \leq \frac{1}{2}$ , we define

$$B^s(\mathcal{S}) = \{u \in S'_h(\mathcal{S}), \quad \|u\|_{B^s} < \infty\}.$$

- For  $s \in ]k - \frac{1}{2}, k + \frac{1}{2}]$ , with  $k \in \mathbb{N}^*$ , we define  $B^s(\mathcal{S})$  as the subset of distributions  $u \in S'_h(\mathcal{S})$  such that  $\partial_x^k u \in B^{s-k}(\mathcal{S})$ .

**Remark 1.4.** For  $u \in B^s$ , there exists a summable sequence of positive numbers  $d_q(u)$  with  $\sum_q d_q(u) = 1$ , such that

$$\|\Delta_q^h u\|_{L^2} \lesssim d_q(u) 2^{-qs} \|u\|_{B^s}.$$

We also need the following time-weighted Chemin-Lerner-type spaces (see [8]).

**Definition 1.5.** Let  $p \in [1, +\infty]$  and  $T \in ]0, \infty]$  and let  $f \in L^1_{loc}(\mathbb{R}^+)$  be non-negative function. We define space  $\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))$  as the closure of  $C([0, T], B^s(\mathcal{S}))$  with respect to the norm

$$\|u\|_{\widetilde{L}^p_{T,f}(B^s(\mathcal{S}))} = \sum_{q \in \mathbb{Z}} 2^{qs} \left( \int_0^T f(t) \|\Delta_q^h u\|_{L^2}^p dt \right)^{\frac{1}{p}}$$

with the usual change if  $p = +\infty$ . In the case of  $f(t) \equiv 1$ , we will simply use the notations  $\widetilde{L}^p_T(B^s(\mathcal{S}))$  and  $\|u\|_{\widetilde{L}^p_T(B^s(\mathcal{S}))}$ .

**Remark 1.6.** For  $u \in \widetilde{L}_T^p(B^s)$ , there exists a summable sequence of positive numbers  $d_q(u, f)$  with  $\sum_q d_q(u, f) = 1$ , such that

$$\|\Delta_q^h u\|_{L_T^p(L^2)} \lesssim d_q(u, f) 2^{-pq_s} \|u\|_{\widetilde{L}_T^p(B^s)}.$$

**1.4. Main results.** The main difficulty consists in controlling the nonlinear terms, using the smoothing effect given by the above function spaces. The idea is to define the following auxiliary functions, using the method introduced by Chemin in [10] (see also [7] or [21]): for any  $f \in L^2(\mathcal{S})$ , we set

$$(1.7) \quad f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) := \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{\phi(t, \xi)} \widehat{f}(t, \xi, y)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda\theta(t))|\xi|,$$

where the quantity  $\theta(t)$  describes the evolution of the analytic band of  $u$ , satisfies

$$(1.8) \quad \theta'(t) = \|\partial_z^2 u_\phi(t)\|_{B^{\frac{1}{2}}} \quad \text{and} \quad \theta(0) = 0.$$

We remark that if we differentiate a function of the type  $e^{\phi(t, D_x)} f(t, x, y)$  with respect to the time variable, we have

$$\partial_t(e^{\phi(t, D_x)} f(t, x, y)) = -\lambda\theta'(t)|D_x|e^{\phi(t, D_x)} f(t, x, y) + e^{\phi(t, D_x)} \partial_t f(t, x, y),$$

where  $-\lambda\theta'(t)|D_x|e^{\phi(t, D_x)} f(t, x, y)$  plays the role of a ‘‘smoothing term’’, provided that  $\theta'(t) > 0$ , and allows to ‘‘absorb’’ the loss of one derivative in the nonlinear terms when we perform energy-type estimates.

Our main result is the following.

**Theorem 1.1.** *Let  $a > 0$  be fixed. We assume that  $e^{a|D_x|}u_0 \in B^{\frac{3}{2}}$  and  $e^{a|D_x|}\partial_z u_0 \in B^{\frac{3}{2}}$ . There exist positive constants  $c$  and  $C$  such that, if we suppose that*

$$(1.9) \quad \|e^{a|D_x|}u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{1}{2}}} \leq \frac{ca}{1 + \|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}}$$

and the compatibility condition  $\int_0^1 u_0 dz = 0$ , then a unique global solution for the system (1.5) exists and satisfies

$$(1.10) \quad \|e^{Rt}u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} + \|e^{Rt}\partial_z^2 u_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})} \\ \leq C(\|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}).$$

Furthermore, if  $e^{a|D_x|}u_0 \in B^{\frac{5}{2}}$ ,  $e^{a|D_x|}\partial_z u_0 \in B^{\frac{5}{2}}$ ,  $e^{a|D_x|}\partial_z^2 u_0 \in B^{\frac{3}{2}}$ , then

$$(1.11) \quad \|e^{Rt}(\partial_t u)_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})}^2 + \|e^{Rt}(\partial_t \partial_z u)_\phi\|_{\widetilde{L}^2(\mathbb{R}^+, B^{\frac{3}{2}})}^2 + \|e^{Rt}\partial_z u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})}^2 + \|e^{Rt}\partial_z^2 u_\phi\|_{\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{3}{2}})}^2 \\ \leq C\left(\|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z^2 u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}u_0\|_{B^{\frac{5}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{5}{2}}}\right).$$

**Remark 1.7.** *The compatibility condition  $\int_0^1 u_0 dz = 0$  ensures that  $\int_0^1 u(t, \cdot) dz$  is conserved by the system (1.5).*

The rest of the paper is arranged as follows : The proof of the main Theorem 1.1 is presented in the next Section 2 and the Appendices 3, 4 and 5 are devoted to the proof of estimates used in Section 2.

## 2. GLOBAL EXISTENCE AND UNIQUENESS OF THE HYDROSTATIC LIMIT SYSTEM

In this section, we prove the existence of a unique global solution of the system (1.5) using the method introduced by Chemin in [6]. We recall that, to be able to deal with the loss of one derivative in the tangential direction  $x$ , we work in the same functional settings as in [21].

**2.1. Energy-type *a priori* estimates.** The aim of the first part of this section is to prove the energy-type estimate (1.10). Our proof is based on the following important estimates of the nonlinear terms. We recall that  $\phi, \theta$  are the auxiliary functions defined as in (1.7) and (1.8) and we set

$$T^* = \sup \left\{ t > 0, \quad \theta(t) < \frac{a}{\lambda} \right\}.$$

**Lemma 2.1.** *Let  $s > 0, 0 < T < T^*, R > 0$ . There exist a generic constant  $C \geq 1$  and some square root summable positive sequences  $(\sum_q \tilde{d}_q^{\frac{1}{2}} = 1, \sum_q \check{d}_q^{\frac{1}{2}} = 1)$  such that*

$$(2.1) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.2) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.3) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.4) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.5) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2$$

and

$$(2.6) \quad \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle \right| dt' \\ \leq C 2^{-2qs} (\tilde{d}_q + \check{d}_q) \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \left( \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \right).$$

The proof of this lemma is given in appendix 3. For  $(u, v)$  solution of (1.5), we define  $(u_\phi, v_\phi)$  be as in (1.7) and (1.8). Direct calculations show that  $(u_\phi, v_\phi)$  satisfies the following system

$$(2.7) \quad \begin{cases} \partial_t \omega_\phi + \lambda \theta'(t) |D_x| \omega_\phi + (u \partial_x \omega)_\phi + (v \partial_z \omega)_\phi + \alpha_1^2 (\partial_z u \partial_x \partial_z u)_\phi - \alpha_1^2 (\partial_z^2 u \partial_x u)_\phi = \partial_z^2 \omega_\phi - \partial_x p_\phi \\ \partial_z p_\phi = 0 \\ \partial_x u_\phi + \partial_y v_\phi = 0 \\ (u_\phi, v_\phi) = 0, \quad \partial_z u_\phi = 0 \quad \text{on } z = 0 \\ (u_\phi, v_\phi) = 0, \quad \partial_z u_\phi = 0 \quad \text{on } z = 1 \\ u_\phi|_{t=0} = e^{a|D_x|} u_0, \end{cases}$$

where  $|D_x|$  denotes the Fourier multiplier of symbol  $|\xi|$ . For any  $q \in \mathbb{Z}$ , applying the dyadic operator  $\Delta_q^h$  to the first equation of the system (2.7) and taking  $L^2(S)$  inner product of the obtained equation with  $\Delta_q^h u_\phi$ , we get

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} (\|\Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2) + \lambda \theta'(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 + \alpha_1^2 \lambda \theta'(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \\ + \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 = - \langle \Delta_q^h (u \partial_x \omega)_\phi, \Delta_q^h u_\phi \rangle - \langle \Delta_q^h (v \partial_z \omega)_\phi, \Delta_q^h u_\phi \rangle \\ - \alpha_1^2 \langle \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h u_\phi \rangle + \alpha_1^2 \langle \Delta_q^h (\partial_z^2 u \partial_x u)_\phi, \Delta_q^h u_\phi \rangle.$$

Multiplying (2.8) by  $e^{2Rt}$  and integrating with respect to the time variable, we obtain

$$\|e^{Rt} \Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + 2\lambda \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \|e^{Rt'} \Delta_q^h \partial_z u_\phi(t')\|_{L^2}^2 dt' \\ + 2\lambda \alpha_1^2 \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' + 2\alpha_1^2 \int_0^t \|e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t')\|_{L^2}^2 dt' \\ = \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + 2D_{1,q} + 2D_{2,q} + 2D_{3,q} + 2D_{4,q},$$

where

$$\begin{aligned} D_{1,q} &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \omega)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{2,q} &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z \omega)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{3,q} &= - \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ D_{4,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'. \end{aligned}$$

Using the definition of  $\omega$ , we can write

$$D_{1,q} = - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' + \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'.$$

Since

$$\Delta_q^h (u \partial_x \partial_z^2 u)_\phi = \partial_z \Delta_q^h (u \partial_x \partial_z u)_\phi - \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi,$$

by integration by parts with respect to  $z$  variable, we have

$$\begin{aligned} & \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' \\ &= \int_0^t \left\langle e^{Rt'} \partial_z \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' - \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt' \\ &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt' - \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt'. \end{aligned}$$

Thus

$$D_{1,q} = A_q + B_q + D_{3,q},$$

where

$$\begin{aligned} A_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ B_q &= - \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'. \end{aligned}$$

In a similar way, we can write

$$D_{4,q} = C_q + D_{3,q},$$

where

$$C_q = - \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt',$$

and

$$D_{2,q} = E_q + F_q + C_q + D_{3,q}$$

and where

$$\begin{aligned} E_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\ F_q &= - \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'. \end{aligned}$$



To summarize, we have

$$\begin{aligned}
(2.9) \quad & \|e^{Rt} \Delta_q^h u_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + 2\lambda \alpha_1^2 \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\
& + 2\lambda \int_0^t \theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\
& + 2\alpha_1^2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t') \right\|_{L^2}^2 dt' \\
& = \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + 2A_q + 2B_q + 2C_q + 8D_{3,q} + 2E_q + 2F_q,
\end{aligned}$$

where

$$\begin{aligned}
A_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\
B_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt', \\
C_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt', \\
D_{3,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\
E_q &= - \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle dt', \\
F_q &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \right\rangle dt'.
\end{aligned}$$

We recall that we set

$$T^* = \sup \left\{ t > 0, \theta(t) < \frac{a}{\lambda} \right\}.$$

Lemma 2.1 yields that, for any  $0 < T < T^*$ , we have

$$\begin{aligned}
& \|e^{Rt} \Delta_q^h u_\phi\|_{L_T^\infty(L^2)}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L_T^\infty(L^2)}^2 + \lambda \alpha_1^2 2^q \int_0^T \theta'(t') \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' \\
& + \alpha_1^2 \int_0^T \left\| e^{Rt'} \Delta_q^h \partial_z^2 u_\phi(t') \right\|_{L^2}^2 dt' + \int_0^T \left\| e^{Rt'} \Delta_q^h \partial_z u_\phi(t') \right\|_{L^2}^2 dt' + \lambda 2^q \int_0^T \theta'(t') \left\| e^{Rt'} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' \\
& \leq \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + \alpha_1^2 \|\Delta_q^h \partial_z u_\phi(0)\|_{L^2}^2 + C 2^{-2qs} d_q^2 \|e^{Rt'} u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^2 \\
& + C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} \left( \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \right).
\end{aligned}$$

Multiplying the previous inequality by  $2^{2qs}$ , taking square root of resulting inequality and then summing with respect to  $q \in \mathbb{Z}$ , we get

$$\begin{aligned}
& \|e^{Rt} u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \alpha_1^2 \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \sqrt{\lambda} \|e^{Rt} u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \alpha_1 \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} \\
& + \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \alpha_1 \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\
& \leq 6 \|u_\phi(0)\|_{B^s} + 6 \alpha_1 \|\partial_z u_\phi(0)\|_{B^s} + C \|e^{Rt} u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + C \alpha_1 \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} \\
& + C \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, there exists a constant  $C_1$  such that

$$\begin{aligned} & \|e^{Rt}u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \sqrt{\lambda} \|e^{Rt'}u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{Rt'}\partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} \\ & + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\ & \leq C_1(\|u_\phi(0)\|_{B^s} + \|\partial_z u_\phi(0)\|_{B^s}) + C_1 \|e^{Rt'}u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + C_1 \|e^{Rt'}\partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} \\ & + C_1 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt'}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt'}\partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}}. \end{aligned}$$

We remark that Young's inequality implies

$$\begin{aligned} C_1 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{Rt'}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}^{\frac{1}{2}} \|e^{Rt'}\partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^{\frac{1}{2}} \\ \leq C_2 \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'}\partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{Rt'}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)}. \end{aligned}$$

So, putting  $C_3 = \max(C_1, C_2)$  and choosing

$$(2.10) \quad \lambda \geq C_3^2(1 + \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}),$$

we have for any  $s > 0$

$$(2.11) \quad \|e^{Rt}u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \\ \leq C_3(\|e^{a|D_x|}u_0\|_{B^s} + \|e^{a|D_x|}\partial_z u_0\|_{B^s}).$$

In particular, under the condition (2.10), we have

$$\|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{3}{2}})} \leq C_3(\|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}).$$

Then, by taking  $\lambda = C_3^2(1 + C_3(\|e^{a|D_x|}u_0\|_{B^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{3}{2}}}))$ , the inequality (2.11) holds for any  $s > 0$ . In particular for  $s = \frac{1}{2}$ , we have

$$(2.12) \quad \|e^{Rt}u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} + \|e^{Rt}\partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \\ \leq C_3(\|e^{a|D_x|}u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{1}{2}}}).$$

Now we want to estimate the function  $\theta(t)$  on  $[0, T^*[$ . Using Cauchy-Schwarz inequality, we can write

$$\begin{aligned} \theta(t) &= \int_0^t e^{-Rt'} \times e^{Rt'} \|\partial_z^2 u_\phi(t')\|_{B^{\frac{1}{2}}} dt' \\ &\leq \left( \int_0^T e^{-2Rt'} dt' \right)^{\frac{1}{2}} \left( \int_0^T \|e^{Rt'}\partial_z^2 u_\phi(t')\|_{B^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C_R \|e^{Rt}\partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \end{aligned}$$

Then, we deduce from (2.12) the existence of a constant  $C_4 > 0$  such that

$$\theta(t) \leq C_4 \left( \|e^{a|D_x|}u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{1}{2}}} \right).$$

Taking  $c$  small enough (see the assumption on the initial data (1.9)), we obtain, for any  $t \in [0, T^*[$  that

$$\theta(t) \leq C_4 \left( \|e^{a|D_x|}u_0\|_{B^{\frac{1}{2}}} + \|e^{a|D_x|}\partial_z u_0\|_{B^{\frac{1}{2}}} \right) \leq \frac{a}{2\lambda}.$$

Using the definition of  $T^*$  and a continuity argument, we deduce  $T^* = +\infty$  and that the inequality (2.11) holds for any  $T > 0$ .

*Conclusion* : Taking  $s = \frac{3}{2}$  in (2.11), we obtain Estimate (1.10) in Theorem 1.1.

**2.2. Estimate of the time derivative of the solution.** In this paragraph, we give brief ideas of the proof of Estimate (1.11). Applying  $\Delta_q^h$  to (2.7) and taking the  $L^2$  inner product of the obtained equation with  $e^{2Rt}\Delta_q^h(\partial_t u)_\phi$ , we have

$$(2.13) \quad \begin{aligned} & \|e^{Rt}\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt}\Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 \\ &= e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - \alpha_1^2 e^{2Rt} \langle \Delta_q^h \partial_z^4 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \\ & \quad - e^{2Rt} \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - e^{2Rt} \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \\ & \quad - e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z^2 u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle. \end{aligned}$$

Since  $(\partial_t u)_\phi = \partial_t u_\phi + \lambda \theta'(t)|D_x|u_\phi$ , then

$$(2.14) \quad \begin{aligned} e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle &= e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h \partial_t u_\phi \rangle + \theta'(t)|D_x| e^{2Rt} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h u_\phi \rangle \\ &= -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 - \lambda \theta'(t) \|e^{Rt} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 \\ &\leq -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z u_\phi\|_{L^2}^2. \end{aligned}$$

Similar calculations give

$$(2.15) \quad -e^{2Rt} \langle \Delta_q^h \partial_z^4 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \leq -\frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2.$$

Plugging the estimates (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} & \|e^{Rt}\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt}\Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|e^{Rt}\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \frac{d}{dt} \|e^{Rt}\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 \\ & \leq -e^{2Rt} \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + \alpha_1^2 e^{2Rt} \langle \Delta_q^h(u \partial_x \partial_z^2 u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \\ & \quad - e^{2Rt} \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle - \alpha_1^2 e^{2Rt} \langle \Delta_q^h(v \partial_z^2 u)_\phi, \Delta_q^h(\partial_t \partial_z u)_\phi \rangle \\ & \quad - e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle + 2e^{2Rt} \alpha_1^2 \langle \Delta_q^h(\partial_z^2 u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle \end{aligned}$$

and so

$$\begin{aligned} & \|e^{Rt}\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt}\Delta_q^h(\partial_t \partial_z u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|e^{Rt}\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \frac{1}{2} \alpha_1^2 \frac{d}{dt} \|e^{Rt}\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^2 \\ & \leq C \left( \|e^{Rt}\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|e^{Rt}\Delta_q^h(u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 + \|e^{Rt}\Delta_q^h(v \partial_z u)_\phi\|_{L^2}^2 \right. \\ & \quad \left. + \|e^{Rt}\Delta_q^h(v \partial_z^2 u)_\phi\|_{L^2}^2 + \|e^{Rt}\Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi\|_{L^2}^2 + \|e^{Rt}\Delta_q^h(\partial_z^2 u \partial_x u)_\phi\|_{L^2}^2 \right). \end{aligned}$$

Integrating with respect to time yields

$$(2.16) \quad \begin{aligned} & \|e^{Rt}\Delta_q^h(\partial_t u)_\phi\|_{L_T^2(L^2)}^2 + \|e^{Rt}\Delta_q^h(\partial_t \partial_z u)_\phi\|_{L_T^2(L^2)}^2 + \|e^{Rt}\Delta_q^h \partial_z u_\phi\|_{L_T^\infty(L^2)}^2 + \|e^{Rt}\Delta_q^h \partial_z^2 u_\phi\|_{L_T^\infty(L^2)}^2 \\ & \leq C \left( \|e^{a|D_x|} \Delta_q^h \partial_z u_0\|_{L^2}^2 + \|e^{a|D_x|} \Delta_q^h \partial_z^2 u_0\|_{L^2}^2 + \|e^{Rt}\Delta_q^h(u \partial_x u)_\phi\|_{L_T^2(L^2)} \right. \\ & \quad \left. + \|e^{Rt}\Delta_q^h(u \partial_x \partial_z^2 u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt}\Delta_q^h(v \partial_z u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt}\Delta_q^h(v \partial_z^2 u)_\phi\|_{L_T^2(L^2)} \right. \\ & \quad \left. + \|e^{Rt}\Delta_q^h(\partial_z u \partial_x \partial_z u)_\phi\|_{L_T^2(L^2)} + \|e^{Rt}\Delta_q^h(\partial_z^2 u \partial_x u)_\phi\|_{L_T^2(L^2)} \right). \end{aligned}$$

To be able to obtain Estimate (1.11), we need the following lemma, the proof of which will be given in the appendix 4.

**Lemma 2.2.** *We have the following estimates for the nonlinear terms :*

$$(2.17) \quad \|e^{Rt}(u \partial_x u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

$$(2.18) \quad \|e^{Rt}(u \partial_x \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})},$$

$$(2.19) \quad \|e^{Rt}(v \partial_z u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

$$(2.20) \quad \|e^{Rt}(v \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})},$$

$$(2.21) \quad \|e^{Rt}(\partial_z u \partial_x \partial_z u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})},$$

$$(2.22) \quad \|e^{Rt} \Delta_q^h (\partial_x u \partial_z^2 u)_\phi\|_{\tilde{L}_T^2(B^{\frac{3}{2}})} \lesssim \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}.$$

*Conclusion :* Multiplying (2.16) by  $2^{\frac{3q}{2}}$ , summing up with respect to  $q \in \mathbb{Z}$ , and then using Lemma 2.2, assumption on initial data (1.9) and Estimate (2.11), we obtain Estimate (1.11) in Theorem 1.1.

**2.3. Uniqueness of the solution.** We suppose that  $u_1, u_2$  are two solutions of (1.5) on  $[0, T]$  with the same initial data. Let  $U = u_1 - u_2$ ,  $V = v_1 - v_2$ ,  $P = p_1 - p_2$  then,  $(U, V, P)$  satisfies the following equations

$$(2.23) \quad \begin{cases} \partial_t(U - \alpha_1^2 \partial_z^2 U) + u_1 \partial_x(U - \alpha_1^2 \partial_z^2 U) + U \partial_x \omega_2 + v_1 \partial_z(U - \alpha_1^2 \partial_z^2 U) + V \partial_z \omega_2 + B \\ \quad \quad \quad = \partial_z^2(U - \alpha_1^2 \partial_z^2 U) - \partial_x P \\ \partial_z P = 0 \\ \partial_x U + \partial_z V = 0 \\ (U, V) = 0, \quad \partial_z U = 0 \quad \text{on } z = 0 \\ (U, V) = 0, \quad \partial_z U = 0 \quad \text{on } z = 1 \\ U|_{t=0} = 0, \end{cases}$$

where we set  $\omega_2 = u_2 - \alpha_1^2 \partial_z^2 u_2$  and where

$$B = \alpha_1^2 \partial_z u_1 \partial_x \partial_z U + \alpha_1^2 \partial_z U \partial_x \partial_z u_2 - \alpha_1^2 \partial_z^2 u_1 \partial_x U - \alpha_1^2 \partial_z^2 U \partial_x u_2.$$

We now consider the following auxiliary functions

$$\Phi(t, \xi) = (a - \mu \Theta(t)) |\xi| \quad \text{with} \quad \Theta'(t) = \|\partial_z^2 u_{1\phi}(t)\|_{B^{\frac{1}{2}}} + \|\partial_z^2 u_{2\phi}(t)\|_{B^{\frac{1}{2}}}$$

and following (1.7) we define

$$f_\Phi(t, x, y) = e^{\Phi(t, D_x)} f(t, x, y) := \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{\Phi(t, \xi)} \hat{f}(t, \xi, y)),$$

where  $\mu \geq \lambda$  will be determined later. From Theorem 1.1, we deduce that

$$\|u_{1\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|u_{2\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|\partial_z u_{1\phi}\|_{L^\infty(B^{\frac{3}{2}})} + \|\partial_z u_{2\phi}\|_{L^\infty(B^{\frac{3}{2}})} \leq M,$$

where  $M \geq 1$  is a constant. Following the proof of (2.9), we get from (2.23) the following estimate

$$(2.24) \quad \begin{aligned} & \|e^{Rt} \Delta_q^h U_\Phi\|_{L^2}^2 + \alpha_1^2 \|e^{Rt} \Delta_q^h \partial_z U_\Phi\|_{L^2}^2 + 2\eta \alpha_1^2 \int_0^t \Theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z U_\Phi(t') \right\|_{L^2}^2 dt' \\ & + 2\mu \int_0^t \Theta'(t') \left\| e^{Rt'} |D_x|^{\frac{1}{2}} \Delta_q^h U_\Phi(t') \right\|_{L^2}^2 dt' + 2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z U_\Phi(t') \right\|_{L^2}^2 dt' \\ & \quad + 2\alpha_1^2 \int_0^t \left\| e^{Rt'} \Delta_q^h \partial_z^2 U_\Phi(t') \right\|_{L^2}^2 dt' \\ & = 2I_{1,q} + 2I_{2,q} + 2I_{3,q} + 2I_{4,q} + 2I_{5,q} + 2I_{6,q} + 2I_{7,q} + 2I_{8,q}, \end{aligned}$$

where

$$\begin{aligned}
I_{1,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u_1 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{2,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (u_1 \partial_x (U - \alpha_1^2 \partial_z^2 U))_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{3,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 u_1 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{4,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (V \partial_z \omega_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{5,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z (U - \alpha_1^2 \partial_z^2 U))_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{6,q} &= \alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z^2 U \partial_x u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{7,q} &= -\alpha_1^2 \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z U \partial_x \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt', \\
I_{8,q} &= -\int_0^t \left\langle e^{Rt'} \Delta_q^h (U \partial_x \omega_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'.
\end{aligned}$$

The terms  $I_{j,q}$ ,  $j \in \{1, \dots, 8\}$  can be controlled as in the following lemma. The proof of this lemma is very close to the proof of Lemma 2.1. We will give a brief version of this proof in the appendix 5.

**Lemma 2.3.** *Let  $s \in ]0, 1[$ ,  $T > 0$ ,  $R > 0$ . There exists a generic constant  $C \geq 1$  such that, for any  $0 < t < T$ , we have*

$$(2.25) \quad |I_{1,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.26) \quad |I_{2,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.27) \quad |I_{3,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.28) \quad |I_{4,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.29) \quad |I_{5,q}| \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.30) \quad |I_{6,q}| \leq C 2^{-2qs} d_q^2 \|u_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

$$(2.31) \quad |I_{7,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})},$$

$$\times \left( \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right),$$

$$(2.32) \quad |I_{8,q}| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}$$

$$\times \left( \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_{2\Phi}\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right).$$

Now, using Lemma 2.3 and choosing  $\mu = C^2 M^2$ , we deduce from (2.24) that, for any  $s \in ]0, 1[$

$$\begin{aligned}
&\|e^{Rt} U_\Phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z U_\Phi\|_{\tilde{L}_T^\infty(B^s)} + \|e^{Rt} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^s)} + \|e^{Rt} \partial_z^2 U_\Phi\|_{\tilde{L}_T^2(B^s)} \\
&\leq C(\|e^{a|D_x|} U_0\|_{B^s} + \|e^{a|D_x|} \partial_z U_0\|_{B^s}) = 0,
\end{aligned}$$

which implies the uniqueness of the solution.

**2.4. Construction of approximate solutions.** Before introducing the construction methods, we remark that if  $(u, v, p)$  is a solution of (1.5) such that all the following expressions make sense, then we have :

**Lemma 2.4.**

$$(2.33) \quad \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2) + \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 = 0.$$

We also remark that Estimate (2.33) is also true for the approximate solutions.

*Proof.* We set  $w = u - \alpha_1^2 \partial_z^2 u$ . Taking  $L^2$  scalar product of (1.5) with  $u$ , we have

$$\langle \partial_t \omega, u \rangle + \langle u \partial_x \omega, u \rangle + \langle v \partial_z \omega, u \rangle + \alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle - \alpha_1^2 \langle \partial_z^2 u \partial_x u, u \rangle = \langle \partial_z^2 \omega, u \rangle - \langle \partial_x p, u \rangle.$$

We first consider the linear terms. Performing integration by parts in  $z$ -variable and remarking that

$$\langle \partial_x p, u \rangle = -\langle p, \partial_x u \rangle = \langle p, \partial_z v \rangle = -\langle \partial_z p, v \rangle = 0.$$

We obtain

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \alpha_1^2 \|\partial_z u\|_{L^2}^2) + \|\partial_z u\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u\|_{L^2}^2 + \text{NL} = 0,$$

where the non-linear term NL is

$$\text{NL} = \langle u \partial_x \omega, u \rangle + \langle v \partial_z \omega, u \rangle + \alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle - \alpha_1^2 \langle \partial_z^2 u \partial_x u, u \rangle.$$

Performing integration by parts in  $x$  or  $z$  variable and using the incompressibility condition, we can write

$$\langle u \partial_x \omega, u \rangle = -\frac{1}{2} \int_{\mathbb{R} \times ]0,1[} (\partial_x u) u^2 + 2\alpha_1^2 \int_{\mathbb{R} \times ]0,1[} (\partial_z^2 u) (\partial_x u) u,$$

$$\begin{aligned} \langle v \partial_z \omega, u \rangle &= \frac{1}{2} \int_{\mathbb{R} \times ]0,1[} v \partial_z (u^2) - \alpha_1^2 \int_{\mathbb{R} \times ]0,1[} \partial_z (\partial_z^2 u) (uv) \\ &= -\frac{1}{2} \int_{\mathbb{R} \times ]0,1[} (\partial_z v) u^2 - \frac{1}{2} \alpha_1^2 \int_{\mathbb{R} \times ]0,1[} (\partial_z u)^2 (\partial_z v) - \alpha_1^2 \int_{\mathbb{R} \times ]0,1[} (\partial_z^2 u) (\partial_x u) u \end{aligned}$$

and

$$\alpha_1^2 \langle \partial_z u \partial_x \partial_z u, u \rangle = -\frac{1}{2} \alpha_1^2 \int_{\mathbb{R} \times ]0,1[} (\partial_z u)^2 \partial_x u.$$

Putting these identities into NL, we get

$$\text{NL} = 0$$

and this concludes the proof of the lemma.  $\square$

We now introduce the following Hilbert spaces

$$\begin{aligned} H_0^2 &= \{f \in H^2(]0,1[) \mid f(0) = f(1) = f'(0) = f'(1) = 0\}, \\ H_0^1 &= \{f \in H^1(]0,1[) \mid f(0) = f(1) = 0\} \end{aligned}$$

equipped with their respective norms

$$\begin{aligned} \|f\|_{H_0^2} &= \left( \alpha_1^2 \int_0^1 (f''(z))^2 dz + \int_0^1 (f'(z))^2 dz \right)^{\frac{1}{2}}, \\ \|f\|_{H_0^1} &= \left( \alpha_1^2 \int_0^1 (f'(z))^2 dz + \int_0^1 (f(z))^2 dz \right)^{\frac{1}{2}} \end{aligned}$$

and let be a common Hilbert basis  $\{\tilde{e}_k\}_{k \geq 1}$  such that

$$\forall v \in H_0^2, \quad \langle \tilde{e}_k, v \rangle_{H_0^2} = \lambda_k \langle \tilde{e}_k, v \rangle_{H_0^1}.$$

For any  $(u, v, p)$  sufficiently smooth on  $[0, T] \times \mathcal{S}$ ,  $T > 0$ , such that

$$\partial_x p = \alpha_1^2 \partial_z^3 u|_{z=0} - \alpha_1^2 \partial_z^3 u|_{z=1} - \partial_x \int_0^1 (u)^2(t, x, y) dz - \alpha_1^2 \partial_x \int_0^1 (\partial_z u)^2(t, x, y) dz,$$

$$v(t, x, z) = - \int_0^z \partial_x u(t, x, \tilde{z}) d\tilde{z}.$$

We set

$$(2.34) \quad R(u) = -(u \partial_x u) + \alpha_1^2 (u \partial_x \partial_z^2 u) - (v \partial_z u) + \alpha_1^2 (v \partial_z \partial_z^2 u) - \alpha_1^2 (\partial_z u \partial_x \partial_z u) + \alpha_1^2 (\partial_z^2 u \partial_x u) - \partial_x p.$$

For any  $n \in \mathbb{N}$ , we will look for an approximate solution  $u^n$  of (1.5) of the form

$$u^n(t, x, z) = \sum_{i=1}^n \tilde{u}_{in}(t, x) \tilde{e}_i(z),$$

where  $\{\tilde{u}_{in}\}$  is solution of the following system of  $n$  equations

$$\begin{cases} \langle \partial_t (u^n - \alpha_1^2 \partial_z^2 u^n), \tilde{e}_k \rangle_{L_z^2} = \langle \partial_z^2 (u^n - \alpha_1^2 \partial_z^2 u^n), \tilde{e}_k \rangle_{L_z^2} + \langle R(u^n), \tilde{e}_k \rangle_{L_z^2}, & k = 1, \dots, n \\ u_n|_{t=0} = \sum_{k=1}^n \langle u_0, \tilde{e}_k \rangle_{H_0^1} \tilde{e}_k, \end{cases}$$

which also means that, for  $1 \leq k \leq n$ ,

$$\partial_t \tilde{u}_{kn}(t, x) = -\lambda_k \tilde{u}_{kn}(t, x) + \langle R(u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L_z^2}.$$

Next, we define the following frequency cut-off operators  $J_n$  in  $x$ -variable. For  $f \in L^2(\mathbb{R})$ , we set

$$J_n f(x) = \mathcal{F}_h^{-1} (\mathbb{1}_{[-n, n]} \mathcal{F}_h f)(x).$$

The operator  $J_n$  is continuous from  $L^2$  to  $L^2$ , satisfies  $J_n^2 f = J_n f$  and for any positive integers  $n$  and  $p$ , we have

$$\begin{aligned} |\partial_x^p J_n f|_{L^2(\mathbb{R})} &\leq n^p |f|_{L^2(\mathbb{R})}, \\ |\partial_x^p J_n f|_{L^\infty(\mathbb{R})} &\leq c_{np} |f|_{L^2(\mathbb{R})}. \end{aligned}$$

Now, we consider the following approximate system

$$(2.35) \quad \begin{cases} \partial_t \tilde{u}_{kn}(t, x) = -\lambda_k \tilde{u}_{kn}(t, x) + \langle J_n R(J_n u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L_z^2}, & k \in \{1, \dots, n\} \\ \tilde{u}_{kn}|_{t=0} = \langle J_n u_0, \tilde{e}_k \rangle_{H_0^1}. \end{cases}$$

Since  $\tilde{u}_{kn} \mapsto \langle J_n R(J_n u^n)(t, x, \cdot), \tilde{e}_k \rangle_{L_z^2}$  is locally Lipschitz in  $L^2$ , (2.35) is a system of ordinary differential equations in  $L^2$ . Then, Picard's theorem implies the existence of a unique maximal solution  $\{\tilde{u}_{kn}(t, x)\}$  of (2.35) on  $[0, T_n[$ . Since  $J_n^2 = J_n$ , we deduce that  $\{J_n \tilde{u}_{kn}\}$  is also a solution of (2.35) on  $[0, T_n[$ . Thus, the uniqueness of the solution implies that  $\tilde{u}_{kn} = J_n \tilde{u}_{kn}$ , for any  $k \in \{1, \dots, n\}$ . We remark that  $\{\tilde{e}_k\}$  is not necessarily orthogonal in  $L_z^2(]0, 1[)$  but using the Gram-Schmidt process, we can always construct a orthogonal family  $\{e_k\}$  of  $L_z^2(]0, 1[)$  such that

$$\text{Vect}\{e_1, \dots, e_n\} = \text{Vect}\{\tilde{e}_1, \dots, \tilde{e}_n\}.$$

We denote  $\mathbb{P}_n$  the orthogonal projection in  $L_z^2(]0, 1[)$  onto  $\text{Vect}\{e_1, \dots, e_n\}$

$$\mathbb{P}_n f = \sum_{k=1}^n \langle f, e_k \rangle_{L_z^2} e_k.$$

Then, we remark that  $u^n$  is also the solution of the system

$$(2.36) \quad \begin{cases} \partial_t (u^n - \alpha_1^2 \partial_z^2 u^n) = \partial_z^2 (u^n - \alpha_1^2 \partial_z^2 u^n) + \mathbb{P}_n R(u^n) \\ u_n|_{t=0} = \mathbb{P}_n u_0. \end{cases}$$

Same calculations as in the proof of Lemma 2.4 and the fact that  $\mathbb{P}_n$  the orthogonal projection in  $L_z^2(]0, 1[)$  onto  $\text{Vect}\{e_1, \dots, e_n\}$  imply that

$$\frac{1}{2} \frac{d}{dt} (\|u^n\|_{L^2}^2 + \alpha_1^2 \|\partial_z u^n\|_{L^2}^2) + \|\partial_z u^n\|_{L^2}^2 + \alpha_1^2 \|\partial_z^2 u^n\|_{L^2}^2 = 0,$$

for any  $t \in [0, T_n[$ , which means  $T_n = +\infty$ .

**Remark 2.5.** Estimates (1.10) and (1.11) also apply to the approximate solutions  $u^n$ .

**2.5. Passage to the limit.** Now, we want to take the limit of the sequence of approximate solutions (2.36). We already see that  $\{u^n\}$  is bounded in  $L^\infty(\mathbb{R}_+, L_{loc}^2)$  and due to (1.11), we remark that  $\{\partial_t u^n\}$  is bounded in  $L^2(\mathbb{R}_+, H_{loc}^{-1})$ . Applying Aubin-Lions lemma, we deduce the existence of a subsequence, always noted by  $\{u^n\}$  for the sake of simplicity, such that  $u^n \rightharpoonup u$  in  $L_{loc}^\infty(\mathbb{R}_+, H_{loc}^{-1})$ . By interpolation, we obtain

$$(2.37) \quad u^n \rightharpoonup u \quad \text{in} \quad L_{loc}^\infty(\mathbb{R}_+, H_{loc}^{-\delta}), \quad \text{for all } 0 < \delta < 1.$$

We recall that  $u^n$  is solution of the following system

$$\begin{cases} \langle \partial_t(u^n - \alpha_1^2 \partial_z^2 u^n), e_k \rangle_{L_z^2} = \langle \partial_z^2(u^n - \alpha_1^2 \partial_z^2 u^n), e_k \rangle_{L_z^2} + \langle R(u^n), e_k \rangle_{L_z^2}, & k = 1, \dots, n \\ u_n|_{t=0} = \sum_{k=1}^n \langle u_0, e_k \rangle_{L_z^2} e_k, \end{cases}$$

where  $R$  is defined in (2.34). So the main point of this paragraph is to prove the following lemma on the convergence of the nonlinear term  $R(u^n)$ .

**Lemma 2.6.** *We have the following convergence as  $n \rightarrow \infty$*

$$R(u^n) \rightharpoonup R(u) \quad \text{in} \quad L_{loc}^2(H_{loc}^{-4}).$$

*Proof of Lemma 2.6.* We will only prove the convergence of the term  $u^n \partial_x u^n$ . The other terms of  $R(u^n)$  can be treated in a similar way. We first write

$$u^n \partial_x u^n - u \partial_x u = (u^n - u) \partial_x u^n + u \partial_x (u^n - u).$$

From Estimates (1.10) and (2.37), we have  $\{\partial_x u^n\}$  is uniformly bounded in  $L_{loc}^2(H_{loc}^{\frac{1}{2}})$ . Choosing  $\delta < \frac{1}{2}$  in (2.37) and using the product law in Sobolev spaces on  $\mathbb{R}^2$ , we get

$$\|(u^n - u) \partial_x u^n\|_{L_{loc}^2(H_{loc}^{-\frac{1}{2}-\delta})} \leq \|u^n - u\|_{L_{loc}^\infty(H_{loc}^{-\delta})} \|\partial_x u^n\|_{L_{loc}^2(H_{loc}^{\frac{1}{2}})} \longrightarrow 0.$$

Now, using again Estimate (1.10), we have that  $u$  is bounded in  $L_{loc}^2(H_{loc}^{\frac{1}{2}})$ . Then, (2.37) and the product law in Sobolev spaces on  $\mathbb{R}^2$  yield

$$\|u \partial_x (u^n - u)\|_{L_{loc}^2(H_{loc}^{-\frac{1}{2}-\delta})} \leq \|u\|_{L_{loc}^2(H_{loc}^{\frac{1}{2}})} \|\partial_x (u^n - u)\|_{L_{loc}^\infty(H_{loc}^{-\delta})} \longrightarrow 0.$$

□

Now, coming back to (2.36), Lemma 2.6 proves that the limit  $u$  is solution of (1.5) in the sense of distributions. This concludes the proof of Theorem 1.1.

### 3. APPENDIX - PROOF OF THE LEMMA 2.1

We first introduce some notations and classical mathematical tools. Then we will prove the estimations of Lemma 2.1. We will only detail the proof of estimates (2.1) and (2.5). The same procedure can be followed to prove the other estimates.

We recall the following Bernstein-type lemma, which states that derivatives act almost as multipliers on distributions whose Fourier transforms are supported in a ball or an annulus. We refer the reader to [2] for a proof of this lemma.

**Lemma 3.1.** *Let  $k \in \mathbb{N}$ ,  $d \in \mathbb{N}^*$  and  $r_1, r_2 \in \mathbb{R}$  satisfy  $0 < r_1 < r_2$ . There exists a constant  $C > 0$  such that, for any  $a, b \in \mathbb{R}$ ,  $1 \leq a \leq b \leq +\infty$ , for any  $\lambda > 0$  and for any  $u \in L^a(\mathbb{R}^d)$ , we have*

$$\text{supp}(\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}$$

and

$$\text{supp}(\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda\} \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$



In order to prove the estimates of Lemma 2.1, we need the Bony decomposition of a product of two functions  $a$  and  $b$  in the horizontal direction (see [2])

$$(3.1) \quad ab = T_a^h b + T_b^h a + R^h(a, b),$$

where

$$T_a^h b = \sum_{q \in \mathbb{Z}} S_{q-1}^h a \Delta_q^h b \quad \text{and} \quad R^h(a, b) = \sum_{|q-q'| \leq 1} \Delta_q^h a \Delta_{q'}^h b = \sum_{q \in \mathbb{Z}} \tilde{\Delta}_q^h a \Delta_q^h b$$

and

$$\tilde{\Delta}_q^h f = \sum_{|q-q'| \leq 1} \Delta_{q'}^h f.$$

From the support properties to the Fourier transform of the terms  $\Delta_q^h f$ , we can verify

$$(3.2) \quad \Delta_q^h (S_{q'-1}^h a \Delta_{q'}^h b) = 0 \quad \text{if} \quad |q' - q| \geq 5 \quad \text{and} \quad \Delta_q^h (S_{q'+2}^h a \Delta_{q'}^h b) = 0 \quad \text{if} \quad q' \leq q - 4.$$

Then, for a function  $f$  (we suppose that all the expressions below make sense), we write

$$(3.3) \quad \int_0^T |\langle e^{Rt'} \Delta_q^h (ab)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q,$$

where

$$\begin{aligned} \mathcal{A}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h (T_a^h b)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt', \\ \mathcal{B}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h (T_b^h a)_\phi, e^{Rt'} \Delta_q^h f \rangle| dt', \\ \mathcal{R}_q &= \int_0^T |\langle e^{Rt'} \Delta_q^h (R^h(a, b))_\phi, e^{Rt'} \Delta_q^h f \rangle| dt'. \end{aligned}$$

Then we have from the definition of the operator  $T_a^h$  and (3.2)

$$\begin{aligned} \mathcal{A}_q &\leq \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} |\langle \Delta_q^h (S_{q'-1}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle| dt', \\ \mathcal{B}_q &\leq \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} |\langle \Delta_q^h (S_{q'-1}^h b \Delta_{q'}^h a)_\phi, \Delta_q^h f_\phi \rangle| dt', \\ \mathcal{R}_q &\leq \sum_{q \geq q-3} \int_0^T e^{2Rt'} |\langle \Delta_q^h (\Delta_{q'}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle| dt'. \end{aligned}$$

Throughout this section, we will keep these notations (which can however be modified when needed). Following the remark 1.4, we define For any  $f \in L^2(\mathbb{R})$ , we define

$$f^+ = \mathcal{F}_h^{-1}(|\hat{f}|).$$

We remark that, for any  $q \in \mathbb{Z}$ , we have

$$(3.4) \quad \Delta_q^h f^+ = (\Delta_q^h f)^+, \quad S_q^h f^+ = (S_q^h f)^+ \quad \text{and} \quad \|f^+\|_{L^2} = \|f\|_{L^2}.$$

We will use the following lemma (see [7]):

**Lemma 3.2.** *For smooth functions we have*

$$\begin{aligned} |\mathcal{F}_h((S_{q'-1}^h a \Delta_{q'}^h b)_\phi)(\xi, y)| &\leq |\mathcal{F}_h(S_{q'-1}^h a_\phi^+ \Delta_{q'}^h b_\phi^+)(\xi, y)|, \\ |\mathcal{F}_h((\Delta_{q'}^h a \Delta_{q'}^h b)_\phi)(\xi, y)| &\leq |\mathcal{F}_h(\Delta_{q'}^h a_\phi^+ \Delta_{q'}^h b_\phi^+)(\xi, y)|. \end{aligned}$$

*Proof.* Setting  $\tilde{a} = S_{q'-1}^h a$  and  $\tilde{b} = \Delta_{q'}^h b$ , we have from the inequality  $\phi(t, \xi) \leq \phi(t, \xi - \eta) + \phi(t, \eta)$

$$\begin{aligned} \left| \mathcal{F}_h \left( (\tilde{a} \tilde{b})_\phi \right) (\xi, y) \right| &= \left| e^{\phi(t, \xi)} \int_{\mathbb{R}} \mathcal{F}_h(\tilde{a})(\xi - \eta, y) \mathcal{F}_h(\tilde{b})(\eta, y) d\eta \right| \\ &\leq \int_{\mathbb{R}} \left| e^{\phi(t, \xi - \eta)} \mathcal{F}_h(\tilde{a})(\xi - \eta, y) \right| \left| e^{\phi(t, \eta)} \mathcal{F}_h(\tilde{b})(\eta, y) \right| d\eta \\ &\leq \int_{\mathbb{R}} \mathcal{F}_h(\tilde{a}_\phi^+) (\xi - \eta, y) \mathcal{F}_h(\tilde{b}_\phi^+) (\eta, y) d\eta, \end{aligned}$$

and this concludes the proof of the first inequality, and the same procedure can be followed to deduce the second inequality.  $\square$

Therefore from this lemma, we derive the following corollary from Plancherel formula, Fubini's theorem and the Hölder's inequality.

**Corollary 3.3.** *For smooth functions we have*

$$\|\Delta_q^h f_\phi\|_{L^2} \lesssim \|f_\phi\|_{L^2},$$

$$\begin{aligned} \left| \langle \Delta_q^h (S_{q'-1}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle \right| &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|S_{q'-1}^h a_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L_z^\infty(L_x^2)} \end{aligned}$$

and

$$\begin{aligned} \left| \langle \Delta_q^h (\Delta_{q'}^h a \Delta_{q'}^h b)_\phi, \Delta_q^h f_\phi \rangle \right| &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h b_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h f_\phi\|_{L^2}, \\ &\lesssim \|\Delta_{q'}^h a_\phi^+\|_{L_z^\infty(L_x^2)} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h f_\phi\|_{L_z^2(L_x^\infty)}. \end{aligned}$$

We will also use the following Poincaré inequality : For a function  $u \in H^2$  such that  $u$  and  $u'$  vanish for  $z = 0, 1$  we have

$$(3.5) \quad \|u\|_{L^\infty} \lesssim \|\partial_z u\|_{L^2} \quad \text{and} \quad \|u\|_{L^2} \lesssim \|\partial_z u\|_{L^2} \lesssim \|\partial_z^2 u\|_{L^2}.$$

*Proof of Estimate (2.1).* We apply the Bony's decomposition with  $a = u$ ,  $b = \partial_x u$  and  $f = u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T \left| \langle e^{Rt'} \Delta_q^h (u \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle \right| dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of  $\mathcal{A}_q = \int_0^T \left| \langle e^{Rt'} \Delta_q^h (T_u^h \partial_x u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle \right| dt'$ . We have from (3.2)

$$\begin{aligned} \mathcal{A}_q &= \int_0^T \left| \left\langle e^{Rt'} \Delta_q^h \left( \sum_{q' \in \mathbb{Z}} S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi, e^{Rt'} \Delta_q^h u_\phi \right\rangle \right| dt' \\ &\leq \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \left| \left\langle \Delta_q^h (S_{q'-1}^h u \Delta_{q'}^h \partial_x u)_\phi, \Delta_q^h u_\phi \right\rangle \right| dt'. \end{aligned}$$

From the previous inequality and corollary 3.3 we get

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt'.$$

Since the support of  $\Delta_q^h u_\phi$  is in a ring and using the first inequality of Lemma 3.1 with  $\lambda = 2^q$ , we get

$$\|\Delta_q^h u_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi^+\|_{L_z^\infty(L_x^2)} = 2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)}.$$

Therefore from Poincaré inequality (3.5) and remark 1.4 we get

$$\begin{aligned} \|\Delta_q^h u_\phi^+\|_{L^\infty} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi\|_{L_x^2(L_z^\infty)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2} \\ &\lesssim d_q (\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}. \end{aligned}$$

Then from the definition 1.2 of  $S_{q'-1}$  and using  $\sum_q d_q(\partial_z^2 u_\phi) = 1$ ,

$$(3.6) \quad \|S_{q'-1}^h u_\phi^+\|_{L^\infty} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

We deduce from the previous inequalities, lemma 3.1 and the cauchy-schwarz inequality that

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left(2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2}\right) \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $\theta'(t) = \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}$ , we have from remark 1.6

$$\left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q'(s+1/2)} d_{q'}(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}$$

and we finally obtain

$$(3.7) \quad \mathcal{A}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

where  $\tilde{d}_q$  is the square root summable sequence (since  $\sum_q d_q(u_\phi) = \sum_{q'} d_{q'}(u_\phi, \theta') = 1$ ) defined by

$$(3.8) \quad \tilde{d}_q = d_q(u_\phi) \left( \sum_{|q'-q|\leq 4} d_{q'}(u_\phi, \theta') 2^{(q-q')(s-1/2)} \right).$$

• Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x}^h u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ . From the notations (3.3),

$$\mathcal{B}_q \lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt'.$$

Then, following the proof of Estimate (3.6) and the definition 1.2 of  $S_q^h$ , we have

$$\|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Therefore using the Cauchy-Schwarz inequality and remark 1.4 we derive

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \left(2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}\right) \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} d_q(u_\phi) \left( \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-1/2)} \right) \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

Then from the definition (3.8) of  $\tilde{d}_q$

$$(3.9) \quad \mathcal{B}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

• Estimate of  $\mathcal{R}_q$ . We have from the definition (3.3)

$$\mathcal{R}_q \leq \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \left| \langle \Delta_q^h (\Delta_{q'}^h u \Delta_{q'}^h \partial_x u)_\phi, \Delta_q^h u_\phi \rangle \right| dt'.$$

Following the proof of (3.6) and using corollary 3.3 and lemma 3.1 we get

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h u_\phi^+\|_{L^\infty(L_x^2)} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L_x^2(L_x^\infty)} dt' \\ &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \left( 2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) \left( 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} \right) \left( 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} \right) dt'. \end{aligned}$$

Them using  $d_{q'}(\partial_z^2 u_\phi) \leq 1$ , the Cauchy-Schwarz inequality and remark 1.4 we get

$$\begin{aligned} \mathcal{R}_q &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left( 2^{-q'(s+1/2)} d_{q'}(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \right) \left( 2^{-q(s+1/2)} d_q(u_\phi, \theta') \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})} \right). \end{aligned}$$

That is

$$(3.10) \quad \mathcal{R}_q \lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2,$$

where  $\check{d}_q$  is the square root summable sequence defined by

$$\check{d}_q = d_q(u_\phi, \theta') \sum_{q' \geq q-3} 2^{(q-q')s} d_{q'}(u_\phi, \theta').$$

• By Summing the estimates (3.7), (3.9) and (3.10) we deduce (2.1).  $\square$

*Proof of Estimate (2.2).* We apply the Bony's decomposition with  $a = u$ ,  $b = \partial_x \partial_z u$  and  $f = \partial_z u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

• Estimate of  $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_u^h \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where

$$(3.11) \quad \check{d}_q = d_q(\partial_z u_\phi, \theta') \left( \sum_{|q'-q| \leq 4} d_{q'}(\partial_z u_\phi, \theta') 2^{(q-q')(s-\frac{1}{2})} \right).$$

- Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x \partial_z u}^h u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned}
\mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt \\
&\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

with the same previous definition (3.11) of  $\check{d}_q$ .

- Estimate of  $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(u, \partial_x \partial_z u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned}
\mathcal{R}_q &\lesssim \sum_{q'\geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h u_\phi^+\|_{L_z^\infty(L_x^2)} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L_z^2(L_x^\infty)} dt' \\
&\lesssim \sum_{q'\geq q-3} \int_0^T e^{2Rt'} 2^{-\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q'\geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$(3.12) \quad \check{d}_q = d_q(\partial_z u_\phi, \theta') \sum_{q'\geq q-3} 2^{(q-q')s} d_{q'}(\partial_z u_\phi, \theta').$$

- By Summing the above estimates for  $\mathcal{A}_q$ ,  $\mathcal{B}_q$  and  $\mathcal{R}_q$ , we deduce (2.2).  $\square$

*Proof of Estimate (2.3).* We apply the Bony's decomposition with  $a = \partial_z u$ ,  $b = \partial_x u$  and  $f = \partial_z u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of  $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h \partial_x u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\mathcal{A}_q \lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

Using Lemma 3.1, Poincaré inequality we have

$$\begin{aligned}
\|\Delta_q^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\
&\lesssim d_q \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.
\end{aligned}$$

Then

$$\|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Using Poincaré inequality and Lemma 3.1, we have

$$\begin{aligned}
\|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)} &\lesssim \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \\
&\lesssim 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}.
\end{aligned}$$

Then

$$\begin{aligned}
 \mathcal{A}_q &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
 &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
 &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})},
 \end{aligned}$$

where  $\tilde{d}_q$  is given by (3.11).

- Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x u}^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned}
 \mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt \\
 &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
 &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
 &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}.
 \end{aligned}$$

- Estimate of  $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(\partial_z u, \partial_x u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned}
 \mathcal{R}_q &\lesssim \sum_{q'\geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi^+\|_{L^2} \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_q^h \partial_z u_\phi\|_{L_z^2(L_x^\infty)} dt' \\
 &\lesssim \sum_{q'\geq q-3} \int_0^T 2^{-\frac{q'}{2}} d_{q'}(\partial_z u_\phi) e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
 &\lesssim 2^{\frac{q}{2}} \sum_{q'\geq q-3} \int_0^T 2^{\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\
 &\lesssim 2^{\frac{q}{2}} \sum_{q'\geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\
 &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})},
 \end{aligned}$$

where  $\check{d}_q$  is given by (3.12).

- Summing up the above estimates for  $\mathcal{A}_q$ ,  $\mathcal{B}_q$  and  $\mathcal{R}_q$ , we obtain (2.3).  $\square$

*Proof of Estimate (2.4).* We apply the Bony's decomposition with  $a = \partial_z u$ ,  $b = \partial_x \partial_z u$  and  $f = u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of  $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ .

$$\mathcal{A}_q \lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)}.$$

Using Lemma 3.1 and Poincaré inequality, we have

$$\begin{aligned}
 \|\Delta_q^h \partial_x \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_x \partial_z u_\phi^+\|_{L^2} \\
 &\lesssim d_q(\partial_x \partial_z u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.
 \end{aligned}$$

Then

$$\|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \lesssim \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

By Poincaré inequality we have

$$\|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

Then

$$\begin{aligned} & \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left( 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \right) \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ & \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where  $\tilde{d}_q$  is defined by (3.11).

- Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ .

$$\mathcal{B}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)} dt'.$$

We have

$$\begin{aligned} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} & \lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\phi^+\|_{L^2} \\ & = \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_z u_\phi\|_{L^2} \\ & \lesssim \sum_{l \leq q'-2} 2^l 2^{\frac{l}{2}} \|\Delta_l^h \partial_z^2 u_\phi\|_{L^2} \\ & \lesssim \sum_{l \leq q'-2} 2^l d_l(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \\ & \lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}, \end{aligned}$$

and by the Poincaré inequality, we get

$$\|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}.$$

From the previous estimates,

$$\begin{aligned} \mathcal{B}_q & \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_x \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L_z^\infty(L_x^2)} dt' \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ & \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

Then

$$\mathcal{B}_q \lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

- Estimate of  $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(\partial_z u, \partial_x \partial_z u))_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ .

$$\begin{aligned}
\mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi^+\|_{L^2} \|\Delta_{q'}^h \partial_x \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L^\infty} dt' \\
&\lesssim \sum_{q' \geq q-3} \int_0^T 2^{-\frac{q'}{2}} d_{q'}(\partial_z u_\phi) e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T 2^{\frac{q'}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\dot{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})},
\end{aligned}$$

where  $\check{d}_q$  is given by (3.12).

- By summing the three estimates for  $\mathcal{A}_q$ ,  $\mathcal{B}_q$  and  $\mathcal{R}_q$ , we obtain (2.4).  $\square$

*Proof of Estimate (2.5).* We apply the Bony's decomposition with  $a = v$ ,  $b = \partial_z u$  and  $f = u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (v \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of  $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_v^h \partial_z u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ . From corollary 3.3,

$$\mathcal{A}_q \lesssim \sum_{|q'-q| \leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h v_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L^2} dt'.$$

From the incompressibility condition (see remark 1.1) and the definition of  $f_\phi$  we get

$$\partial_x u_\phi + \partial_z v_\phi = 0.$$

Since  $v_\phi|_{z=0} = 0$ , then

$$v_\phi(t, x, z) = - \int_0^z \partial_x u_\phi(t, x, z') dz' \quad \text{and} \quad \Delta_q^h v_\phi(t, x, z) = - \int_0^z \partial_x \Delta_q^h u_\phi(t, x, z') dz'.$$

Therefore from lemma 3.1 we get

$$\begin{aligned}
\|\Delta_q^h v_\phi\|_{L^\infty(L_x^2)} &\leq \int_0^1 \|\partial_x \Delta_q^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\
&\lesssim \int_0^1 2^q \|\Delta_q^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\
(3.13) \quad &\lesssim 2^q \|\Delta_q^h u_\phi\|_{L^2}.
\end{aligned}$$

Again by Lemma 3.1 and the properties of (3.4) of  $f^+$ , and the previous inequality we have

$$(3.14) \quad \|\Delta_q^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h v_\phi\|_{L^\infty(L_x^2)} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h u_\phi\|_{L^2}.$$

Then from the Poincaré inequality and the remark 1.4,

$$\|\Delta_q^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2} \lesssim 2^q d_q(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$

Then

$$\|S_{q'-1}^h v_\phi^+\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^l d_l(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \lesssim 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}.$$



From the previous estimates and lemma 3.1,

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \left( 2^{q'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where  $\tilde{d}_q$  is given by (3.11).

- Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z}^h u)_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ .

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_z^2(L_x^\infty)} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\Delta_{q'}^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \left( 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} \right) \|\Delta_{q'}^h u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

- Estimate of  $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(v, \partial_z u))_\phi, e^{Rt'} \Delta_q^h u_\phi \rangle | dt'$ .

$$\mathcal{R}_q \lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h u_\phi\|_{L_z^2(L_x^\infty)} dt'.$$

From Estimate (3.13) we derive

$$\begin{aligned} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} &\leq \int_0^1 \|\partial_x \Delta_{q'}^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\ &\lesssim 2^{q'} \int_0^1 \|\Delta_{q'}^h u_\phi(t, \cdot, z')\|_{L_x^2} dz' \\ &\lesssim 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2}. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L^2} 2^{\frac{q}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} 2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where  $\tilde{d}_q$  is given by (3.12).

- By summing up the above three estimates for  $\mathcal{A}_q$ ,  $\mathcal{B}_q$  and  $\mathcal{R}_q$ , we obtain (2.5). □

*Proof of Estimate (2.6).* We apply the Bony's decomposition with  $a = v$ ,  $b = \partial_z^2 u$  and  $f = \partial_z u_\phi$ . Then following the notations of (3.3) we can write

$$\int_0^T | \langle e^{Rt'} \Delta_q^h (v \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt' \leq \mathcal{A}_q + \mathcal{B}_q + \mathcal{R}_q.$$

- Estimate of  $\mathcal{A}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_v^h \partial_z^2 u)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ . By Poincaré inequality we have

$$\|\Delta_q^h u_\phi\|_{L^2} \lesssim \|\Delta_q^h \partial_z u_\phi\|_{L^2}^{\frac{1}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^{\frac{1}{2}}.$$

Then from (3.14) and the previous estimate, and remark 1.4 we get

$$\begin{aligned} \|\Delta_q^h v_\phi^+\|_{L^\infty} &\lesssim 2^{\frac{3q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^{\frac{1}{2}} \|\Delta_q^h \partial_z^2 u_\phi\|_{L^2}^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q}{2}} d_q (\partial_z u_\phi)^{\frac{1}{2}} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} d_q (\partial_z^2 u_\phi)^{\frac{1}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$\|S_{q'-1}^h v_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q'}{2}} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} \mathcal{A}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h v_\phi^+\|_{L^\infty} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \int_0^T e^{Rt'} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2} e^{Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z u_\phi\|_{B^{\frac{3}{2}}} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \left( \int_0^T e^{2Rt'} \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \hat{d}_q \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^s)} \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$\hat{d}_q = d_q(\partial_z^2 u_\phi, 1) \left( \sum_{|q'-q|\leq 4} d_{q'}(\partial_z u_\phi, \theta') 2^{(q-q')(s-\frac{1}{2})} \right).$$

- Estimate of  $\mathcal{B}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ .

$$\begin{aligned} \mathcal{B}_q &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|S_{q'-1}^h \partial_z^2 u_\phi^+\|_{L_x^2(L_x^\infty)} \|\Delta_{q'}^h v_\phi\|_{L_x^\infty(L_x^2)} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^{2q'} \left( 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \right) \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \tilde{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T,\theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where  $\tilde{d}_q$  is given by (3.11).

- Estimate of  $\mathcal{R}_q = \int_0^T | \langle e^{Rt'} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi, e^{Rt'} \Delta_q^h \partial_z u_\phi \rangle | dt'$ . From corollary 3.3, we have

$$\mathcal{R}_q \lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} \|\Delta_{q'}^h v_\phi\|_{L_x^\infty(L_x^2)} \|\tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L^2} \|\Delta_q^h \partial_z u_\phi\|_{L_x^2(L_x^\infty)} dt'.$$

From (3.13), we have

$$\|\Delta_{q'}^h v_\phi\|_{L^\infty(L_x^2)} \lesssim 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}.$$

Then, using remark 1.4 and lemma 3.1, remark 1.6, we have for any  $s > 0$

$$\begin{aligned} \mathcal{R}_q &\lesssim \sum_{q' \geq q-3} \int_0^T e^{2Rt'} 2^{q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \left( 2^{-\frac{q'}{2}} d_{q'}(\partial_z^2 u_\phi) \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \right) 2^{\frac{q}{2}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \int_0^T e^{2Rt'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^T e^{2Rt'} \|\partial_z^2 u_\phi\|_{B^{\frac{1}{2}}} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} \check{d}_q \|e^{Rt'} \partial_z u_\phi\|_{\tilde{L}_{T, \theta'(t)}^2(B^{s+\frac{1}{2}})}^2, \end{aligned}$$

where  $\check{d}$  is given by (3.12).

- Summing up the above three estimates, we obtain (2.6). □

#### 4. APPENDIX - PROOF OF THE LEMMA 2.2

*Proof of Estimate (2.17).* We first recall the Bony's decomposition (3.1)

$$u \partial_x u = T_u^h \partial_x u + T_{\partial_x u}^h u + R^h(u, \partial_x u).$$

Then

$$\begin{aligned} &\left( \int_0^T \|e^{Rt} \Delta_q^h (u \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

- Estimate of  $\|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}$ . From the definition of  $T_h$  we have

$$\|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2} = \|e^{Rt} \Delta_q^h \left( \sum_{q' \in \mathbb{Z}} S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi\|_{L^2} \lesssim \sum_{|q-q'| \leq 4} \left\| e^{Rt} \left( S_{q'-1}^h u \Delta_{q'}^h \partial_x u \right)_\phi \right\|_{L^2}.$$

From corollary 3.3, Poincarre inequality and lemma 3.1 we get

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt &\lesssim \sum_{|q-q'| \leq 4} \int_0^T \|S_{q'-1}^h u_\phi^+\|_{L^\infty}^2 \|e^{Rt} \Delta_{q'}^h \partial_x u_\phi\|_{L^2}^2 dt \\ &\lesssim \sum_{|q-q'| \leq 4} \int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 \left( 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right) dt \\ &\lesssim \sum_{|q-q'| \leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left( 2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right). \end{aligned}$$

Then from the remark 1.6

$$\begin{aligned} &\left( \int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \sum_{|q-q'| \leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left( \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ (4.1) \quad &\lesssim \sum_{|q-q'| \leq 4} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left( 2^{-\frac{5q'}{2}} d_{q'}(\partial_z u_\phi, 1) \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right) \\ &\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \end{aligned}$$

where

$$(4.2) \quad \tilde{d}_q = \sum_{|q-q'|\leq 4} d_{q'} (\partial_z u_\phi, 1) 2^{\frac{3}{2}(q-q')}.$$

- Estimate of  $\|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}$ . Following the previous estimates we get

$$(4.3) \quad \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}.$$

- Estimate of  $\|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}$ . From corollary 3.3, we have

$$\begin{aligned} \left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left( \int_0^T \|\tilde{\Delta}_{q'}^h u_\phi\|_{L_x^2(L_x^\infty)}^2 \|e^{Rt} \Delta_{q'}^h \partial_x u_\phi\|_{L_x^\infty(L_x^2)}^2 dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \left( \int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 \left( 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 \right) dt \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \|\Delta_{q'}^h \partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})} 2^{q'} \left( \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Then from remark 1.6

$$(4.4) \quad \left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

where

$$(4.5) \quad \check{d}_q = \sum_{q' \geq q-3} d_{q'} (\partial_z u_\phi, 1) 2^{\frac{3}{2}(q-q')}.$$

- By summing estimates (4.1), (4.3) and (4.4) we get

$$\left( \int_0^T \|e^{Rt} \Delta_q^h (u \partial_x u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim (2\tilde{d}_q + \check{d}_q) 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}.$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$  we obtain (2.17).  $\square$

*Proof of Estimate (2.18).* We have

$$\begin{aligned} \left( \int_0^T \|e^{Rt} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \left( \int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z^2 u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

By adapting the proof of Estimate (4.1) we get

$$\begin{aligned} \left( \int_0^T \|e^{Rt} \Delta_q^h (T_u^h \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \\ \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z^2 u}^h u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}, \end{aligned}$$

where  $\tilde{d}_q$  is given by (4.2). By adapting the proof of Estimate (4.4) we get

$$\left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(u, \partial_x \partial_z^2 u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

where  $\tilde{d}_q$  is given by (4.5). Thus by summing the previous estimates we derive

$$\begin{aligned} & \left( \int_0^T \|e^{Rt} \Delta_q^h (u \partial_x \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-\frac{3q}{2}} \left( (\tilde{d}_q + \check{d}_q) \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} + \tilde{d}_q \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$ , we obtain (2.18).  $\square$

*Proof of Estimate (2.19).* We have by Bony's decomposition in  $x$  variable

$$\begin{aligned} \left( \int_0^T \|e^{Rt} \Delta_q^h (v \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} & \lesssim \left( \int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \quad + \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

- Estimate of  $\left( \int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}$ .

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt & \leq \sum_{|q-q'|\leq 4} \int_0^T \|S_{q'-1}^h v_\phi^+\|_{L^\infty}^2 \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left( 2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right). \end{aligned}$$

Similar to (4.1) we get

$$(4.6) \quad \left( \int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

where  $\tilde{d}_q$  is given by (4.2).

- Estimate of  $\left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}}$ .

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|S_{q'-1}^h \partial_z u_\phi^+\|_{L_x^\infty(L_x^\infty)}^2 \|e^{Rt} \Delta_{q'}^h v_\phi\|_{L_x^\infty(L_x^\infty)}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \int_0^T \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \\ & \lesssim \sum_{|q-q'|\leq 4} \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2 \left( 2^{2q'} \int_0^T \|e^{Rt} \Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt \right). \end{aligned}$$

Similar to (4.1) we get

$$(4.7) \quad \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h v)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})},$$

where  $\tilde{d}_q$  is given by (4.2).

- Estimate of  $\|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}$ .

$$\begin{aligned} \int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt &\lesssim \sum_{q' \geq q-3} \int_0^T \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)}^2 \|e^{Rt} \tilde{\Delta}_{q'}^h \partial_z u_\phi\|_{L_z^2(L_x^\infty)}^2 dt \\ &\lesssim \sum_{q' \geq q-3} \int_0^T \left(2^{2q'} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2\right) \|\partial_z u_\phi\|_{B^{\frac{1}{2}}}^2 dt \\ &\lesssim \sum_{q' \geq q-3} 2^{2q'} \left(\int_0^T \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2}^2 dt\right) \|\partial_z u_\phi\|_{L_T^\infty(B^{\frac{1}{2}})}^2. \end{aligned}$$

Similar to (4.1) we get

$$(4.8) \quad \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z u))_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})},$$

where  $\check{d}_q$  is given by (4.5).

- By summing the previous estimates (4.6), (4.7) and (4.8) we get

$$\left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z u)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \lesssim (2\check{d}_q + \check{d}_q) 2^{-\frac{3q}{2}} \left(\|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}\right).$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$ , we obtain (2.19).  $\square$

*Proof of Estimate (2.20).* We have by Bony's decomposition in  $x$  variable

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (v \partial_z^2 u)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim \left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z^2 u)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\quad + \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

By adapting the proof of Estimate (4.6) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_v^h \partial_z^2 u)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim \sum_{|q'-q| \leq 4} \left(\int_0^T \|S_{q'-1}^h v_\phi^+\|_{L^\infty}^2 \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (4.7) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h v)_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim \sum_{|q-q'| \leq 4} \left(\|S_{q'-1}^h e^{Rt} \partial_z^2 u_\phi^+\|_{L_z^2(L_x^\infty)}^2 \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)}^2\right)^{\frac{1}{2}} \\ &\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt} \partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (4.8) we obtain

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(v, \partial_z^2 u))_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^T \|\Delta_{q'}^h v_\phi\|_{L_z^\infty(L_x^2)}^2 2^{\frac{q'}{2}} \|e^{Rt} \tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \end{aligned}$$

Then by summing the previous estimates we get

$$\begin{aligned} & \left( \int_0^T \|e^{Rt} \Delta_q^h (v \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-\frac{3q}{2}} \left( (\tilde{d}_q + \check{d}_q) \left( \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \right) + \tilde{d}_q \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$ , we obtain (2.20).  $\square$

*Proof of Estimate (2.21).* We have by Bony's decomposition in  $x$  variable

$$\begin{aligned} & \|e^{Rt} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi\|_{L^2} \\ & \leq \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (R^h (\partial_z u, \partial_x \partial_z u))_\phi\|_{L^2}. \end{aligned}$$

By adapting the proof of Estimate (4.1) we get

$$\left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z u}^h \partial_x \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}$$

and

$$\left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x \partial_z u}^h \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}.$$

By adapting the proof of Estimate (4.4) we get

$$\left( \int_0^T \|e^{Rt} \Delta_q^h (R^h (\partial_z u, \partial_x \partial_z u))_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}.$$

Then by summing the previous estimates we get

$$\begin{aligned} & \left( \int_0^T \|e^{Rt} \Delta_q^h (\partial_z u \partial_x \partial_z u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim 2^{-\frac{3q}{2}} \left( 2\tilde{d}_q \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} + \check{d}_q \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})} \right). \end{aligned}$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$ , we obtain (2.21).  $\square$

*Proof of Estimate (2.22).* We have by Bony's decomposition in  $x$  variable

$$\|e^{Rt} \Delta_q^h (\partial_x u \partial_z^2 u)_\phi\|_{L^2} \leq \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h \partial_z^2 u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} + \|e^{Rt} \Delta_q^h (R^h (\partial_x u, \partial_z^2 u))_\phi\|_{L^2}.$$

By adapting the proof of Estimate (4.1) we get

$$\begin{aligned} \left( \int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_x u}^h \partial_z^2 u)_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} & \lesssim \sum_{|q'-q| \leq 4} \left( \int_0^T \|S_{q'-1}^h \partial_x u_\phi^+\|_{L^\infty}^2 \|\Delta_{q'}^h \partial_z^2 u_\phi\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ & \lesssim \tilde{d}_{q,2} 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}, \end{aligned}$$

where  $\tilde{d}_{q,2} = \sum_{|q-q'| \leq 4} d_{q'} (\partial_z^2 u_\phi, 1) 2^{\frac{3}{2}(q-q')}$ .

- Estimate of  $\left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} dt\right)^{\frac{1}{2}}$ .

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (T_{\partial_z^2 u}^h \partial_x u)_\phi\|_{L^2} dt\right)^{\frac{1}{2}} &\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^T \|S_{q'-1}^h e^{Rt} \partial_z^2 u_\phi^+\|_{L_z^2(L_x^\infty)}^2 \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^T \|e^{Rt} \partial_z^2 u_\phi\|_{B^{\frac{1}{2}}}^2 2^{2q'} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \tilde{d}_q 2^{-\frac{3q}{2}} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|\partial_x u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})}. \end{aligned}$$

By adapting the proof of Estimate (4.4) we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (R^h(\partial_x u, \partial_z^2 u))_\phi\|_{L^2} dt\right)^{\frac{1}{2}} &\lesssim \sum_{q'\geq q-3} \left(\int_0^T \|\Delta_{q'}^h \partial_x u_\phi\|_{L_z^\infty(L_x^2)}^2 \|e^{Rt} \tilde{\Delta}_{q'}^h \partial_z^2 u_\phi\|_{L_z^2(L_x^\infty)}^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \check{d}_q 2^{-\frac{3q}{2}} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})}. \end{aligned}$$

Then by summing up the previous estimates, we get

$$\begin{aligned} \left(\int_0^T \|e^{Rt} \Delta_q^h (\partial_x u \partial_z^2 u)_\phi\|_{L^2} dt\right)^{\frac{1}{2}} &\lesssim 2^{-\frac{3q}{2}} \left(\tilde{d}_{q,2} \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{5}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} + (\tilde{d}_q + \check{d}_q) \|\partial_z u_\phi\|_{\tilde{L}_T^\infty(B^{\frac{1}{2}})} \|e^{Rt} \partial_z^2 u_\phi\|_{\tilde{L}_T^2(B^{\frac{5}{2}})}\right). \end{aligned}$$

Multiplying the previous inequality by  $2^{\frac{3q}{2}}$  and taking the sum over  $\mathbb{Z}$ , we obtain (2.22).  $\square$

## 5. APPENDIX - PROOF OF THE LEMMA 2.3

In this Appendix, we give a brief proof of estimates used to prove the uniqueness of the solution.

- Estimate of  $I_{1,q}$ . From Lemma 3.1 in [21] we obtain (2.25).
- Estimate of  $I_{2,q}$ . From Lemma 3.1 in [21] we have

$$\left|\int_0^t \left\langle e^{Rt'} \Delta_q^h (u_1 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| \leq C 2^{-2qs} d_q^2 \|e^{Rt'} U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

By using integration by parts, and from Lemma 3.1 in [21] we have

$$\begin{aligned} &\left|\int_0^t \left\langle e^{Rt'} \Delta_q^h (u_1 \partial_x \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| \\ &\leq \left|\int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_z u_1 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt'\right| + \left|\int_0^t \left\langle e^{Rt'} \Delta_q^h (u_1 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle dt'\right| \\ &\leq C 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2. \end{aligned}$$

By summing up the above estimates we obtain (2.26).

- Estimate of  $I_{3,q}$ . By integration by parts in  $z$  variable and from Lemma 3.1 in [21], we obtain (2.27).
- Estimate of  $I_{4,q}$ . It follows from the proof of Estimate (5.14) in [21] that

$$(5.1) \quad \int_0^t \left|\left\langle e^{Rt'} \Delta_q^h (V \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle\right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T,\Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$



By integration by parts

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^3 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_x U \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' \\
& \quad + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt'.
\end{aligned}$$

Following the prove of Lemma 3.2 in [21] we deduce for any  $s \in ]0, 1[$  that

$$\int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

From Lemma 3.1 in [21] we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z u_2 \partial_x \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

Then

$$(5.2) \quad \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (V \partial_z^3 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.$$

By summing the estimates (5.1) and (5.2) we obtain (2.28).

• Estimate of  $I_{5,q}$ . Following the proof of Estimate (5.13) in [21], we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

By integration by parts in  $z$  variable, we have

$$\begin{aligned}
& \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z^3 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt' \right| \\
& \leq \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (v_1 \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle dt' \right| + \left| \int_0^t \left\langle e^{Rt'} \Delta_q^h (\partial_x u_1 \partial_z^2 U)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle dt' \right| \\
& \leq C 2^{-2qs} d_q^2 \|u_{1\Phi}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z^2 U_\Phi\|_{\tilde{L}_T^2(B^{\frac{1}{2}})} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})}^2.
\end{aligned}$$

This conclude the proof of (2.29).

• Estimate of  $I_{6,q}$ . Following the proof of Estimate (5.13) in [21], we get (2.30).

• Estimate of  $I_{7,q}$ . Following the proof of Estimate (5.11) in [21], we obtain (2.31).

• Estimate of  $I_{8,q}$ . Following the proof of Estimate (5.11) in [21], we deduce that

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} \left( \|e^{Rt'} U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|u_{2\Theta}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right).
\end{aligned}$$

By integration by parts

$$\begin{aligned}
& \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x \partial_z^2 u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \leq \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (U \partial_x \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h \partial_z U_\Phi \right\rangle \right| dt' + \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h (\partial_z U \partial_x \partial_z u_2)_\Phi, e^{Rt'} \Delta_q^h U_\Phi \right\rangle \right| dt' \\
& \lesssim 2^{-2qs} d_q^2 \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} \\
& \quad \times \left( \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_{T, \Theta'(t)}^2(B^{s+\frac{1}{2}})} + \|\partial_z u_{2\Theta}\|_{L^\infty(B^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{Rt'} \partial_z U_\Phi\|_{\tilde{L}_T^2(B^{s+\frac{1}{2}})} \right).
\end{aligned}$$

Then by summing up the previous estimates, we get (2.32).

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