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ALMOST EVERYWHERE NON-UNIQUENESS OF INTEGRAL CURVES FOR DIVERGENCE-FREE SOBOLEV VECTOR FIELDS

J. PITCHO, M. SORELLA

ABSTRACT. We construct divergence-free Sobolev vector fields in $C([0, 1]; W^{1,r}(\mathbb{T}^d; \mathbb{R}^d))$ with $r < d$ and $d \geq 2$ which simultaneously admit any finite number of distinct positive solutions to the continuity equation. We then show that the vector fields we produce have at least as many integral curves starting from \mathcal{L}^d -a.e. point of \mathbb{T}^d as the number of distinct positive solutions to the continuity equation these vector fields admit. Our work uses convex integration techniques introduced in [4, 20] to study non-uniqueness for positive solutions of the continuity equation. We then infer non-uniqueness for integral curves from Ambrosio's superposition principle.

Keywords: Sobolev vector fields, generalized flows, continuity equation, ODE, integral curves.

MSC (2020): 35A02 - 35D30 - 35Q49 -34A12.

1. INTRODUCTION

In this paper we study positive solutions of the continuity equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\cdot, t) = \rho_0(\cdot) \end{cases} \quad (1.1)$$

where $u : [0, 1] \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ is a prescribed vector field on the d -dimensional torus and $\rho_0 : \mathbb{T}^d \rightarrow \mathbb{R}$ is the initial datum. Throughout this work, (1.1) will be understood in the sense of distributions which only requires that ρ and ρu be integrable. We then study integral curves of the vector field u .

In the smooth setting, the Cauchy-Lipschitz theory guarantees the existence of a unique flow $\mathbf{X} : [0, 1] \times \mathbb{T}^d \rightarrow \mathbb{T}^d$ of the vector field u satisfying

$$\begin{cases} \partial_t \mathbf{X}(t, x) = u(t, \mathbf{X}(t, x)), \\ \mathbf{X}(0, x) = x. \end{cases} \quad (1.2)$$

The classical Liouville theorem then gives a representation of solutions of (1.1) in terms of the flow \mathbf{X} of the vector field u through the formula

$$\rho(t, \cdot)_{\mathcal{L}^d} = \mathbf{X}(t, \cdot)_{\#}(\rho_0_{\mathcal{L}^d}). \quad (1.3)$$

For rough vector fields, the relationship between the continuity equation and the corresponding flow is an active field of research since the foundational work of DiPerna and Lions in [16]. By means of a regularization scheme, they showed that if $u \in L^1((0, 1); W^{1,r}(\mathbb{T}^d))$ and $\operatorname{div} u \in L^1((0, 1) \times \mathbb{T}^d)$, then (1.1) is well-posed in the class $L^\infty((0, 1); L^p(\mathbb{T}^d))$, where $p, r \geq 1$ satisfy the relation

$$\frac{1}{p} + \frac{1}{r} \leq 1.$$

In [2], Ambrosio extended the work of DiPerna and Lions to the setting of BV vector fields.

We now gather some useful definitions to investigate the relation of the ODE (1.2) and the PDE (1.1) in the non-smooth setting.

Definition 1.1. Let $u : (0, 1) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ be a Borel map. We say that $\gamma \in AC([0, 1]; \mathbb{T}^d)$ is an *integral curve* of u starting at x if $\gamma(0) = x$ and $\gamma'(t) = u(t, \gamma(t))$ for a.e. $t \in (0, 1)$.

The *regular Lagrangian flow* is then a suitable selection of integral curves of u by a compressibility condition (introduced in [2, 16]).

Definition 1.2 (Regular Lagrangian flow). Let $u : (0, 1) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ be Borel. We say that a Borel map $\mathbf{X} : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a *regular Lagrangian flow* of u if

- (i) for \mathcal{L}^d -a.e. $x \in \mathbb{T}^d$, $t \mapsto \mathbf{X}(t, x)$ is integral curve of u with $\mathbf{X}(0, x) = x$,
- (ii) there is a constant $C > 0$ such that for every $t \in [0, 1]$, $\mathbf{X}(t, \cdot)_{\#} \mathcal{L}^d \leq C \mathcal{L}^d$.

The well-posedness of the regular Lagrangian flow for vector fields u in $L^1((0, 1); W^{1,1}(\mathbb{T}^d))$ with the negative part of the divergence satisfying $[\operatorname{div} u]^- \in L^1((0, 1); L^\infty(\mathbb{T}^d))$ was first derived from the well-posedness of (1.1) for bounded densities, and for such densities the formula (1.2) holds using as \mathbf{X} the regular Lagrangian flow (see [16] and see [2] for the BV vector fields case). Later in [12], Crippa and De Lellis proved well-posedness of the regular Lagrangian flow without resorting to the PDE (1.1), but their approach only works for vector fields in $L^1((0, 1); W^{1,r}(\mathbb{T}^d))$ with $r > 1$. At any rate, the uniqueness of the regular Lagrangian flow does not imply \mathcal{L}^d -a.e. uniqueness of *integral curves*. Indeed, Bruè, Colombo and De Lellis recently produced divergence-free Sobolev vector fields – uniqueness of the regular Lagrangian flow associated to these vector fields therefore holds – for which almost everywhere uniqueness of integral curves fails (see [4, Theorem 1.3]). We note that the case of continuous vector field is still open (question posed in [1, Section 2.3]) although in [7], Crippa and Caravenna proved almost everywhere uniqueness of the trajectories when $u \in C((0, 1); W^{1,r})$ when $r > d$.

In this work we show that, for divergence-free Sobolev vector fields, uniqueness of integral curves of the ODE (1.2) can fail for a set of initial data with full measure. In fact, we show that the non-uniqueness for integral curves is even worse: for any natural number N we produce divergence-free Sobolev vector fields with at least N integral curves starting almost everywhere. We highlight that our result demonstrates the power of the selection principle of the regular Lagrangian flow ((ii) in Definition 1.2) for integral curves of Sobolev vector fields. Indeed, amongst at least N integral curves starting from \mathcal{L}^d -a.e. point of \mathbb{T}^d , the regular Lagrangian flow selects a *single* integral curve for \mathcal{L}^d -a.e. starting point.

Theorem 1.3. *For every $d, N \in \mathbb{N}$, $d \geq 2$, $r \in [1, d)$ and $s < \infty$ there is a divergence-free vector field $u \in C((0, 1); W^{1,r}(\mathbb{T}^d; \mathbb{R}^d) \cap L^s)$ such that the following holds for every Borel map v with $u = v$ \mathcal{L}^{d+1} -a.e.: (NU) For \mathcal{L}^d -a.e. $x \in \mathbb{T}^d$ there are at least N integral curves of v starting at x .*

Ambrosio’s superposition principle [3, Theorem 3.2] bridges the gap between positive solutions of the continuity equation (1.1) for a vector field u and the integral curves of u (as in Definition 1.1): it gives a way of representing positive solutions of the continuity equation in terms of integral curves of the vector field without any differentiability assumption, i.e. under more general assumptions than DiPerna-Lions theory. Using Ambrosio’s superposition principle, we will derive Theorem 1.3 from a non-uniqueness result for positive solutions of (1.1), which in turn will be proved using a convex integration iterative procedure. The term convex integration is generic to designate iterative techniques by which *wild solutions* of PDEs are constructed. Such techniques were introduced in the study of the continuity equation in the groundbreaking work of Modena and Székelyhidi [20, 21] (see also [5, 6, 8–11, 13–15, 17, 18] for interesting results using the convex integration methods).

Theorem 1.4. *Let $d, N \in \mathbb{N}$ with $d \geq 2$. Let $p \in (1, \infty)$, $r \in [1, \infty]$ be such that*

$$\frac{1}{p} + \frac{1}{r} > 1 + \frac{1}{d},$$

and denote by p' the dual exponent of p , i.e. $1/p + 1/p' = 1$. Then there exists a divergence-free vector field $u \in C^0([0, 1], W^{1,r}(\mathbb{T}^d) \cap L^{p'}(\mathbb{T}^d))$ and a family of nonnegative densities $\{\rho_i\}_{1 \leq i \leq N} \subset C^0([0, 1], L^p(\mathbb{T}^d))$ such that the following holds

- (i) the couple (u, ρ_i) weakly solves (1.1),
- (ii) for each time $t \in [0, 1/3]$, $\rho_i(t, \cdot) \equiv 1$ for any i ,
- (iii) for each time $t \in [2/3, 1]$, $\text{supp}(\rho_i(t, \cdot)) \cap \text{supp}(\rho_j(t, \cdot))$ is negligible for any $i \neq j$. Furthermore $\text{supp}(\rho_i(t, \cdot))$ has non-empty interior for any $t \in [0, 1]$ and any i .

In order to prove Theorem 1.4, we adapt the convex integration scheme for positive solutions of the continuity equation introduced in [4]: our proof makes use of two new ideas. Firstly, we keep track of a fixed number of densities and one *single* vector field throughout the iteration scheme. Each density is perturbed using a distinct family of building blocks¹. Each of these building blocks then interacts *only* with one term of the perturbation to the vector field (see the key identity (4.11)). Secondly, we localize in space the corrector parts of the perturbation to the densities (see (3.15)) which will be negative, in order to preserve the positivity of the solutions. We also note that to prove Theorem 1.4 in dimension $d = 2$ the ideas of [4, Section 7] need to be adapted for technical reasons. This will be explained in Section 6.

2. PRELIMINARY LEMMAS

In this section, we gather some useful lemmas from [4, 20]. We will write \mathbb{T}^d for $\mathbb{R}^d/\mathbb{Z}^d$.

Lemma 2.1. *Let $d, N \in \mathbb{N}^+$. Then, there exist disjoint families, Λ^i , of finite sets $\{\xi\}_{\xi \in \Lambda^i} \subseteq \partial B_1 \cap \mathbb{Q}^d$ for $i = 1, \dots, N$ and smooth nonnegative coefficients $a_\xi(R)$ such that for every $R \in \partial B_1$*

$$R = \sum_{\xi \in \Lambda^i} a_\xi(R) \xi$$

for any $i = 1, \dots, N$.

2.1. Antidivergences. We recall that the operator $\nabla \Delta^{-1}$ is an antidivergence when applied to smooth vector fields of zero mean. The following lemma proven in [20, Lemma 2.3] and [19, Lemma 3.5] gives an improved antidivergence operator for functions with a particular structure.

Lemma 2.2. *(Cp. with [19, Lemma 3.5]) Let $\lambda \in \mathbb{N}$ and $f, g : \mathbb{T}^d \rightarrow \mathbb{R}$ be smooth functions, and $g_\lambda = g(\lambda x)$. Assume that $\int g = 0$. Then if we set $\mathcal{R}(fg_\lambda) = f \nabla \Delta^{-1} g_\lambda - \nabla \Delta^{-1} (\nabla f \cdot \nabla \Delta^{-1} g_\lambda + \int f g_\lambda)$, we have that $\text{div} \mathcal{R}(fg_\lambda) = f g_\lambda - \int f g_\lambda$ and for some $C := C(k, p)$*

$$\|D^k \mathcal{R}(fg_\lambda)\|_{L^p} \leq C \lambda^{k-1} \|f\|_{C^{k+1}} \|g\|_{W^{k,p}} \quad \text{for every } k \in \mathbb{N}, p \in [1, \infty]. \quad (2.1)$$

Proof. It is enough to combine [19, Lemma 3.5] and the remark in [19, page 12]. \square

2.2. Slow and fast variables. Finally we recall the following improved Hölder inequality, stated as in [20, Lemma 2.6] (see also [6, Lemma 3.7]). If $\lambda \in \mathbb{N}$ and $f, g : \mathbb{T}^d \rightarrow \mathbb{R}$ are smooth functions, then we have

$$\|f(x)g(\lambda x)\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^p} + \frac{C(p)\sqrt{d}\|f\|_{C^1}\|g\|_{L^p}}{\lambda^{1/p}} \quad (2.2)$$

and

$$\left| \int f(x)g(\lambda x) dx \right| \leq \left| \int f(x)(g(\lambda x) - \int g) dx \right| + \left| \int f \cdot \int g \right| \leq \frac{\sqrt{d}\|f\|_{C^1}\|g\|_{L^1}}{\lambda} + \left| \int f \cdot \int g \right|. \quad (2.3)$$

¹The term building block refers to smooth functions or vector fields which are fixed before iteration, and which are used to construct perturbations in a convex integration iterative scheme.

2.3. Building blocks. The building blocks are the same as those of [4, Section 4]. We recall them here for the convenience of the reader.

Let $0 < \rho < \frac{1}{4}$ be a constant. We consider $\varphi \in C_c^\infty(B_\rho)$ and $\psi \in C_c^\infty(B_{2\rho})$ which satisfy

$$\int \varphi = 1, \quad \varphi \geq 0, \quad \psi \equiv 1 \text{ on } B_\rho.$$

Given $\mu \ll 1$ we define the 1-periodic functions

$$\begin{aligned} \bar{\varphi}_\mu(x) &:= \sum_{k \in \mathbb{Z}^d} \mu^{d/p} \varphi(\mu(x+k)) \\ \bar{\psi}_\mu(x) &:= \sum_{k \in \mathbb{Z}^d} \mu^{d/p'} \psi(\mu(x+k)). \end{aligned}$$

Let $\omega : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth 1-periodic function such that $\omega(x) = x \cdot \xi'$ on $B_{2\rho}(0)$.

Given Λ as in Lemma 2.1, for any $\xi \in \Lambda$ we chose $\xi' \in \partial B_1$ such that $\xi \cdot \xi' = 0$ and we define

$$\Omega_\xi^\mu(x) := \mu^{-1} \omega(\mu x) (\xi \otimes \xi' - \xi' \otimes \xi).$$

Notice that $\operatorname{div} \Omega_\xi^\mu$ is divergence-free since Ω_ξ^μ is skew-symmetric and $\operatorname{div} \Omega_\xi^\mu = \xi$ on $\operatorname{supp}(\bar{\psi}_\mu)$ and $\operatorname{supp}(\bar{\varphi}_\mu)$.

For $\sigma > 0$ we set

$$\tilde{W}_{\xi, \mu, \sigma}(t, x) := \sigma^{1/p'} \operatorname{div} \left[(\Omega_\xi^\mu \bar{\psi}_\mu)(x - \mu^{d/p'} \sigma^{1/p'} t \xi) \right] \quad (2.4)$$

$$\tilde{\Theta}_{\xi, \mu, \sigma}(t, x) := \sigma^{1/p} \bar{\varphi}_\mu(x - \mu^{d/p'} \sigma^{1/p'} t \xi). \quad (2.5)$$

Notice that $\tilde{W}_{\xi, \mu, \sigma}$ is divergence-free since it is also the divergence of the skew-symmetric matrix $\Omega_\xi^\mu \bar{\psi}_\mu$. By construction we have

$$\tilde{W}_{\xi, \mu, \sigma}(t, x) = \sigma^{1/p'} \left[\bar{\psi}_\mu \xi + \Omega_\xi^\mu \cdot \nabla \bar{\psi}_\mu \right] (x - \mu^{d/p'} \sigma^{1/p'} t \xi),$$

hence the following properties are easily verified. If we consider the translations

$$W_{\xi, \mu, \sigma}(t, x) = \tilde{W}_{\xi, \mu, \sigma}(t, x - v_\xi), \quad \tilde{\Theta}_{\xi, \mu, \sigma}(t, x) = \tilde{\Theta}_{\xi, \mu, \sigma}(t, x - v_\xi),$$

we can prove the following result.

Lemma 2.3. *Let $d \geq 3$, $\Lambda \subset \partial B_1 \cap \mathbb{Q}^d$ be a finite set. Then there exists $\mu_0 > 0$ such that the following holds.*

There exist two families of functions $\{\Theta_{\xi, \mu, \sigma}\}_{\xi, \mu, \sigma} \subset C^\infty(\mathbb{T}^d)$, $\{W_{\xi, \mu, \sigma}\}_{\xi, \mu, \sigma} \subset C^\infty(\mathbb{T}^d; \mathbb{R}^d)$, where $\xi \in \Lambda$, $\sigma, \mu \in \mathbb{R}$ such that for any $\mu \geq \mu_0$, $\sigma > 0$ we have

$$\partial_t \Theta_{\xi, \mu, \sigma} + \operatorname{div}(W_{\xi, \mu, \sigma} \Theta_{\xi, \mu, \sigma}) = 0, \quad (2.6)$$

$$\operatorname{div} W_{\xi, \mu, \sigma} = 0,$$

$$\int W_{\xi, \mu, \sigma} = 0, \quad (2.7)$$

$$\int W_{\xi, \mu, \sigma} \Theta_{\xi, \mu, \sigma} = \sigma \xi. \quad (2.8)$$

For any $k \in \mathbb{N}$ and any $s \in [1, \infty]$ one has

$$\|D^k \Theta_{\xi, \mu, \sigma}\|_{L^s} \leq C(d, k, s) \sigma^{1/p} \mu^{k+d(1/p-1/s)}, \quad \|\partial_t^k \Theta_{\xi, \mu, \sigma}\|_{L^s} \leq C(d, k, s) \sigma^{1+\frac{k-1}{p'}} \mu^{k+d(\frac{k-1}{p'}+1-\frac{1}{s})} \quad (2.9)$$

$$\|D^k W_{\xi, \mu, \sigma}\|_{L^s} \leq C(d, k, s) \sigma^{1/p'} \mu^{k+d(1/p'-1/s)}, \quad \|\partial_t^k W_{\xi, \mu, \sigma}\|_{L^s} \leq C(d, k, s) \sigma^{\frac{k+1}{p'}} \mu^{k+d(\frac{k+1}{p'}-\frac{1}{s})}. \quad (2.10)$$

Finally, they have pairwise compact disjoint supports for any $\xi \neq \xi'$, namely

$$\operatorname{supp} W_{\xi, \mu, \sigma} \cap \operatorname{supp} \Theta_{\xi', \mu, \sigma} = \operatorname{supp} W_{\xi, \mu, \sigma} \cap \operatorname{supp} W_{\xi', \mu, \sigma} = \operatorname{supp} \Theta_{\xi', \mu, \sigma} \cap \operatorname{supp} \Theta_{\xi, \mu, \sigma} = \emptyset, \quad (2.11)$$

for any $\xi \neq \xi'$.

The proof of the previous lemma follows combining [4, Lemma 4.1 and Lemma 4.2].

3. ITERATION SCHEME

The convex integration scheme to construct solutions of the continuity equation was first introduced in [20]. The scheme was later adapted in [4] to construct positive solutions of the continuity equation. To prove Theorem 1.3 we will adapt the convex integration scheme of [4].

First, we define the notion of a family of a -open sets. This notion is useful because of the convolution step in the iteration scheme.

Definition 3.1. Let $N \in \mathbb{N}$, $a \in \mathbb{R}^+$ and $\{A_i\}_{1 \leq i \leq N}$ be a finite family of open sets of \mathbb{T}^d . We say that the family $\{A_i\}_{1 \leq i \leq N}$ is a -open if for any $i = 1, \dots, N$ there exists a ball B_i of radius $\frac{1}{4aN}$ such that $B_i \subset A_i$.

As in [20] we consider the following system of equations in $[0, 1] \times \mathbb{T}^d$, where $d \geq 2$,

$$\begin{cases} \partial_t \rho_{q,i} + \operatorname{div}(\rho_{q,i} u_q) = -\operatorname{div} R_{q,i} \\ \operatorname{div} u_q = 0, \end{cases} \quad (3.1)$$

where the indices are $i, q \in \mathbb{N}$ and $1 \leq i \leq N$. We then fix three parameters $a_0, b > 0$ and $\beta > 0$, to be chosen later only in terms of d, p, r , and for any choice of $a > a_0$ we define

$$\lambda_0 = a, \quad \lambda_{q+1} = \lambda_q^b \quad \text{and} \quad \delta_q = \lambda_q^{-2\beta}.$$

The following proposition builds a converging sequence of functions with the inductive estimates

$$\max_t \|R_{q,i}(t, \cdot)\|_{L^1} \leq \delta_{q+1} \quad (3.2)$$

$$\max_t (\|\rho_{q,i}(t, \cdot)\|_{C^1} + \|\partial_t \rho_{q,i}(t, \cdot)\|_{C^0} + \|u_q(t, \cdot)\|_{W^{1,p'}} + \|u_q(t, \cdot)\|_{W^{2,r}} + \|\partial_t u_q(t, \cdot)\|_{L^1}) \leq \lambda_q^\alpha, \quad (3.3)$$

for any $i = 1, \dots, N$, where α is yet another positive parameter which will be specified later.

Proposition 3.2. Let $d, N \in \mathbb{N}$, $d \geq 3$. There exist $\alpha, b, a_0, M > 5$, $0 < \beta < (2b)^{-1}$ such that the following holds. For every family $\{A_i\}_{1 \leq i \leq N}$ of a_0 -open in \mathbb{T}^d and for every $a \geq a_0$, if $\{(\rho_{q,i}, u_q, R_{q,i})\}_{1 \leq i \leq N}$ solve (3.1) and enjoy the estimates (3.2), (3.3), then there exist $\{(\rho_{q+1,i}, u_{q+1}, R_{q+1,i})\}_{1 \leq i \leq N}$ which solve (3.1) and enjoy the estimates (3.2), (3.3) with q replaced by $q+1$. Moreover, for any $i = 1, \dots, N$, the following hold:

- (a) $\sup_{[0,1]} [\|(\rho_{q+1,i} - \rho_{q,i})(t, \cdot)\|_{L^p}^p + \|(u_{q+1} - u_q)(t, \cdot)\|_{W^{1,r}}^r + \|(u_{q+1} - u_q)(t, \cdot)\|_{L^{p'}}^{p'}] \leq M\delta_{q+1}$;
- (b) the following properties

$$\inf_{[0,1] \times (\mathbb{T}^d \setminus A_i)} \rho_{q,i} \geq 0, \quad \inf_{[0,1] \times A_i} \rho_{q,i} \geq c > 0,$$

imply

$$\inf_{[0,1] \times (\mathbb{T}^d \setminus A_i)} \rho_{q+1,i} \geq 0, \quad \inf_{[0,1] \times A_i} \rho_{q+1,i} \geq c - \delta_{q+1},$$

- (c) if for some $t_0 > 0$ we have that $\rho_{q,i}(t, \cdot) = 1$, $R_{q,i}(t, \cdot) = 0$ and $u_q(t, \cdot) = 0$ for every $t \in [0, t_0]$, then $\rho_{q+1,i}(t, \cdot) = 1$, $R_{q+1,i}(t, \cdot) = 0$ and $u_{q+1}(t, \cdot) = 0$ for every $t \in [0, t_0 - \lambda_q^{-1-\alpha}]$,
- (d) if for some $t_0 > 0$ we have that $\operatorname{supp} \rho_{q,i}(t, \cdot) \subset B_i$, $R_{q,i}(t, \cdot) = 0$ and $u_q(t, \cdot) = 0$ for every $t \in [t_0, 1]$, then $\operatorname{supp} \rho_{q+1,i}(t, \cdot) \subset B_{i, \lambda_q^{-1-\alpha}}$, $R_{q+1,i}(t, \cdot) = 0$ and $u_{q+1}(t, \cdot) = 0$ for every $t \in [t_0 + \lambda_q^{-1-\alpha}, 1]$, where $B_{i, \lambda_q^{-1-\alpha}} := \{x \in \mathbb{T}^d : d(x, B_i) < \lambda_q^{-1-\alpha}\}$.

Remark 3.3. We highlight that the constant a_0 in the proposition above does not depend on the sets A_i but only on the number N . Therefore, when we apply this proposition (to prove Theorem 1.4), we choose the sets A_i after having fixed a_0 .

To prove Proposition 3.2 we use a convex integration scheme similar to the one in [4, Proposition 2.1]. However, the end products of our scheme are different from those of [4]. Indeed, we seek to produce a *single* vector field u and N densities ρ_i with mutually disjoint compact supports for some time such that (ρ_i, u) weakly solves (1.1). Accordingly, we modified the iterative proposition of [4] in two essential ways: we index N distinct densities $\rho_{q,i}$ by the parameter $i = 1, \dots, N$; we have refined the control $\inf \rho_{q+1,i}$ over the subregion $\mathbb{T}^d \setminus A_i$ thanks to (b). The former is achieved by taking N disjoint families of building blocks $\{\Lambda_i\}_{1 \leq i \leq N}$, and the latter by localizing to A_i the corrector part of the perturbation to $\rho_{q,i}$.

3.1. Choice of the parameters. The choice of parameter is the same of [4, Section 5.1]. We define first the constant

$$\gamma := \left(1 + \frac{1}{p}\right) \left(\min \left\{ \frac{d}{p}, \frac{d}{p'}, -1 - d \left(\frac{1}{p'} - \frac{1}{r} \right) \right\}\right)^{-1} > 0,$$

Notice that, up to enlarging r , we can assume that the quantity in the previous line is less than $1/2$, namely that $\gamma > 2$. Hence we set $\alpha := 4 + \gamma(d+1)$,

$$b := \max\{p, p'\}(3(1+\alpha)(d+2)+2), \quad (3.4)$$

and

$$\beta := \frac{1}{2b} \min \left\{ p, p', r, \frac{1}{b+1} \right\} = \frac{1}{2b(b+1)}. \quad (3.5)$$

Finally, we choose a_0 and M sufficiently large (possibly depending on all previously fixed parameters) to absorb numerical constants in the inequalities. We set

$$\ell := \lambda_q^{-1-\alpha}, \quad (3.6)$$

$$\mu_{q+1} := \lambda_{q+1}^\gamma. \quad (3.7)$$

3.2. Convolution. The convolution step is the same of [4, Section 5.2]. We just write here the definitions. We first perform a convolution of ρ_q and u_q to have estimates on more than one derivative of these objects and of the corresponding error. Let $\phi \in C_c^\infty(B_1)$ be a standard convolution kernel in space-time, ℓ as in (3.6) and define

$$\rho_{\ell,i} := \rho_{q,i} * \phi_\ell, \quad u_\ell := u_q * \phi_\ell, \quad R_{\ell,i} := R_{q,i} * \phi_\ell.$$

We observe that $(\rho_{\ell,i}, u_\ell, R_{\ell,i} + (\rho_{q,i} u_q)_\ell - \rho_{\ell,i} u_\ell)$ solves system (3.1) for any $i = 1, \dots, N$ and by (3.2), (3.5) enjoys the following estimates

$$\|R_{\ell,i}\|_{L^1} \leq \delta_{q+1}, \quad (3.8)$$

$$\|\rho_{\ell,i} - \rho_{q,i}\|_{L^p} \leq \ell \|\rho_{q,i}\|_{C^1} \leq \ell \lambda_q^\alpha \leq \lambda_q^{-1} \leq \frac{\delta_{q+1}^{1/p}}{2}, \quad (3.9)$$

$$\|u_\ell - u_q\|_{L^{p'}} \leq C \ell \lambda_q^\alpha \leq C \delta_{q+1}^{1/p'},$$

$$\|u_\ell - u_q\|_{W^{1,r}} \leq C \ell \lambda_q^\alpha \leq C \delta_{q+1}^{1/r}.$$

Indeed note that by (3.5)

$$\ell \lambda_q^\alpha = \lambda_q^{-1} = \delta_{q+1}^{\frac{1}{2b\beta}} \leq \delta_{q+1}^{\max\{1/p, 1/p', 1/r\}}.$$

Next observe that

$$\|\partial_t^S \rho_{\ell,i}\|_{C^0} + \|\rho_{\ell,i}\|_{C^S} + \|u_\ell\|_{W^{1+S,r}} + \|\partial_t^S u_\ell\|_{W^{1,r}} \leq C(S) \ell^{-S+1} (\|\rho_{q,i}\|_{C^1} + \|u_q\|_{W^{2,r}}) \leq C(S) \ell^{-S+1} \lambda_q^\alpha$$

for every $S \in \mathbb{N} \setminus \{0\}$ and for every $i = 1, \dots, N$. Using the Sobolev embedding $W^{d,r} \subset W^{d,1} \subset C^0$ we then conclude

$$\|\partial_t^S u_\ell\|_{C^0} + \|u_\ell\|_{C^S} \leq C(S) \ell^{-S-d+2} \lambda_q^\alpha.$$

By Young's inequality we estimate the higher derivatives of $R_{\ell,i}$ in terms of $\|R_{q,i}\|_{L^1}$ to get

$$\|R_{\ell,i}\|_{C^S} + \|\partial_t^S R_{\ell,i}\|_{C^0} \leq \|D^S \rho_{\ell,i}\|_{L^\infty} \|R_{q,i}\|_{L^1} \leq C(S) \ell^{-S-d} \leq C(S) \lambda_q^{(1+\alpha)(d+S)} \quad (3.10)$$

for every $L \in \mathbb{N}$ and $i = 1, \dots, N$. Finally, thanks to [4, Lemma 5.1] for the last part of the error we have

$$\|(\rho_{q,i} u_q)_\ell - \rho_{\ell,i} u_\ell\|_{L^1} \leq C \ell^2 \lambda_q^{2\alpha} \leq \frac{1}{4} \delta_{q+2}, \quad (3.11)$$

where we have assumed that a is sufficiently large.

3.3. Definition of the perturbation. Let $\mu_{q+1} > 0$ be as in (3.7) and let $\chi \in C_c^\infty(-\frac{3}{4}, \frac{3}{4})$ such that $\sum_{n \in \mathbb{Z}} \chi(\tau - n) = 1$ for every $\tau \in \mathbb{R}$. Let $\bar{\chi} \in C_c^\infty(-\frac{4}{5}, \frac{4}{5})$ be a nonnegative function satisfying $\bar{\chi} = 1$ on $[-\frac{3}{4}, \frac{3}{4}]$. Notice that $\sum_{n \in \mathbb{Z}} \bar{\chi}(\tau - n) \in [1, 2]$ and $\chi \cdot \bar{\chi} = \chi$.

Fix a parameter $\kappa = 40p'/\delta_{q+2}$ and consider $2N$ disjoint families $\{\Lambda_i^1\}_{1 \leq i \leq N}$, $\{\Lambda_i^2\}_{1 \leq i \leq N}$ as in Lemma 2.1. Next, for $n \in \mathbb{N}$, define $[n]$ to be 1 or 2 depending on the congruence class of n . Finally, we take our building blocks according to Lemma 2.3 with $\Lambda = \cup_{i=1}^2 \cup_{j=1}^N \Lambda_j^i$ and observe that their spatial supports are disjoint. We define the new density and vector field by adding to ρ_ℓ and u_ℓ principal terms and correctors, namely we set

$$\begin{aligned} \rho_{q+1,i} &:= \rho_{\ell,i} + \theta_{q+1,i}^{(p)} + \theta_{q+1,i}^{(c)}, \\ u_{q+1} &:= u_\ell + \sum_{i=1}^N (w_{q+1,i}^{(p)} + w_{q+1,i}^{(c)}). \end{aligned}$$

The principal perturbations are given, respectively, by

$$w_{q+1,i}^{(p)}(t, x) = \sum_{n \geq 12} \bar{\chi}(\kappa |R_{\ell,i}(t, x)| - n) \sum_{\xi \in \Lambda_i^{[n]}} W_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x), \quad (3.12)$$

$$\theta_{q+1,i}^{(p)}(t, x) = \sum_{n \geq 12} \chi(\kappa |R_{\ell,i}(t, x)| - n) \sum_{\xi \in \Lambda_i^{[n]}} a_\xi \left(\frac{R_{\ell,i}(t, x)}{|R_{\ell,i}(t, x)|} \right) \Theta_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x), \quad (3.13)$$

where it is understood that the terms in the second sum vanish at points where $R_{\ell,i}$ vanishes. In the definition of $w_{q+1,i}^{(p)}$ and $\theta_{q+1,i}^{(p)}$ the first sum runs for n in the range

$$12 \leq n \leq C \ell^{-d} \delta_{q+2}^{-1} \leq C \lambda_q^{d(1+\alpha)+2\beta b^2} \leq C \lambda_q^{d(1+\alpha)+1} \leq \lambda_q^{d(1+\alpha)+2}, \quad (3.14)$$

where the last holds providing $a_0 \geq C$. Indeed $\chi(\kappa |R_{\ell,i}(t, x)| - n) = 0$ if $n \geq 20\delta_{q+2}^{-1} \|R_{\ell,i}\|_{C^0} + 1$ and by (3.10) we obtain an upper bound for n .

The aim of the corrector term for the density is to ensure that the overall perturbation has zero average. So we set

$$\theta_{q+1,i}^{(c)}(t, x) := -g_i(x) \int \theta_{q+1,i}^{(p)}(t, x) dx, \quad (3.15)$$

where $\{g_i\}_{1 \leq i \leq N} \subset C^\infty(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d} g_i = 1$, $g_i \geq 0$ and $\text{supp}(g_i)$ is compactly contained on A_i and $\|g_i\|_{C^s} \lesssim (\lambda_0 N)^{d+s}$ (where \lesssim means inequality up to a geometric constant depending only on d). Here we used the property that the family $\{A_i\}_{1 \leq i \leq N}$ is λ_0 -open. We observe that the functions $\{g_i\}_{1 \leq i \leq N}$ do not depend on q , they only depend only on the fixed open sets $\{A_i\}_{1 \leq i \leq N}$. The aim of the corrector term for the vector field is to ensure that the overall perturbation has zero divergence. Thanks to (2.7), we can apply Lemma 2.2 to define

$$w_{q+1,i}^{(c)} := - \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \mathcal{R} [\nabla \bar{\chi}(\kappa |R_{\ell,i}(t, x)| - n) \cdot W_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x)]$$

Moreover, since $W_{\xi, \mu_{q+1}, n/\kappa}$ is divergence-free, the argument inside \mathcal{R} has 0 average for every $t \geq 0$, and so $w_{q+1,i}^{(c)}$ is indeed well defined. Notice finally that the perturbations equals 0 on every time interval where $R_{\ell,i}$ vanishes identically for any $i = 1, \dots, N$.

4. PROOF OF PROPOSITION 3.2

For the sake of readability, the quantifier “for every $i = 1, \dots, N$ ” will be implicit in the rest of this paper. Before coming to the main arguments, we recall [4, Lemma 6.1] for the “slowly varying coefficients”.

Lemma 4.1. *For $m \in \mathbb{N}$, $S \in \mathbb{N} \setminus \{0\}$ and $n \geq 2$ we have*

$$\begin{aligned} \|\partial_t^m \chi(\kappa |R_{\ell,i}| - n)\|_{C^S} + \|\partial_t^m \bar{\chi}(\kappa |R_{\ell,i}| - n)\|_{C^S} &\leq C(m, S) \delta_{q+2}^{-2(S+m)} \ell^{-(S+m)(1+d)} \leq C(m, S) \lambda_q^{(S+m)(d+2)(1+\alpha)} \\ \|\partial_t^m (a_\xi(\frac{R_{\ell,i}}{|R_{\ell,i}|}))\|_{C^S} &\leq C(m, S) \delta_{q+2}^{-S-m} \ell^{-(S+m)(1+d)} \leq C(m, S) \lambda_q^{(S+m)(d+2)(1+\alpha)} \quad \text{on } \{\chi(\kappa |R_{\ell,i}| - n) > 0\}. \end{aligned}$$

4.1. Estimate on $\|\theta_{q+1,i}\|_{L^p}$ and on $\inf_{\mathbb{T}^d} \theta_{q+1,i}$. We apply the improved Hölder inequality of (2.2), Lemma 4.1 and (3.14) to get

$$\begin{aligned} \|\theta_{q+1,i}^{(p)}\|_{L^p} &\leq \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\chi(\kappa |R_{\ell,i}(t,x)| - n) a_\xi \left(\frac{R_{\ell,i}(t,x)}{|R_{\ell,i}(t,x)|} \right)\|_{L^p} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x)\|_{L^p} \\ &\quad + \frac{1}{\lambda_{q+1}^{1/p}} \sum_{n \geq 12} \sum_{\xi \in \Lambda^{[n]}} \|\chi(\kappa |R_{\ell,i}(t,x)| - n) a_\xi \left(\frac{R_{\ell,i}(t,x)}{|R_{\ell,i}(t,x)|} \right)\|_{C^1} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x)\|_{L^p} \\ &\leq C \sum_{n \geq 12} \|(n/k)^{1/p} \chi(\kappa |R_{\ell,i}(t,x)| - n)\|_{L^p} + C \lambda_{q+1}^{-1/p} \delta_{q+2}^{1/p} \lambda_q^{(d+2)(1+\alpha) + (1+1/p)((d(1+\alpha)+1))} \\ &\leq C \|R_{\ell,i}\|_{L^1}^{1/p} + C \lambda_{q+1}^{-1/p} \delta_{q+2}^{1/p} \lambda_q^{3(d+2)(1+\alpha)} \\ &\leq C \delta_{q+1}^{1/p}, \end{aligned} \tag{4.1}$$

provided that in the second last inequality we use

$$(d+2)(1+\alpha) + (1+1/p)((d(1+\alpha)+1)) \leq 3(1+\alpha)(d+2)$$

and in the last inequality we use (3.8).

Next, we use

$$\|\Theta_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \leq \left(\frac{n}{\kappa}\right)^{1/p} \mu_{q+1}^{-d/p'}$$

from (2.9), (2.3) applied to the λ_{q+1}^{-1} -periodic function $(\Theta_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x))$, (3.14) and Lemma 4.1 to get

$$\begin{aligned} |\theta_{q+1,i}^{(c)}(t,x)| &\leq \sqrt{d} \lambda_{q+1}^{-1} \sum_{n \geq 12} \sum_{\xi \in \Lambda^{[n]}} \|g_i\|_{L^\infty} \|\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{C^1} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \\ &\quad + \sum_{n \geq 12} \sum_{\xi \in \Lambda^{[n]}} \|g_i\|_{L^\infty} \|\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{L^1} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \\ &\leq \|g_i\|_{L^\infty} \lambda_{q+1}^{-1} \mu_{q+1}^{-d/p'} \lambda_q^{3(1+\alpha)(d+2)} + \|g_i\|_{L^\infty} \mu_{q+1}^{-d/p'} \sum_{n \geq 12} (n/\kappa)^{1/p} \|\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{L^1} \\ &\leq \|g_i\|_{L^\infty} \mu_{q+1}^{-d/p'} + \|g_i\|_{L^\infty} \mu_{q+1}^{-d/p'} \|R_{\ell,i}\|_{L^1}^{1/p} \lesssim (N \lambda_0)^d \lambda_{q+1}^{-1} \leq \frac{\delta_{q+1}}{2}, \end{aligned} \tag{4.2}$$

where in the second to last inequality, we enlarge a to absorb the constant N . Now, recall that $\theta_{q+1}^{(p)}$ is nonnegative by definition. Therefore,

$$\inf_{[0,1] \times \mathbb{T}^d} [\theta_{q+1,i}^{(p)} + \theta_{q+1,i}^{(c)}] \geq \inf_{[0,1] \times \mathbb{T}^d} \theta_{q+1,i}^{(c)} \geq -\frac{\delta_{q+1}}{2}.$$

Also

$$\inf_{[0,1] \times (\mathbb{T}^d \setminus A_i)} [\theta_{q+1,i}^{(p)} + \theta_{q+1,i}^{(c)}] \geq 0.$$

Since $\rho_{\ell,i}$ is nonnegative whenever $\rho_{q,i}$ is nonnegative, by (3.9) we get property (b) of Proposition 3.2.

4.2. Estimate on $\|w_{q+1,i}\|_{L^{p'}}$ and $\|Dw_{q+1,i}\|_{L^r}$. Exactly with the same computation as in (4.1), replacing p with p' , we have that

$$\|w_{q+1,i}^{(p)}\|_{L^{p'}} \leq C\|R_{\ell,i}\|_{L^1}^{1/p'} + C\lambda_{q+1}^{-1/p'}\delta_{q+2}^{1/p'}\lambda_q^{3(d+2)(1+\alpha)} \leq C\delta_{q+1}^{1/p'} \quad (4.3)$$

Concerning the corrector term $w_{q+1}^{(c)}$, we use (2.10) (precisely $\|W_{\xi,\mu_{q+1},n/\kappa}\|_{L^{p'}} \leq (\frac{n}{\kappa})^{1/p'} \leq \lambda_q^{(d+2)(1+\alpha)}$) Lemma 2.2 and (3.14) to get

$$\begin{aligned} \|w_{q+1,i}^{(c)}\|_{L^{p'}} &\leq \frac{1}{\lambda_{q+1}} \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\bar{\chi}(\kappa|R_{\ell,i}| - n)\|_{C^2} \|W_{\xi,\mu_{q+1},n/\kappa}\|_{L^{p'}} \\ &\leq C\lambda_{q+1}^{-1} \sum_{n \geq 12} \lambda_q^{2(d+2)(1+\alpha)} (n/\kappa)^{1/p'} \\ &\leq C\lambda_{q+1}^{-1} \delta_{q+2}^{1/p'} \lambda_q^{4(d+2)(1+\alpha)} \leq \delta_{q+2}^{1/p'} \leq \delta_{q+1}^{1/p'}. \end{aligned} \quad (4.4)$$

Computing the gradient of $w_{q+1}^{(p)}$ and combining Lemma 4.1 with (2.10) we have

$$\|Dw_{q+1,i}^{(p)}\|_{L^r} \leq \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\bar{\chi}(\kappa|R_{\ell,i}| - n)\|_{C^1} \|W_{\xi,\mu_{q+1},n/\kappa}\|_{L^r} + \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \lambda_{q+1} \|DW_{\xi,\mu_{q+1},n/\kappa}\|_{L^r} \quad (4.5)$$

$$\leq C\delta_{q+2}^{1/p'} \lambda_q^{3(1+\alpha)(d+2)+2+b\gamma d(1/p'-1/r)} + C\delta_{q+2}^{1/p'} \lambda_q^{b+3(1+\alpha)(d+2)+b\gamma(1+d(1/p'-1/r))} \quad (4.6)$$

$$\leq \delta_{q+2}^{1/p'} \leq \delta_{q+1}^{1/r}. \quad (4.7)$$

Concerning the corrector, by Lemma 2.2 and similar computations as above,

$$\|Dw_{q+1,i}^{(c)}\|_{L^r} \leq C \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\bar{\chi}(\kappa|R_{\ell,i}| - n)\|_{C^3} \|W_{\xi,\mu_{q+1},n/\kappa}\|_{L^r} \quad (4.8)$$

$$\leq C\delta_{q+2}^{1/p'} \lambda_q^{5(1+\alpha)(d+2)} \mu_{q+1}^{d(\frac{1}{r}-\frac{1}{p'})} \leq \delta_{q+2}^{1/p'} \lambda_q^{5(1+\alpha)(d+2)-b(1+1/p)} \leq \delta_{q+1}^{1/r}. \quad (4.9)$$

4.3. Definition of the new error $R_{q+1,i}$. This part is similar to [4], however here $\theta_{q+1,i}^{(c)}$ is not constant in space and we have to adapt the argument accordingly: we will pick up a new error term which we call R^{space} .

By definition the new error $R_{q+1,i}$ must satisfy

$$\begin{aligned} -\operatorname{div} R_{q+1,i} &= \partial_t \rho_{q+1,i} + \operatorname{div}(\rho_{q+1,i} u_{q+1}) \\ &= \operatorname{div}(\theta_{q+1,i}^{(p)} w_{q+1}^{(p)} - R_{\ell,i}) + \partial_t \theta_{q+1,i}^{(p)} + \partial_t \theta_{q+1,i}^{(c)} \\ &\quad + \operatorname{div}(\theta_{q+1,i}^{(p)} u_{\ell} + \rho_{\ell,i} w_{q+1} + \theta_{q+1,i}^{(p)} w_{q+1}^{(c)}) \\ &\quad + \operatorname{div}((u_{\ell} + w_{q+1}) \theta_{q+1,i}^{(c)}) + \operatorname{div}((\rho_{q,i} u_q)_{\ell} - \rho_{\ell,i} u_{\ell}) \end{aligned} \quad (4.10)$$

In the second equality above we have used that $(\rho_{\ell}, u_{\ell}, R_{\ell,i} + (\rho_{q,i} u_q)_{\ell} - \rho_{\ell,i} u_{\ell})$ solves (3.1).

We now decompose

$$\begin{aligned} \partial_t \theta_{q+1,i}^{(p)} &= \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \chi(\kappa|R_{\ell,i}| - n) a_{\xi} \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \partial_t [\Theta_{\xi,\mu_{q+1},n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x)] \\ &\quad + \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \partial_t \left[\chi(\kappa|R_{\ell,i}| - n) a_{\xi} \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \Theta_{\xi,\mu_{q+1},n/\kappa}(\lambda_{q+1} t, \lambda_{q+1} x) \\ &=: (\partial_t \theta_{q+1,i}^{(p)})_1 + (\partial_t \theta_{q+1,i}^{(p)})_2, \end{aligned}$$

We now observe that

$$\begin{aligned} \theta_{q+1,i}^{(p)} \sum_{k=1}^N w_{q+1,k}^{(p)} &= \theta_{q+1,i}^{(p)} w_{q+1,i}^{(p)} \\ &= \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) (\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x), \end{aligned} \quad (4.11)$$

where the first equality holds because $\Lambda_i^k \cap \Lambda_j^p = \emptyset$ for any p, k and $i \neq j$; the second equality follows from (2.11) and the definitions of χ and $\bar{\chi}$. Also, $\Theta_{\xi, \mu_{q+1}, n/\kappa}$ and $W_{\xi, \mu_{q+1}, n/\kappa}$ solve the transport equation (2.6). These observations in conjunction with Lemma 2.1 yield the cancellation of the error $R_{\ell,i}$ up to lower order terms

$$\begin{aligned} &\operatorname{div}(\theta_{q+1,i}^{(p)} \sum_{k=1}^N w_{q+1,k}^{(p)}) + (\partial_t \theta_{q+1,i}^{(p)})_1 - \operatorname{div} R_{\ell,i} \\ &= \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \nabla \left[\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] (\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \operatorname{div} R_{\ell,i} \\ &+ \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \lambda_{q+1} \left[\partial_t \Theta_{\xi, \mu_{q+1}, n/\kappa} + \operatorname{div}(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa}) \right] (\lambda_{q+1} t, \lambda_{q+1} x) \\ &= \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \nabla \left[\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \left[(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{\kappa} \xi \right] \\ &+ \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \nabla \left[\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \frac{n}{\kappa} \xi - \operatorname{div} R_{\ell,i} \\ &= \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \nabla \left[\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \left[(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{\kappa} \xi \right] \\ &\quad + \operatorname{div}(\tilde{R}_{\ell,i} - R_{\ell,i}), \end{aligned} \quad (4.12)$$

where

$$\tilde{R}_{\ell,i} := \sum_{n \geq 12} \chi(\kappa |R_{\ell,i}| - n) \frac{R_{\ell,i}}{|R_{\ell,i}|} \frac{n}{\kappa}.$$

We have

$$\begin{aligned} |R_{\ell,i} - \tilde{R}_{\ell,i}| &\leq \left| \sum_{n=-1}^{11} \chi(\kappa |R_{\ell,i}| - n) R_{\ell,i} \right| + \left| \sum_{n \geq 12} \chi(\kappa |R_{\ell,i}| - n) \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \frac{n}{\kappa} - R_{\ell,i} \right) \right| \\ &\leq \frac{13}{\kappa} + \sum_{n \geq 12} \chi(\kappa |R_{\ell,i}| - n) \left| |R_{\ell,i}| - \frac{n}{\kappa} \right| \end{aligned} \quad (4.13)$$

$$\text{(by definition of } \kappa \text{ and (3.14))} \leq \frac{13}{20} \delta_{q+2} + \frac{3}{40} \delta_{q+2} \leq \frac{15}{20} \delta_{q+2}.$$

We can now define $R_{q+1,i}$ which satisfies (4.10) as

$$-R_{q+1,i} := R_i^{quadr} + (\tilde{R}_{\ell,i} - R_{\ell,i}) + R_i^{time} + R_i^{space} + \theta_{q+1,i}^{(p)} u_\ell + \rho_{\ell,i} w_{q+1} + \theta_{q+1,i}^{(p)} w_{q+1}^{(c)} + [(\rho_{q,i} u_q)_\ell - \rho_{\ell,i} u_\ell], \quad (4.14)$$

where

$$R_i^{quadr} := \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \mathcal{R} \left[\nabla \left(\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right) \cdot \left((\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{\kappa} \xi \right) \right], \quad (4.15)$$

$$R_i^{time} := \nabla \Delta^{-1} ((\partial_t \theta_{q+1,i}^{(p)})_2 + \partial_t \theta_{q+1,i}^{(c)} + m_i), \quad (4.16)$$

$$m_i := \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \int \nabla \left[\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \left[(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{\kappa} \xi \right] dx,$$

$$R_i^{space} := (u_\ell + w_{q+1}) \theta_{q+1,i}^{(c)}. \quad (4.17)$$

Property (d) is now clear from the definition of $R_{q+1,i}$ and the definition of $\rho_{q+1,i}$. Notice that R_i^{quadr} is well defined since by (2.8) the function $(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{\kappa} \xi$ has 0 mean. From the second equality in (4.10) and since the average of $(\partial_t \theta_{q+1,i}^{(p)})_1$ is m_i by integration by parts, we deduce that $(\partial_t \theta_{q+1,i}^{(p)})_2 + \partial_t \theta_{q+1,i}^{(c)} + m_i$ has 0 mean, so that R_i^{time} is well defined.

4.4. Estimate on $\|R_{q+1,i}\|_{L^1}$. Recall that the estimate on $\|(\rho_{q,i} u_q)_\ell - \rho_{\ell,i} u_\ell\|_{L^1}$ has been already established in (3.11). By the property (2.1) of the antidivergence operator \mathcal{R} , Lemma 4.1 and (3.14) we have

$$\begin{aligned} \|R_i^{quadr}\|_{L^1} &\leq \frac{C}{\lambda_{q+1}} \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{C^2} \|\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \\ &\leq C \delta_{q+2} \frac{\lambda_q^{4(1+\alpha)(d+2)+2}}{\lambda_{q+1}} \leq \frac{\delta_{q+2}}{20}. \end{aligned}$$

To estimate the terms which are linear with respect to the fast variables, we take advantage of the concentration parameter μ_{q+1} . First of all, by Calderon-Zygmund estimates we get

$$\|R_i^{time}\|_{L^1} \leq C \|(\partial_t \theta_{q+1,i}^{(p)})_2 + \partial_t \theta_{q+1,i}^{(c)} - m_i\|_{L^1} \leq \|(\partial_t \theta_{q+1,i}^{(p)})_2\|_{L^1} + |\partial_t \theta_{q+1,i}^{(c)}| + |m_i|.$$

Next, notice that

$$\|(\partial_t \theta_{q+1,i}^{(p)})_2\|_{L^1} \leq C \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\partial_t [\chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)]\|_{C^0} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \quad (4.18)$$

$$\leq C \delta_{q+2}^{1/p} \lambda_q^{3(1+\alpha)(d+2)} \mu_{q+1}^{-d/p'} \leq \frac{\delta_{q+2}}{20 \lambda_{q+1}}. \quad (4.19)$$

If $a_0(N)$ is sufficiently large, from (4.2), (4.18), (2.6) and (2.3) we get

$$\begin{aligned} &|\partial_t \theta_{q+1,i}^{(c)}| + |m_i| \\ &\leq \|g_i\|_{L^\infty} \left| \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \int \chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \partial_t [(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x)] dx \right| + |m_i| \\ &\quad + \|g_i\|_{L^\infty} \|(\partial_t \theta_{q+1,i}^{(p)})_2\|_{L^1} \\ &\leq \|g_i\|_{L^\infty} \left| \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \int \chi(\kappa |R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \operatorname{div} [(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x)] dx \right| + |m_i| \\ &\quad + \frac{\delta_{q+2}}{20} \end{aligned}$$

$$\begin{aligned}
&= 2\|g_i\|_{L^\infty} \left| \sum_{n \geq 12} \sum_{\xi \in \Lambda^{[n]}} \int \nabla \left[\chi(\kappa|R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right) \right] \cdot \left[(\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa})(\lambda_{q+1} t, \lambda_{q+1} x) - \frac{n}{k} \xi \right] dx \right| \\
&\quad + \frac{\delta_{q+2}}{20} \\
&\leq \frac{\delta_{q+2}}{20} + \frac{2\|g_i\|_{L^\infty} \sqrt{d}}{\lambda_{q+1}} \sum_{n \geq 12} \sum_{\xi \in \Lambda^{[n]}} \|\chi(\kappa|R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{C^2} \|\Theta_{\xi, \mu_{q+1}, n/\kappa} W_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \\
&\leq \frac{\delta_{q+2}}{20} + C\|g_i\|_{L^\infty} \lambda_{q+1}^{-1} \delta_{q+2} \lambda_q^{4(1+\alpha)(d+2)} \leq \frac{1}{10} \delta_{q+2},
\end{aligned}$$

and using also (4.2) and (3.3), we estimate the error R_i^{space}

$$\|R_i^{space}\|_{L^1} \leq \|(u_\ell + w_{q+1})\theta_{q+1,i}^{(c)}\|_{L^1} \leq (\|u_\ell\|_{L^1} + \|w_{q+1}\|_{L^1}) \|\theta_{q+1,i}^{(c)}\|_{L^\infty} \leq 2\lambda_q^\alpha (N\lambda_0)^d \lambda_{q+1}^{-1} \leq \frac{\delta_{q+2}}{20},$$

where the last inequality holds up to enlarging a_0 depending on N .

We also have that

$$\begin{aligned}
&\|\theta_{q+1,i}^{(p)} u_\ell + \rho_{\ell,i} w_{q+1}^{(p)}\|_{L^1} \leq \|\theta_{q+1,i}^{(p)}\|_{L^1} \|u_\ell\|_{L^\infty} + \|\rho_{\ell,i}\|_{L^\infty} \|w_{q+1}^{(p)}\|_{L^1} \\
&\leq \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \|\chi(\kappa|R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{L^\infty} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \|u_\ell\|_{L^\infty} \\
&\quad + \|\rho_{\ell,i}\|_{L^\infty} \|\bar{\chi}(\kappa|R_{\ell,i}| - n)\|_{L^\infty} \|W_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} \\
&\leq C\delta_{q+2}^{1/p} \lambda_q^{2(1+\alpha)(d+2)} \mu_{q+1}^{-d/p'} + C\delta_{q+2}^{1/p'} \lambda_q^{2(1+\alpha)(d+2)} \mu_{q+1}^{-d/p} \leq \frac{\delta_{q+2}}{20}.
\end{aligned}$$

In the last inequality we used $2\beta b^2 \leq 1$, the definition of γ , and $b(1+1/p) \geq 2(1+\alpha)(d+2) + 1$.

Finally, from (4.1) and (4.4)

$$\begin{aligned}
&\|(\rho_{\ell,i} + \theta_{q+1,i}^{(p)}) w_{q+1}^{(c)}\|_{L^1} \leq (\|\rho_{\ell,i}\|_{C^1} + \|\theta_{q+1,i}^{(p)}\|_{L^p}) \|w_{q+1}^{(c)}\|_{L^{p'}} \\
&\leq C\lambda_q^{4(1+\alpha)(d+2)+\alpha} \lambda_{q+1}^{-1} \leq \frac{1}{20} \delta_{q+2}.
\end{aligned} \tag{4.20}$$

4.5. Estimates on higher derivatives.

$$\begin{aligned}
&\|\rho_{q+1,i}\|_{C^1} \leq \|\rho_{\ell,i}\|_{C^1} + \|\theta_{q+1,i}\|_{C^1} \\
&\leq \|\rho_{q,i}\|_{C^1} + \sum_{n \geq 10} \sum_{\xi \in \Lambda_i^{[n]}} \|\chi(\kappa|R_{\ell,i}| - n) a_\xi \left(\frac{R_{\ell,i}}{|R_{\ell,i}|} \right)\|_{C^1} \|\Theta_{\xi, \mu_{q+1}, n/\kappa}(\lambda_{q+1} x)\|_{C^1} \\
&\leq C\lambda_q^\alpha + C\lambda_q^{3(1+\alpha)(d+2)} \lambda_{q+1} \mu_{q+1}^{1+d/p} \leq \lambda_{q+1}^\alpha.
\end{aligned} \tag{4.21}$$

An entirely similar estimate is valid for $\|\partial_t \rho_{q+1,i}\|_{C^0}$ and the one for $\|u_\ell + w_{q+1,i}^{(p)}\|_{W^{2,r}}$ is analogous. Concerning $w_{q+1,i}^{(c)}$, we use Lemma 2.2 and (3.14)

$$\begin{aligned}
&\|w_{q+1,i}^{(c)}\|_{W^{2,r}} \leq \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \lambda_{q+1} \|\bar{\chi}(\kappa|R_{\ell,i}| - n)\|_{C^4} \|W_{\xi, \mu_{q+1}, n/\kappa}\|_{W^{2,r}} \\
&\leq C\lambda_q^{6(1+\alpha)(d+2)} \lambda_{q+1}^2 \mu_{q+1}^{2+d(1/p'-1/r)} \leq \lambda_{q+1}^\alpha.
\end{aligned}$$

It remains just to estimate

$$\|\partial_t u_{q+1}\|_{L^1} \leq \|\partial_t u_\ell\|_{L^1} + \sum_{i=1}^N \|\partial_t w_{q+1,i}^{(p)}\|_{L^1} + \|\partial_t w_{q+1,i}^{(c)}\|_{L^1}.$$

From (2.10) and Lemma 4.1

$$\begin{aligned} \|\partial_t w_{q+1,i}^{(p)}\|_{L^1} &\leq \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \lambda_{q+1} \|\partial_t W_{\xi, \mu_{q+1}, n/\kappa}\|_{L^1} + \|\partial_t \bar{\chi}(\kappa |R_{\ell,i}| - n)\|_{L^\infty} \|W_{\xi, \mu_{q+1}, \kappa/n}\|_{L^1} \\ &\leq C \delta_{q+2}^{2/p'} \lambda_q^{(1+2/p')(d(1+\alpha)+1)} \lambda_{q+1}^{1+\gamma(1+d(2/p'-1))} \leq \lambda_{q+1}^{2+\gamma(d+1)} \leq \lambda_{q+1}^\alpha. \end{aligned}$$

A similar computation is valid for $\|\partial_t w_{q+1,i}^{(c)}\|_{L^1}$.

5. PROOF OF MAIN RESULTS

5.1. Proof of Theorem 1.4.

Proof of Theorem 1.4 assuming Proposition 3.2. Let $\alpha, b, a_0, M > 5$, $\beta > 0$ be fixed as in Proposition 3.2. Let $a \geq a_0$ be chosen such that

$$\begin{aligned} \sum_{q=0}^{+\infty} \delta_{q+1}^{1/p} &< \frac{1}{32M}, \\ \sum_{q=0}^{+\infty} \lambda_q^{-1-\alpha} &< \frac{1}{32N}, \end{aligned}$$

Let $\{\phi_i\}_{1 \leq i \leq N} \subset C^\infty(\mathbb{T}^d)$ be nonnegative functions with mutually disjoint compact supports such that $\int_{\mathbb{T}^d} \phi_i(x) dx = 1$, $\{x \in \mathbb{T}^d : \phi_i(x) > 1\}$ contains a ball of radius $\frac{1}{4N}$, and $d(\text{supp } \phi_i, \text{supp } \phi_j) \geq 1/4N$ for $i \neq j$. We also require that $\|\phi_i\|_{C^S} \leq (100N)^{d+S}$ for any $S \in \mathbb{N}$. Let $\chi : [0, 1] \rightarrow [0, 1]$ be a smooth function such that $\chi \equiv 0$ on $[0, 2/5]$, $\chi \equiv 1$ on $[3/5, 1]$ with χ' compactly supported on $(2/5, 3/5)$, and $\|\partial_t \chi\|_{L^\infty} \leq 20$. Define $\rho_{0,i}(t, x) := (1 - \chi(t)) + \chi(t)\phi_i(\lambda_0 x)$ and set $u_0 \equiv 0$. We also set

$$R_{0,i}(t) := -\nabla \Delta^{-1} \left(\partial_t \rho_{0,i}(t) + \text{div}(\rho_{0,i}(t) u_0(t)) \right) = -\nabla \Delta^{-1} \left(\partial_t \rho_{0,i}(t) \right) = -\partial_t \chi \nabla \Delta^{-1} (\phi_i(\lambda_0 \cdot) - 1).$$

We then have N starting triples $\{(\rho_{0,i}, u_0, R_{0,i})\}_{1 \leq i \leq N}$ for our iteration scheme which enjoy (3.1) with $q = 0$ for any $i = 1, \dots, N$. Moreover, thanks to Lemma 2.2, we have $\|R_{0,i}\|_{L^1} \leq C \lambda_0^{-1}$. Thus (3.2) is satisfied because $2\beta < 1$ (here we have taken $\lambda_0 = a_0$ sufficiently large to absorb the constant C). Next, we have $\|\partial_t \rho_{0,i}\|_{C^0} + \|\rho_{0,i}\|_{C^1} \leq C \lambda_0$. Since $u_0 \equiv 0$ and $\alpha > 1$ we conclude that (3.3) is satisfied as well.

Finally we observe that the family of sets $A_i := \{x \in \mathbb{T}^d : \phi_i(\lambda_0 x) > 1\}$ for $i = 1, \dots, N$ form a a_0 -open family.

We can recursively apply Proposition 3.2 to obtain a family of sequences $\{(\rho_{q,i}, u_q, R_{q,i})_{q \in \mathbb{N}}\}_{1 \leq i \leq N}$ of smooth solutions to (3.1) and such that

- the sequences $\{\rho_{q,i}\}_{q \in \mathbb{N}}$ is Cauchy in $C(L^p)$ and we denote by ρ_i its limit for any $i = 1, \dots, N$,
- the sequence of divergence-free $\{u_q\}_{q \in \mathbb{N}}$ is Cauchy in $C(L^{p'} \cap W^{1,r})$ and we denote by u its limit (whose divergence understood in the sense of distribution vanishes).

Thanks to property (3.2) we get that (u, ρ_i) solve the continuity equation for any $i = 1, \dots, N$. Property (b) and $\inf_{A_i} \rho_{0,i}(t, \cdot) \geq 1$ also yield

$$\inf_{A_i} \rho_i(t, \cdot) \geq 1 - \sum_{q=0}^{+\infty} \delta_{q+1}^{1/p} \geq \frac{1}{2}.$$

This implies that $A_i \subset \text{supp}(\rho_i(t, \cdot))$ for any $t \in [0, 1]$ and any $i = 1, \dots, N$. Thus $\text{supp}(\rho_i(t, \cdot))$ has non-empty interior, and

$$\inf_{\mathbb{T}^d \setminus A_i} \rho_i \geq 0.$$

So ρ_i are nonnegative.

Finally, since $\rho_{0,i}(t, \cdot) \equiv 1$ for $t \in [0, 2/5]$ and $\sum_{q=0}^{+\infty} \lambda_q^{-1-\alpha} < \frac{1}{\lambda_0^{1+\alpha}} < \frac{1}{15}$, and by property (c) and (d) of Proposition 3.2, we get that $\rho_i(t, \cdot) \equiv 1$ for $t \in [0, 1/3]$ for any $i = 1, \dots, N$. Also, by property (d), and since $\sum_{q=0}^{+\infty} \lambda_q^{-1-\alpha} < \frac{1}{\lambda_0^N}$ and $d(\text{supp } \rho_{0,i}(1, \cdot), \text{supp } \rho_{0,j}(1, \cdot)) \geq \frac{1}{\lambda_0^N}$ for $i \neq j$, we must have that $\text{supp}(\rho_i) \cap \text{supp}(\rho_j)$ is negligible for $i \neq j$. \square

5.2. Proof of Theorem 1.3.

Proof of Theorem 1.3 assuming Theorem 1.4. Let $\{\rho_0\}_{1 \leq i \leq N} \subset C_t L^p$ be nonnegative densities and $u \in C_t(L^{p'} \cap W^{1,r})$ a divergence-free vector field given by Theorem 1.4. Then (ρ_i, u) solves (1.1) for $i = 1, \dots, N$. Thanks to the Ambrosio's superposition principle (see [3, Theorem 3.2]), each nonnegative $L^1([0, 1] \times \mathbb{T}^d)$ solution is transported by a generalized flow η^i of the vector field u . More precisely, $\eta^i \in \mathcal{M}_+(AC([0, 1]; \mathbb{T}^d) \times \mathbb{T}^d)$ is concentrated on pairs (γ, x) such that γ an integral curve of u starting from x , and we have $\rho_i(x, t) \mathcal{L}^d = (e_t)_\# \eta^i$ for every $t \in [0, 1]$.

Observe that the family of probability measures $\{\eta^i\}_{1 \leq i \leq N}$ does not depend on the pointwise representative of u . Indeed, given two pointwise representative v and w of u (u and v are two Borel maps such that $v = u = w$ \mathcal{L}^{d+1} -a.e.), by Fubini and by the superposition principle, we have for each integer $1 \leq i \leq N$

$$\begin{aligned} & \int_{AC([0,1], \mathbb{T}^d) \times \mathbb{T}^d} \left(\int_0^1 |v(\gamma(s)) - w(\gamma(s))| ds \right) d\eta^i(\gamma, x) \\ &= \int_0^1 \left(\int_{\mathbb{T}^d} |v(y) - w(y)| \rho_i(s, x) d\mathcal{L}^d(x) \right) ds = 0. \end{aligned} \quad (5.1)$$

Thus, η^i is concentrated on integral curves of v if and only if η^i is concentrated on integral curves of w .

By the superposition principle we have

$$\int_{\mathbb{T}^d} \psi(x) \rho_i(1, x) d\mathcal{L}^d(x) = \int_{\mathbb{T}^d} \int_{AC([0,1], \mathbb{T}^d)} \psi(\gamma(1)) d\eta_x^i(\gamma) d\mathcal{L}^d(x), \quad (5.2)$$

and for every $\psi \in C(\mathbb{T}^d)$. Therefore, for \mathcal{L}^d -a.e. $x \in \mathbb{T}^d$ and η_x^i -a.e. $\gamma \in AC([0, 1]; \mathbb{T}^d)$, we have $\gamma(1) \in A_i$. Since $A_i \cap A_j = \emptyset$ for $i \neq j$, it follows that for \mathcal{L}^d -a.e. $x \in \mathbb{T}^d$, the measures $\{\eta_x^i\}_{1 \leq i \leq N}$ have mutually disjoint supports. Therefore, for \mathcal{L}^d -a.e. $x \in \mathbb{T}^d$, there are at least N integral curves starting from x . \square

6. DIMENSION $d = 2$

The two dimensional case (i.e. for $d = 2$) is slightly more technical. We can no longer use Lemma 4.2 of [4] to translate in space the tubes supporting the building blocks and thereby make these tubes disjoint. In [4], the authors found a way around this issue. They are able make the building blocks in the case $d = 2$ disjoint. They take advantage of the presence of a single error to argue that only building blocks with comparable speeds – that is building blocks for which the speed ratio is of order $\sim 10^{-2}$ – need to have disjoint supports. Indeed in [4], the speeds at the inductive step $q \in \mathbb{N}$ of the convex integration scheme are $w_n = \mu_{q+1}^{d/p'} \left(\frac{n}{\kappa}\right)^{1/p'}$ for $n = 1, \dots, \lambda_q^{d(1+\alpha)+2}$. The supports of the building blocks are then translated in space suitably, the speeds $\{w_n\}_{n=1, \dots, \lambda_q^{d(1+\alpha)+2}}$ are approximated by $\{v_n\}_{n=1, \dots, \lambda_q^{d(1+\alpha)+2}}$, and at the price of a small error the authors obtain building blocks satisfying

$$W_{\xi, \mu_{q+1}, v_n} \cdot \Theta_{\xi', \mu_{q+1}, v_m}(t, x) = 0 \quad \text{for any } (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (6.1)$$

for any $\xi \neq \xi'$, $|n - m| \leq 1$, $n, m = 1, \dots, \lambda_q^{d(1+\alpha)+2}$.

However, they deal with only two distinct families Λ_1 and Λ_2 of directions for the building blocks in and a single error R_q at each step q of the iteration, whereas in our setting there are $2N$ distinct families of directions $\{\Lambda_i\}_{1 \leq i \leq 2N}$ for the building blocks and N errors $\{R_{q,i}\}_{1 \leq i \leq N}$. We therefore need that any

²this hypothesis is in [4, Lemma 7.2] where it is required that $w < 10$

building block with direction in Λ_i has disjoint support with any other building block with direction in Λ_j for any $i \neq j$ because in the convex integration scheme we need the key identity (4.11) to hold. More precisely we will need

$$W_{\xi, \mu_{q+1}, v_n} \cdot \Theta_{\xi', \mu_{q+1}, v_m}(t, x) = 0 \quad \text{for any } (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (6.2)$$

whenever $\xi \neq \xi' \in \Lambda$, $n, m = 1, \dots, \lambda_q^{d(1+\alpha)+2}$. This identity is achievable because the speed ratios of the building blocks are at most of order $\lambda_q^{d(1+\alpha)+2}$, typically a very small number compared to μ_{q+1} in the iterative proposition (see Section 3.1). So we will prove that we can find $\sim \lambda_q^{d(1+\alpha)+2}$ balls of radius $\sim \mu_{q+1}^{-1}$ which are moving with speed ratio at most $\sim \lambda_q^{d(1+\alpha)+2}$ and which don't intersect at any time. We will proceed similarly to [4, Section 7], although our argument differs in some parts for reasons which were outlined above.

Lemma 6.1. *Let $\xi, \xi' \in \mathbb{S}^1 \cap \mathbb{Q}^2$ be two distinct vectors and let $w = \frac{A}{N} < \lambda_q^{d(1+\alpha)+2}$ where A and N are positive, coprime integers such that $N < \lambda_q^{d(1+\alpha)+2}$. Then there exists $C = C(\xi, \xi')$ such that for any $\varepsilon > 0$*

$$\mathcal{L}^1([0, 1] \setminus \{s : d_{\mathbb{T}^2}(t\xi, (tw + s)\xi') \geq \varepsilon \quad \forall t \geq 0\}) < CN\varepsilon\lambda_q^{d(1+\alpha)+2}.$$

Proof. Let $\varepsilon > 0$. Set $T_{int} := \{(t, t') : \xi t = \xi' t' \text{ on } \mathbb{T}^2\}$ and observe that $T_{int} \subset \mathbb{Q}^2$ since the matrix with columns ξ and ξ' is invertible with rational coefficients. Moreover T_{int} is an additive discrete subgroup of \mathbb{R}^2 , hence it is a free group of rank $k \in \{0, 1, 2\}$. Denoting by T and T' the period of, respectively, $t \rightarrow \xi t$ and $t \rightarrow \xi' t$ one has that $(T, 0), (0, T') \in T_{int}$. This implies that the rank of T_{int} is two, hence we can find two generators $(t_1, t'_1), (t_2, t'_2) \in T_{int}$. Let us finally introduce

$$A := \{\xi t \in \mathbb{T}^2 : (t, s) \in T_{int} \text{ for some } s \in \mathbb{R}\}$$

to denote the set of points in \mathbb{T}^2 where the supports of the curves $t \rightarrow t\xi$ and $t \rightarrow t\xi'$ intersect.

Let $s \in [0, 1]$ be such that $d_{\mathbb{T}^2}(t\xi, (t + s)w\xi') < \varepsilon$ for some $t \geq 0$. There exists $q \in A$ such that $d_{\mathbb{T}^2}(t\xi, q) \leq \bar{c}\varepsilon$, where $\bar{c} = \bar{c}(\xi, \xi') > 1$, hence up to modifying t we can assume that $t\xi =: q \in A$ and $d_{\mathbb{T}^2}(q, (tw + s)\xi') \leq 3\bar{c}\varepsilon\lambda_q^{d(1+\alpha)+2}$. Since $t\xi \in A$ there exists t' such that $(t, t') \in T_{int}$ and, exploiting the fact that $(t_1, t'_1), (t_2, t'_2) \in T_{int}$ are generators, we can find $k_1, k_2 \in \mathbb{Z}$ such that $t = k_1 t_1 + k_2 t_2$ and $t' = k_1 t'_1 + k_2 t'_2$. The following identity holds on \mathbb{T}^2

$$\begin{aligned} (tw + s)\xi' &= t'\xi' - t'\xi' + (tw + s)\xi' = q - (k_1 t'_1 + k_2 t'_2)\xi' + ((k_1 t_1 + k_2 t_2)w + s)\xi' \\ &= q + (k_1(t_1 w - t'_1) + k_2(t_2 w - t'_2) + s)\xi' \end{aligned}$$

therefore $d_{\mathbb{T}^2}((k_1(t_1 w - t'_1) + k_2(t_2 w - t'_2) + s)\xi', 0) \leq 3\bar{c}\varepsilon\lambda_q^{d(1+\alpha)+2}$ this implies that $-s \in B_{3\bar{c}\varepsilon\lambda_q^{d(1+\alpha)+2}}((k_1(wt_1 - t'_1) + k_2(wt_2 - t'_2))) + \mathbb{Z}T'$.

Notice now that the set $E := \{k_1(wt_1 - t'_1) + k_2(wt_2 - t'_2) : k_1, k_2 \in \mathbb{Z}\}$ is discrete, so any two neighbouring points in E are at least a distance $c \geq c'(\xi, \xi')N^{-1} > 0$ from each other, and $E + \mathbb{Z}T' = E$. In particular

$$\begin{aligned} \mathcal{L}^1([0, 1] \setminus \{s : d_{\mathbb{T}^2}(t\xi, (tw + s)\xi') \geq \varepsilon \quad \forall t \geq 0\}) &\leq \mathcal{L}^1\left([0, 1] \cap \bigcup_{r \in E} B_{3\bar{c}\varepsilon\lambda_q^{d(1+\alpha)+2}}(r)\right) \\ &\leq \frac{2}{c'(\xi, \xi')N^{-1}} 3\bar{c}\varepsilon\lambda_q^{d(1+\alpha)+2} \leq \frac{6\bar{c}}{c'(\xi, \xi')} \varepsilon (\lambda_q^{d(1+\alpha)+2})^2, \quad \square \end{aligned} \quad (6.3)$$

where in the last we used the inequality $N < \lambda_q^{d(1+\alpha)+2}$.

We now need a number theory lemma, it is just a property on real numbers, but we state it for a sequence of real numbers, since we will apply it for a sequence.

Lemma 6.2. *Let $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ such that $\alpha_n \leq \lambda_q^{d(1+\alpha)+2}$, then there exists $\{\tilde{v}_n\}_{n \in \mathbb{N}} \subset \mathbb{Q}$ such that the following holds:*

- $\tilde{v}_n = a_n + \frac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{N}$,
- $a_n = \lfloor \alpha_n \rfloor \leq \lambda_q^\alpha$,
- $q_n, p_n \leq \lambda_q^{d(1+\alpha)+2}$,
- $0 \leq \alpha_n - \tilde{v}_n \leq \frac{2}{\lambda_q^{d(1+\alpha)+2}}$,

for any $n \in \mathbb{N}$.

Proof. Fix α_n , we define $a_n := \lfloor \alpha_n \rfloor$ and $\bar{\alpha}_n := \alpha_n - a_n \in [0, 1)$. We want to approximate $\bar{\alpha}_n$ with dyadic numbers. We define $\ell := \max\{N \in \mathbb{N} : 2^N \leq \lambda_q^{d(1+\alpha)+2}\}$. Since the dyadic intervals are such that

$$\bigcup_{i=0}^{2^\ell-1} \left[\frac{i}{2^\ell}, \frac{i+1}{2^\ell} \right) = [0, 1)$$

there exists $i = 0, \dots, 2^\ell$ such that $\bar{\alpha}_n \in \left[\frac{i}{2^\ell}, \frac{i+1}{2^\ell} \right)$, defining $p_n = i$ and $q_n = 2^\ell$, we get the thesis. \square

Proposition 6.3. *Consider a finite number of disjoint sets Λ_i^j for $i = 1, \dots, N$, $j = 1, 2$, as in Lemma 2.1 and their union $\Lambda := \bigcup_{i=1}^N \bigcup_{j=1}^2 \Lambda_i^j \subset \mathbb{R}^2$. Let $\{w_n\}_{n=1, \dots, \lambda_q^{d(1+\alpha)+2}} \subset \mathbb{R}$ satisfy*

$$w_n = a_n + \frac{p_n}{q_n}$$

where a_n, q_n, p_n are positive integers and they are less or equal than $\lambda_q^{d(1+\alpha)+2}$. Then there exists a constant $c_0 := c_0(\bar{C}, \Lambda) > 0$ with the following property: for every $\xi \in \Lambda$ and $n \in \mathbb{N}$ there exists $a_{\xi, n} \in [0, 1]$ such that the family of curves

$$x_{\xi, n}(t) := (w_n t + a_{\xi, n})\xi \quad \text{with } \xi \in \Lambda, n = 1, \dots, \lambda_q^{d(1+\alpha)+2} \quad (6.4)$$

satisfies

$$d_{\mathbb{T}^2}(x_{\xi, n}(t), x_{\xi', m}(t)) \geq \frac{c_0}{(\lambda_q^{d(1+\alpha)+2})^4} \quad \text{for every } t \geq 0, \text{ when } \xi \neq \xi'. \quad (6.5)$$

Proof. We fix c_0 such that

$$C c_0 |\Lambda| < 1.$$

We define the following sets

$$A_{\xi, \xi', n, m} := \left\{ s \in [0, 1] : d_{\mathbb{T}^2}((w_n t + s)\xi, (w_m t + s)\xi') \geq \frac{c_0}{(\lambda_q^{d(1+\alpha)+2})^4} \right\}$$

for $\xi, \xi' \in \Lambda$ and $n, m = 1, \dots, \lambda_q^{d(1+\alpha)+2}$.

We define

$$A := \bigcap_{n, m=1}^{\lambda_q^{d(1+\alpha)+2}} \bigcap_{\xi \neq \xi' \in \Lambda} A_{\xi, \xi', n, m}$$

and the thesis will follow by proving that A is not empty. We claim that $\mathcal{L}^1(A) > 0$. Using Lemma 6.1 we notice that the measure of the complement of the set $A_{\xi, \xi', n, m}$ satisfies $\mathcal{L}^1(A_{\xi, \xi', n, m}^c) \leq \frac{C c_0 (\lambda_q^{d(1+\alpha)+2})^2}{(\lambda_q^{d(1+\alpha)+2})^4} = \frac{C c_0}{(\lambda_q^{d(1+\alpha)+2})^2}$. Then

$$\begin{aligned} \mathcal{L}^1(A^c) &= \mathcal{L}^1 \left(\bigcup_{n, m=1}^{\lambda_q^{d(1+\alpha)+2}} \bigcup_{\xi \neq \xi' \in \Lambda} A_{\xi, \xi', n, m}^c \right) \\ &\leq \sum_{n, m=1}^{\lambda_q^{d(1+\alpha)+2}} \sum_{\xi \neq \xi' \in \Lambda} \mathcal{L}^1(A_{\xi, \xi', n, m}^c) \leq (\lambda_q^{d(1+\alpha)+2})^2 |\Lambda| \frac{C c_0}{(\lambda_q^{d(1+\alpha)+2})^2} \leq C c_0 |\Lambda| < 1, \end{aligned}$$

and so A is not empty. □

6.1. Disjointness of the supports. Set $w_n := \mu_{q+1}^{d/p'} v_n^{1/p'}$, where the sequence $\{\tilde{v}_n\}_{n=1, \dots, \lambda_q^{d(1+\alpha)+2}}$ is given by Lemma 6.2 applied to the sequence $\alpha_n = (\frac{n}{k})^{1/p'}$ and $v_n = \tilde{v}_n^{p'}$ (notice that the assumption $\alpha_n \leq \lambda_q^{d(1+\alpha)+2}$ is satisfied thanks to the bound (3.14)). We apply Proposition 6.3 to $\{w_n\}_{n=1, \dots, \lambda_q^{d(1+\alpha)+2}}$ (notice that the assumptions are satisfied in view of Lemma 6.2) obtaining the family $\{a_{\xi, n} : \xi \in \Lambda, n = 1, \dots, \lambda_q^{d(1+\alpha)+2}\}$. Finally, starting from the building blocks introduced in Section 2.3, we define

$$W_{\xi, \mu_{q+1}, v_n}(t, x) := \tilde{W}_{\xi, \mu_{q+1}, v_n}(t, x - a_{\xi, n} \xi), \quad \Theta_{\xi, \mu_{q+1}, v_n}(t, x) := \tilde{\Theta}_{\xi, \mu_{q+1}, v_n}(t, x - a_{\xi, n} \xi), \quad (6.6)$$

for any $n = 1, \dots, \lambda_q^{d(1+\alpha)+2}$ and $\xi \in \Lambda$.

We now show that

$$W_{\xi, \mu_{q+1}, v_n} \cdot \Theta_{\xi', \mu_{q+1}, v_m}(t, x) = 0 \quad \text{for any } (x, t) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad (6.7)$$

for any $\xi \neq \xi' \in \Lambda$, $n, m = 1, \dots, \lambda_q^{d(1+\alpha)+2}$.

Indeed for any fixed $t \geq 0$ one has the inclusions

$$\text{supp } W_{\xi, \mu_{q+1}, v_n}(t, \cdot) \subset B_{2\rho\mu_{q+1}^{-1}}(tw_n\xi + a_{\xi, n}\xi), \quad \text{supp } \Theta_{\xi', \mu_{q+1}, v_m}(t, \cdot) \subset B_{\rho\mu_{q+1}^{-1}}(tw_m\xi' + a_{\xi', m}\xi'), \quad (6.8)$$

hence we just need to check that $B_{2\rho\mu_{q+1}^{-1}}(tw_n\xi + a_{\xi, n}\xi) \cap B_{\rho\mu_{q+1}^{-1}}(tw_m\xi' + a_{\xi', m}\xi') = \emptyset$. Proposition 6.3 guarantees

$$d_{\mathbb{T}^2}(tw_n\xi + a_{\xi, n}\xi, tw_m\xi' + a_{\xi', m}\xi') \geq \frac{c_0}{(\lambda_q^{d(1+\alpha)+2})^4},$$

hence the claim is proved provided

$$\frac{3}{4}\mu_{q+1}^{-1} \leq \frac{c_0}{(\lambda_q^{d(1+\alpha)+2})^4}. \quad (6.9)$$

(6.9) follows from our choice of $\mu_{q+1} = \lambda_q^{b\gamma}$, because $\gamma > 1$ and $b > 4(d(1+\alpha)+2)$.

6.2. Proof of the Proposition 3.2 in the case $d=2$. The estimates up to Section 4.2 are done in the same way by observing that $\tilde{v}_n = v_n^{1/p'}$ and $(n/\kappa)^{1/p'}$ are comparable up to a factor 2. In Section 4.3, we computed the product $\theta_{q+1, i}^{(p)} w_{q+1}^{(p)}$ in (4.12) with which we were able to compensate the old error $R_{\ell, i}$ (for $i = 1, \dots, N$). Now this product has the form

$$\theta_{q+1, i}^{(p)} \sum_{k=1}^N w_{q+1, k}^{(p)} = \theta_{q+1, i}^{(p)} w_{q+1, i}^{(p)} = \sum_{n \geq 12} \sum_{\xi \in \Lambda_i^{[n]}} \chi(\kappa|R_{\ell, i}| - n) a_{\xi} \left(\frac{R_{\ell, i}}{|R_{\ell, i}|} \right) (\Theta_{\xi, \mu_{q+1}, v_n} W_{\xi, \mu_{q+1}, v_n})(\lambda_{q+1} t, \lambda_{q+1} x), \quad (6.10)$$

as a consequence of (6.2), (3.14), the fact that $\chi \cdot \bar{\chi} = \chi$ and $\chi(\kappa|R_{\ell, i}| - n) \cdot \chi(\kappa|R_{\ell, i}| - m) = 0$ when $|n - m| > 1$.

Since the average of $\Theta_{\xi, \mu_{q+1}, v_n} W_{\xi, \mu_{q+1}, v_n}$ which appears from the forth line of formula (4.12), in the definition of R^{quadr} and in m is now $v_n \xi$ rather than $n/\kappa \xi$, the definition of $\tilde{R}_{\ell, i}$ should now be replaced by

$$\tilde{R}_{\ell, i} := \sum_{n \geq 12} \chi(\kappa|R_{\ell, i}| - n) \frac{R_{\ell, i}}{|R_{\ell, i}|} v_n,$$

and the obvious modification takes place for the definition of R^{quadr} and m . Observing that $|v_n - \frac{n}{k}| \leq p' \frac{n}{k} |v_n^{1/p'} - (\frac{n}{k})^{1/p'}| \leq p' \lambda_q^{d(1+\alpha)+2} \frac{\delta_{q+2}}{40p'} \frac{2}{\lambda_q^{d(1+\alpha)+2}} = \frac{\delta_{q+2}}{20}$ the estimate (4.13) now works analogously to give $|R_{\ell, i} - \tilde{R}_{\ell, i}| \leq \frac{16}{20} \delta_{q+2}$. The rest of the estimates work as in Sections 4.3, 4.4 and 4.5.

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