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Abstract This rather theoretical study with practical applications for engineers and physicists is essentially based on both the use of the topological notion of the Euler characteristic and the Gauss-Bonnet theorem to establish various analytical formulas concerning firstly the **axisymmetric capillary bridges distortions**. In differential geometry, the Gauss-Binet-Bonnet formula connects the integral of the Gauss curvature of a compact two-dimensional surface with boundaries to its topology. It is a suitable and elegant mathematical tool for calculating exactly the value of the bending stress and highlighting its parameters and their respective influence.
We establish rigorously in particular the fact that the bending effects measured *via* the bending stress are proportional to the sum of the cosine values of the two contact angles (possibly distinct, with the same sign or opposite signs depending on the nature of the supports and the affinity between the liquid and the solids). We specify the exact expression of the proportionality constant: a way to structure, on the basis of this quantified criterion, a classification, the right cylinders being then the borderline cases, without bending stress.

This returns to the delicate question of the formation of the contact angles, often spontaneous values and *a priori* unknowns of the problem. The importance of the contact angles is thereby mathematically evident in the writing of the boundary conditions associated to the generalized Young-Laplace equation and in the expression of the bending stress.
The method can be generalized beyond the axisymmetric capillary bridges case. For that, one has to associate topological and analytical geometries to calculate the total signed geodesic curvature of the boundaries, not necessarily circular. The value of the contact angles again plays an essential role.

Keywords: Distortion of capillary bridges · Mean and Gaussian curvatures impact · Generalized Young-Laplace equation · Bending effects · Gauss-Bonnet Theorem.

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1 Foreword and various definitions

1.1 About the generalized Young-Laplace equation

The generalized Young-Laplace equation concerns the strong distorsions for which the bending effects are modeled by an additional curvature-related term: the introduction of C_K, a multiplier coefficient of the Gaussian curvature K, at the dimension of a force and standing for the bending stress [1][9][13][37]. Under appropriate boundary conditions, the shape of the free interface is then described by the so-called generalized Young-Laplace equation, thus involving both mean and Gaussian curvatures. The resulting strongly nonlinear differential equation, at the downward vertical measurement x linked to the value Δp_0 at $x = 0$ (a spontaneous unknown value) comes in the following form [15][21][31][35][36]:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c \ N} = \Delta p_0 - \Delta \rho \ g x \ . \tag{1}$$

In (1), the force C_K divided by the area $\rho_c N$ stands *via* a pressure for the local bending stress, ρ_c and N being the principal radii of curvature (evaluated algebraically, positively when the curvature is turned into the interior of the capillary bridge) and the pressure deficiency is Δp_0 at $x = 0$.

The bending stress over the free surface Σ is then represented in the following integral form, at the dimension of a force

$$\mathcal{E}_{bending \ stress} = C_K \int_{\Sigma} K \ d\Sigma$$

where K is the Gaussian curvature of the free surface Σ, intrinsic value, in particular independent of the choice of the unit normal vector.

We want to obtain an explicit expression of this integral, of simple and immediate use for the experimenter, and highlight the determining parameters and their respective influence.

Let us limit ourselves here, temporarily, for didactic purposes, to the very particular case of **axisymmetric bridges**.

In the particular case of **surfaces of revolution**, taking then if necessary into account the effects of gravity, *via* an over-pressure, results conventionally in the modified nonlinear differential equation for the distorted profile $x \to Y(x)$, according to the densities difference between the liquid and the surrounding fluid $\Delta \rho = \rho_{int} - \rho_{ext}$, a quantitated balance between the surface tension, the bending stress, often ignored without justification, and gravity forces (see [9] for example):

$$\frac{Y''(x)}{\left(1 + Y'^2(x)\right)^{3/2}} - \frac{C_K}{\gamma} \frac{1}{Y(x) \sqrt{1 + Y'^2(x)}} = -\frac{\Delta p_0}{\gamma} + \frac{g \ \Delta \rho}{\gamma} x. \tag{2}$$
$$=: H + Bx \ , \ x \in I.$$

Recall [10][34], that this strongly nonlinear differential equation is mathematically isomorphic (the same structure) but with different variables and physical units, to the **Gullstrand equation** of geometrical optics, which relates the optic power P'_{op} of a thick lens (in diopters,

the reciprocal of the equivalent focal length) to its geometry and the properties of the media. For example, the superficial tension γ is equivalent to the refractivity $\frac{n_1}{n_2} - 1$, where n_i is a refractive index, C_K is analogous to the expression $-\left(\frac{n_1}{n_2} - 1\right)^2 \frac{n_2}{n_1} d$, where d is the lens thickness and Δp corresponds to P'_{op}.

Shear or free energy problems and the longitudinal bending stress of ship hulls have also an analogy with the subject [4][30][33][40].

1.2 Reminder on a behavioral classification of the common free capillary surfaces

Synclastic surfaces (dome shape) are those in which the centers or curvature are on the same side of the surface (dome shape or elliptic surface). The Gaussian curvature is everywhere strictly positive. For examples among the Delaunay constant mean curvature surfaces of revolution (see *in* a synoptic table for identifying the capillary bridges of revolution [8][23]): a portion of unduloid, catenoid or nodoid with concave upper meridian, the axis of the bridge being horizontal.

This is opposed to **anticlastic surfaces** (saddle shape), which are those in which the centers of curvature are located on opposing sides of the surface (saddle shape or hyperbolic surface for the confined liquid). In that case, the Gaussian curvature is everywhere strictly negative. As an example, we may consider a portion of unduloid, catenoid, nodoid, or sphere with convex upper meridian, the axis of the bridge being horizontal. It is then not mathematically correct to say without further information that a nodoid is an anticlastic surface [10].

Under the condition $\frac{1}{\rho_c} + \frac{1}{N} \neq 0$, excluding the minimal surfaces, the quotients that occur in the generalized Young-Laplace equation is

$$\left| \frac{\frac{1}{\rho_c\, N}}{\frac{1}{\rho_c} + \frac{1}{N}} \right| \; i.e. \; \frac{1}{|\rho_c + N|} \text{ and the scalar } \frac{|C_K|}{\gamma} \frac{1}{|\rho_c + N|} \; ,$$

allow also in a certain way to assess the relative importance of bending effects in relation to the consequences of the surface tension [10]. It is then a question of studying the variations of the function $(\rho_c, N) \rightarrow |\rho_c + N|$.

The Gaussian curvature of a right circular cylinder is everywhere equal to zero and therefore accordingly this case is outside the scope of this study.

By placing oneself out of gravity for a simple illustration, it appears that, as expected by experience, the bending is to be considered in the case of the anticlastic surfaces and are of little importance for synclastic surfaces.

Hence, the common horizontal axis nodoid with convex upper meridian is certainly sensitive to bending effects compared to the tension effects related to the small free surface (it is the opposite for a convex nodoid, *i.e.* with concave upper meridian). The Gaussian curvature of a right circular cylinder is everywhere equal to zero and therefore, this case is a borderline case for this study.

1.3 Homotopic surfaces and Euler characteristic

Recall that the Euler characteristic is a topological invariant, an integer that describes, according to precise axiomatic principles, the shape or a structure of a topological space regardless of how it is bent (invariance by homeomorphy). It is commonly denoted by χ, $\chi(M)$... As examples for surfaces: $\chi = 2$ for a sphere, $\chi = 4$ for two spheres (not connected), $\chi = 0$ for a torus, and $\chi = -2$ for a two-holed torus...)

The Euler characteristic of the right cylinder is $\chi = 0$, so that also for a cylinder with one or two boundaries.

This invariance property makes it a providential tool in the context of this study on the bending effects, associated to the Gauss–Bonnet theorem, a deep relationship between surfaces in differential geometry, connecting the Gaussian curvature of a surface to its Euler characteristic.

These following free surfaces with two circular boundaries and whose meridian is an arc of Delaunay roulette are considered topologically equivalent (same common topological genus), because it is possible to continuously move one to obtain the other: portion of convex nodoid, portion of convex catenoid or unduloid, right circular cylinder (transition convex-concave), portion of concave nodoid or unduloid, portion of a sphere.

It is the same for their continuous axisymmetric smooth deformations by distorting effect of bending or gravity [21][24].

Accordingly, these axisymmetric surfaces have in common the same Euler characteristic, in this case, the value zero (indeed, the cylindrical surface of a circular cylinder with two open boundaries has Euler characteristic 0).

In the case of the consideration of bending effects, one must also introduce the expression of the variation of the bending stress *via* the Gaussian curvature K in the integral form, over the free surface, at the dimension of a force:

$$\mathcal{E}_{bending\ stress} = C_k \int_{\Sigma} K \ d\Sigma \ .$$

2 Practical exact expression of bending stress

2.1 The Gauss-Bonnet theorem

The Gauss-Bonnet theorem is reputed to be one of the most profound and elegant results of the study of surfaces [2][3][5]. It is used in sectors of activity where the problems of bending beams surely arises (civil engineering, naval architecture, shell theory to predict the stress and the displacement arising in an elastic shell [5][16][17][18], *etc...*).

In fact, it unexpectedly links two completely different ways of studying a surface: one geometric, the other topological. It is formulated as follows:

For any compact, boundaryless two-dimensional Riemannian manifold Σ, the integral of the Gaussian curvature K over the entire manifold with respect to area measure is 2π times the Euler characteristic of Σ, also called the Euler number of the manifold, i.e.

$$\int_{\Sigma} K \; d\Sigma = 2\pi\chi\left(\Sigma\right).$$

For example, for a sphere Σ of radius R in $\mathcal{R}^3$, it comes:

$$\int_{\Sigma} K \; d\Sigma = \frac{1}{R^2} \; 4\pi R^2 = 4\pi \quad \text{and here } \chi\left(\Sigma\right) = 2.$$

Suppose now that M is a compact two-dimensional Riemannian manifold with a boundary δM and let k_g be the signed geodesic curvature of δM. Then, in nondimensional writing,

$$\int_{M} K \; dM + \int_{\delta M} k_g ds = 2\pi\chi\left(M\right). \tag{3}$$

The **geodesic curvature** k_g of an arbitrary curve at a point P on a smooth surface is defined as the curvature at P of the orthogonal projection of the curve onto the plane tangent to the surface at P. A curve whose geodesic curvature is zero is a geodesic.

2.2 Expression of the Gauss-Bonnet formula for capillary bridges

2.2.1 The concave capillary bridges.

Such a liquid bridge is concave if it does not contain all the line segments connecting any pair of its points. The Gaussian curvature is then strictly negative (for example, some portions of nodoids, catenoids, unduloids with convex meridian).

We introduce at any point P of the circular edge of radius R, the contact angles values after distortion effects, eventually distinct θ_1 and θ_2. In the case of a concave bridge, θ_1 and $\theta_2 \in \left[0, \frac{\pi}{2}\right]$.

The use of the Gauss-Bonnet theorem is thus greatly simplified, reduced to considering orthogonal projections of circles (representing the contact lines) on an inclined tangent plane, to the knowledge of the curvature at the vertices of an ellipse in order to take into account the total geodesic signed curvatures of the boundaries.

Recall that in an ellipse with major axis $2a$ and minor axis $2b$, the vertices on the major axis have the smallest radius of curvature of any points, $R_{\min} = \dfrac{b^2}{a}$, and the vertices on the minor axis have the largest radius of curvature of any points, $R_{\max} = \dfrac{a^2}{b}$.

Consequently, the geodesic curvature at a point P on the boundary of the capillary bridge is equal to $\dfrac{1}{R}\cos\theta_i$, and, by integration along each circle, the **total geodesic curvature** of the boundaries δM is then

$$\int_{\delta M} k_g ds = 2\pi\left(\cos\theta_1 + \cos\theta_2\right)$$

According to (3) with $\chi(M) = 0$, the bending stress can be expressed very easily as:

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = -2\pi C_k \left(\cos\theta_1 + \cos\theta_2\right). \tag{4}$$

For symmetric profiles with $\theta_1 = \theta_2 = \theta$, we obtain then a relationship between the contact angle θ and the bending stress:

$$\theta = -\arccos \frac{\displaystyle\int_M K\ dM}{4\pi}.$$

Coming back to (4) with the same contact angle θ, the relative finite variation of the bending stress, as function of the contact angle θ, is then given by the formula:

$$\frac{\delta\left(\mathcal{E}_{bending\ stress}\right)}{\mathcal{E}_{bending\ stress}} = -\tan\theta \quad \delta\theta \quad , \theta \neq \tfrac{\pi}{2}.$$

2.2.2 The convex capillary bridges.

Such a liquid bridge is convex if it contains all the line segments connecting any pair of its points. The Gaussian curvature is then strictly positive (portion of sphere, some portions of nodoid or unduloid with concave meridian, *etc...*). At any point P, the free surface is bending away from the tangent plane in all tangent directions at P (local dome shape).

In the case of a convex bridge, we introduce at any point P of the circular edge of radius R the contact angles, eventually distinct θ_1 and θ_2 ; then θ_1 and $\theta_2 \in \left[\frac{\pi}{2}, \pi\right]$.

The use of the Gauss-Bonnet theorem is so again greatly simplified, reduced to introduce orthogonal projections of circles on a plane, to the knowledge of the curvature at the vertices of an ellipse in order to take into account the total geodesic curvatures of the boundaries. Consequently, the curvature at a point P on the boundary of the capillary bridge is still equal to $\dfrac{1}{R}\cos\theta_i$ and the total geodesic curvature is then $2\pi\left(\cos\theta_1 + \cos\theta_2\right)$, as for concave capillary bridges.

The bending stress along the free surface Σ is then presented in the following form, very easy to use for the experimenter:

$$C_k \int_\Sigma K\ d\Sigma = -2\pi C_k \left(\cos\theta_1 + \cos\theta_2\right).$$

2.2.3 Axisymmetric bridges with changes of concavity

Such a situation requires that the profile has one or more inflection points. The methods of Euler's characteristic associated to the Gauss-Bonnet theorem apply immediately to these cases. These axisymmetric surfaces have in common the same Euler characteristic, in this case, the value zero. It then comes again with adjusted data:

$$\mathcal{E}_{bending\ stress} = C_k \int_\Sigma K\ d\Sigma = -2\pi C_k \left(\cos\theta_1 + \cos\theta_2\right)$$

2.2.4 The prevailing role of the contact angles

It is very important to quote that, for any profile of capillary bridges, all the factors determining the contact angle have an influence on the bending stress (in particular surface roughness and heterogeneity, influence of gravity, contact angle hysteresis) [14][38][39].

It is well known that the contact angle value is determined by the balance between adhesive and cohesive forces on the rigid supports. As the tendency of a drop to spread out over a flat solid surface increases, the contact angle decreases. Thus, the contact angle provides an inverse measure of wettability.

A low contact angle less than $\frac{\pi}{2}$(low contact angle) usually indicates that wetting of the surface is very favorable, and the fluid will spread over a large area of the surface. Contact angles greater than $\frac{\pi}{2}$ (high contact angle) generally mean that wetting of the surface is unfavorable. In this context, the case of the right cylinders is still a borderline case. It should be observed that a certain number of terms of the generalized Young-Laplace equation are spontaneous values, resulting from instantaneous equilibrium, and are therefore implicit unknowns. This is a difficulty for the mathematical resolution of this nonlinear differential boundary problem.

3 Explicit parameterization of the distorted profile

The qualitative results elaborated in the framework of the constant mean curvature theory [8] are essentially based on the existence of an exact invariant (in fact, a first integral for the second order nonlinear differential equation which reveals the conservation law for the total energy of the free surface). With minor adaptations, they are immediately applicable to the situation where the Gaussian curvature and bending effects are taken into account. Indeed, as we will see, we still highlight in this case a first integral for the generalized Young-Laplace equation by limiting ourselves to a presentation concerning essentially any bridge with strictly negative Gaussian curvature.

For the spontaneous but *a prior* unknown value of H, $H = H_B$, the generalized Young-Laplace equation can be rewritten, with intuitive notations, in the differential form:

$$-\frac{1}{Y}\frac{d}{dY}\left(\frac{Y}{\sqrt{1+Y'^2}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2}\right)=H+Bx^{\pm}\left(Y\right).$$

Hence, along each concerned branch of the distorted profile, the two expressions are constant and equal, at the dimension of a force. For an example, for any $x \geq X^*$, (X^*, Y^*) being the coordinates of the moved neck,

$$\mathcal{F}^{+}_{C_K}=2\pi\gamma\left(\frac{Y\left(x\right)}{\sqrt{1+Y'^2\left(x\right)}}-\frac{C_K}{2\gamma}\frac{1}{1+Y'^2\left(x\right)}+\frac{H}{2}Y^2\left(x\right)+B\int_{Y^*}^{Y(x)}x^{+}\left(y\right)ydy\right).$$

is constant at over the profile $x \geq X^*$. Such a first integral allows to obtain parameterizations of the distorted profiles and information on constant functional expressions [9].

For the convenience of the reader and for a fairly comprehensive presentation, it is interesting to note that when considering, *via* a first integral, the only bending effects (*i.e.* $C_K \neq 0$, $B = 0$), then for any axisymmetric capillary bridge, the interparticle capillary force

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma \left(\frac{Y}{\sqrt{1+Y'^2}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2} + \frac{H}{2} Y^2 \right) \tag{5}$$

is constant at all points of the profile. Then, as proved in [10],

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma Y(x) \cos\Theta(x) - \pi C_K \cos^2\Theta(x) + \pi\gamma H Y^2(x)$$

where $\Theta(x)$ is the easily calculable angle made by the tangent vector to the meniscus with the $x-$axis at the generic point $(x, Y(x))$.

This latter relationship is interesting because it allows to easily obtain a parameterization of the profile *via* Θ by generalizing formulas established in an other way (by pure geometry) by C.H. Delaunay in 1841 [6], p. 313. By a direct calculation and solving a quadratic equation if $H \neq 0$, we get at $H > 0$ (see for example [9][10]):

$$Y(\Theta) = Y\left(\Theta, \frac{C_K}{\gamma}\right), \text{ the positive root such that } Y\left(0, \frac{C_K}{\gamma}\right) = Y^*,$$

i.e.

$$Y(\Theta) = \frac{1}{H}\left(-\cos\Theta + \sqrt{(HY^*+1)^2 - \left(1 + H\frac{C_K}{\gamma}\right)\sin^2\Theta} \right),$$

$$x(\Theta) = X^* + \int_0^{\Theta} dY(\theta)\cot(\theta)\, d\theta, \quad i.e.,$$

$$x(\Theta) = X^* + \frac{1}{H}\left(\sin\Theta + \left(1 + H\tfrac{C_K}{\gamma}\right) \int_0^{\Theta} \frac{\cos^2\theta \; d\theta}{\sqrt{(HY^*+1)^2 - \left(1 + H\frac{C_K}{\gamma}\right)\sin^2\theta}} \right).$$

4 The general case and its implementation

In the general cases of non-axisymmetric capillary bridges between two supports, possibly of distinct natures, the method remains applicable in principle. The difficulty is not conceptual in dealing with the general case but rather calculative. We must then, in any given case, engage in a delicate exercise in differential and analytical geometries to explicitly calculate the total signed geodesic curvature of the boundaries by the classical methods of analytical geometry.

The calculation procedure is as follows.

- At any point P of the border liquid-solid, one considers the tangent plane in P to the free surface (that supposes an adequate local regularity). One then considers the orthogonal projection of each edge into this tangent plane. The curvature in P of the

projected curve is then calculated, what introduces the important role of the cosine of the local contact angle[3]. The geodesics correspond to zero values. The academic literature provides various classical formulas for calculating geodesic curvature (*cf.* also [19] on the theme of interpolation and blending on surfaces, [22] for discrete curvature estimation methods).

- Multiplied by the coeffient $(-C_K)$ at the dimension of a force, the dimensionless integral of these curvature values along the reunion of the two contact edges gives the value of the resulting bending stress.
- When the dimensioned constant term C_K is considered as a missing value to be restored, it is possible to obtain the result by missing data recovery using the first integral [10], or by identification of constants by inverse problem or experimental measurement [23][25]-[29].

5 Conclusion

For didactic purposes, we limited ourselves to the detailed case of axisymmetric capillary bridges to show the interest of the simultaneous use of the Gauss-Binet-Bonnet formula and of the topological notion of Euler characteristic to evaluate the importance of the bending stress. The obtained result clearly shows in an explicit way, the major role of the contact angles values (eventually distinct) after distortion effects. The value of the bending stress depends, besides physical constants, only on the observed values of the contact angles, whereas these angles result in part implicitly from the final equilibrium of the device.

On the other hand, let us insist on the factual interdependence between gravity and bending distortions, for a bridge between two horizontal plates, to fix ideas. When the gravity distortions have significant effects, the low contact line progresses over dry areas while the upper contact line recedes over wet areas, which generally according to the dynamics of contact lines on surfaces, creates an accentuated asymmetry between the contact angles with the adjacent solid surfaces (canthotaxis). It follows then a significant quantitative modification of the bending stress by gravity effect. The evolution of the new equilibrium is not very predictable but easy to observe and measure.

It would be interesting to reconsider, in taking into account the bending effects, the resulting contact angles and the elegant Gauss-Bonnet theorem, a new analytical framework for evaluating the cohesion effects of coalescence between saddle shaped capillary bridges [12]. There are no major issues for generalization of the bending stress calculation to the non-axisymmetric cases.

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