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► **To cite this version:**

Emmanuel Roy. POISSON-ORLICZ NORM AND INFINITE ERGODIC THEORY. 2023. hal-03362759v2

**HAL Id: hal-03362759**

**<https://hal.science/hal-03362759v2>**

Preprint submitted on 23 Jun 2023

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# POISSON-ORLICZ NORM AND INFINITE ERGODIC THEORY

EMMANUEL ROY

ABSTRACT. Urbanik's theorem for a Poisson process on an infinite measure space  $(X, \mathcal{A}, \mu)$  relates integrability of stochastic integrals to a particular Orlicz function space  $L^\Phi(\mu)$  on which the  $L^1$ -norm of the Poisson process induces a norm (called *Poisson-Orlicz* in the sequel) that is shown to be equivalent to the classical gauge and Orlicz norms.

We obtain a full characterization of stochastic integrals using difference operators that, together with a simple duality argument, allows to derive Urbanik's theorem as well as an optimal inequality between the Orlicz and the Poisson-Orlicz norm.

In a second part, we show that the Poisson-Orlicz norm plays a role in infinite Ergodic Theory where it is seen as an alternative to the  $L^1$ -norm to identify several dynamical invariants that the latter fails to identify. We also show that, whereas the  $L^1$ -norm fully characterizes *exact* endomorphisms (Lin's theorem), Poisson-Orlicz norm fully characterizes *remotely infinite* endomorphisms.

## 1. INTRODUCTION

**1.1. The context: Integrability of stochastic integrals, Urbanik's theorem.** Let  $(X, \mathcal{A}, \mu)$  be a non-atomic infinite measure Borel space. To this space, we can always associate a probability space  $(X^*, \mathcal{A}^*, \mu^*)$  (Section 2.1) that naturally supports a Poisson point process with intensity  $\mu$ . This space can be considered as a companion space to  $(X, \mathcal{A}, \mu)$  that offers a probabilistic point of view to this infinite measure object. Our starting point is the observation that some elementary integrability properties on  $(X, \mathcal{A}, \mu)$  are seen through integrability properties of the so-called *stochastic integrals* (Section 2.3) with respect to the Poisson point process seen as a random element of  $(X^*, \mathcal{A}^*, \mu^*)$ , it can be formulated this way: Which functions  $f \in L^0(\mu)$  can be integrated against the Poisson point process? This question is well understood (see [12] for example): upon compensating this point process, a stochastic integral  $I(f)$  is well defined if and only if  $f$  satisfies  $\int_X f^2 \wedge 1 d\mu < +\infty$ . It happens that the set

$$L^X(\mu) := \left\{ f \in L^0(\mu), \int_X f^2 \wedge 1 d\mu < +\infty \right\}$$

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2000 *Mathematics Subject Classification.* 37A40, 37A50, 28D05, 60G55.

*Key words and phrases.* Infinite ergodic theory, Poisson process, stochastic integral, Orlicz space.

is a so-called *generalized Orlicz linear space* (see [15], page 400). It illustrates the following idea: some function spaces over  $(X, \mathcal{A}, \mu)$  have a very natural interpretation in terms of Poisson space. This idea culminates when looking at  $L^2(\mu)$  (Section 3) since we have the equivalence

$$I(f) \in L^2(\mu^*) \iff f \in L^2(\mu),$$

and, setting  $I_1(f) := I(f) - \mathbb{E}_{\mu^*}[I(f)]$ , we have

$$\|I_1(f)\|_{L^2(\mu^*)} = \|f\|_{L^2(\mu)}.$$

This has tremendous consequences as it yields the decomposition of  $L^2(\mu^*)$  as the Fock space over  $L^2(\mu)$  and this plays a great role in quantum physics, stochastic geometry, Poisson-Malliavin calculus, etc... Less neglected and our main focus in this paper is the integrable case (Section 4): Letting  $\Phi$  be defined on  $\mathbb{R}_+$  by

$$x \mapsto \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases},$$

$\Phi$  is then a so-called *Young function* of the Orlicz space  $L^\Phi(\mu)$

$$L^\Phi(\mu) := \left\{ f \in L^0(\mu), \int_X \Phi(|f|) d\mu < \infty \right\}$$

that turns to be a Banach space when endowed with one of the two naturally defined equivalent norms, the *gauge* norm  $N_\Phi$  or the *Orlicz* norm  $\|\cdot\|_\Phi$  associated to  $\Phi$ . It can then be verified that

$$I(f) \in L^1(\mu^*) \iff f \in L^\Phi(\mu),$$

and we have the remarkable result of Urbanik that we rephrase in our context:

**Theorem 1.** (*Urbanik, [18]*) *The formula*

$$\|f\|_* := \|I_1(f)\|_{L^1(\mu^*)}$$

*defines a norm on  $L^\Phi(\mu)$  that is equivalent to the gauge or Orlicz norm. In particular  $(L^\Phi(\mu), \|\cdot\|_*)$  is a Banach space.*

The Young function defining  $L^\Phi(\mu)$  is highly non-unique and so are the gauge or Orlicz norms, that is, replacing  $\Phi$  by another appropriately chosen *Young function* would lead to another equivalent gauge or Orlicz norm on the same space  $L^\Phi(\mu)$ . In contrast,  $\|\cdot\|_*$  only depends on the measure  $\mu$ , we propose to call this norm the *Poisson-Orlicz* norm.

**1.2. A characterization of stochastic integrals, a proof of Urbanik's theorem with an optimal bound.** Our first result consists into giving a complete characterization of stochastic integrals (Theorem 15) by means of *difference operators* (which is a popular tool used in the quantum physics usage of Poisson processes) that is the full generalization of a result in [8] that was restricted to the  $L^2$ -case and might have an interest on its own.

As a nearly immediate consequence, we get the “Banach space part” of Urbanik’s theorem. The equivalence of norms being derived by very simple duality considerations, leading to an optimal inequality: for any  $f \in L^\Phi(\mu)$ ,

$$\|f\|_* \leq \|f\|_\Phi.$$

We observe also that from the general theory of Orlicz space ([15]), for any  $f \in L^\Phi(\mu)$ ,

$$\|f\|_\Phi \leq 2N_\Phi(f)$$

that therefore yields

$$\|f\|_* \leq 2N_\Phi(f),$$

improving a result of Marcus and Rosinski ([11]) that obtained

$$\|f\|_* \leq (2.125) N_\Phi(f).$$

**1.3. Poisson-Orlicz norm and infinite Ergodic theory.** We then turn to Ergodic Theory, adding a measure preserving transformation  $T$  on  $(X, \mathcal{A}, \mu)$  (and the corresponding  $T_*$  on  $(X^*, \mathcal{A}^*, \mu^*)$ ) to the picture (Section 2.2). Our aim is to show the interest of the Poisson-Orlicz norm within infinite Ergodic Theory allowing to characterize some of the main dynamical invariants.

**1.3.1.  $L^1$ -Birkhoff sums convergence in infinite measure.** We recall this classical fact: if  $(X, \mathcal{A}, \mu, T)$  is an ergodic dynamical system then, whereas the Birkhoff sums,  $\frac{1}{n} \sum_{k=1}^n f \circ T^k(x)$ , associated to any  $f \in L^1(\mu)$  converge pointwise to 0 as  $n$  tends to  $+\infty$  for  $\mu$ -a.e.  $x \in X$  (see [6]), the convergence cannot hold for all  $f$  in  $L^1(\mu)$  with respect to  $\|\cdot\|_1$ . The same result occurs when we replace the ergodicity requirement by the absence of absolutely continuous  $T$ -invariant probability measure. As such the  $(L^1(\mu), \|\cdot\|_1)$ -isometry  $T$  doesn’t belong to the class of “*mean ergodic operators*”:

**Definition 2.** (see Section 8.4, page 136 in [5]) Let  $\varphi$  be a continuous linear operator defined on a Hausdorff topological vector space  $H$ .  $\varphi$  is said to be *mean ergodic* if, for any  $h \in H$ ,  $\frac{1}{n} \sum_{k=1}^n \varphi^k h$  converges as  $n$  tends to  $+\infty$ .

In contrast, replacing  $(L^1(\mu), \|\cdot\|_1)$  by  $(L^p(\mu), \|\cdot\|_p)$ ,  $1 < p < \infty$ ,  $T$  becomes a mean ergodic operator and one has:

**Theorem.** *The dynamical system  $(X, \mathcal{A}, \mu, T)$  has no absolutely continuous  $T$ -invariant probability measure if and only if, for every  $f \in L^p(\mu)$ :*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \rightarrow_{\|\cdot\|_p} 0,$$

as  $n$  tends to  $+\infty$ .

The fact that  $T$  is a mean ergodic operator on  $(L^p(\mu), \|\cdot\|_p)$ ,  $1 < p < \infty$ , follows, for example, by the fact that they are, unlike  $(L^1(\mu), \|\cdot\|_1)$ , reflexive spaces, by a result due to Lorch (1939).

Let us illustrate another instance where  $\|\cdot\|_1$  can't help in identifying another invariant:

**Definition.**  $(X, \mathcal{A}, \mu, T)$  is of *zero type* if for every sets  $A$  and  $B$  in  $\mathcal{A}_f$ :

$$\mu(A \cap T^{-n}B) \rightarrow 0$$

as  $n$  tends to  $+\infty$ ,

We recall this classical result:

**Theorem.** (Blum-Hanson, [3]) *Let  $(\Omega, \mathcal{F}, \mathbb{P}, S)$  be a probability measure-preserving dynamical system and let  $1 \leq p < \infty$ . Then  $(\Omega, \mathcal{F}, \mathbb{P}, S)$  is mixing if and only if, for every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of integers and every  $F \in L^p(\mathbb{P})$ :*

$$\frac{1}{n} \sum_{k=1}^n F \circ S^{n_k} \xrightarrow{\|\cdot\|_p} \mathbb{E}_{\mathbb{P}}[F]$$

as  $n$  tends to  $+\infty$ .

As shown in [7], replacing  $(\Omega, \mathcal{F}, \mathbb{P})$  by an infinite-measure space, “mixing” by “zero type” and the limit by 0, then the theorem holds for  $1 < p < \infty$  but fails of course again for  $p = 1$ .

1.3.2.  $\|\cdot\|_*$  as an intrinsic alternative to  $\|\cdot\|_1$ . Replacing  $\|\cdot\|_1$  by  $\|\cdot\|_*$ , we are then able to prove (Theorem 26):

**Theorem.** *The dynamical system  $(X, \mathcal{A}, \mu, T)$  has no absolutely continuous  $T$ -invariant probability measure if and only if, for every  $f \in L^1(\mu)$  (or  $f \in L^\Phi(\mu)$ ):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \xrightarrow{\|\cdot\|_*} 0$$

as  $n$  tends to  $+\infty$ .

And (Theorem 28):

**Theorem.** *The dynamical system  $(X, \mathcal{A}, \mu, T)$  is of zero type if and only if for every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of integers and every  $f \in L^1(\mu)$  (or  $f \in L^\Phi(\mu)$ ):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \xrightarrow{\|\cdot\|_*} 0$$

as  $n$  tends to  $+\infty$ .

Thus  $\|\cdot\|_*$  helps to get the “correct rate” for the above convergence to characterize these two dynamical invariants.

Then, mimicking the construction of the transfer operator  $\widehat{T}$  on  $L^1(\mu)$  obtained as the predual operator of  $T$  acting on  $L^\infty(\mu)$ , we get an operator  $\widehat{T}_{\mathcal{P}}$  acting on  $L^\Phi(\mu)$  (Definition 33) and we show that not only does it preserve  $L^1(\mu)$ , but it coincides with  $\widehat{T}$  on it (Proposition 34).

We now need to recall these two other definitions (see [7]) to explain our results involving the behavior of  $\widehat{T}$  relatively to  $\|\cdot\|_1$  and  $\|\cdot\|_*$ .

**Definition.**  $(X, \mathcal{A}, \mu, T)$  is said to be

- *remotely infinite* if  $\bigcap_{n \geq 0} T^{-n} \mathcal{A}$  contains only sets of zero or infinite  $\mu$ -measure.
- *exact* if  $\bigcap_{n \geq 0} T^{-n} \mathcal{A} = \{\emptyset, X\}$  mod.  $\mu$ .

In particular, as  $\mu(X) = +\infty$ , we have the chain of implications:

exact  $\implies$  remotely infinite  $\implies$  zero type  $\implies$  absence of a.c.  $T$ -invariant measure.

Set  $L_0^1(\mu) := \{f \in L^1(\mu), \int_X f d\mu = 0\}$ . Lin showed the following characterization of exactness:

**Theorem.** ([9]) *The dynamical system  $(X, \mathcal{A}, \mu, T)$  is exact if and only if for every  $f \in L_0^1(\mu)$ :*

$$\left\| \widehat{T^n f} \right\|_1 \rightarrow 0$$

as  $n$  tends to  $+\infty$ .

Then, adapting the rate with  $\|\cdot\|_*$  we get (Theorem 39):

**Theorem.** *The dynamical system  $(X, \mathcal{A}, \mu, T)$  is remotely infinite if and only if for every  $f \in L^\Phi(\mu)$ :*

$$\left\| \widehat{T_p^n f} \right\|_* \rightarrow 0$$

or equivalently for every  $f \in L_0^1(\mu)$  (or  $f \in L^1(\mu)$ )

$$\left\| \widehat{T^n f} \right\|_* \rightarrow 0$$

as  $n$  tends to  $+\infty$ .

## 2. BACKGROUND ON THE POISSON SPACE AND STOCHASTIC INTEGRALS

**2.1. Poisson space.** Throughout this paper  $(X, \mathcal{A}, \mu)$  is a non-atomic infinite Borel space, that is, measurably isomorphic to the real line endowed with Borel sets (for the usual topology) and Lebesgue measure.

We recall a possible definition of the *Poisson space* (also called *Poisson measure* or *Poisson point process*)  $(X^*, \mathcal{A}^*, \mu^*)$  over  $(X, \mathcal{A}, \mu)$ :

- $X^*$  is the collection of measures of the form  $\nu := \sum_{i \in I} \delta_{x_i}$ ,  $x_i \in X$ ,  $I$  countable.
- $\mathcal{A}^* := \sigma\{N(A), A \in \mathcal{A}\}$  where  $N(A)$  is the map  $\omega \in X^* \mapsto \omega(A)$ .
- $\mu^*$  is the only probability measure such that, for any  $k \in \mathbb{N}$ , any collection  $A_1, \dots, A_k$  of pairwise disjoint sets in  $\mathcal{A}_f$ , the random variables  $N(A_1), \dots, N(A_k)$  are independent and Poisson distributed with parameter  $\mu(A_1), \dots, \mu(A_k)$  respectively.

It can be checked out that, once  $\mathcal{A}^*$  is completed with respect to  $\mu^*$ , up to a negligible set, the Poisson space becomes a Lebesgue probability space (a standard Borel space with the completion of the Borel  $\sigma$ -algebra). In other words, it is a “nice” probability space.

**2.2. Poisson suspension.** If  $T$  is an endomorphism of  $(X, \mathcal{A}, \mu)$  then the map  $T_*$  defined by

$$\omega \in X^* \mapsto \omega \circ T^{-1},$$

is an endomorphism of  $(X^*, \mathcal{A}^*, \mu^*)$ .

The dynamical system  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is the *Poisson suspension* over the base  $(X, \mathcal{A}, \mu, T)$  and we can present the most basic ergodic results, obtained by Marchat ([10]):

**Theorem 3.** *Let  $T$  be an automorphism of  $(X, \mathcal{A}, \mu)$ .*

- *$(X^*, \mathcal{A}^*, \mu^*, T_*)$  is ergodic (and weakly mixing) if and only if  $(X, \mathcal{A}, \mu, T)$  doesn't possess any  $T$ -invariant set of positive and finite  $\mu$ -measure, or equivalently, if there is no absolutely continuous  $T$ -invariant probability measure.*
- *$(X^*, \mathcal{A}^*, \mu^*, T_*)$  is mixing if and only if  $(X, \mathcal{A}, \mu, T)$  is of zero type.*

In particular, as we suppose  $\mu$  infinite,  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is ergodic as soon as  $(X, \mathcal{A}, \mu, T)$  is.

*Remark 4.* Most references on Poisson suspensions deal with automorphisms (see for example [16]) however Zweimüller ([19]) dealt with endomorphisms and proved that the natural extension of the Poisson suspension over some system is the Poisson suspension of the natural extension of this same system. The above theorem is thus valid for endomorphisms as well.

**2.3. Stochastic integrals.** In the following, depending on the context, when dealing with the integral of a function  $g$  with respect to a measure  $\rho$  on the space  $X$ , we'll use the following equivalent notation:

- $\int_X g(x) \rho(dx)$
- $\int_X g d\rho$
- $\rho(g)$

However, we'll stick to the probabilistic notation  $\mathbb{E}_{\mu^*}[F]$  for the expectation of a random variable  $F$  on the probability space  $(X^*, \mathcal{A}^*, \mu^*)$ .

We detail the notion of *stochastic integral* in the Poisson setting whose complete construction can be found in ([12]).

The starting point is the following formula readily available from the definition: for any  $A \in \mathcal{A}_f$

$$\mathbb{E}_{\mu^*}[N(A)] = \mu(A),$$

This allows to safely define such quantity as  $N(f)$  on  $X^*$  for an  $L^1(\mu)$ -function  $f$  on  $(X, \mathcal{A})$ :

$$N(f) : \omega \mapsto \omega(f),$$

at least for  $\mu^*$ -a.e.  $\omega \in X^*$ .

Indeed, if  $f \in L^1(\mu)$ , using standard monotone approximation arguments,  $N(|f|)$  is easily proven to be in  $L^1(\mu^*)$  and satisfies:

$$\mathbb{E}_{\mu^*} [N(|f|)] = \mu(|f|).$$

Therefore, for any  $f \in L^1(\mu)$ ,  $N(|f|)$  is finite  $\mu^*$ -a.e. and thus  $f$  is  $\omega$ -integrable for  $\mu^*$ -a.e.  $\omega \in X^*$  and we have:

$$(2.1) \quad \mathbb{E}_{\mu^*} [N(f)] = \mu(f).$$

Observe that replacing  $f$  by some function  $\tilde{f}$  such that  $f = \tilde{f}$   $\mu$ -a.e. yields two random variables  $N(f)$  and  $N(\tilde{f})$  that are equal  $\mu^*$ -a.e..

We define, for any  $f \in L^1(\mu)$

$$(2.2) \quad I_1(f) := N(f) - \mu(f).$$

It is a centered and integrable random variable.

It is however possible to get further. Set  $\chi(x) = x^2 \wedge 1$ ,  $x \geq 0$  and consider the *generalized Orlicz space* (see [15], page 400):

$$L^\chi(\mu) := \left\{ f \in L^0(\mu), \int_X \chi(|f|) d\mu < +\infty \right\}.$$

Then the *stochastic integral*  $I(f)$ , for  $f \in L^\chi(\mu)$ , is defined as

$$\begin{aligned} I(f) &:= \lim_{\epsilon \rightarrow 0} N(f 1_{|f| > \epsilon}) - \mu(f 1_{|f| \leq 1} 1_{|f| > \epsilon}) \\ &= \lim_{\epsilon \rightarrow 0} I_1\left(f 1_{\epsilon < |f| \leq \frac{1}{\epsilon}}\right) + \mu\left(f 1_{1 < |f| \leq \frac{1}{\epsilon}}\right), \end{aligned}$$

the limit taking place in  $\mu^*$ -measure.

*Remark 5.* Pay attention to the fact that  $I$  is not linear, nevertheless we have:

$$I(\alpha f + g) = \alpha I(f) + I(g) + c$$

for some constant  $c$  depending on  $\alpha$ ,  $f$  and  $g$ .

In particular, we have

$$I(f) = I(\Re f) + iI(\Im f) + d$$

for some constant  $d$  depending on  $f$ .

However, the effect of an endomorphism doesn't involve the addition of a constant, thanks to the fact it preserves the measure:

$$\begin{aligned} I(f) \circ T_* &= \lim_{\epsilon \rightarrow 0} N(f 1_{|f| > \epsilon}) \circ T_* - \mu(f 1_{|f| \leq 1} 1_{|f| > \epsilon}) \\ &= \lim_{\epsilon \rightarrow 0} N(f \circ T 1_{|f \circ T| > \epsilon}) - \mu(f 1_{|f| \leq 1} 1_{|f| > \epsilon}) \\ &= \lim_{\epsilon \rightarrow 0} N(f \circ T 1_{|f \circ T| > \epsilon}) - \mu(f \circ T 1_{|f \circ T| \leq 1} 1_{|f \circ T| > \epsilon}) \\ &= I(f \circ T) \end{aligned}$$



Lastly,  $I(f)$  is constant if and only if  $f$  vanishes  $\mu$ -a.e..

**Definition 6.** A *stochastic integral* defined on  $(X^*, \mathcal{A}^*, \mu^*)$  is a random variable of the form

$$I(f) + c,$$

with  $f$  in  $L^X(\mu)$  and  $c \in \mathbb{C}$ .

In particular, when  $f \in L^1(\mu)$ ,  $I_1(f)$  is a stochastic integral since  $I_1(f) = I(f) - \mu(f1_{|f|>1})$ .

*Remark 7.* It is worth mentioning that a stochastic integral  $I(f) + c$  is always an *infinitely divisible* random variable (on  $\mathbb{C} \simeq \mathbb{R}^2$ ) variable without Gaussian part (see [12] together with Theorem 8.1, page 37 in [17]) whose Lévy measure is the image measure of  $\mu$  by  $f$ , restricted to  $\mathbb{C} \setminus \{0\}$ . Up to the addition of a constant, the Lévy measure completely characterizes the distribution of a stochastic integral.

2.3.1. *Campbell measure and difference operators.* To derive useful properties of stochastic integrals, it is convenient to consider a representation of them in the product space  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*, \mu \otimes \mu^*)$  as it is done in [8] for the  $L^2$ -case. We need however to go beyond  $L^2$  and we have been unable to find all the results we want in the existing literature, we thereafter give proofs of what is, as far as we know, new material.

**Definition 8.** (see for example [4], page 269) The *Campbell measure*  $Q$  of  $(X^*, \mathcal{A}^*, \mu^*)$  is defined on  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$ , for any positive measurable  $\varphi$  by

$$\int_{X \times X^*} \varphi(x, \omega) Q(d(x, \omega)) := \int_{X^*} \left( \int_X \varphi(x, \omega) \omega(dx) \right) \mu^*(d\omega).$$

The important formula is the following:

**Theorem 9.** (*Mecke Formula*, [13]) *For any positive measurable function  $\varphi$  on  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$ :*

$$\int_{X \times X^*} \varphi(x, \omega) Q(d(x, \omega)) = \int_{X \times X^*} \varphi(x, \omega + \delta_x) \mu \otimes \mu^*(d(x, \omega)).$$

This formula simply says that  $Q$  is the image measure of  $\mu \otimes \mu^*$  under the mapping:

$$(x, \omega) \mapsto (x, \omega + \delta_x).$$

*Remark 10.* It is worth noting that, when  $\mu$  is infinite, the projection of  $Q$  along the second coordinate is equivalent to  $\mu^*$  (of course, the same applies to  $\mu \otimes \mu^*$ ). To see that, take  $f > 0$  in  $L^1(\mu)$ , then, as  $\mu$  is an infinite measure, for  $\mu^*$ -almost all  $\omega \in X^*$ ,  $\omega \neq 0$  and thus  $\int_X f(x) \omega(dx) > 0$  for  $\mu^*$ -almost all  $\omega \in X^*$ . We now consider the measure  $\tilde{Q}$  on  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$  defined by  $\frac{d\tilde{Q}}{dQ}(x, \omega) := \frac{f(x)}{\int_X f(s)\omega(ds)}$  so that  $\tilde{Q} \sim Q$ . Now consider its projection  $\bar{Q}$  on  $X^*$ :

$$\begin{aligned}
 \overline{Q}(\mathfrak{A}) &= \int_{X \times X^*} 1_{\mathfrak{A}}(\omega) \frac{f(x)}{\int_X f(s) \omega(ds)} Q(d(x, \omega)) \\
 &= \int_{X^*} 1_{\mathfrak{A}}(\omega) \frac{1}{\int_X f(s) \omega(ds)} \left( \int_X f(t) \omega(dt) \right) \mu^*(d\omega) \\
 &= \mu^*(\mathfrak{A}).
 \end{aligned}$$

Thus  $\overline{Q} = \mu^*$ .

For any measurable  $F$  on  $(X, \mathcal{A}^*)$ , define  $F_\delta$  on  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$  by

$$(x, \omega) \mapsto F(\omega + \delta_x)$$

**Definition 11.** (See [8]) The *difference operator*  $D$  associates a measurable  $F$  on  $(X^*, \mathcal{A}^*)$  to a measurable  $DF$  on  $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$  by

$$(x, \omega) \mapsto (D_x F)(\omega) := F_\delta(x, \omega) - F(\omega).$$

It has to be noted that, taking  $F = \tilde{F}$   $\mu^*$ -a.e. implies that  $DF = D\tilde{F}$   $\mu \otimes \mu^*$ -a.e., indeed, consider

$$(D_x F)(\omega) - (D_x \tilde{F})(\omega) = (F - \tilde{F})_\delta(x, \omega) - (F - \tilde{F})(\omega)$$

But  $F = \tilde{F}$   $\mu \otimes \mu^*$ -a.e. and, thanks to Remark 10,  $(F - \tilde{F})_\delta$  is distributed, under  $\tilde{Q} \sim \mu \otimes \mu^*$ , as  $F - \tilde{F}$  under  $\mu^*$  and therefore vanishes  $\mu \otimes \mu^*$ -a.e.

This proves  $DF = D\tilde{F}$   $\mu \otimes \mu^*$ -a.e. and, in particular  $DF$  can be safely defined on  $F$  that are only defined  $\mu^*$ -a.e.

Those operators are well suited for stochastic integrals as we get:

**Proposition 12.** Let  $f$  be in  $L^X(\mu)$ . For  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$ :

$$D_x I(f)(\omega) = f(x)$$

*Proof.* Set  $f_n := f 1_{|f| > \frac{1}{n}}$ ,  $n \geq 1$ , we have  $\mu \{ |f| > \frac{1}{n} \} < \infty$ , hence, for  $\mu^*$ -a.e.  $\omega$ ,  $\omega \{ |f| > \frac{1}{n} \} < \infty$  and we can safely write, for  $\mu \otimes \mu^*$ -a.e.  $(\omega, x)$ :

$$\begin{aligned}
 (I(f_n))_\delta(\omega, x) &= \int_{|f| > \frac{1}{n}} f d(\omega + \delta_x) - \int_{|f| > \frac{1}{n}} f 1_{|f| \leq 1} d\mu \\
 &= \int_{|f| > \frac{1}{n}} f d(\omega) + f(x) 1_{|f| > \frac{1}{n}}(x) - \int_{|f| > \frac{1}{n}} f 1_{|f| \leq 1} d\mu \\
 &= I(f_n)(\omega) + f_n(x)
 \end{aligned}$$

By definition  $I(f_n)$  tends to  $I(f)$  in  $\mu^*$ -measure. In particular, there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  such that  $I(f_{n_k})$  tends to  $I(f)$   $\mu^*$ -a.e. as  $k$  tends to  $+\infty$ .

But this implies, thanks to Remark 10 once again, that  $(I(f_{n_k}))_\delta$  tends to  $(I(f))_\delta$   $\mu \otimes \mu^*$ -a.e. as  $k$  tends to  $+\infty$ .

Therefore

$$(I(f))_\delta(\omega, x) = I(f)(\omega) + f(x),$$

hence the result.

Our next goal is to show the converse, to achieve this we need the following intermediate results.  $\square$

**Proposition 13.** (See [8]) *If  $F \in L^2(\mu^*)$  then  $x \mapsto \mathbb{E}_{\mu^*}[D_x F]$  belongs to  $L^2(\mu)$  and, if there exists a measurable  $f$  on  $(X, \mathcal{A})$  such that  $D_x F(\omega) = f(x)$  for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$  then there exists  $c \in \mathbb{C}$  such that*

$$F = I(f) + c.$$

And:

**Lemma 14.** *Let  $F$  be a measurable on  $(X^*, \mathcal{A}^*, \mu^*)$  such that,  $\mu \otimes \mu^*$ -a.e.,*

$$DF = 0,$$

*then  $F$  is constant  $\mu^*$ -a.e..*

*Proof.* For  $\mu \otimes \mu^*$ -almost all  $(x, \omega) \in X \times X^*$  we have

$$F_\delta(x, \omega) = F(\omega),$$

thus, for every Borel set  $A$ , we get

$$1_A \circ F_\delta(x, \omega) = 1_A \circ F(\omega),$$

thus,  $\mu \otimes \mu^*$ -a.e:

$$D(1_A \circ F) = 0.$$

But  $1_A \circ F \in L^2(\mu^*)$ , therefore it is constant by Proposition 13. Since this is true for all Borel sets, this gives the result.  $\square$

We can now formulate our characterization of stochastic integrals:

**Theorem 15.** *A measurable function  $F$  on  $(X^*, \mathcal{A}^*)$  is a stochastic integral if and only if there exists a measurable function  $f$  on  $(X, \mathcal{A})$  such that for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$ ,*

$$(D_x F)(\omega) = f(x),$$

*in particular there exists  $c \in \mathbb{C}$  such that  $F = I(f) + c$ .*

*Proof.* Only one direction has to be proved, thanks to Proposition 12. Assume  $F$  is a measurable, real valued function  $F$  on  $(X^*, \mathcal{A}^*)$  and there exists a measurable, real valued function  $f$  on  $(X, \mathcal{A})$  such that for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$ ,

$$(D_x F)(\omega) = f(x).$$

For any  $t \in \mathbb{R}$  and for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$  we have

$$\begin{aligned} \exp itF_\delta(x, \omega) - \exp itF(\omega) &= \exp itf(x) \exp itF(\omega) - \exp itF(\omega) \\ &= \exp itF(\omega) (\exp itf(x) - 1). \end{aligned}$$

Therefore

$$\mathbb{E}_{\mu^*}[D_x(\exp itF)] = \mathbb{E}_{\mu^*}[\exp itF](\exp itf(x) - 1).$$

Now from Proposition 13, since  $\exp itF \in L^2(\mu^*)$ , then  $x \mapsto \mathbb{E}_{\mu^*} [D_x(\exp itF)] \in L^2(\mu)$  and therefore  $\exp itf - 1 \in L^2(\mu)$  and:

$$\int_X |\exp itf(x) - 1|^2 \mu(dx) = 4 \int_X \sin^2\left(\frac{1}{2}tf(x)\right) \mu(dx) < +\infty.$$

We can find  $t_1$  and  $t_2$  such that there exists  $\epsilon > 0$ , such that, for any  $|y| > 1$ ,  $\sin^2\left(\frac{1}{2}t_1y\right) + \sin^2\left(\frac{1}{2}t_2y\right) > \epsilon$ . In particular,

$$\int_X \epsilon 1_{|f|>1} \mu(dx) \leq \int_X \sin^2\left(\frac{1}{2}t_1f(x)\right) + \sin^2\left(\frac{1}{2}t_2f(x)\right) \mu(dx) < \infty$$

that is

$$\mu(|f| > 1) < \infty$$

and, as  $\sin^2(1)z^2 \leq \sin^2(z)$  on  $[-1, 1]$ ,

$$\int_X f^2 1_{|f|\leq 1} d\mu \leq \int_X \frac{4}{t_1^2 \sin^2(1)} \sin^2\left(\frac{1}{2}t_1f(x)\right) \mu(dx) < \infty.$$

It follows that  $f \in L^X(\mu)$  and thus the stochastic integral  $I(f)$  is well defined and satisfies, for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$   $D_x(I(f))(\omega) = f(x)$ .

Therefore,  $\mu \otimes \mu^*$ -a.e.  $D(F - I(f)) = 0$  and then  $F = I(f) + c$  for some  $c \in \mathbb{R}$ , thanks to Lemma 14.

We get the complex case by considering real and imaginary parts.  $\square$

**2.4. The set of stochastic integrals.** Denote by  $\mathcal{I}_{\mu^*} \subset L^0(\mu^*)$  the set of stochastic integrals. As a first consequence of Theorem 15, we get:

**Proposition 16.**  $\mathcal{I}_{\mu^*}$  is a closed subspace of  $L^0(\mu^*)$  with respect to convergence in measure.

*Proof.* The fact that it is a linear space follows immediately from Remark 5.

Let  $\{F_n\}_{n \in \mathbb{N}} := \{I(f_n) + c_n\}_{n \in \mathbb{N}}$  be a sequence of stochastic integrals converging to some random variable  $F$  with respect to  $\mu^*$ . There exists a subsequence  $\{F_{n_k}\}_{k \in \mathbb{N}}$  converging to  $F$   $\mu^*$ -a.e.

For the same reason already explained earlier  $\{F_{n_k \delta}\}_{k \in \mathbb{N}}$  converges  $\tilde{Q}$ -a.e. and thus  $\mu \otimes \mu^*$ -a.e. to  $F_\delta$ . Now from Proposition 12, for  $\mu \otimes \mu^*$ -a.e.  $(x, \omega) \in X \times X^*$ :

$$f_{n_k}(x) = D_x F_{n_k}(\omega) = F_{n_k \delta}(x, \omega) - F_{n_k}(\omega).$$

Therefore  $\{f_{n_k}\}_{k \in \mathbb{N}}$  converges  $\mu$ -a.e. to some measurable  $f$  satisfying

$$f(x) = F_\delta(x, \omega) - F(\omega) = D_x F(\omega).$$

Thanks to Proposition 15,  $F = I(f) + c$  for some  $c \in \mathbb{C}$  as expected.  $\square$

3. SQUARE-INTEGRABLE STOCHASTIC INTEGRALS, FOCK SPACE  
STRUCTURE

In this section, we recall the classical  $L^2$ -case that relies almost entirely on the Hilbertian structure (the details can be found in [14], Chapter 10, Section 4 and also [8]).

The starting point is the following isometry identity: for any  $A$  and  $B$  in  $\mathcal{A}_f$ ,

$$\mathbb{E}_{\mu^*} [(N(A) - \mu(A))(N(B) - \mu(B))] = \mu(A \cap B).$$

And we get, for any functions  $f$  and  $g$  in  $L^1(\mu) \cap L^2(\mu)$ :

$$\langle I_1(f), I_1(g) \rangle_{L^2(\mu^*)} = \langle f, g \rangle_{L^2(\mu)},$$

where  $I_1$  was defined by (2.2).

Moreover, from Remark 7 together with the square integrability criterion found in [17], page 163, a stochastic integral  $I(h)$  is square integrable if and only if  $h \in L^2(\mu)$ .

Setting  $\mathcal{I}_2 := \mathcal{I} \cap L_0^2(\mu^*)$  this leads immediately to:

**Theorem 17.**  $I_1$  extends to an isometric isomorphism between the Hilbert spaces  $(L^2(\mu), \|\cdot\|_2)$  and  $(\mathcal{I}_2, \|\cdot\|_{L^2(\mu^*)})$ . In particular, for any  $f$  and  $g$  in  $L^2(\mu)$ :

$$\langle I_1(f), I_1(g) \rangle_{L^2(\mu^*)} = \langle f, g \rangle_{L^2(\mu)},$$

and

$$\|I_1(f)\|_{L^2(\mu^*)} = \|f\|_2.$$

The above theorem is actually the ground for the following fundamental result (see [14], Chapter 10, Section 4):

**Theorem 18.** There is a natural isometric identification of  $L^2(\mu^*)$  as the Fock space  $F(L^2(\mu))$  over  $L^2(\mu)$ :

$$L^2(\mu^*) \simeq F(L^2(\mu)) := \mathbb{C} \oplus L^2(\mu) \oplus L^2(\mu)^{\odot 2} \oplus \dots \oplus L^2(\mu)^{\odot n} \oplus \dots$$

The latter space is understood as the completion of the infinite orthogonal sum of the symmetric tensor products of  $L^2(\mu)$  with the appropriately scaled scalar product.

Let us succinctly describe how this identification is actually implemented:

Set, for any  $n \geq 1$ , the *stochastic integral of order  $n$*  by the following formula:

$$I_n(f^{\otimes n}) := \int_{\Delta_n^c} f^{\otimes n} d(N - \mu)^{\otimes n} \in L^2(\mu^*).$$

where  $\Delta_n$  is the subset of  $X^n$  where at least two coordinates coincide,  $f$  is a simple function in  $L^2(\mu)$  and  $f^{\otimes}$  its tensor product, an element of  $L^2(\mu^{\otimes n})|_{sym} \simeq L^2(\mu)^{\odot n}$ .

The following formula holds for simple functions  $f$  and  $g$  in  $L^2(\mu)$ :

$$\langle I_n(f^{\otimes n}), I_n(g^{\otimes n}) \rangle_{L^2(\mu^*)} = n! \langle f^{\otimes n}, g^{\otimes n} \rangle_{L^2(\mu)^{\otimes n}}$$

and if  $n \neq m$ ,

$$\langle I_n(f^{\otimes n}), I_n(g^{\otimes m}) \rangle_{L^2(\mu^*)} = 0.$$

The formula  $I_n(f^{\otimes n})$  extends isometrically to any  $f \in L^2(\mu)$  and if we set  $\mathcal{I}_2^{(0)} = \mathbb{C} \cdot 1_{X^*}$ ,  $\mathcal{I}_2^{(1)} = \mathcal{I}_2$  and  $\mathcal{I}_2^{(n)} = \text{Span} \langle I_n(f^{\otimes n}), f \in L^2(\mu) \rangle$ , one gets the orthogonal sum:

$$L^2(\mu^*) = \overline{\mathcal{I}_2^{(0)} \oplus \mathcal{I}_2^{(1)} \oplus \mathcal{I}_2^{(2)} \oplus \dots \oplus \mathcal{I}_2^{(n)} \oplus \dots}$$

where each  $\mathcal{I}_2^{(n)}$  is isometrically identified to  $(L^2(\mu)^{\otimes n}, n! \langle \cdot, \cdot \rangle_{L^2(\mu)^{\otimes n}})$  through  $I_n$  and called the *chaos of order  $n$* .

Observe now that if  $T$  is an endomorphism of  $(X, \mathcal{A}, \mu)$ , then, for any  $f \in L^2(\mu)$ :

$$I_n(f) \circ T_* = I_n((f \circ T)^{\otimes n}),$$

in particular,  $T_*$  preserves each chaos.

We end this section by the following lemma that we shall need later, it is obvious given the above discussion:

**Lemma 19.** *On  $L^2(\mu^*)$ , let  $P$  be the orthogonal projection on  $\mathcal{I}_2$ . Then  $P$  and  $T_*$  commute.*

#### 4. INTEGRABLE STOCHASTIC INTEGRALS, THE ORLICZ SPACE $L^\Phi(\mu)$

The integrable case has deserved considerably less attention than the square-integrable case, the latter being very handy to work with given its Hilbertian nature.

Remark 7 together with the integrability criterion found in [17], page 163 yield:

$$(4.1) \quad I(f) \in L^1(\mu^*) \iff \int_X \min(|f|, |f|^2) d\mu < \infty.$$

And it turns out that if this condition is satisfied,  $\mathbb{E}_{\mu^*}[I(f)] = \mu(f1_{|f|>1})$  and this allows to extend the definition of  $I_1$  by setting

$$I_1(f) = \lim_{\epsilon \rightarrow 0} N(f1_{|f|>\epsilon}) - \mu(f1_{|f|>\epsilon}),$$

in measure.

As a result, we get

$$(4.2) \quad I(f) \in L_0^1(\mu^*) \iff \int_X \min(|f|, |f|^2) d\mu < \infty \text{ and } I(f) = I_1(f).$$

We set  $\mathcal{I}_1 := \mathcal{I} \cap L_0^1(\mu^*)$ .

4.1. **The Orlicz space**  $L^\Phi(\mu)$ . To understand the set  $L^\Phi(\mu)$  of functions  $f \in L^0(\mu)$  satisfying  $\int_X \min(|f|, |f|^2) d\mu < \infty$ , it was natural (see [18]) to replace it by the equivalent condition

$$\int_X \Phi(|f|) d\mu < \infty,$$

$\Phi$  being the function on  $\mathbb{R}_+$  given by:

$$x \mapsto \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

The consequences are decisive, we list them thereafter, following the first four chapters of [15]:

$\Phi$  is a *Young function*, that is

- $\Phi(0) = 0$
- $\lim_{x \rightarrow \infty} \Phi(x) = +\infty$
- $\Phi$  is convex

Moreover,  $\Phi$  is continuous, strictly increasing and  $\Delta_2$ -regular, that is, for all  $x \geq 0$ :

$$\Phi(2x) \leq 4\Phi(x).$$

A number of objects associated to such a function come naturally :

One defines  $N_\Phi$  on  $L^\Phi(\mu)$  by:

$$N_\Phi(f) := \inf \left\{ \lambda > 0, \int_X \Phi\left(\left|\frac{f}{\lambda}\right|\right) d\mu \leq 1 \right\}.$$

Also the *complementary function* to  $\Phi$  is the function  $\Psi$  defined on  $\mathbb{R}_+$  by

$$\Psi(y) = \sup \{xy - \Phi(x)\}$$

and in this case, we have

$$\Psi(y) = \begin{cases} y^2 & \text{if } y \leq 2 \\ +\infty & \text{if } y > 2 \end{cases}.$$

This yields another set

$$L^\Psi(\mu) := \left\{ g \in L^0(\mu), \exists \alpha > 0 \int_X \Psi(|\alpha g|) d\mu < \infty \right\},$$

with  $N_\Psi$  being defined similarly on  $L^\Psi(\mu)$  by

$$N_\Psi(g) := \inf \left\{ \lambda > 0, \int_X \Psi\left(\left|\frac{g}{\lambda}\right|\right) d\mu \leq 1 \right\}.$$

The complementary function  $\Psi$  allows to define the quantity  $\|f\|_\Phi$  for  $f \in L^\Phi(\mu)$  by:

$$\|f\|_\Phi := \sup \left\{ \int_X |fg| d\mu, \int_X \Psi(|g|) d\mu \leq 1 \right\}.$$

We can now sum up the properties of these objects, thanks to the well understood theory of Orlicz spaces (see [15] again) as well as simple considerations:

**Theorem 20.** *The following holds:*

- $L^\Phi(\mu)$  and  $L^\Psi(\mu)$  are Orlicz vector spaces.
- $N_\Phi$  and  $\|\cdot\|_\Phi$  are both norms on  $L^\Phi(\mu)$  ( $N_\Phi$  is called the gauge or Luxemburg norm and  $\|\cdot\|_\Phi$  the Orlicz norm).
- For all  $f \in L^\Phi(\mu)$ ,

$$(4.3) \quad N_\Phi(f) \leq \|f\|_\Phi \leq 2N_\Phi(f).$$

- $(L^\Phi(\mu), N_\Phi)$  is a non-reflexive separable Banach space.
- $L^1(\mu) \subset L^\Phi(\mu)$  and  $\overline{L^1_0(\mu)} = \overline{L^1(\mu)} = L^\Phi(\mu)$ .
- $L^2(\mu) \subset L^\Phi(\mu)$  and  $\overline{L^2(\mu)} = L^\Phi(\mu)$ .
- $L^\Psi(\mu) = L^2(\mu) \cap L^\infty(\mu)$ .
- $N_\Psi = \max(\|\cdot\|_2, \frac{1}{2}\|\cdot\|_\infty)$ .
- For all  $f \in L^\Phi(\mu)$ ,  $\|f\|_\Phi := \sup\{\int_X |fg| d\mu, N_\Psi(g) \leq 1\}$ .
- $(L^\Psi(\mu), N_\Psi)$  is (a representation of) the topological dual  $(L^\Phi(\mu))'$ ,  $\|\cdot\|'_\Phi$  of  $(L^\Phi(\mu), \|\cdot\|_\Phi)$ .
- If  $\{f_n\}_{n \in \mathbb{N}}$  is a sequence of elements of  $L^\Phi(\mu)$  converging to  $f \in L^\Phi(\mu)$  with respect to  $\|\cdot\|_\Phi$ , then there exists a  $\mu$ -a.e. converging subsequence.

**4.2. The Poisson-Orlicz norm and Urbanik's theorem.** Set  $\mathcal{I}_1 := \mathcal{I} \cap L^1_0(\mu^*)$ .  $I_1$  is an onto linear map between  $L^\Phi(\mu)$  and  $\mathcal{I}_1$ , it is also one-to-one thanks to Remark 7, for example. This leads naturally to the definition:

**Definition 21.** For all  $f \in L^\Phi(\mu)$ ,

$$\|f\|_* := \|I_1(f)\|_{L^1(\mu^*)},$$

defines a norm. We call it the *Poisson-Orlicz norm* of  $L^\Phi(\mu)$ .

As such,  $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$  is isometric to  $(L^\Phi(\mu), \|\cdot\|_*)$ .

The natural question is therefore to compare this norm with  $N_\Phi$  and  $\|\cdot\|_\Phi$ . The answer has been given by Urbanik [18]:

**Theorem 22.** *The norms  $\|\cdot\|_*$ ,  $N_\Phi$  and  $\|\cdot\|_\Phi$  are equivalent.*

In [11], explicit bounds were obtained:

$$(4.4) \quad (0.125) N_\Phi(f) \leq \|f\|_* \leq (2.125) N_\Phi(f),$$

which gives another proof of Urbanik's result. The purpose of the authors in [11] was to get good approximations of the  $L^1$ -norm of integrable infinitely divisible random vectors (which can always be obtained as a particular (multidimensional) stochastic integral).

For the sake of completeness and in an effort to better understand this space, we will give our own short proof of Theorem 22. This will allow us



to get an optimal upper bound with respect to  $\|\cdot\|_\Phi$  (and thus improve the right-hand part of Eq (4.4):  $\|f\|_* \leq 2N_\Phi(f)$ ).

Let us see how  $\|\cdot\|_*$  behaves:

**Proposition 23.** *We have*

- For any  $f \in L^1(\mu)$ ,

$$\|f\|_* \leq 2\|f\|_1.$$

- For any  $f \in L^2(\mu)$ ,

$$\|f\|_* \leq \|f\|_2.$$

- For any  $f \in L^\Phi(\mu)$ ,

$$\|f\|_* \leq \|f\|_\Phi.$$

Moreover,  $\|\cdot\|_*$  and  $\|\cdot\|_\Phi$  are different and the bound is optimal.

*Proof.* If  $f \in L^1(\mu)$ , then

$$\begin{aligned} \|f\|_* &= \|I_1(f)\|_{L^1(\mu^*)} \\ &= \mathbb{E}_{\mu^*}[|I_1(f)|] \\ &= \mathbb{E}_{\mu^*}[|N(f) - \mu(f)|] \\ &\leq \mathbb{E}_{\mu^*}[|N(f)|] + |\mu(f)| \\ &\leq \mathbb{E}_{\mu^*}[N(|f|)] + \mu(|f|) \\ &= 2\|f\|_1. \end{aligned}$$

And if  $f \in L^2(\mu)$ ,

$$\begin{aligned} \|f\|_* &= \|I_1(f)\|_{L^1(\mu^*)} \\ &\leq \|I_1(f)\|_{L^2(\mu^*)} \\ &= \|f\|_2, \end{aligned}$$

thanks to Theorem 17.

Denote by  $(L^\Phi(\mu)^{t*}, \|\cdot\|'_*)$  the topological dual of  $(L^\Phi(\mu), \|\cdot\|_*)$  and let  $\phi \in L^\Phi(\mu)^{t*}$ . From the beginning of the proof, for any  $f \in L^1(\mu)$ :

$$(4.5) \quad |\phi(f)| \leq \|\phi\|'_* \|f\|_* \leq 2\|\phi\|'_* \|f\|_1,$$

and for any  $f \in L^2(\mu)$ :

$$(4.6) \quad |\phi(f)| \leq \|\phi\|'_* \|f\|_* \leq \|\phi\|'_* \|f\|_2.$$

Thus  $\phi$  is a continuous linear form on both  $(L^1(\mu), \|\cdot\|_1)$  and  $(L^2(\mu), \|\cdot\|_2)$ , and there exists a unique  $g_1 \in L^\infty(\mu)$  such that, for any  $f \in L^1(\mu)$ :

$$\phi(f) = \int_X g_1 f d\mu,$$

and a unique  $g_2 \in L^2(\mu)$  such that, for any  $f \in L^2(\mu)$ :

$$\phi(f) = \int_X g_2 f d\mu.$$

Of course  $g_1 = g_2$   $\mu$ -a.e..

Let  $g := g_1 \in L^2(\mu) \cap L^\infty(\mu)$  and take  $f$  in  $L^\Phi(\mu)$ . We have  $f = f1_{|f| \leq 1} + f1_{|f| > 1}$  with  $f1_{|f| \leq 1} \in L^2(\mu)$  and  $f1_{|f| > 1} \in L^1(\mu)$  therefore

$$\begin{aligned} \phi(f) &= \phi(f1_{|f| \leq 1}) + \phi(f1_{|f| > 1}) \\ &= \int_X gf1_{|f| \leq 1} d\mu + \int_X gf1_{|f| > 1} d\mu \\ &= \int_X gf d\mu. \end{aligned}$$

In particular,  $\phi \in L^\Phi(\mu)'$ , which yields  $L^\Phi(\mu)'^* \subset L^\Phi(\mu)' \simeq L^2(\mu) \cap L^\infty(\mu)$ . From now on, we identify  $L^\Phi(\mu)'^*$  as a subspace of  $L^2(\mu) \cap L^\infty(\mu) \simeq L^\Phi(\mu)'$ .

From the inequalities (4.5) and (4.6), for any  $g \in L^\Phi(\mu)'^*$ ,

$$\max\left(\|g\|_2, \frac{1}{2}\|g\|_\infty\right) \leq \|g\|'_*,$$

thus, for any  $g \in L^\Phi(\mu)'^*$ ,

$$N_\Psi(g) \leq \|g\|'_*.$$

Hence for any  $f \in L^\Phi(\mu)$ :

$$\begin{aligned} \|f\|_* &= \sup_{g \in L^\Phi(\mu)'^*} \frac{\int_X gf d\mu}{\|g\|'_*} \\ &\leq \sup_{g \in L^\Phi(\mu)'^*} \frac{\int_X gf d\mu}{N_\Psi(g)} \\ &\leq \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X gf d\mu}{N_\Psi(g)} \\ &= \|f\|_\Phi. \end{aligned}$$

Now observe that for any  $f \in L^\Phi(\mu)$ , we have

$$\| \|f\| \|f\|_\Phi = \|f\|_\Phi.$$

This is not the case with  $\|\cdot\|_*$ . To see this, let  $A_n$  and  $B_n$ ,  $n \geq 2$  be measurable sets such that, for all  $n \geq 2$ ,  $A_n \cap B_n = \emptyset$  and  $\mu(A_n) = \mu(B_n) = \frac{1}{2n}$ , and set  $f_n := 1_{A_n} - 1_{B_n}$ , so that  $f_n \in L^1_0(\mu)$  and  $\|f_n\|_1 = \frac{1}{n}$ .

We have:

$$\begin{aligned}
\|f_n\|_* &= \|1_{A_n \cup B_n}\|_* \\
&= \mathbb{E}_{\mu^*} [|N(A_n \cup B_n) - \mu(A_n \cup B_n)|] \\
&= \mathbb{E}_{\mu^*} [\mu(A_n \cup B_n) 1_{N(A_n \cup B_n)=0}] \\
&\quad + \mathbb{E}_{\mu^*} [(N(A_n \cup B_n) - \mu(A_n \cup B_n)) 1_{N(A_n \cup B_n) \geq 1}] \\
&= \frac{1}{n} e^{-\frac{1}{n}} + \mathbb{E}_{\mu^*} [N(A_n \cup B_n) - \mu(A_n \cup B_n)] \\
&\quad - \mathbb{E}_{\mu^*} [(N(A_n \cup B_n) - \mu(A_n \cup B_n)) 1_{N(A_n \cup B_n)=0}] \\
&= \frac{2}{n} e^{-\frac{1}{n}},
\end{aligned}$$

whereas

$$\begin{aligned}
\|f_n\|_* &= \mathbb{E}_{\mu^*} [|I_1(f_n)|] \\
&= \mathbb{E}_{\mu^*} [|N(f_n) - \mu(f_n)|] \\
&= \mathbb{E}_{\mu^*} [|N(f_n)|] \\
&\leq \mathbb{E}_{\mu^*} [N(|f_n|)] \\
&= \|f_n\|_1 = \frac{1}{n}.
\end{aligned}$$

To prove the optimality, observe that, since for all  $g \in L^2(\mu) \cap L^\infty(\mu)$ ,  $N_\Psi(g) = \max(\|g\|_2, \frac{1}{2}\|g\|_\infty) \geq \frac{1}{2}\|g\|_\infty$ , we have, for any  $f \in L^1(\mu)$ :

$$\begin{aligned}
\|f\|_\Phi &= \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X g f d\mu}{N_\Psi(g)} \\
&\leq 2 \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X g f d\mu}{\|g\|_\infty} \\
&\leq 2 \sup_{g \in L^\infty(\mu)} \frac{\int_X g f d\mu}{\|g\|_\infty} \\
&= 2\|f\|_1.
\end{aligned}$$

Thus  $\|f_n\|_\Phi \leq \frac{2}{n}$ . □

We have just seen that, in general,  $\|f\|_* = \|f\|_\Phi$  doesn't hold. Nevertheless we can compare the two quantities:

**Proposition 24.** *Let  $f \in L^\Phi(\mu)$ , real valued. Then*

$$\|f\|_* \leq 2 \|f\|_\Phi,$$

and

$$\|f\|_\Phi \leq 2 \|f\|_*.$$

*Proof.* Write  $f = f^+ - f^-$ . Since  $f^+$  and  $f^-$  are supported on disjoint sets,  $I_1(f^+)$  and  $I_1(f^-)$  are independent thanks to the independence properties

of a Poisson measure. Therefore, taking the conditional expectation with respect to  $\sigma(I_1(f^-))$ :

$$\begin{aligned} \mathbb{E}_{\mu^*} [|I_1(f)|] &= \mathbb{E}_{\mu^*} [\mathbb{E}_{\mu^*} [|I_1(f^+) - I_1(f^-)| \mid \sigma(I_1(f^-))]] \\ &\geq \mathbb{E}_{\mu^*} [|\mathbb{E}_{\mu^*} [I_1(f^+) - I_1(f^-) \mid \sigma(I_1(f^-))]|] \\ &= \mathbb{E}_{\mu^*} [|\mathbb{E}_{\mu^*} [I_1(f^+) \mid \sigma(I_1(f^-))] - \mathbb{E}_{\mu^*} [I_1(f^-) \mid \sigma(I_1(f^-))]|] \\ &= \mathbb{E}_{\mu^*} [|\mathbb{E}_{\mu^*} [I_1(f^+)] - I_1(f^-)|] \\ &= \mathbb{E}_{\mu^*} [|I_1(f^-)|]. \end{aligned}$$

Thus  $\|f\|_* \geq \|f^-\|_*$  and of course  $\|f\|_* \geq \|f^+\|_*$ . This yields

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \|f\|_*,$$

and the triangular inequality implies

$$\|f\|_* \leq \|f^+\|_* + \|f^-\|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*),$$

Hence

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \|f\|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*).$$

Writing now  $|f| = f^+ + f^-$ , the same reasoning gives

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \| |f| \|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*),$$

and we get the result.  $\square$

We can now end this section with our proof of Urbanik's result, Theorem 22:

*Proof. (of Theorem 22)* From Proposition 16, it follows that  $\mathcal{I}_1$  is closed in the Banach space  $(L^1(\mu^*), \|\cdot\|_{L^1(\mu^*)})$ . But  $(L^\Phi(\mu), \|\cdot\|_*)$  is isometrically isomorphic to  $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$  therefore  $(L^\Phi(\mu), \|\cdot\|_*)$  is a Banach space. Thanks to Proposition 23 and the open map theorem applied to the identity on  $L^\Phi(\mu)$ , the norms  $\|\cdot\|_*$  and  $\|\cdot\|_\Phi$  are equivalent.  $\square$

## 5. $L^\Phi(\mu)$ -ERGODIC RESULTS

**5.1. Mean ergodic theorem in  $L^\Phi(\mu)$ .** We now introduce an endomorphism  $T$  on  $(X, \mathcal{A}, \mu)$ . The next observation follows from Remark 5.

**Proposition 25.**  $T_*$  satisfies, for any  $f \in L^\Phi(\mu)$ :

$$I_1(f) \circ T_* = I_1(f \circ T).$$

In particular,  $T$  is an isometry of  $(L^\Phi(\mu), \|\cdot\|_*)$ .

We can state:

**Theorem 26.** *The dynamical system  $(X, \mathcal{A}, \mu, T)$  has no absolutely continuous  $T$ -invariant probability measure if and only if, for every  $f \in L^1(\mu)$  (or  $f \in L^\Phi(\mu)$ ):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \rightarrow_{\|\cdot\|_*} 0,$$

as  $n$  tends to  $+\infty$ .

*Proof.* Assume  $T$  has no absolutely continuous  $T$ -invariant probability measure (in particular  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is ergodic by Theorem 3) and let  $f \in L^1(\mu)$ :

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* &= \left\| I_1 \left( \frac{1}{n} \sum_{k=1}^n f \circ T^k \right) \right\|_{L^1(\mu^*)} \\ &= \left\| \frac{1}{n} \sum_{k=1}^n I_1(f) \circ T_*^k \right\|_{L^1(\mu^*)} \\ &\rightarrow 0 \end{aligned}$$

as  $n$  tends to infinity thanks to Birkhoff's ergodic theorem.

Let now  $f \in L^1(\mu)$  and  $g \in L^\Phi(\mu)$ :

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n g \circ T^k \right\|_* &= \left\| \frac{1}{n} \sum_{k=1}^n (g - f) \circ T^k + \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \\ &\leq \frac{1}{n} \sum_{k=1}^n \left\| (g - f) \circ T^k \right\|_* + \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \\ &= \|g - f\|_* + \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \end{aligned}$$

since  $T$  is an isometry of  $L^\Phi(\mu)$  (Proposition 25).

Thus

$$\limsup \left\| \frac{1}{n} \sum_{k=1}^n g \circ T^k \right\|_* \leq \|g - f\|_*$$

And the result follows by density of  $L^1(\mu)$  in  $L^\Phi(\mu)$  (Theorem 20).

Conversely, assume there exists an absolutely continuous  $T$ -invariant probability measure  $\nu \ll \mu$ . This implies the existence of a non-trivial  $T$ -invariant set  $A \in \mathcal{A}_f$  and we get:

$$\left\| \frac{1}{n} \sum_{k=1}^n 1_A \circ T^k \right\|_* = \|1_A\|_* > 0.$$

□

*Remark 27.* Mean ergodic theorems are often associated to pointwise convergence. In our context, the result is immediate, for any  $f \in L^\Phi(\mu)$ :

$$\frac{1}{n} \sum_{k=1}^n f(T^k x)$$

converges as  $n$  tends to infinity, for  $\mu$ -almost every  $x \in X$  since, as we saw above, we can write  $f = f1_{|f| \leq 1} + f1_{|f| > 1}$  with  $f1_{|f| \leq 1} \in L^2(\mu)$  and  $f1_{|f| > 1} \in L^1(\mu)$  and the  $\mu$ -almost sure convergence holds for both  $L^1(\mu)$  ([6]) and  $L^2(\mu)$  ([2]). In particular, if  $T$  has no absolutely continuous  $T$ -invariant probability measure, then for  $\mu$ -almost every  $x \in X$ :

$$\frac{1}{n} \sum_{k=1}^n f(T^k x) \rightarrow 0,$$

as  $n$  tends to infinity.

## 5.2. Blum-Hanson type Theorem.

**Theorem 28.** *The following assertions are equivalent:*

- (1)  $(X, \mathcal{A}, \mu, T)$  is of zero type
- (2)  $\{T^n\}_{n \in \mathbb{N}}$  tends weakly to 0 in  $L^\Phi(\mu)$
- (3) for every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of integers and every  $f \in L^1(\mu)$  (or equivalently  $f \in L^\Phi(\mu)$ ):

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \rightarrow_{\|\cdot\|_*} 0,$$

as  $n$  tends to infinity.

*Proof.* Observe that  $(X, \mathcal{A}, \mu, T)$  is of zero type is equivalent to the fact that  $\{T^n\}_{n \in \mathbb{N}}$  tends weakly to 0 in  $L^2(\mu)$ . But since  $L^\Phi(\mu)' \simeq L^2(\mu) \cap L^\infty(\mu)$  and  $\overline{L^2(\mu)} = L^\Phi(\mu)$  (Theorem 20), we easily obtain that zero type is equivalent to the fact that  $\{T^n\}_{n \in \mathbb{N}}$  tends weakly to 0 in  $L^\Phi(\mu)$ .

If  $(X, \mathcal{A}, \mu, T)$  is of zero type then  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is mixing (Theorem 3) and Blum-Hanson theorem ([3]) applies: for every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of integers, every  $f \in L^\Phi(\mu)$ ,

$$\frac{1}{n} \sum_{k=1}^n I_1(f) \circ T_*^{n_k} \rightarrow_{\|\cdot\|_{L^1(\mu^*)}} 0,$$

but

$$\left\| \frac{1}{n} \sum_{k=1}^n I_1(f) \circ T_*^{n_k} \right\|_{L^1(\mu^*)} = \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \right\|_*,$$

that is,

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \rightarrow_{\|\cdot\|_*} 0.$$

Conversely, assume the convergence holds for every  $f \in L^1(\mu)$  and every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of integers and consider sets  $A$  and  $B$  in  $\mathcal{A}_f$ .

We want to prove  $\mu(A \cap T^{-n}B) \rightarrow 0$  as  $n$  tends to infinity. By a standard exercise, it is equivalent to show the Cesàro convergence to zero of its subsequences i.e., for every strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$ :

$$\frac{1}{n} \sum_{k=1}^n \mu(A \cap T^{-n_k}B) \rightarrow 0,$$

as  $n$  tends to infinity, but this will indeed be the case, since

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=1}^n \mu(A \cap T^{-n_k}B) \right| &= \left| \int_X 1_A \cdot \frac{1}{n} \sum_{k=1}^n 1_B \circ T^{n_k} d\mu \right| \\ &\leq \|1_A\|'_* \left\| \frac{1}{n} \sum_{k=1}^n 1_B \circ T^{n_k} \right\|_* \end{aligned}$$

as  $1_A \in L^\infty(\mu) \cap L^2(\mu)$  and  $1_B \in L^1(\mu) \subset L^\Phi(\mu)$ .

Hence  $T$  is of zero type.  $\square$

## 6. DUALITY AND TRANSFER OPERATORS

**6.1. Duality.** It will be useful to have another representation of  $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$ , the dual of  $(L^\Phi(\mu), \|\cdot\|_*)$ , and that requires the introduction of a few objects.

Recall that, we denoted by  $\mathcal{I}_2 := \{I_1(f), f \in L^2(\mu)\}$  the first chaos and we pointed out the fact that  $I_1$  is implementing an isometric isomorphism between  $(\mathcal{I}_2, \|\cdot\|_{L^2(\mu^*)})$  and  $(L^2(\mu), \|\cdot\|_2)$ . Recall that we have introduced  $P$  in Lemma 19 as the orthogonal projection on  $\mathcal{I}_2$ ; we can then define  $\pi$  from  $L^2(\mu^*)$  to  $L^2(\mu)$  by

$$\pi := I_1^{-1} \circ P.$$

Consider now the restriction of  $\pi$  to  $L^\infty(\mu^*) \subset L^2(\mu^*)$ . We will show:

**Proposition 29.**  $\pi$  is a surjective bounded operator from  $(L^\infty(\mu^*), \|\cdot\|_{L^\infty(\mu^*)})$  to  $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$ .

*Proof.* If  $Z \in L^\infty(\mu^*)$ , then the formula

$$L^\Phi(\mu) \ni f \mapsto \mathbb{E}_{\mu^*} [ZI_1(f)]$$

defines a continuous linear form on  $(L^\Phi(\mu), \|\cdot\|_*)$  since

$$\begin{aligned} |\mathbb{E}_{\mu^*} [ZI_1(f)]| &\leq \|Z\|_{L^\infty(\mu^*)} \|I_1(f)\|_{L^1(\mu^*)} \\ (6.1) \qquad \qquad &= \|Z\|_{L^\infty(\mu^*)} \|f\|_*. \end{aligned}$$

Therefore, there exists  $g \in L^2(\mu) \cap L^\infty(\mu)$  such that, for any  $f \in L^\Phi(\mu)$ :

$$\mathbb{E}_{\mu^*} [ZI_1(f)] = \int_X gf d\mu$$

but, if  $f \in L^2(\mu)$ , then, if  $P(Z) = I_1(h)$  for some  $h \in L^2(\mu)$ , then

$$\begin{aligned} \mathbb{E}_{\mu^*} [ZI_1(f)] &= \mathbb{E}_{\mu^*} [I_1(g) I_1(f)] \\ &= \int_X hf d\mu. \end{aligned}$$

Therefore  $h = g$  and  $\pi(Z) = g \in L^2(\mu) \cap L^\infty(\mu)$ .  
Conversely, for any  $g \in L^2(\mu) \cap L^\infty(\mu)$ , the map

$$F \mapsto \int_X gI_1^{-1}(F) d\mu = \mathbb{E}_{\mu^*} [I_1(g) F]$$

defines a continuous linear form on  $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$  that extends, thanks to Hahn-Banach theorem to  $(L^1(\mu^*), \|\cdot\|_{L^1(\mu^*)})$ , that is, there exists  $Z \in L^\infty(\mu^*)$  such that, for, any  $F \in L^1(\mu^*)$ :

$$\mathbb{E}_{\mu^*} [ZF] = \mathbb{E}_{\mu^*} [I_1(g) F].$$

In particular, for any  $f \in L^2(\mu)$ ,

$$\mathbb{E}_{\mu^*} [ZI_1(f)] = \mathbb{E}_{\mu^*} [I_1(g) I_1(f)].$$

And this proves that  $\pi(Z) = g$ . Therefore the range of  $\pi$  is  $L^2(\mu) \cap L^\infty(\mu)$  and from (6.1), we get

$$\|\pi(Z)\|'_* \leq \|Z\|_{L^\infty(\mu^*)}.$$

□

**Proposition 30.** *In the duality  $\sigma(L^1(\mu^*), L^\infty(\mu^*))$ , the orthogonal  $\mathcal{I}_1^\perp$  of  $\mathcal{I}_1$  is the subspace  $\ker \pi$ . In particular, there is a canonical isometric identification*

$$\left( L^\infty(\mu^*) / \ker \pi, \|\cdot\|_{/} \right) \simeq \left( L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_* \right),$$

where  $\|\cdot\|_{/}$  is the usual quotient norm induced by  $\|\cdot\|_{L^\infty(\mu^*)}$ .

*Proof.* If  $f \in L^2(\mu)$  and  $Z \in L^\infty(\mu^*)$ , then

$$\begin{aligned} \mathbb{E}_{\mu^*} [ZI_1(f)] &= \mathbb{E}_{\mu^*} [I_1(\pi(Z)) I_1(f)] \\ &= \int_X \pi(Z) f d\mu. \end{aligned}$$

This formula extends by density of  $L^2(\mu)$  in  $L^\Phi(\mu)$  with respect to  $\|\cdot\|_*$ , and thus of  $\mathcal{I}_2$  in  $\mathcal{I}_1$  with respect to  $\|\cdot\|_{L^1(\mu^*)}$ .

Therefore  $\mathcal{I}_1^\perp = \ker \pi$ .

The map  $\pi$  induces an isometric isomorphism between  $(L^\infty(\mu^*) / \ker \pi, \|\cdot\|_{/})$  and  $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$ . □

*Remark 31.* We could have appeal to general results on Banach spaces to get that  $L^\infty(\mu^*) / \mathcal{I}_1^\perp$  is canonically isomorphic (and isometric) to the dual of  $\mathcal{I}_1$ , however we get an explicit description through the operator  $\pi$  as above.



**6.2. Transfer operators.** Let  $T$  be an endomorphism on  $(X, \mathcal{A}, \mu)$ . The *transfer operator* is the operator  $\widehat{T}$  acting on  $L^1(\mu)$  defined as the predual operator of  $T$  seen as acting as an isometry on  $L^\infty(\mu)$ , characterized by the following relation, for any  $f \in L^1(\mu)$  and  $g \in L^\infty(\mu)$ :

$$\int_X \widehat{T}f \cdot g d\mu = \int_X f \cdot g \circ T d\mu.$$

Similarly, the same object exists at the level of the Poisson suspension  $(X^*, \mathcal{A}^*, \mu^*, T_*)$ :  $\widehat{T}_*$  is the transfer operator of  $T_*$ .

We will mimic the same procedure to define a “transfer” operator acting on  $L^\Phi(\mu)$ .

**Proposition 32.** *The isometry  $T_*$  on  $(L^\infty(\mu^*), \|\cdot\|_{L^\infty(\mu^*)})$  acts also on the quotient space  $(L^\infty(\mu^*) / \ker \pi, \|\cdot\|_{\prime})$  as an isometry and corresponds to  $T$  through the identification with  $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$ . In particular,  $T$  is an isometry of  $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$ .*

*Proof.* The first step consists in showing that  $T_*$  preserves  $\ker \pi$ :

Take  $Z \in \ker \pi$ , in particular  $Z$  belongs to  $\mathcal{I}_2^\perp \subset L^2(\mu^*)$ . But  $T_*$  commutes with  $P$  thanks to Lemma 19 and thus  $P(Z \circ T_*) = P(Z) \circ T_* = 0$  and thus  $Z \circ T_* \in \ker \pi$ .

Now observe that for any  $Z \in L^\infty(\mu^*)$ :

$$\begin{aligned} \pi(Z \circ T_*) &= I_1^{-1}(P(Z \circ T_*)) \\ &= I_1^{-1}(P(Z) \circ T_*) \\ &= I_1^{-1}(P(Z)) \circ T \\ &= \pi(Z) \circ T \end{aligned}$$

and this proves the result.  $\square$

**Definition 33.** We define the *Poisson-transfer operator*  $\widehat{T}_{\mathcal{P}}$  acting on  $L^\Phi(\mu)$  as the dual operator of  $T$  acting on  $L^\Phi(\mu)'$ . It satisfies, for any  $f$  in  $L^\Phi(\mu)$ ,  $g \in L^2(\mu) \cap L^\infty(\mu)$ :

$$\int_X \widehat{T}_{\mathcal{P}}(f) \cdot g d\mu = \int_X f g \circ T d\mu.$$

**Proposition 34.**  $\widehat{T}_*$  preserves  $\mathcal{I}_1$  and satisfies, for all  $f \in L^\Phi(\mu)$ :

$$\widehat{T}_* \mathcal{I}_1(f) = \mathcal{I}_1(\widehat{T}_{\mathcal{P}} f)$$

Moreover  $\widehat{T}_{\mathcal{P}}$  preserves  $L^1(\mu)$  and coincide with  $\widehat{T}$  on it.

*Proof.* The first part of the proof follows immediately from Proposition 32 and the fact that  $\ker \pi = \mathcal{I}_1^\perp$  from Proposition 30.

Now taking  $f \in L^1(\mu) \subset L^\Phi(\mu)$ , for any set  $A \in \mathcal{A}_f$ :

$$\int_A \widehat{T}_{\mathcal{P}}(f) d\mu = \int_{T^{-1}A} f d\mu = \int_A \widehat{T}(f) d\mu.$$

and this implies  $\widehat{T}_{\mathcal{P}}f = \widehat{T}f$ , that is  $\widehat{T}_{\mathcal{P}}$  coincide with  $\widehat{T}$  on  $L^1(\mu)$ .  $\square$

*Remark 35.* Being a positive operator defined on  $L^1(\mu)$ ,  $\widehat{T}$  extends in a unique (mod.  $\mu$ ) way to any non-negative measurable function. The same applies to  $\widehat{T}_{\mathcal{P}}$  and since they coincide on  $L^1(\mu)$ , the extension of  $\widehat{T}$  to  $L^{\Phi}(\mu)$  is in fact  $\widehat{T}_{\mathcal{P}}$ .

## 7. EXACT AND REMOTELY INFINITE TRANSFORMATIONS

We recall the two definitions that we will deal with in this section:

**Definition 36.** (See [7])  $(X, \mathcal{A}, \mu, T)$  is said to be

- *exact* if  $\bigcap_{n \geq 0} T^{-n}\mathcal{A} = \{\emptyset, X\}$  mod.  $\mu$ .
- *remotely infinite*<sup>1</sup> if  $\bigcap_{n \geq 0} T^{-n}\mathcal{A}$  contains only 0 or infinite measure sets.

We recall the chain of implications we already gave in the introduction:

$T_{\text{exact}} \Rightarrow T_{\text{remotely infinite}} \Rightarrow T_{\text{of zero type}} \implies$  absence of a.c.  $T$ -invariant measure.

Note also that exactness implies ergodicity whereas it is not the case for the other three properties.

**Example 37.** It is be proven (see Corollary 3.7.7 in [1]) that pointwise dual ergodic endomorphisms (see also [1] for the definition and examples) are always remotely infinite. Null recurrent aperiodic Markov chains under their infinite invariant measure yields unilateral shifts that are exact endomorphisms.

We are interested in transfer operator characterization of these properties, we also recall Lin's result:

**Theorem 38.** ([9])  $(X, \mathcal{A}, \mu, T)$  is exact if and only if, for all  $f \in L_0^1(\mu)$ :

$$\left\| \widehat{T}^n f \right\|_{L^1(\mu)} \rightarrow 0,$$

as  $n \rightarrow +\infty$ .

We will prove the following:

**Theorem 39.**  $(X, \mathcal{A}, \mu, T)$  is remotely infinite if and only if, for all  $f \in L_0^1(\mu)$ :

$$\left\| \widehat{T}^n f \right\|_* \rightarrow 0,$$

as  $n \rightarrow +\infty$ .

To this end, we first need to establish the following characterization of remotely infinite systems:

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<sup>1</sup>We warn the reader that the authors of [7] also define, *remotely infinite automorphisms* (which cannot exist with the above definition...) in the same paper. This is the same difference that occurs between exact endomorphisms and  $K$ -automorphisms.

**Proposition 40.**  $(X, \mathcal{A}, \mu, T)$  is remotely infinite if and only if  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is exact.

*Proof.* Recall (see [16]) that if  $\mathcal{B} \subset \mathcal{A}$  is a  $\sigma$ -algebra then  $\mathcal{B}^* := \sigma\{N(A), A \in \mathcal{B}\}$  is said to be a *Poisson  $\sigma$ -algebra*.

The proof of the Proposition is then a straightforward consequence of the following facts  $\square$

- $(T^{-n}\mathcal{A})^* = T^{-n}\mathcal{A}^*$
- $(\cap_{n \geq 0} T^{-n}\mathcal{A})^* = \cap_{n \geq 0} T_*^{-n}\mathcal{A}^*$  (see Lemma 3.3 in [16] for a proof)
- A Poisson  $\sigma$ -algebra  $\mathcal{G}^*$  for  $\mathcal{G} \subset \mathcal{A}$  is trivial if and only if  $\mathcal{G}$  contains only zero or infinite measure sets (the Poisson random measure  $N$  cannot distinguish between zero measure sets nor between infinite measure sets).

We can then prove our theorem:

*Proof. (of Theorem 39).* Assume  $(X, \mathcal{A}, \mu, T)$  is remotely infinite. Then  $(X^*, \mathcal{A}^*, \mu^*, T_*)$  is exact from Proposition 40, and thus, for any  $f \in L_0^1(\mu)$ ,

$$\left\| \widehat{T}_*^n I_1(f) \right\|_{L^1(\mu^*)} \rightarrow 0$$

But, we have seen in Proposition 34 that  $\widehat{T}_*^n I_1(f) = I_1(\widehat{T}^n f)$ , therefore

$$\begin{aligned} \left\| \widehat{T}_*^n I_1(f) \right\|_{L^1(\mu^*)} &= \left\| I_1(\widehat{T}^n f) \right\|_{L^1(\mu^*)} \\ &= \left\| \widehat{T}^n f \right\|_* . \end{aligned}$$

For the converse, assume that for all  $f \in L_0^1(\mu)$ ,  $\left\| \widehat{T}^n f \right\|_* \rightarrow 0$  as  $n \rightarrow +\infty$ .

Recall that  $\widehat{T}^n_{\mathcal{P}}$  is an isometry on  $L^\Phi(\mu)$  and from Proposition 34,  $\widehat{T}^n = \widehat{T}^n_{\mathcal{P}}$  on  $L_0^1(\mu)$ , thus, for any  $g \in L^\Phi(\mu)$  and  $f \in L_0^1(\mu)$ :

$$\begin{aligned} \left\| \widehat{T}^n_{\mathcal{P}} g \right\|_* &= \left\| \widehat{T}^n_{\mathcal{P}}(g - f) + \widehat{T}^n_{\mathcal{P}} f \right\|_* \\ &\leq \left\| \widehat{T}^n_{\mathcal{P}}(g - f) \right\|_* + \left\| \widehat{T}^n_{\mathcal{P}} f \right\|_* \\ &\leq \|g - f\|_* + \left\| \widehat{T}^n f \right\|_* \end{aligned}$$

That is

$$\limsup_{n \rightarrow \infty} \left\| \widehat{T}^n_{\mathcal{P}} g \right\|_* \leq \|g - f\|_* .$$

We conclude by density of  $L_0^1(\mu)$  in  $L^\Phi(\mu)$  (see Theorem 20).

In other words, from Proposition 34 we have proved that, for any  $F \in \mathcal{I}_1$ :

$$\left\| \widehat{T}_*^n F \right\|_{L^1(\mu^*)} \rightarrow 0$$

as  $n \rightarrow +\infty$ .

For the converse we follow the same lines of arguments used in the proof of Theorem 38: assume that  $(X, \mathcal{A}, \mu, T)$  is not remotely infinite, this means that there exists  $A \in \bigcap_{n \geq 0} T^{-n} \mathcal{A}$  such that  $0 < \mu(A) < \infty$  and we can set, for any  $n \in \mathbb{N}$ ,  $A_n$  such that  $A = T^{-n} A_n$ .

Consider  $1_{N(A)=0} \in L^\infty(\mu^*)$ , we have  $1_{N(A_n)=0} \circ T_*^n = 1_{N(T^{-n} A_n)=0} = 1_{N(A)=0}$ .

Then, for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left\| \widehat{T}_*^n I_1(-1_A) \right\|_{L^1(\mu^*)} &\geq \mathbb{E}_{\mu^*} \left[ \left| 1_{N(A_n)=0} \cdot \widehat{T}_*^n I_1(-1_A) \right| \right] \\ &\geq \mathbb{E}_{\mu^*} \left[ 1_{N(A_n)=0} \cdot \widehat{T}_*^n I_1(-1_A) \right] \\ &\geq \mathbb{E}_{\mu^*} \left[ 1_{N(A_n)=0} \circ T_*^n \cdot I_1(-1_A) \right] \\ &= \mathbb{E}_{\mu^*} \left[ 1_{N(A)=0} \cdot I_1(-1_A) \right] \\ &= \mathbb{E}_{\mu^*} \left[ 1_{N(A)=0} \cdot (-N(A) + \mu(A)) \right] \\ &= \mu(A) \mathbb{E}_{\mu^*} \left[ 1_{N(A)=0} \right] \\ &= \mu(A) e^{-\mu(A)} > 0. \end{aligned}$$

This contradicts that for any  $F \in \mathcal{I}_1$ ,  $\left\| \widehat{T}_*^n F \right\|_{L^1(\mu^*)} \rightarrow 0$ .  $\square$

Combining Proposition 39 and the last point of Theorem 20, we obtain:

**Corollary 41.**  *$(X, \mathcal{A}, \mu, T)$  is not remotely infinite if there exists a non-negative  $f \in L^1(\mu)$  such that*

$$\liminf_{n \rightarrow \infty} \widehat{T}^n f \neq 0 \text{ } \mu\text{-a.e.}$$

Observe that Fatou lemma gives, if  $\liminf_{n \rightarrow \infty} \widehat{T}^n f \neq 0$ :

$$0 < \int_X \left( \liminf_{n \rightarrow \infty} \widehat{T}^n f \right) d\mu \leq \liminf_{n \rightarrow \infty} \left\| \widehat{T}^n f \right\|_1$$

which only implies that  $(X, \mathcal{A}, \mu, T)$  is not exact.

## 8. CONCLUSION

We aimed to have illustrated the canonical character of  $L^\Phi(\mu)$  and think that this space could be a new reference space when studying infinite measure dynamical systems. If the Poisson-Orlicz norm is by far the preferred choice when looking at the interplay between a system and its Poisson suspension as we did, the actual computation of the norm of a particular function is a mathematical challenge on its own! However the equivalent Orlicz norm is always available and much easier to compute in concrete situations, making the use of  $L^\Phi(\mu)$  quite flexible.

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