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POISSON-ORLICZ NORM IN INFINITE ERGODIC THEORY

EMMANUEL ROY

ABSTRACT. Starting with an infinite measure Lebesgue space (X, \mathcal{A}, μ) , we consider two norms on $L^1(\mu)$ that are both weaker than $\|\cdot\|_1$. The first one is associated to a particular Orlicz space and allows to obtain general ergodic theorems for a measure preserving transformation, involving the convergence of its Birkhoff sums, that are true for $(L^p(\mu), \|\cdot\|_p)$, $1 < p < \infty$, but fail for $p = 1$ as the measure is infinite. With the second, we obtain non-trivial T -invariant vectors in the completion of $L^1(\mu)$ with respect to this new norm and this leads to a $L^1(\mu)$ -characterization of ergodicity. Both norms are in fact related to the Poisson process over (X, \mathcal{A}, μ) and their family of stochastic integrals whose inherent integrability constraints are the main ingredient in the proofs. A new characterization of stochastic integrals is obtained and additional results on coboundaries come out as a natural byproduct.

1. INTRODUCTION

1.1. **The setup.** In the sequel (X, \mathcal{A}, μ) will be a non-atomic infinite Lebesgue space (i.e. measurably null-isomorphic to the real line endowed with the Lebesgue σ -algebra and Lebesgue measure). $L^0(\mu)$ denotes the linear space of (equivalence classes of) measurable functions and \mathcal{A}_f denotes the collection of sets $A \in \mathcal{A}$ such that $\mu(A) < +\infty$.

A measure-preserving transformation T will be called an *endomorphism* or *automorphism* in case it is invertible. A *dynamical system* is a quadruplet (X, \mathcal{A}, μ, T) where T is an endomorphism.

1.2. Failure of L^1 -Birkhoff sums convergence in infinite measure.

The principal motivation behind this work was the following classical observation: if (X, \mathcal{A}, μ, T) is an ergodic dynamical system then, whereas the Birkhoff sums, $\frac{1}{n} \sum_{k=1}^n f \circ T^k(x)$, associated to any $f \in L^1(\mu)$ converge pointwise to 0 as n tends to $+\infty$ for μ -a.e. $x \in X$ (see [7]), the convergence cannot hold for all f in $L^1(\mu)$ with respect to $\|\cdot\|_1$. The same result occurs when we replace the ergodicity requirement by the absence of absolutely continuous T -invariant probability measure. This puts the $(L^1(\mu), \|\cdot\|_1)$ -isometry T outside the nice category of “*mean ergodic operators*”:

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Definition 1. (see Section 8.4, page 136 in [6]) Let φ be a continuous linear operator defined on a Hausdorff topological vector space H . φ is said to be *mean ergodic* if, for any $h \in H$, $\frac{1}{n} \sum_{k=1}^n \varphi^k h$ converges as n tends to $+\infty$.

In contrast, replacing $(L^1(\mu), \|\cdot\|_1)$ by $(L^p(\mu), \|\cdot\|_p)$, $1 < p < \infty$, T becomes a mean ergodic operator and one has:

Theorem. *The dynamical system (X, \mathcal{A}, μ, T) has no absolutely continuous T -invariant probability measure if and only if, for every $f \in L^p(\mu)$:*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \xrightarrow{\|\cdot\|_p} 0,$$

as n tends to $+\infty$.

The fact that T is a mean ergodic operator on $(L^p(\mu), \|\cdot\|_p)$, $1 < p < \infty$, follows, for example, by the fact that they are, unlike $(L^1(\mu), \|\cdot\|_1)$, reflexive spaces, by a result due to Lorch (1939).

Let us illustrate once more this “defectiveness” of T with respect to $\|\cdot\|_1$. To this end we first recall the definition of a *zero type* system,

Definition 2. (X, \mathcal{A}, μ, T) is of *zero type* if for every sets A and B in \mathcal{A}_f :

$$\mu(A \cap T^{-n}B) \rightarrow 0$$

as n tends to $+\infty$,

and recall also this old and classical result:

Theorem 3. (Blum-Hanson, [3]) *Let $(\Omega, \mathcal{F}, \mathbb{P}, S)$ be a probability measure-preserving dynamical system and let $1 \leq p < \infty$. Then $(\Omega, \mathcal{F}, \mathbb{P}, S)$ is mixing if and only if, for every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers and every $F \in L^p(\mathbb{P})$:*

$$\frac{1}{n} \sum_{k=1}^n F \circ S^{n_k} \xrightarrow{\|\cdot\|_p} \mathbb{E}_{\mathbb{P}}[F]$$

as n tends to $+\infty$.

As shown in [9], replacing $(\Omega, \mathcal{F}, \mathbb{P})$ by an infinite-measure space, “mixing” by “zero type” and the limit by 0, then the theorem holds for $1 < p < \infty$ but fails of course again for $p = 1$.

One way to circumvent this unfortunate bad behavior of the Birkhoff sums has been to weight them non-homogeneously and this has proven very fruitful, leading to a considerable amount of deep convergence results and theoretical insights (see [1]). In this paper though, we stick to Birkhoff sums.

1.3. An alternative to $\|\cdot\|_1$. We propose to introduce another norm on $L^1(\mu)$ for which the above theorems hold. More precisely, we will consider the *Orlicz space* $L^\Phi(\mu)$ where Φ is defined on \mathbb{R}_+ by

$$x \mapsto \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases},$$

that is, $L^\Phi(\mu)$ is the vector space measurable functions satisfying

$$\int_X \Phi(|f|) d\mu < \infty \left(\iff \int_X \min(|f|, |f|^2) d\mu < \infty \right).$$

Endowed with its *Orlicz norm* $\|\cdot\|_\Phi$ (see its definition in Section 5.1), $(L^\Phi(\mu), \|\cdot\|_\Phi)$ is a Banach space and $(L^1(\mu), \|\cdot\|_1)$ is continuously included and $\|\cdot\|_\Phi$ -dense in it.

We are then able to prove (Theorem 37):

Theorem. *The dynamical system (X, \mathcal{A}, μ, T) has no absolutely continuous T -invariant probability measure if and only if, for every $f \in L^1(\mu)$ (or $f \in L^\Phi(\mu)$):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \xrightarrow{\|\cdot\|_\Phi} 0$$

as n tends to $+\infty$.

And (Theorem 40):

Theorem. *The dynamical system (X, \mathcal{A}, μ, T) is of zero type if and only if for every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers and every $f \in L^1(\mu)$ (or $f \in L^\Phi(\mu)$):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \xrightarrow{\|\cdot\|_\Phi} 0$$

as n tends to $+\infty$.

Thus $\|\cdot\|_\Phi$ (or any equivalent norm) helps to get the “correct rate” for the above convergence to characterize these two dynamical invariants.

Then, mimicking the construction of the transfer operator \widehat{T} on $L^1(\mu)$ obtained as the predual operator of T acting on $L^\infty(\mu)$, we get an operator $\widehat{T}_\mathcal{P}$ acting on $L^\Phi(\mu)$ (Definition 45) and we show that not only does it preserve $L^1(\mu)$, but it coincides with \widehat{T} on it (Proposition 46).

We now need to recall these two other definitions (see [9]) to explain our results involving the behavior of \widehat{T} relatively to $\|\cdot\|_1$ and $\|\cdot\|_\Phi$.

Definition 4. (X, \mathcal{A}, μ, T) is said to be

- *remotely infinite* if $\bigcap_{n \geq 0} T^{-n} \mathcal{A}$ contains only sets of zero or infinite μ -measure.
- *exact* if $\bigcap_{n \geq 0} T^{-n} \mathcal{A} = \{\emptyset, X\}$ mod. μ .

In particular, as $\mu(X) = +\infty$, we have the chain of implications:

exact \implies remotely infinite \implies zero type \implies absence of a.c. T -invariant measure.

Set $L_0^1(\mu) := \{f \in L^1(\mu), \int_X f d\mu = 0\}$. Lin showed the following characterization of exactness:

Theorem 5. ([11]) *The dynamical system (X, \mathcal{A}, μ, T) is exact if and only if for every $f \in L_0^1(\mu)$:*

$$\left\| \widehat{T^n f} \right\|_1 \rightarrow 0$$

as n tends to $+\infty$.

Then, adapting the rate with $\|\cdot\|_\Phi$ we get (Theorem 50):

Theorem. *The dynamical system (X, \mathcal{A}, μ, T) is remotely infinite if and only if for every $f \in L^\Phi(\mu)$:*

$$\left\| \widehat{T_{\mathcal{P}}^n f} \right\|_\Phi \rightarrow 0$$

or equivalently for every $f \in L_0^1(\mu)$ (or $f \in L^1(\mu)$)

$$\left\| \widehat{T^n f} \right\|_\Phi \rightarrow 0$$

as n tends to $+\infty$.

The fact that one can replace $L_0^1(\mu)$ by $L^1(\mu)$ follows from the fact that $L_0^1(\mu)$ is actually $\|\cdot\|_\Phi$ -dense in $L^\Phi(\mu)$ (Proposition 39).

1.4. Poisson space, stochastic integrals and the Poisson-Orlicz norm.

To prove the above results, we'll actually not rely on $(L^\Phi(\mu), \|\cdot\|_\Phi)$ itself but rather transfer it to another space, the L^1 -space of the *Poisson space* $(X^*, \mathcal{A}^*, \mu^*)$ (also called *Poisson measure*, or *Poisson process*).

On this probability space is defined a family of stochastic integrals $I(f)$ for any function f such that $\int_X f^2 \wedge 1 d\mu < +\infty$ and the key result for us is that those stochastic integrals $I(f)$ are integrable on the Poisson space if and only if f belongs to $L^\Phi(\mu)$ and the norm

$$\|f\|_* := \|I_1(f)\|_{L^1(\mu^*)},$$

where $I_1(f)$ is the centered version of $I(f)$, is equivalent to $\|\cdot\|_\Phi$. This gives way to an isometric embedding of $(L^\Phi(\mu), \|\cdot\|_*)$ into $(L^1(\mu^*), \|\cdot\|_{L^1(\mu^*)})$. This structure result is originally due to Urbanik ([21]).

Given its canonical character, we think $\|\cdot\|_*$ deserves to be named the *Poisson-Orlicz norm*.

We will recall the previously known results on this topic however, our presentation of stochastic integrals departs from the traditional ones and we actually obtain a new characterization of those objects that we hope, has a probabilistic interest on its own (Theorem 21).

This theorem allows first to recover the above structural result by very simple considerations (and one actually gets an optimal upper bound). The ergodic theory results presented above are then derived by introducing the transformation T_* on $(X^*, \mathcal{A}^*, \mu^*)$, canonically associated to T on (X, \mathcal{A}, μ) . The interplay between the *Poisson suspension* $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and its *base* (X, \mathcal{A}, μ, T) has already been exploited in previous works (see for example [5], [16], [19] among others), although mainly focusing on L^2 -results whose

remarkable Fock space structure makes it very handy to work with (as recalled in Theorem 28). Here we deal with the L^1 -structure which is bit more mysterious.

As another consequence of Theorem 21, we obtain a full characterization of coboundaries for stochastic integrals linked to coboundaries on the base space, that once again illustrates the integrability constraints coming from the Poisson space (Theorem 25).

1.5. Another norm. Coming back to the initial observation that the Birkhoff sums doesn't converge under the L^1 -norm when there is no absolutely continuous invariant probability measure, observe it also doesn't allow to determine if a transformation is ergodic or not as it is the case when the measure is finite.

We propose to get a second look at this question by considering a completion of $L^1(\mu)$ with respect to yet another norm, though very close to $\|\cdot\|_1$ (and $\|\cdot\|_*$).

More precisely we will consider the norm $\|\cdot\|_{**}$ on $L^1(\mu)$ defined by

$$\|f\|_{**} := \left\| \int_X f dN \right\|_{L^1(\mu^*)},$$

where N denotes the underlying Poisson random measure.

It satisfies (Proposition 56):

- For all $f \in L^1(\mu)$: $\|f\|_{**} \leq \|f\|_1$.
- $\|f\|_* = \|f\|_1$ if and only if $f = \lambda|f|$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.
- If T is an endomorphism then it is an isometry of $(L^1(\mu), \|\cdot\|_{**})$, and thus extends to the completion $(\overline{L^1(\mu)}, \|\cdot\|_{**})$.
- If moreover there is no absolutely continuous T -invariant probability measure, the following convergence holds with respect to $\|\cdot\|_{**}$, for any $f \in L^1(\mu)$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f \circ T^k = \left(\int_X f d\mu \right) \cdot \bar{1},$$

where $\bar{1}$ is a T -invariant vector in $(\overline{L^1(\mu)}, \|\cdot\|_{**})$ (Theorem 59).

In particular this allows to obtain non-trivial T -invariant vectors in $(\overline{L^1(\mu)}, \|\cdot\|_{**})$.

This latter observation is further exploited to obtain a L^1 -characterization of ergodicity (also valid if μ is finite) (Theorem 61):

Theorem. *Let T be a measure-preserving automorphism of (X, \mathcal{A}, μ) . Then (X, \mathcal{A}, μ, T) is ergodic if and only if*

$$\sup \left(\dim \left(\text{fix} \left(T, \left(\overline{L^1(\mu)}, \|\cdot\| \right) \right) \right) \right) = 1,$$

the supremum being taken over the norms $\|\cdot\|$ on $L^1(\mu)$ (later called T -norms) satisfying:

- T is a mean ergodic contraction on $(L^1(\mu), \|\cdot\|)$
- There exists $K > 0$ such that for all $f \in L^1(\mu)$, $\|f\| \leq K \|f\|_1$.

It T is an endomorphism, we have to replace contractions by isometries to get the equivalence.

2. INFINITE DIVISIBILITY

In this section we recall a few probabilistic features that will be shared by stochastic integrals to be introduced in Section 3.

2.1. Infinite divisibility. We refer to the book [20] for the notion detailed in this section. It is customary to use the language of probability to present it.

Here $d = 1$ or 2 , $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on \mathbb{R}^d , $|\cdot|$ the Euclidean norm and $B_{1,d}$ the unit ball.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Y a random vector defined on it, with distribution σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ (when $d = 1$, we'll prefer to use the term *random variable* for Y).

Definition 6. Y (or its distribution σ) is said to be *infinitely divisible* (abr. ID) if, for any $k \geq 1$ there exists a probability measure σ_k on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\sigma = \sigma_k^{*k},$$

where σ_k^{*k} is the k -th convolution power of σ_k .

The structure of these ID distributions is transparent through the *Lévy-Khintchine representation* (Theorem 8.1, page 37 in [20]):

There exists a $d \times d$ symmetric nonnegative-definite matrix A and a σ -finite measure Q_σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying $Q_\sigma(\{0_{\mathbb{R}^d}\}) = 0$, $\int_{\mathbb{R}^d} |y|^2 \wedge 1 Q_\sigma(dy) < \infty$, and $b \in \mathbb{R}^d$ such that, for all $\gamma \in \mathbb{R}^d$,

$$\mathbb{E} \left[e^{i\langle \gamma, Y \rangle} \right] = \exp \left[\frac{1}{2} i \langle \gamma, A \gamma \rangle + i \langle \gamma, b \rangle + \int_{\mathbb{R}^d} e^{i\langle \gamma, y \rangle} - 1 - \langle \gamma, y \rangle 1_D(y) Q_\sigma(dy) \right].$$

(A, Q_σ, b) is called the *generating triplet* of σ , A is the *Gaussian covariance matrix* and Q_σ is the *Lévy measure* of σ .

Moreover, the above formula defines the characteristic function of an ID distribution as soon as A is $d \times d$ symmetric nonnegative-definite matrix and Q is a σ -finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $\int_{\mathbb{R}^d} |y|^2 \wedge \delta Q_\sigma(dy) < \infty$, for some (and then for all) $\delta > 0$.

There are many choices for the *cut-off* function 1_D above (see [20], page 38), and another legitimate choice would only affect the constant term b , neither A nor Q_σ being altered.

If Q_σ vanishes, then σ is the distribution of a Gaussian random vector, whereas if A vanishes, then it is the distribution of what we will call an *IDp* random vector, the “p” referring to “Poisson” as we shall explain later on (these are also frequently referred to as *ID without Gaussian part*). The random vectors that are both Gaussian and IDp are constant.

If $Y = (Y_1, Y_2)$ is an IDp random vector with distribution σ and Lévy measure Q_σ , then it is easy to see that both coordinates are IDp random variables and if p_i , $i = 1, 2$ is the projection on the i -th coordinate, one has

$$Q_{\sigma \circ p_i^{-1}} = (Q_\sigma \circ p_i^{-1})|_{\mathbb{R} \setminus \{0\}}.$$

i.e., the Lévy measures of the projections are the projections of the Lévy measure, after having removed the eventual mass at 0.

2.2. Integrability.

Proposition 7. (See [20], Example 25.12 page 163) *Let Y be an IDp random vector with distribution σ , then*

$$(2.1) \quad \mathbb{E}[|X|] < \infty \iff \int_{\mathbb{R}^d} |y| 1_{|y| \geq \delta} Q_\sigma(dy) < \infty,$$

for some (and then for all) $\delta > 0$.

If (2.1) holds and, for all $\beta \in \mathbb{R}^d$, $\mathbb{E}[\langle \beta, Y \rangle] = 0$ then, for all $\gamma \in \mathbb{R}^d$:

$$\mathbb{E}[e^{i\langle \gamma, Y \rangle}] = \exp \left[\int_{\mathbb{R}^d} e^{i\langle \gamma, y \rangle} - 1 - \langle \gamma, y \rangle Q_\sigma(dy) \right],$$

and

$$(2.2) \quad \mathbb{E}[|Y|^2] = \int_{\mathbb{R}^d} |y|^2 Q_\sigma(dy).$$

As we can see, the L^2 -norm has a particularly nice form, this is definitely not the case for the L^1 form although the general formula (which we won't use in the paper) is of course available (see [8]): if Z is any real integrable random variable:

$$\mathbb{E}[|Z|] = \frac{2}{\pi} \int_0^{+\infty} \frac{1 - \Re \mathbb{E}[e^{itZ}]}{t^2} dt.$$

Let us finish this section by mentioning this result that is a combination of Theorem 24.7, page 149 in [20] and the equivalence (2.1):

Proposition 8. *Let Y be an integrable IDp random variable with distribution σ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then*

$$\exists m \in \mathbb{R}, m \leq Y \text{ a.e.} \iff Q_\sigma(\mathbb{R}_-) = 0, \text{ and } \int_{\mathbb{R}_+} y Q_\sigma(dy) < \infty.$$

3. BACKGROUND ON THE POISSON SPACE AND STOCHASTIC INTEGRALS

3.1. Poisson space. We recall the definition of the *Poisson space* (also called *Poisson measure* or *Poisson point process*) $(X^*, \mathcal{A}^*, \mu^*)$ over (X, \mathcal{A}, μ) :

- X^* is the collection of measures of the form $\nu := \sum_{i \in I} \delta_{x_i}$, $x_i \in X$, I countable.
- $\mathcal{A}^* := \sigma\{N(A), A \in \mathcal{A}\}$ where $N(A)$ is the map $\omega \in X^* \mapsto \omega(A)$.

- μ^* is the only probability measure such that, for any $k \in \mathbb{N}$, any collection A_1, \dots, A_k of pairwise disjoint sets in \mathcal{A}_f , the random variables $N(A_1), \dots, N(A_k)$ are independent and Poisson distributed with parameter $\mu(A_1), \dots, \mu(A_k)$ respectively.

It can be checked out that, once \mathcal{A}^* is completed with respect to μ^* , then the Poisson space becomes a Lebesgue probability space.

3.2. Poisson suspension. If T is an endomorphism of (X, \mathcal{A}, μ) then the map T_* defined by

$$\omega \in X^* \mapsto \omega \circ T^{-1},$$

is an endomorphism of $(X^*, \mathcal{A}^*, \mu^*)$.

The dynamical system $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is the *Poisson suspension* over the base (X, \mathcal{A}, μ, T) and we can present the most basic ergodic results, obtained by Marchat ([12]):

Theorem 9. *Let T be an automorphism of (X, \mathcal{A}, μ) .*

- $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is ergodic (and weakly mixing) if and only if (X, \mathcal{A}, μ, T) doesn't possess any T -invariant set of positive and finite μ -measure, or equivalently, if there is no absolutely continuous T -invariant probability measure.
- $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is mixing if and only if (X, \mathcal{A}, μ, T) is of zero type.

In particular, as we suppose μ infinite, $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is ergodic as soon as (X, \mathcal{A}, μ, T) is.

Remark 10. Most references on Poisson suspensions deal with automorphisms (see for example [19]) however Zweimüller ([22]) dealt with endomorphisms and proved that the natural extension of the Poisson suspension over some system is the Poisson suspension of the natural extension of this same system. The above theorem is thus valid for endomorphisms as well.

3.3. Stochastic integrals. In the following, depending on the context, when dealing with the integral of a function g with respect to a measure ρ on the space X , we'll use the following equivalent notation:

- $\int_X g(x) \rho(dx)$
- $\int_X g d\rho$
- $\rho(g)$

However, we'll stick to the probabilistic notation $\mathbb{E}_{\mu^*}[F]$ for the expectation of a random variable F on the probability space $(X^*, \mathcal{A}^*, \mu^*)$.

We detail the notion of *stochastic integral* in the Poisson setting whose construction can be found in ([14]).

The starting point is the following formula readily available from the definition: for any $A \in \mathcal{A}_f$

$$\mathbb{E}_{\mu^*}[N(A)] = \mu(A),$$

This allows to safely define such quantity as $N(f)$ on X^* for an $L^1(\mu)$ -function f on (X, \mathcal{A}) :

$$N(f) : \omega \mapsto \omega(f),$$

at least for μ^* -a.e. $\omega \in X^*$.

Indeed, if $f \in L^1(\mu)$, using standard monotone approximation arguments, $N(|f|)$ is easily proven to be in $L^1(\mu^*)$ and satisfies:

$$\mathbb{E}_{\mu^*} [N(|f|)] = \mu(|f|).$$

Therefore, for any $f \in L^1(\mu)$, $N(|f|)$ is finite μ^* -a.e. and thus f is ω -integrable for μ^* -a.e. $\omega \in X^*$ and we have:

$$(3.1) \quad \mathbb{E}_{\mu^*} [N(f)] = \mu(f).$$

Observe that replacing f by some function \tilde{f} such that $f = \tilde{f}$ μ -a.e. yields two random variables $N(f)$ and $N(\tilde{f})$ that are equal μ^* -a.e..

We define, for any $f \in L^1(\mu)$

$$(3.2) \quad I_1(f) := N(f) - \mu(f).$$

It is a centered and integrable random variable.

It is however possible to get further. Set $\chi(x) = x^2 \wedge 1$, $x \geq 0$ and consider the *generalized Orlicz space* (see [18], page 400):

$$L^\chi(\mu) := \left\{ f \in L^0(\mu), \int_X \chi(|f|) d\mu < +\infty \right\}.$$

Then the *stochastic integral* $I(f)$, for $f \in L^\chi(\mu)$, is defined as

$$\begin{aligned} I(f) &:= \lim_{\epsilon \rightarrow 0} N(f 1_{|f| > \epsilon}) - \mu(f 1_{|f| \leq 1} 1_{|f| > \epsilon}) \\ &= \lim_{\epsilon \rightarrow 0} I_1\left(f 1_{\epsilon < |f| \leq \frac{1}{\epsilon}}\right) + \mu\left(f 1_{1 < |f| \leq \frac{1}{\epsilon}}\right), \end{aligned}$$

the limit taking place in μ^* -measure.

The random vector $(\Re I(f), \Im I(f))$ has characteristic function given by

$$\begin{aligned} &\mathbb{E}_{\mu^*} [\exp(i(a\Re I(f) + b\Im I(f)))] \\ &= \exp\left(\int_X e^{i(a\Re f + b\Im f)} - 1 + (a\Re f + b\Im f) 1_{|f| \leq 1} d\mu\right). \end{aligned}$$

In particular, it is infinitely divisible without Gaussian part, and its Lévy measure is the image measure of μ by $(\Re f, \Im f)$, restricted to $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Remark 11. Pay attention to the fact that I is not linear, nevertheless we have:

$$I(\alpha f + g) = \alpha I(f) + I(g) + c$$

for some constant c depending on α , f and g .

In particular, we have

$$I(f) = I(\Re f) + iI(\Im f) + d$$

for some constant d depending on f .

However, the effect of an endomorphism doesn't involve the addition of a constant, thanks to the fact it preserves the measure:

$$\begin{aligned}
 I(f) \circ T_* &= \lim_{\epsilon \rightarrow 0} N(f1_{|f|>\epsilon}) \circ T_* - \mu(f1_{|f|\leq 1}1_{|f|>\epsilon}) \\
 &= \lim_{\epsilon \rightarrow 0} N(f \circ T1_{|f \circ T|>\epsilon}) - \mu(f1_{|f|\leq 1}1_{|f|>\epsilon}) \\
 &= \lim_{\epsilon \rightarrow 0} N(f \circ T1_{|f \circ T|>\epsilon}) - \mu(f \circ T1_{|f \circ T|\leq 1}1_{|f \circ T|>\epsilon}) \\
 &= I(f \circ T)
 \end{aligned}$$

Lastly, $I(f)$ is constant if and only if f vanishes μ -a.e..

Definition 12. A *stochastic integral* defined on $(X^*, \mathcal{A}^*, \mu^*)$ is a random variable of the form

$$I(f) + c,$$

with f in $L^X(\mu)$ and $c \in \mathbb{C}$.

Remark 13. In particular, when $f \in L^1(\mu)$, $I_1(f)$ is a stochastic integral since $I_1(f) = I(f) - \mu(f1_{|f|>1})$.

3.3.1. *Campbell measure and difference operators.* To derive useful properties of stochastic integrals, it is convenient to consider a representation of them in the product space $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*, \mu \otimes \mu^*)$ as it is done in [10] for the L^2 -case. We need however to go beyond L^2 and we have been unable to find all the results we want in the existing literature, we thereafter give proofs of what we believe is, as far as we know, new material.

Definition 14. (see for example [4], page 269) The *Campbell measure* Q of $(X^*, \mathcal{A}^*, \mu^*)$ is defined on $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$, for any positive measurable φ by

$$\int_{X \times X^*} \varphi(x, \omega) Q(d(x, \omega)) := \int_{X^*} \left(\int_X \varphi(x, \omega) \omega(dx) \right) \mu^*(d\omega).$$

The important formula is the following:

Theorem 15. (*Mecke Formula*, [15]) *For any positive measurable function φ on $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$:*

$$\int_{X \times X^*} \varphi(x, \omega) Q(d(x, \omega)) = \int_{X \times X^*} \varphi(x, \omega + \delta_x) \mu \otimes \mu^*(d(x, \omega)).$$

This formula says that Q is the image measure of $\mu \otimes \mu^*$ under the mapping:

$$(x, \omega) \mapsto (x, \omega + \delta_x).$$

Remark 16. It is worth noting that, when μ is infinite, the projection of Q along the second coordinate is equivalent to μ^* (of course, the same applies to $\mu \otimes \mu^*$). To see that, take $f > 0$ in $L^1(\mu)$, then, as μ is an infinite measure, for μ^* -almost all $\omega \in X^*$, $\omega \neq 0$ and thus $\int_X f(x) \omega(dx) > 0$ for μ^* -almost all $\omega \in X^*$. We now consider the measure \tilde{Q} on $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$ defined

by $\frac{d\tilde{Q}}{dQ}(x, \omega) := \frac{f(x)}{\int_X f(s)\omega(ds)}$ so that $\tilde{Q} \sim Q$. Now consider its projection \bar{Q} on X^* :

$$\begin{aligned}\bar{Q}(\mathfrak{A}) &= \int_{X \times X^*} 1_{\mathfrak{A}}(\omega) \frac{f(x)}{\int_X f(s)\omega(ds)} Q(d(x, \omega)) \\ &= \int_{X^*} 1_{\mathfrak{A}}(\omega) \frac{1}{\int_X f(s)\omega(ds)} \left(\int_X f(t)\omega(dt) \right) \mu^*(d\omega) \\ &= \mu^*(\mathfrak{A}).\end{aligned}$$

Thus $\bar{Q} = \mu^*$.

For any measurable F on (X, \mathcal{A}^*) , define F_δ on $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$ by

$$(x, \omega) \mapsto F(\omega + \delta_x)$$

Definition 17. (See [10]) The *difference operator* D associates a measurable F on (X^*, \mathcal{A}^*) to a measurable DF on $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*)$ by

$$(x, \omega) \mapsto (D_x F)(\omega) := F_\delta(x, \omega) - F(\omega).$$

It has to be noted that, taking $F = \tilde{F}$ μ^* -a.e. implies that $DF = D\tilde{F}$ $\mu \otimes \mu^*$ -a.e., indeed, consider

$$(D_x F)(\omega) - (D_x \tilde{F})(\omega) = (F - \tilde{F})_\delta(x, \omega) - (F - \tilde{F})(\omega)$$

But $F = \tilde{F}$ $\mu \otimes \mu^*$ -a.e. and, thanks to Remark 16 $(F - \tilde{F})_\delta$ is distributed, under $\tilde{Q} \sim \mu \otimes \mu^*$, as $F - \tilde{F}$ under μ^* and therefore vanishes $\mu \otimes \mu^*$ -a.e.

This proves $DF = D\tilde{F}$ $\mu \otimes \mu^*$ -a.e.. and, in particular DF can be safely defined on F that are only defined μ^* -a.e.

Those operators are well suited for stochastic integrals as we get:

Proposition 18. *Let f be in $L^X(\mu)$. For $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$:*

$$D_x I(f)(\omega) = f(x)$$

Proof. Set $f_n := f 1_{|f| > \frac{1}{n}}$, $n \geq 1$, we have $\mu \{ |f| > \frac{1}{n} \} < \infty$, hence, for μ^* -a.e. ω , $\omega \{ |f| > \frac{1}{n} \} < \infty$ and we can safely write, for $\mu \otimes \mu^*$ -a.e. (ω, x) :

$$\begin{aligned}(I(f_n))_\delta(\omega, x) &= \int_{|f| > \frac{1}{n}} f d(\omega + \delta_x) - \int_{|f| > \frac{1}{n}} f 1_{|f| \leq 1} d\mu \\ &= \int_{|f| > \frac{1}{n}} f d(\omega) + f(x) 1_{|f| > \frac{1}{n}}(x) - \int_{|f| > \frac{1}{n}} f 1_{|f| \leq 1} d\mu \\ &= I(f_n)(\omega) + f_n(x)\end{aligned}$$

By definition $I(f_n)$ tends to $I(f)$ in μ^* -measure. In particular, there exists an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $I(f_{n_k})$ tends to $I(f)$ μ^* -a.e. as k tends to $+\infty$.

But this implies, thanks to Remark 16 once again, that $(I(f_{n_k}))_\delta$ tends to $(I(f))_\delta$ $\mu \otimes \mu^*$ -a.e. as k tends to $+\infty$.

Therefore

$$(I(f))_\delta(\omega, x) = I(f)(\omega) + f(x),$$

hence the result.

Our next goal is to show the converse, to achieve this we need the following intermediate results. \square

Proposition 19. (See [10]) *If $F \in L^2(\mu^*)$ then $x \mapsto \mathbb{E}_{\mu^*}[D_x F]$ belongs to $L^2(\mu)$ and, if there exists a measurable f on (X, \mathcal{A}) such that $D_x F(\omega) = f(x)$ for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$ then there exists $c \in \mathbb{C}$ such that*

$$F = I(f) + c.$$

And:

Lemma 20. *Let F be a measurable on (X^*, \mathcal{A}^*) such that, $\mu \otimes \mu^*$ -a.e.,*

$$DF = 0,$$

then F is constant μ^ -a.e..*

Proof. For $\mu \otimes \mu^*$ -almost all $(x, \omega) \in X \times X^*$ we have

$$F_\delta(x, \omega) = F(\omega),$$

thus, for every Borel set A , we get

$$1_A \circ F_\delta(x, \omega) = 1_A \circ F(\omega),$$

thus, $\mu \otimes \mu^*$ -a.e:

$$D(1_A \circ F) = 0.$$

But $1_A \circ F \in L^2(\mu^*)$, therefore it is constant by Proposition 19. Since this is true for all Borel sets, this gives the result. \square

We can now formulate a characterization of stochastic integrals:

Theorem 21. *A measurable function F on (X^*, \mathcal{A}^*) is a stochastic integral if and only if there exists a measurable function f on (X, \mathcal{A}) such that for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$,*

$$(D_x F)(\omega) = f(x),$$

in particular there exists $c \in \mathbb{R}$ such that $F = I(f) + c$.

Proof. It is sufficient to deal with real valued functions as we get the complex case by considering real and imaginary parts.

For any $t \in \mathbb{R}$ and for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$ we have

$$\begin{aligned} \exp itF_\delta(x, \omega) - \exp itF(\omega) &= \exp itf(x) \exp itF(\omega) - \exp itF(\omega) \\ &= \exp itF(\omega) (\exp itf(x) - 1). \end{aligned}$$

Therefore

$$\mathbb{E}_{\mu^*}[D_x(\exp itF)] = \mathbb{E}_{\mu^*}[\exp itF](\exp itf(x) - 1).$$

Now from Proposition 19, since $\exp itF \in L^2(\mu^*)$, then $x \mapsto \mathbb{E}_{\mu^*}[D_x(\exp itF)] \in L^2(\mu)$ and therefore $\exp itf - 1 \in L^2(\mu)$ and:

$$\int_X |\exp itf(x) - 1|^2 \mu(dx) = 4 \int_X \sin^2\left(\frac{1}{2}tf(x)\right) \mu(dx) < +\infty.$$

We can find t_1 and t_2 such that there exists $\epsilon > 0$, such that, for any $|y| > 1$, $\sin^2\left(\frac{1}{2}t_1y\right) + \sin^2\left(\frac{1}{2}t_2y\right) > \epsilon$. In particular,

$$\int_X \epsilon 1_{|f|>1} \mu(dx) \leq \int_X \sin^2\left(\frac{1}{2}t_1f(x)\right) + \sin^2\left(\frac{1}{2}t_2f(x)\right) \mu(dx) < \infty$$

that is

$$\mu(|f| > 1) < \infty$$

and, as $\sin^2(1)z^2 \leq \sin^2(z)$ on $[-1, 1]$,

$$\int_X f^2 1_{|f|\leq 1} d\mu \leq \int_X \frac{4}{t_1^2 \sin^2(1)} \sin^2\left(\frac{1}{2}t_1f(x)\right) \mu(dx) < \infty.$$

It follows that $f \in L^X(\mu)$ and thus the stochastic integral $I(f)$ is well defined and satisfies, for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$ $D_x(I(f))(\omega) = f(x)$.

Therefore, $\mu \otimes \mu^*$ -a.e. $D(F - I(f)) = 0$ and then $F = I(f) + c$ for some $c \in \mathbb{R}$, thanks to Lemma 20. \square

3.4. The set of stochastic integrals. Denote by $\mathcal{I}_{\mu^*} \subset L^0(\mu^*)$ the set of stochastic integrals. As a first consequence of Theorem 21, we get:

Proposition 22. *\mathcal{I}_{μ^*} is a closed subspace of $L^0(\mu^*)$ with respect to convergence in measure.*

Proof. The fact that it is a linear space follows immediately from Remark 11.

Let $\{F_n\}_{n \in \mathbb{N}} := \{I(f_n) + c_n\}_{n \in \mathbb{N}}$ be a sequence of stochastic integrals converging to some random variable F with respect to μ^* . There exists a subsequence $\{F_{n_k}\}_{k \in \mathbb{N}}$ converging to F μ^* -a.e.

For the same reason already explained earlier $\{F_{n_k\delta}\}_{k \in \mathbb{N}}$ converges \tilde{Q} -a.e. and thus $\mu \otimes \mu^*$ -a.e. to F_δ . Now from Proposition 18, for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$:

$$f_{n_k}(x) = D_x F_{n_k}(\omega) = F_{n_k\delta}(x, \omega) - F_{n_k}(\omega).$$

Therefore $\{f_{n_k}\}_{k \in \mathbb{N}}$ converges μ -a.e. to some measurable f satisfying

$$f(x) = F_\delta(x, \omega) - F(\omega) = D_x F(\omega).$$

Thanks to Proposition 21, $F = I(f) + c$ for some $c \in \mathbb{C}$ as expected. \square

Denote by $\|\cdot\|_0$ on $L^0(\mu^*)$ the F -norm defined by

$$\|F\|_0 := \mathbb{E}_{\mu^*}[|F| \wedge 1],$$

It metrizes convergence in probability.

Let \mathfrak{C} be the closed subspace of constant random variables, Proposition 22, together with the above discussion and Remark 11 yield:

Corollary 23. *We have*

$$L^\chi(\mu) \simeq \mathcal{I}_{\mu^*} / \mathcal{C}$$

and $\|\cdot\|_0$ induces an F -norm $\|\cdot\|'_0$

$$\|f\|'_0 = \min_{c \in \mathbb{C}} \|I(f) + c\|_0,$$

that makes $(L^\chi(\mu), \|\cdot\|'_0)$ a complete F -space. If T is an endomorphism, then it is an isometry of $(L^\chi(\mu), \|\cdot\|'_0)$.

3.5. Coboundaries. We recall that a S -coboundary of a dynamical system (Y, \mathcal{B}, ν, S) is a measurable function $g \in L^0(\mu)$ such that there exists $h \in L^0(\mu)$ satisfying

$$g = h \circ S - h.$$

h is then called a *transfer function*.

Definition 24. A T_* -coboundary of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is said to be Poisson if any transfer function is a stochastic integral.

As another application of Theorem 21 we can identify Poisson coboundaries:

Theorem 25. *Assume (X, \mathcal{A}, μ, T) is conservative ergodic and let $F \in L^0(\mu^*)$. The following assertions are equivalent:*

- (1) F is a Poisson T_* -coboundary
- (2) F is a T_* -coboundary and a stochastic integral of a T -coboundary
- (3) F is a T_* -coboundary and a stochastic integral

Proof. Let $F \in L^0(\mu^*)$ and $f \in L^\chi(\mu)$ and $c \in \mathbb{C}$ such that:

$$F = (I(f) + c) \circ T_* - (I(f) + c).$$

Then $f \circ T - f \in L^\chi(\mu)$ and, thanks to Remark 11, there exists $d \in \mathbb{C}$ such that

$$F = I(f \circ T - f) + d,$$

that is, F is a stochastic integral of a T -coboundary

Thus 1. \Rightarrow 2. \Rightarrow 3.

Assume there exists $f \in L^\chi(\mu)$, real-valued and $c \in \mathbb{R}$ and $G \in L^0(\mu^*)$ also real-valued such that $F = I(f) + c = G \circ T_* - G$.

Consider the function H on $L^0(\mu \otimes \mu^*)$:

$$H(x, \omega) := G_\delta(T_*\omega, x) - G_\delta(\omega, x) - F(\omega)$$

and observe that

$$\begin{aligned} G_\delta(Tx, T_*\omega) &= G(T_*\omega + \delta_{Tx}) \\ &= G(T_*(\omega + \delta_x)) \\ &= (G \circ T_*)(\omega + \delta_x) \\ &= (G \circ T_*)_\delta(x, \omega) \end{aligned}$$

We have

$$\begin{aligned}
 H(Tx, T_*\omega) &= G_\delta(T_*^2\omega, Tx) - G_\delta(T_*\omega, Tx) - F(T_*\omega) \\
 &= (G \circ T_*)_\delta(x, T_*\omega) - (G \circ T_*)_\delta(x, \omega) - F(T_*\omega) \\
 &= (G \circ T_* - G)_\delta(x, T_*\omega) + G_\delta(x, T_*\omega) \\
 &\quad - ((G \circ T_* - G)_\delta(x, \omega) + G_\delta(x, \omega)) - (F(T_*\omega) - F(\omega) + F(\omega)) \\
 &= F_\delta(x, T_*\omega) + G_\delta(x, T_*\omega) - (F_\delta(x, \omega) + G_\delta(x, \omega)) \\
 &\quad - (F(T_*\omega) - F(\omega) + F(\omega)) \\
 &= F_\delta(x, T_*\omega) - F(T_*\omega) - (F_\delta(x, \omega) - F(\omega)) \\
 &\quad + G_\delta(x, T_*\omega) - G_\delta(x, \omega) - F(\omega) \\
 &= D_x F(T_*\omega) - D_x F(\omega) + G_\delta(x, T_*\omega) - G_\delta(x, \omega) - F(\omega) \\
 &= f(x) - f(x) + H(x, \omega) \\
 &= H(x, \omega)
 \end{aligned}$$

Hence H is $T \times T_*$ -invariant.

Now, from Meyerovitch's theorem (see [16]), $(X \times X^*, \mathcal{A} \otimes \mathcal{A}^*, \mu \otimes \mu^*, T \times T_*)$ is ergodic since (X, \mathcal{A}, μ, T) is conservative ergodic. Therefore H is constant, that is, there exists $\gamma \in \mathbb{R}$ such that, for $\mu \otimes \mu^*$ -a.e. $(x, \omega) \in X \times X^*$:

$$G_\delta(T_*\omega, x) - G_\delta(x, \omega) - F(\omega) = \gamma,$$

or

$$G_\delta(T_*\omega, x) - G_\delta(x, \omega) - G(T_*\omega) - G(\omega) = \gamma,$$

that is

$$D_x G(T_*\omega) - D_x G(\omega) = \gamma.$$

Thanks to Fubini's Theorem, for μ -a.e. $x \in X$, we have

$$D_x G(T_*\omega) - D_x G(\omega) = \gamma,$$

for μ^* -a.e. $\omega \in X^*$

This implies that, for μ^* -a.e. $\omega \in X^*$, for any $t \in \mathbb{R}$:

$$\exp it D_x G(T_*\omega) = \exp it \gamma \exp it D_x G(\omega),$$

for μ^* -a.e. $\omega \in X^*$ which means that, for μ -a.e. $x \in X$, $\omega \mapsto \exp it D_x G(\omega)$ is an eigenfunction of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ for the eigenvalue $\exp it \gamma$. But, as $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is ergodic, it is also weakly mixing and thus $\exp it \gamma = 1$, and this for all $t \in \mathbb{R}$. This implies $\gamma = 0$.

We deduce that, for μ -a.e. $x \in X$:

$$D_x G(T_*\omega) = D_x G(\omega),$$

for μ^* -a.e. $\omega \in X^*$. Thus, for μ -a.e. $x \in X$, there exists $f(x)$ such that $D_x G(\omega) = f(x)$ for μ^* -a.e. $\omega \in X^*$ by ergodicity of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ once again. Thanks to Fubini's theorem again, one can choose f to be measurable. It remains to apply Theorem 21, to obtain that G is a stochastic integral. \square

We can rephrase one implication of the theorem into a coboundary existence criterion:

Corollary 26. *Let (X, \mathcal{A}, μ, T) be conservative ergodic and f in $L^X(\mu)$. If $I(f)$ is a T_* -coboundary then f is a T -coboundary with a transfer function in $L^X(\mu)$.*

4. SQUARE-INTEGRABLE STOCHASTIC INTEGRALS, FOCK SPACE STRUCTURE

In this section, we recall the classical L^2 -case that relies almost entirely on the Hilbertian structure (the details can be found in [17], Chapter 10, Section 4 and also [10]).

The starting point is the following isometry identity: for any A and B in \mathcal{A}_f ,

$$\mathbb{E}_{\mu^*} [(N(A) - \mu(A))(N(B) - \mu(B))] = \mu(A \cap B).$$

And we get, for any functions f and g in $L^1(\mu) \cap L^2(\mu)$:

$$\langle I_1(f), I_1(g) \rangle_{L^2(\mu^*)} = \langle f, g \rangle_{L^2(\mu)},$$

where I_1 was defined by (3.2).

Moreover, from (Eq (2.2)), a stochastic integral $I(h)$ is square integrable if and only if $h \in L^2(\mu)$.

Setting $\mathcal{I}_2 := \mathcal{I} \cap L_0^2(\mu^*)$ this leads immediately to:

Theorem 27. *I_1 extends to an isometric isomorphism between the Hilbert spaces $(L^2(\mu), \|\cdot\|_2)$ and $(\mathcal{I}_2, \|\cdot\|_{L^2(\mu^*)})$. In particular, for any f and g in $L^2(\mu)$:*

$$\langle I_1(f), I_1(g) \rangle_{L^2(\mu^*)} = \langle f, g \rangle_{L^2(\mu)},$$

and

$$\|I_1(f)\|_{L^2(\mu^*)} = \|f\|_2.$$

The above theorem is actually the ground for the following fundamental result (see [17], Chapter 10, Section 4):

Theorem 28. *There is a natural isometric identification of $L^2(\mu^*)$ as the Fock space $F(L^2(\mu))$ over $L^2(\mu)$:*

$$L^2(\mu^*) \simeq F(L^2(\mu)) := \mathbb{C} \oplus L^2(\mu) \oplus L^2(\mu)^{\odot 2} \oplus \cdots \oplus L^2(\mu)^{\odot n} \oplus \cdots$$

The latter space is understood as the completion of the infinite orthogonal sum of the symmetric tensor products of $L^2(\mu)$ with the appropriately scaled scalar product.

Let us succinctly describe how this identification is actually implemented:

Set, for any $n \geq 1$, the *stochastic integral of order n* by the following formula:

$$I_n(f^{\otimes n}) := \int_{\Delta_n^c} f^{\otimes n} d(N - \mu)^{\otimes n} \in L^2(\mu^*).$$

where Δ_n is the subset of X^n where at least two coordinates coincide, f is a simple function in $L^2(\mu)$ and $f^{\otimes n}$ its tensor product, an element of $L^2(\mu^{\otimes n})|_{sym} \simeq L^2(\mu)^{\odot n}$.

The following formula holds for simple functions f and g in $L^2(\mu)$:

$$\langle I_n(f^{\otimes n}), I_n(g^{\otimes n}) \rangle_{L^2(\mu^*)} = n! \langle f^{\otimes n}, g^{\otimes n} \rangle_{L^2(\mu)^{\odot n}}$$

and if $n \neq m$,

$$\langle I_n(f^{\otimes n}), I_m(g^{\otimes m}) \rangle_{L^2(\mu^*)} = 0.$$

The formula $I_n(f^{\otimes n})$ extends isometrically to any $f \in L^2(\mu)$ and if we set $\mathcal{I}_2^{(0)} = \mathbb{C} \cdot 1_{X^*}$, $\mathcal{I}_2^{(1)} = \mathcal{I}_2$ and $\mathcal{I}_2^{(n)} = \text{Span} \langle I_n(f^{\otimes n}), f \in L^2(\mu) \rangle$, one gets the orthogonal sum:

$$L^2(\mu^*) = \overline{\mathcal{I}_2^{(0)} \oplus \mathcal{I}_2^{(1)} \oplus \mathcal{I}_2^{(2)} \oplus \cdots \oplus \mathcal{I}_2^{(n)} \oplus \cdots}$$

where each $\mathcal{I}_2^{(n)}$ is isometrically identified to $(L^2(\mu)^{\odot n}, n! \langle \cdot, \cdot \rangle_{L^2(\mu)^{\odot n}})$ through I_n and called the *chaos of order n*.

Observe now that if T is an endomorphism of (X, \mathcal{A}, μ) , then, for any $f \in L^2(\mu)$:

$$I_n(f) \circ T_* = I_n((f \circ T)^{\otimes n}),$$

in particular, T_* preserves each chaos.

We end this section by the following lemma that we shall need later, it is obvious given the above discussion:

Lemma 29. *On $L^2(\mu^*)$, let P be the orthogonal projection on \mathcal{I}_2 . Then P and T_* commute.*

5. INTEGRABLE STOCHASTIC INTEGRALS, THE ORLICZ SPACE $L^\Phi(\mu)$

The integrable case has deserved considerably less attention than the square-integrable case, the latter being very handy to work with given its Hilbertian nature.

Urbanik in [21] first obtained the representation result below (Theorem 33) that is the key ingredient to our work. For the sake of completeness and clarity, we will prove this result again, using very simple arguments.

The starting point is the equivalence, thanks to 2.1), combined with the condition for a stochastic integral to be defined, we get:

$$(5.1) \quad I(f) \in L^1(\mu^*) \iff \int_X \min(|f|, |f|^2) d\mu < \infty.$$

And it turns out that if this condition is satisfied, $\mathbb{E}_{\mu^*}[I(f)] = \mu(f1_{|f|>1})$ and this allows to extend the definition of I_1 by setting

$$I_1(f) = \lim_{\epsilon \rightarrow 0} N(f1_{|f|>\epsilon}) - \mu(f1_{|f|>\epsilon}),$$

in measure.

The cut-off term is thus no longer needed in the characteristic function: for a and b in \mathbb{R} ,

$$\begin{aligned} & \mathbb{E}_{\mu^*} [\exp (i (a \Re I_1 (f) + b \Im I_1 (f)))] \\ &= \exp \left(\int_X e^{i(a \Re f + b \Im f)} - 1 + (a \Re f + b \Im f) d\mu \right). \end{aligned}$$

As a result, we get

$$(5.2) \quad I(f) \in L_0^1(\mu^*) \iff \int_X \min(|f|, |f|^2) d\mu < \infty \text{ and } I(f) = I_1(f).$$

We set $\mathcal{I}_1 := \mathcal{I} \cap L_0^1(\mu^*)$.

We mention here the following remarkable distributional property as a consequence of Proposition 8:

Proposition 30. *Let f be real valued in $L^\Phi(\mu)$, then:*

$$\exists m \in \mathbb{R}, I(f) \geq m \iff f \in L^1(\mu), f \geq 0.$$

In particular, we can write

$$I_1(f) = N(f) - \mu(f).$$

5.1. The Orlicz space $L^\Phi(\mu)$. To understand the set $L^\Phi(\mu)$ of functions $f \in L^0(\mu)$ satisfying $\int_X \min(|f|, |f|^2) d\mu < \infty$, it was natural (see [21]) to replace it by the equivalent condition

$$\int_X \Phi(|f|) d\mu < \infty,$$

Φ being the function on \mathbb{R}_+ given by:

$$x \mapsto \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

The consequences are decisive, we list them thereafter, following the first four chapters of [18]:

Φ is a *Young function*, that is

- $\Phi(0) = 0$
- $\lim_{x \rightarrow \infty} \Phi(x) = +\infty$
- Φ is convex

Moreover, Φ is continuous, strictly increasing and Δ_2 -regular, that is, for all $x \geq 0$:

$$\Phi(2x) \leq 4\Phi(x).$$

A number of objects associated to such a function come naturally :

One defines N_Φ on $L^\Phi(\mu)$ by:

$$N_\Phi(f) := \inf \left\{ \lambda > 0, \int_X \Phi \left(\left| \frac{f}{\lambda} \right| \right) d\mu \leq 1 \right\}.$$

Also the *complementary function* to Φ is the function Ψ defined on \mathbb{R}_+ by

$$\Psi(y) = \sup \{xy - \Phi(x)\}$$

and in this case, we have

$$\Psi(y) = \begin{cases} y^2 & \text{if } y \leq 2 \\ +\infty & \text{if } y > 2 \end{cases}.$$

This yields another set

$$L^\Psi(\mu) := \left\{ g \in L^0(\mu), \exists \alpha > 0 \int_X \Psi(|\alpha g|) d\mu < \infty \right\},$$

with N_Ψ being defined similarly on $L^\Psi(\mu)$ by

$$N_\Psi(g) := \inf \left\{ \lambda > 0, \int_X \Psi\left(\left|\frac{g}{\lambda}\right|\right) d\mu \leq 1 \right\}.$$

The complementary function Ψ allows to define the quantity $\|f\|_\Phi$ for $f \in L^\Phi(\mu)$ by:

$$\|f\|_\Phi := \sup \left\{ \int_X |fg| d\mu, \int_X \Psi(|g|) d\mu \leq 1 \right\}.$$

We can now sum up the properties of those objects, thanks to the well understood theory of Orlicz spaces (see [18] again):

Theorem 31. *The following holds:*

- $L^\Phi(\mu)$ and $L^\Psi(\mu)$ are Orlicz vector spaces.
- N_Φ and $\|\cdot\|_\Phi$ are both norms on $L^\Phi(\mu)$ (N_Φ is called the gauge or Luxemburg norm and $\|\cdot\|_\Phi$ the Orlicz norm).
- For all $f \in L^\Phi(\mu)$,

$$(5.3) \quad N_\Phi(f) \leq \|f\|_\Phi \leq 2N_\Phi(f).$$

- $(L^\Phi(\mu), N_\Phi)$ is a non-reflexive separable Banach space.
- $L^1(\mu) \subset L^\Phi(\mu)$ and $\overline{L^1(\mu)} = L^\Phi(\mu)$.
- $L^2(\mu) \subset L^\Phi(\mu)$ and $\overline{L^2(\mu)} = L^\Phi(\mu)$.
- $L^\Psi(\mu) = L^2(\mu) \cap L^\infty(\mu)$.
- $N_\Psi = \max(\|\cdot\|_2, \frac{1}{2}\|\cdot\|_\infty)$.
- For all $f \in L^\Phi(\mu)$, $\|f\|_\Phi := \sup \left\{ \int_X |fg| d\mu, N_\Psi(g) \leq 1 \right\}$.
- $(L^\Psi(\mu), N_\Psi)$ is (a representation of) the topological dual $(L^\Phi(\mu))', \|\cdot\|'_\Phi$ of $(L^\Phi(\mu), \|\cdot\|_\Phi)$.
- If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of elements of $L^\Phi(\mu)$ converging to $f \in L^\Phi(\mu)$ with respect to $\|\cdot\|_\Phi$, then there exists a μ -a.e. converging subsequence.

5.2. The Poisson-Orlicz norm and Urbanik's theorem. Set $\mathcal{I}_1 := \mathcal{I} \cap L_0^1(\mu^*)$. I_1 is an onto linear map between $L^\Phi(\mu)$ and \mathcal{I}_1 , it is also one-to-one thanks to Proposition 30, for example. This leads naturally to the definition:

Definition 32. For all $f \in L^\Phi(\mu)$,

$$\|f\|_* := \|I_1(f)\|_{L^1(\mu^*)},$$

defines a norm. We call it the *Poisson-Orlicz norm* of $L^\Phi(\mu)$.

As such, $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$ is isometric to $(L^\Phi(\mu), \|\cdot\|_*)$.

The natural question is therefore to compare this norm with N_Φ and $\|\cdot\|_\Phi$. The answer has been given by Urbanik [21]:

Theorem 33. *The norms $\|\cdot\|_*$, N_Φ and $\|\cdot\|_\Phi$ are equivalent.*

The strategy developed by Urbanik was first to show that $(L^\Phi(\mu), \|\cdot\|_*)$ is a Banach space, then to show that convergence with respect to $\|\cdot\|_*$ implies convergence with respect to N_Φ and vice versa.

In [13], explicit bounds were obtained:

$$(0.125) N_\Phi(f) \leq \|f\|_* \leq (2.125) N_\Phi(f),$$

which gives yet another proof of Urbanik's result.

For the sake of completeness and in an effort to better understand this space, we will give our own, quick, proof of Theorem 33. This will allow us to get an optimal upper bound with respect to $\|\cdot\|_\Phi$.

Let us see how $\|\cdot\|_*$ behaves:

Proposition 34. *We have*

- For any $f \in L^1(\mu)$,

$$\|f\|_* \leq 2 \|f\|_1.$$

- And for any $f \in L^2(\mu)$,

$$\|f\|_* \leq \|f\|_2.$$

Consequently, for any $f \in L^\Phi(\mu)$,

$$\|f\|_* \leq \|f\|_\Phi,$$

Moreover, $\|\cdot\|_*$ and $\|\cdot\|_\Phi$ are different and the bound is optimal.

Proof. If $f \in L^1(\mu)$, then

$$\begin{aligned} \|f\|_* &= \|I_1(f)\|_{L^1(\mu^*)} \\ &= \mathbb{E}_{\mu^*}[|I_1(f)|] \\ &= \mathbb{E}_{\mu^*}[|N(f) - \mu(f)|] \\ &\leq \mathbb{E}_{\mu^*}[|N(f)|] + |\mu(f)| \\ &\leq \mathbb{E}_{\mu^*}[N(|f|)] + \mu(|f|) \\ &= 2 \|f\|_1. \end{aligned}$$

And if $f \in L^2(\mu)$,

$$\begin{aligned} \|f\|_* &= \|I_1(f)\|_{L^1(\mu^*)} \\ &\leq \|I_1(f)\|_{L^2(\mu^*)} \\ &= \|f\|_2, \end{aligned}$$

thanks to Theorem 27.

Denote by $(L^\Phi(\mu)^*, \|\cdot\|'_*)$ the topological dual of $(L^\Phi(\mu), \|\cdot\|_*)$ and let $\phi \in L^\Phi(\mu)^*$. From Proposition 34, for any $f \in L^1(\mu)$:

$$(5.4) \quad |\phi(f)| \leq \|\phi\|'_* \|f\|_* \leq 2 \|\phi\|'_* \|f\|_1,$$

and similarly, for any $f \in L^2(\mu)$:

$$(5.5) \quad |\phi(f)| \leq \|\phi\|'_* \|f\|_* \leq \|\phi\|'_* \|f\|_2.$$

Thus ϕ is a continuous linear form on both $(L^1(\mu), \|\cdot\|_1)$ and $(L^2(\mu), \|\cdot\|_2)$, and there exists a unique $g_1 \in L^\infty(\mu)$ such that, for any $f \in L^1(\mu)$:

$$\phi(f) = \int_X g_1 f d\mu,$$

and a unique $g_2 \in L^2(\mu)$ such that, for any $f \in L^2(\mu)$:

$$\phi(f) = \int_X g_2 f d\mu.$$

Of course $g_1 = g_2$ μ -a.e..

Let $g := g_1 \in L^2(\mu) \cap L^\infty(\mu)$ and take f in $L^\Phi(\mu)$. We have $f = f1_{|f| \leq 1} + f1_{|f| > 1}$ with $f1_{|f| \leq 1} \in L^2(\mu)$ and $f1_{|f| > 1} \in L^1(\mu)$ therefore

$$\begin{aligned} \phi(f) &= \phi(f1_{|f| \leq 1}) + \phi(f1_{|f| > 1}) \\ &= \int_X g f 1_{|f| \leq 1} d\mu + \int_X g f 1_{|f| > 1} d\mu \\ &= \int_X g f d\mu. \end{aligned}$$

In particular, $\phi \in L^\Phi(\mu)'$, which yields $L^\Phi(\mu)^* \subset L^\Phi(\mu)' \simeq L^2(\mu) \cap L^\infty(\mu)$. From now on, we identify $L^\Phi(\mu)^*$ as a subspace of $L^2(\mu) \cap L^\infty(\mu) \simeq L^\Phi(\mu)'$.

From the inequalities (5.4) and (5.5), for any $g \in L^\Phi(\mu)^*$,

$$\max\left(\|g\|_2, \frac{1}{2} \|g\|_\infty\right) \leq \|g\|'_*,$$

thus, for any $g \in L^\Phi(\mu)^*$,

$$N_\Psi(g) \leq \|g\|'_*.$$

Hence for any $f \in L^\Phi(\mu)$:

$$\begin{aligned}
 \|f\|_* &= \sup_{g \in L^\Phi(\mu)^{*}} \frac{\int_X g f d\mu}{\|g\|_*'} \\
 &\leq \sup_{g \in L^\Phi(\mu)^{*}} \frac{\int_X g f d\mu}{N_\Psi(g)} \\
 &\leq \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X g f d\mu}{N_\Psi(g)} \\
 &= \|f\|_\Phi.
 \end{aligned}$$

Now observe that for any $f \in L^\Phi(\mu)$, we have

$$\| \|f\| \| \Phi = \|f\|_\Phi.$$

This is not the case with $\|\cdot\|_*$. To see this, let A_n and B_n , $n \geq 2$ be measurable sets such that, for all $n \geq 2$, $A_n \cap B_n = \emptyset$ and $\mu(A_n) = \mu(B_n) = \frac{1}{2n}$, and set $f_n := 1_{A_n} - 1_{B_n}$, so that $f_n \in L_0^1(\mu)$ and $\|f_n\|_1 = \frac{1}{n}$.

We have:

$$\begin{aligned}
 \| \|f_n\| \|_* &= \| \|1_{A_n \cup B_n}\| \|_* \\
 &= \mathbb{E}_{\mu^*} [|N(A_n \cup B_n) - \mu(A_n \cup B_n)|] \\
 &= \mathbb{E}_{\mu^*} [\mu(A_n \cup B_n) 1_{N(A_n \cup B_n)=0}] \\
 &\quad + \mathbb{E}_{\mu^*} [(N(A_n \cup B_n) - \mu(A_n \cup B_n)) 1_{N(A_n \cup B_n) \geq 1}] \\
 &= \frac{1}{n} e^{-\frac{1}{n}} + \mathbb{E}_{\mu^*} [N(A_n \cup B_n) - \mu(A_n \cup B_n)] \\
 &\quad - \mathbb{E}_{\mu^*} [(N(A_n \cup B_n) - \mu(A_n \cup B_n)) 1_{N(A_n \cup B_n)=0}] \\
 &= \frac{2}{n} e^{-\frac{1}{n}},
 \end{aligned}$$

whereas

$$\begin{aligned}
 \|f_n\|_* &= \mathbb{E}_{\mu^*} [|I_1(f_n)|] \\
 &= \mathbb{E}_{\mu^*} [|N(f_n) - \mu(f_n)|] \\
 &= \mathbb{E}_{\mu^*} [|N(f_n)|] \\
 &\leq \mathbb{E}_{\mu^*} [N(|f_n|)] \\
 &= \|f_n\|_1 = \frac{1}{n}.
 \end{aligned}$$

To prove the optimality, observe that, since for all $g \in L^2(\mu) \cap L^\infty(\mu)$, $N_\Psi(g) = \max(\|g\|_2, \frac{1}{2}\|g\|_\infty) \geq \frac{1}{2}\|g\|_\infty$, we have, for any $f \in L^1(\mu)$:

$$\begin{aligned} \|f\|_\Phi &= \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X g f d\mu}{N_\Psi(g)} \\ &\leq 2 \sup_{g \in L^2(\mu) \cap L^\infty(\mu)} \frac{\int_X g f d\mu}{\|g\|_\infty} \\ &\leq 2 \sup_{g \in L^\infty(\mu)} \frac{\int_X g f d\mu}{\|g\|_\infty} \\ &= 2\|f\|_1. \end{aligned}$$

Thus $\|f_n\|_\Phi \leq \frac{2}{n}$. \square

We have just seen that, in general, $\|f\|_* = \|f\|_\Phi$ doesn't hold. Nevertheless we can compare the two quantities:

Proposition 35. *Let $f \in L^\Phi(\mu)$, real valued. Then*

$$\|f\|_* \leq 2\|f\|_\Phi,$$

and

$$\|f\|_\Phi \leq 2\|f\|_*.$$

Proof. Write $f = f^+ - f^-$. Since f^+ and f^- are supported on disjoint sets, $I_1(f^+)$ and $I_1(f^-)$ are independent thanks to the independence properties of a Poisson measure. Therefore, taking the conditional expectation with respect to $\sigma(I_1(f^-))$:

$$\begin{aligned} \mathbb{E}_{\mu^*}[|I_1(f)|] &= \mathbb{E}_{\mu^*}[\mathbb{E}_{\mu^*}[|I_1(f^+) - I_1(f^-)| \mid \sigma(I_1(f^-))]] \\ &\geq \mathbb{E}_{\mu^*}[\mathbb{E}_{\mu^*}[|I_1(f^+) - I_1(f^-)| \mid \sigma(I_1(f^-))]] \\ &= \mathbb{E}_{\mu^*}[\mathbb{E}_{\mu^*}[I_1(f^+) \mid \sigma(I_1(f^-))] - \mathbb{E}_{\mu^*}[I_1(f^-) \mid \sigma(I_1(f^-))]] \\ &= \mathbb{E}_{\mu^*}[\mathbb{E}_{\mu^*}[I_1(f^+)] - I_1(f^-)] \\ &= \mathbb{E}_{\mu^*}[|I_1(f^-)|]. \end{aligned}$$

Thus $\|f\|_* \geq \|f^-\|_*$ and of course $\|f\|_* \geq \|f^+\|_*$. This yields

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \|f\|_*,$$

and the triangular inequality implies

$$\|f\|_* \leq \|f^+\|_* + \|f^-\|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*),$$

Hence

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \|f\|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*).$$

Writing now $|f| = f^+ + f^-$, the same reasoning gives

$$\max(\|f^-\|_*, \|f^+\|_*) \leq \|f\|_* \leq 2 \max(\|f^-\|_*, \|f^+\|_*),$$

and we get the result. \square

We can now end this section with our proof of Theorem 33:

Proof. (of Theorem 33) From Proposition 22, it follows that \mathcal{I}_1 is closed in the Banach space $(L^1(\mu^*), \|\cdot\|_{L^1(\mu^*)})$. But $(L^\Phi(\mu), \|\cdot\|_*)$ is isometrically isomorphic to $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$ therefore $(L^\Phi(\mu), \|\cdot\|_*)$ is a Banach space. Thanks to Proposition 34 and the open map theorem applied to the identity on $L^\Phi(\mu)$, the norms $\|\cdot\|_*$ and $\|\cdot\|_\Phi$ are equivalent. \square

6. $L^\Phi(\mu)$ -ERGODIC RESULTS

6.1. Mean ergodic theorem in $L^\Phi(\mu)$. We now introduce an endomorphism T on (X, \mathcal{A}, μ) . The next observation follows from Remark 11.

Proposition 36. T_* satisfies, for any $f \in L^\Phi(\mu)$:

$$I_1(f) \circ T_* = I_1(f \circ T).$$

In particular, T is an isometry of $(L^\Phi(\mu), \|\cdot\|_*)$.

We can state:

Theorem 37. *The dynamical system (X, \mathcal{A}, μ, T) has no absolutely continuous T -invariant probability measure if and only if, for every $f \in L^1(\mu)$ (or $f \in L^\Phi(\mu)$):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \xrightarrow{\|\cdot\|_*} 0,$$

as n tends to $+\infty$.

Proof. Assume T has no absolutely continuous T -invariant probability measure (in particular $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is ergodic by Theorem 9) and let $f \in L^1(\mu)$:

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* &= \left\| I_1 \left(\frac{1}{n} \sum_{k=1}^n f \circ T^k \right) \right\|_{L^1(\mu^*)} \\ &= \left\| \frac{1}{n} \sum_{k=1}^n I_1(f) \circ T_*^k \right\|_{L^1(\mu^*)} \\ &\rightarrow 0 \end{aligned}$$

as n tends to infinity thanks to Birkhoff's ergodic theorem.

Let now $f \in L^1(\mu)$ and $g \in L^\Phi(\mu)$:

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n g \circ T^k \right\|_* &= \left\| \frac{1}{n} \sum_{k=1}^n (g - f) \circ T^k + \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \\ &\leq \frac{1}{n} \sum_{k=1}^n \left\| (g - f) \circ T^k \right\|_* + \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \\ &= \|g - f\|_* + \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \end{aligned}$$

since T is an isometry of $L^\Phi(\mu)$ (Proposition 36).

Thus

$$\limsup \left\| \frac{1}{n} \sum_{k=1}^n g \circ T^k \right\|_* \leq \|g - f\|_*$$

And the result follows by density of $L^1(\mu)$ in $L^\Phi(\mu)$ (Theorem 31).

Conversely, assume there exists an absolutely continuous T -invariant probability measure $\nu \ll \mu$. This implies the existence of a non-trivial T -invariant set $A \in \mathcal{A}_f$ and we get:

$$\left\| \frac{1}{n} \sum_{k=1}^n 1_A \circ T^k \right\|_* = \|1_A\|_* > 0.$$

□

Remark 38. Mean ergodic theorems are often associated to pointwise convergence. In our context, the result is immediate, for any $f \in L^\Phi(\mu)$:

$$\frac{1}{n} \sum_{k=1}^n f(T^k x)$$

converges as n tends to infinity, for μ -almost every $x \in X$ since, as we saw above, we can write $f = f1_{|f| \leq 1} + f1_{|f| > 1}$ with $f1_{|f| \leq 1} \in L^2(\mu)$ and $f1_{|f| > 1} \in L^1(\mu)$ and the μ -almost sure convergence holds for both $L^1(\mu)$ ([7]) and $L^2(\mu)$ ([2]). In particular, if T has no absolutely continuous T -invariant probability measure, then for μ -almost every $x \in X$:

$$\frac{1}{n} \sum_{k=1}^n f(T^k x) \rightarrow 0,$$

as n tends to infinity.

As an application we can get:

Proposition 39. $L_0^1(\mu)$ is dense in $L^\Phi(\mu)$.

Proof. We already know (Theorem 31) that $L^1(\mu)$ is dense in $L^\Phi(\mu)$ with respect to $\|\cdot\|_*$, it is thus enough to show that $L_0^1(\mu)$ is dense in $L^1(\mu)$ with respect to this same norm.

Indeed, let $f \in L^1(\mu)$, choose an ergodic endomorphism T of (X, \mathcal{A}, μ) and consider the sequence $\{f_n\}_{n \in \mathbb{N}}$ defined by

$$f_n := f - \frac{1}{n} \sum_{k=1}^n f \circ T^k.$$

For any $n \in \mathbb{N}$, $f_n \in L^1_0(\mu)$ and we have

$$\|f_n - f\|_* = \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^k \right\|_* \rightarrow 0$$

as n tends to infinity, thanks to Theorem 37. \square

6.2. Blum-Hanson type Theorem.

Theorem 40. *(X, \mathcal{A}, μ, T) is of zero type if and only if, for every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers and every $f \in L^1(\mu)$ (or equivalently $f \in L^\Phi(\mu)$):*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \rightarrow_{\|\cdot\|_*} 0,$$

as n tends to infinity.

Proof. If (X, \mathcal{A}, μ, T) is of zero type then $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is mixing (Theorem 9) and Blum-Hanson theorem applies: for every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers, every $f \in L^\Phi(\mu)$,

$$\frac{1}{n} \sum_{k=1}^n I_1 \circ T_*^{n_k} \rightarrow_{\|\cdot\|_{L^1(\mu^*)}} 0,$$

but

$$\left\| \frac{1}{n} \sum_{k=1}^n I_1 \circ T_*^{n_k} \right\|_{L^1(\mu^*)} = \left\| \frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \right\|_*,$$

that is,

$$\frac{1}{n} \sum_{k=1}^n f \circ T^{n_k} \rightarrow_{\|\cdot\|_*} 0.$$

Conversely, assume the convergence holds for every $f \in L^1(\mu)$ and every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of integers and consider sets A and B in \mathcal{A}_f .

We want to prove $\mu(A \cap T^{-n}B) \rightarrow 0$ as n tends to infinity. By a standard exercise, it is equivalent to show the Cesàro convergence to zero of its subsequences i.e., for every strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$:

$$\frac{1}{n} \sum_{k=1}^n \mu(A \cap T^{-n_k}B) \rightarrow 0,$$

as n tends to infinity, but this will indeed be the case, since

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=1}^n \mu(A \cap T^{-n_k} B) \right| &= \left| \int_X 1_A \cdot \frac{1}{n} \sum_{k=1}^n 1_B \circ T^{n_k} d\mu \right| \\ &\leq \|1_A\|'_* \left\| \frac{1}{n} \sum_{k=1}^n 1_B \circ T^{n_k} \right\|_* \end{aligned}$$

as $1_A \in L^\infty(\mu) \cap L^2(\mu)$ and $1_B \in L^1(\mu) \subset L^\Phi(\mu)$.

Hence T is of zero type. \square

7. DUALITY AND TRANSFER OPERATORS

7.1. Duality. It will be useful to have another representation of $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$, the dual of $(L^\Phi(\mu), \|\cdot\|_*)$, and that requires the introduction of a few objects.

Recall that, we denoted by $\mathcal{I}_2 := \{I_1(f), f \in L^2(\mu)\}$ the first chaos and we pointed out the fact that I_1 is implementing an isometric isomorphism between $(\mathcal{I}_2, \|\cdot\|_{L^2(\mu^*)})$ and $(L^2(\mu), \|\cdot\|_2)$. Recall that we have introduced P in Lemma 29 as the orthogonal projection on \mathcal{I}_2 ; we can then define π from $L^2(\mu^*)$ to $L^2(\mu)$ by

$$\pi := I_1^{-1} \circ P.$$

Consider now the restriction of π to $L^\infty(\mu^*) \subset L^2(\mu^*)$. We will show:

Proposition 41. π is a surjective bounded operator from $(L^\infty(\mu^*), \|\cdot\|_{L^\infty(\mu^*)})$ to $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$.

Proof. If $Z \in L^\infty(\mu^*)$, then the formula

$$L^\Phi(\mu) \ni f \mapsto \mathbb{E}_{\mu^*} [ZI_1(f)]$$

defines a continuous linear form on $(L^\Phi(\mu), \|\cdot\|_*)$ since

$$\begin{aligned} |\mathbb{E}_{\mu^*} [ZI_1(f)]| &\leq \|Z\|_{L^\infty(\mu^*)} \|I_1(f)\|_{L^1(\mu^*)} \\ (7.1) \qquad &= \|Z\|_{L^\infty(\mu^*)} \|f\|_* . \end{aligned}$$

Therefore, there exists $g \in L^2(\mu) \cap L^\infty(\mu)$ such that, for any $f \in L^\Phi(\mu)$:

$$\mathbb{E}_{\mu^*} [ZI_1(f)] = \int_X g f d\mu$$

but, if $f \in L^2(\mu)$, then, if $P(Z) = I_1(h)$ for some $h \in L^2(\mu)$, then

$$\begin{aligned} \mathbb{E}_{\mu^*} [ZI_1(f)] &= \mathbb{E}_{\mu^*} [I_1(g) I_1(f)] \\ &= \int_X h f d\mu. \end{aligned}$$

Therefore $h = g$ and $\pi(Z) = g \in L^2(\mu) \cap L^\infty(\mu)$.

Conversely, for any $g \in L^2(\mu) \cap L^\infty(\mu)$, the map

$$F \mapsto \int_X g I_1^{-1}(F) d\mu = \mathbb{E}_{\mu^*} [I_1(g) F]$$

defines a continuous linear form on $(\mathcal{I}_1, \|\cdot\|_{L^1(\mu^*)})$ that extends, thanks to Hahn-Banach theorem to $(L^1(\mu^*), \|\cdot\|_{L^1(\mu^*)})$, that is, there exists $Z \in L^\infty(\mu^*)$ such that, for, any $F \in L^1(\mu^*)$:

$$\mathbb{E}_{\mu^*} [ZF] = \mathbb{E}_{\mu^*} [I_1(g)F].$$

In particular, for any $f \in L^2(\mu)$,

$$\mathbb{E}_{\mu^*} [ZI_1(f)] = \mathbb{E}_{\mu^*} [I_1(g)I_1(f)].$$

And this proves that $\pi(Z) = g$. Therefore the range of π is $L^2(\mu) \cap L^\infty(\mu)$ and from (7.1), we get

$$\|\pi(Z)\|'_* \leq \|Z\|_{L^\infty(\mu^*)}.$$

□

Proposition 42. *In the duality $\sigma(L^1(\mu^*), L^\infty(\mu^*))$, the orthogonal \mathcal{I}_1^\perp of \mathcal{I}_1 is the subspace $\ker \pi$. In particular, there is a canonical isometric identification*

$$(L^\infty(\mu^*) / \ker \pi, \|\cdot\|_{/}) \simeq (L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*),$$

where $\|\cdot\|_{/}$ is the usual quotient norm induced by $\|\cdot\|_{L^\infty(\mu^*)}$.

Proof. If $f \in L^2(\mu)$ and $Z \in L^\infty(\mu^*)$, then

$$\begin{aligned} \mathbb{E}_{\mu^*} [ZI_1(f)] &= \mathbb{E}_{\mu^*} [I_1(\pi(Z))I_1(f)] \\ &= \int_X \pi(Z) f d\mu. \end{aligned}$$

This formula extends by density of $L^2(\mu)$ in $L^\Phi(\mu)$ with respect to $\|\cdot\|_*$, and thus of \mathcal{I}_2 in \mathcal{I}_1 with respect to $\|\cdot\|_{L^1(\mu^*)}$.

Therefore $\mathcal{I}_1^\perp = \ker \pi$.

The map π induces an isometric isomorphism between $(L^\infty(\mu^*) / \ker \pi, \|\cdot\|_{/})$ and $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$. □

Remark 43. We could have appeal to general results on Banach spaces to get that $L^\infty(\mu^*) / \mathcal{I}_1^\perp$ is canonically isomorphic (and isometric) to the dual of \mathcal{I}_1 , however we believe it is much more intuitive to get an explicit description through the operator π as above.

7.2. Transfer operators. Let T be an endomorphism on (X, \mathcal{A}, μ) . The *transfer operator* is the operator \widehat{T} acting on $L^1(\mu)$ defined as the predual operator of T seen as acting as an isometry on $L^\infty(\mu)$, characterized by the following relation, for any $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$:

$$\int_X \widehat{T}f \cdot g d\mu = \int_X f \cdot g \circ T d\mu.$$

Similarly, the same object exists at the level of the Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$: \widehat{T}_* is the transfer operator of T_* .

We will mimic the same procedure to define a “transfer” operator acting on $L^\Phi(\mu)$.

Proposition 44. *The isometry T_* on $(L^\infty(\mu^*), \|\cdot\|_{L^\infty(\mu^*)})$ acts also on the quotient space $(L^\infty(\mu^*) / \ker \pi, \|\cdot\|_/)$ as an isometry and corresponds to T through the identification with $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$. In particular, T is an isometry of $(L^2(\mu) \cap L^\infty(\mu), \|\cdot\|'_*)$.*

Proof. The first step consists in showing that T_* preserves $\ker \pi$:

Take $Z \in \ker \pi$, in particular Z belongs to $\mathcal{I}_2^1 \subset L^2(\mu^*)$. But T_* commutes with P thanks to Lemma 29 and thus $P(Z \circ T_*) = P(Z) \circ T_* = 0$ and thus $Z \circ T_* \in \ker \pi$.

Now observe that for any $Z \in L^\infty(\mu^*)$:

$$\begin{aligned} \pi(Z \circ T_*) &= I_1^{-1}(P(Z \circ T_*)) \\ &= I_1^{-1}(P(Z) \circ T_*) \\ &= I_1^{-1}(P(Z)) \circ T \\ &= \pi(Z) \circ T \end{aligned}$$

and this proves the result. \square

Definition 45. We define the *Poisson-transfer operator* $\widehat{T}_{\mathcal{P}}$ acting on $L^\Phi(\mu)$ as the dual operator of T acting on $L^\Phi(\mu)'$. It satisfies, for any f in $L^\Phi(\mu)$, $g \in L^2(\mu) \cap L^\infty(\mu)$:

$$\int_X \widehat{T}_{\mathcal{P}}(f) \cdot g d\mu = \int_X fg \circ T d\mu.$$

Observe that taking $f \in L^\Phi(\mu)$, $f \geq 0$, for set $A \in \mathcal{A}_f$:

$$\int_A \widehat{T}_{\mathcal{P}}(f) d\mu = \int_{T^{-1}A} f d\mu \geq 0,$$

$\widehat{T}_{\mathcal{P}}$ is thus a positive operator.

Proposition 46. \widehat{T}_* preserves \mathcal{I}_1 and satisfies, for all $f \in L^\Phi(\mu)$:

$$\widehat{T}_* I_1(f) = I_1(\widehat{T}_{\mathcal{P}} f)$$

Moreover $\widehat{T}_{\mathcal{P}}$ preserves $L^1(\mu)$ and coincide with \widehat{T} on it.

Proof. The first part of the proof follows immediately from Proposition 44 and the fact that $\ker \pi = \mathcal{I}_1^1$ from Proposition 42.

Now since \widehat{T}_* is a positive operator, if $f \in L^1(\mu)$, $f \geq 0$, then $I_1(f) = N(f) - \mu(f) \geq -\mu(f)$ and therefore $I_1(\widehat{T}_{\mathcal{P}} f) = \widehat{T}_*(I_1(f)) \geq -\mu(f)$.

Thanks to Proposition 30, this implies that $\widehat{T}_{\mathcal{P}} f \in L^1(\mu)$.

By considering real and imaginary parts of $f \in L^1(\mu)$ and then their respective positive and negative parts, we get that $\widehat{T}_{\mathcal{P}}$ preserves $L^1(\mu)$.

Now, for any $f \in L^1(\mu)$ and $A \in \mathcal{A}_f$, 1_A belongs to $L^2(\mu) \cap L^\infty(\mu) \subset L^\infty(\mu)$, thus:

$$\int_A \widehat{T}_{\mathcal{P}} f d\mu = \int_X \widehat{T}_{\mathcal{P}} f \cdot 1_A d\mu = \int_X f \cdot 1_A \circ T d\mu = \int_X \widehat{T} f \cdot 1_A d\mu = \int_A \widehat{T} f d\mu.$$

This implies $\widehat{T}_{\mathcal{P}} f = \widehat{T} f$, that is $\widehat{T}_{\mathcal{P}}$ coincide with \widehat{T} on $L^1(\mu)$. \square

Remark 47. Being a positive operator defined on $L^1(\mu)$, \widehat{T} extends in a unique (mod. μ) way to any non-negative measurable function. The same applies to $\widehat{T}_{\mathcal{P}}$ and since they coincide on $L^1(\mu)$, the extension of \widehat{T} to $L^\Phi(\mu)$ is in fact $\widehat{T}_{\mathcal{P}}$.

8. EXACT AND REMOTELY INFINITE TRANSFORMATIONS

We recall the two definitions that we will deal with in this section:

Definition 48. (See [9]) (X, \mathcal{A}, μ, T) is said to be

- *exact* if $\bigcap_{n \geq 0} T^{-n} \mathcal{A} = \{\emptyset, X\}$ mod. μ .
- *remotely infinite*¹ if $\bigcap_{n \geq 0} T^{-n} \mathcal{A}$ contains only 0 or infinite measure sets.

We recall the chain of implications we already gave in the introduction:

$T_{\text{exact}} \Rightarrow T_{\text{remotely infinite}} \Rightarrow T_{\text{of zero type}} \implies$ absence of a.c. T -invariant measure.

Note also that exactness implies ergodicity whereas it is not the case for the other three properties.

We are interested in transfer operator characterization of these properties, we also recall Lin's result:

Theorem 49. ([11]) (X, \mathcal{A}, μ, T) is exact if and only if, for all $f \in L^1_0(\mu)$:

$$\left\| \widehat{T}^n f \right\|_{L^1(\mu)} \rightarrow 0,$$

as $n \rightarrow +\infty$.

We will prove the following:

Theorem 50. (X, \mathcal{A}, μ, T) is remotely infinite if and only if, for all $f \in L^1_0(\mu)$:

$$\left\| \widehat{T}^n f \right\|_* \rightarrow 0,$$

as $n \rightarrow +\infty$.

To this end, we first need to establish the following characterization of remotely infinite systems:

¹We warn the reader that the authors of [9] also define, *remotely infinite automorphisms* (which cannot exist with the above definition...) in the same paper. This is the same difference that occurs between exact endomorphisms and K -automorphisms.

Proposition 51. (X, \mathcal{A}, μ, T) is remotely infinite if and only if $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is exact.

Proof. Recall (see [19]) that if $\mathcal{B} \subset \mathcal{A}$ is a σ -algebra then $\mathcal{B}^* := \sigma\{N(A), A \in \mathcal{B}\}$ is said to be a *Poisson σ -algebra*.

The proof of the Proposition is then a straightforward consequence of the following facts □

- $(T^{-n}\mathcal{A})^* = T_*^{-n}\mathcal{A}^*$
- $(\cap_{n \geq 0} T^{-n}\mathcal{A})^* = \cap_{n \geq 0} T_*^{-n}\mathcal{A}^*$ (see Lemma 3.3 in [19] for a proof)
- A Poisson σ -algebra \mathcal{G}^* for $\mathcal{G} \subset \mathcal{A}$ is trivial if and only if \mathcal{G} contains only zero or infinite measure sets (the Poisson random measure N cannot distinguish between zero measure sets nor between infinite measure sets).

We can then prove our theorem:

Proof. (of Theorem 50). Assume (X, \mathcal{A}, μ, T) is remotely infinite. Then $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is exact from Proposition 51, and thus, for any $f \in L_0^1(\mu)$,

$$\left\| \widehat{T}_*^n I_1(f) \right\|_{L^1(\mu^*)} \rightarrow 0$$

But, we have seen in Proposition 46 that $\widehat{T}_*^n I_1(f) = I_1(\widehat{T}^n f)$, therefore

$$\begin{aligned} \left\| \widehat{T}_*^n I_1(f) \right\|_{L^1(\mu^*)} &= \left\| I_1(\widehat{T}^n f) \right\|_{L^1(\mu^*)} \\ &= \left\| \widehat{T}^n f \right\|_* . \end{aligned}$$

For the converse, assume that for all $f \in L_0^1(\mu)$, $\left\| \widehat{T}^n f \right\|_* \rightarrow 0$ as $n \rightarrow +\infty$.

Recall that $\widehat{T}^n_{\mathcal{P}}$ is an isometry on $L^\Phi(\mu)$ and from Proposition 46, $\widehat{T}^n = \widehat{T}^n_{\mathcal{P}}$ on $L_0^1(\mu)$, thus, for any $g \in L^\Phi(\mu)$ and $f \in L_0^1(\mu)$:

$$\begin{aligned} \left\| \widehat{T}^n_{\mathcal{P}} g \right\|_* &= \left\| \widehat{T}^n_{\mathcal{P}}(g - f) + \widehat{T}^n_{\mathcal{P}} f \right\|_* \\ &\leq \left\| \widehat{T}^n_{\mathcal{P}}(g - f) \right\|_* + \left\| \widehat{T}^n_{\mathcal{P}} f \right\|_* \\ &\leq \|g - f\|_* + \left\| \widehat{T}^n f \right\|_* \end{aligned}$$

That is

$$\limsup_{n \rightarrow \infty} \left\| \widehat{T}^n_{\mathcal{P}} g \right\|_* \leq \|g - f\|_* .$$

We conclude by density of $L_0^1(\mu)$ in $L^\Phi(\mu)$ (Proposition 39).

In other words, from Proposition 46 we have proved that, for any $F \in \mathcal{I}_1$:

$$\left\| \widehat{T}_*^n F \right\|_{L^1(\mu^*)} \rightarrow 0$$

as $n \rightarrow +\infty$.

For the converse we follow the same lines of arguments used in the proof of Theorem 49: assume that (X, \mathcal{A}, μ, T) is not remotely infinite, this means that there exists $A \in \bigcap_{n \geq 0} T^{-n} \mathcal{A}$ such that $0 < \mu(A) < \infty$ and we can set, for any $n \in \mathbb{N}$, A_n such that $A = T^{-n} A_n$.

Consider $1_{N(A)=0} \in L^\infty(\mu^*)$, we have $1_{N(A_n)=0} \circ T_*^n = 1_{N(T^{-n} A_n)=0} = 1_{N(A)=0}$.

Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \widehat{T}_*^n I_1(-1_A) \right\|_{L^1(\mu^*)} &\geq \mathbb{E}_{\mu^*} \left[\left| 1_{N(A_n)=0} \cdot \widehat{T}_*^n I_1(-1_A) \right| \right] \\ &\geq \mathbb{E}_{\mu^*} \left[1_{N(A_n)=0} \cdot \widehat{T}_*^n I_1(-1_A) \right] \\ &\geq \mathbb{E}_{\mu^*} \left[1_{N(A_n)=0} \circ T_*^n \cdot I_1(-1_A) \right] \\ &= \mathbb{E}_{\mu^*} \left[1_{N(A)=0} \cdot I_1(-1_A) \right] \\ &= \mathbb{E}_{\mu^*} \left[1_{N(A)=0} \cdot (-N(A) + \mu(A)) \right] \\ &= \mu(A) \mathbb{E}_{\mu^*} \left[1_{N(A)=0} \right] \\ &= \mu(A) e^{-\mu(A)} > 0. \end{aligned}$$

This contradicts that for any $F \in \mathcal{I}_1$, $\left\| \widehat{T}_*^n F \right\|_{L^1(\mu^*)} \rightarrow 0$. \square

Example 52. It can be proven (see Corollary 3.7.7 in [1]) that pointwise dual ergodic endomorphisms (see also [1] for the definition) are remotely infinite.

Combining Proposition 50 and the last point of Theorem 31, we obtain:

Corollary 53. (X, \mathcal{A}, μ, T) is not remotely infinite if there exists a non-negative $f \in L^1(\mu)$ such that

$$\liminf_{n \rightarrow \infty} \widehat{T}^n f \neq 0.$$

Observe that Fatou Lemma gives, if $\liminf_{n \rightarrow \infty} \widehat{T}^n f \neq 0$:

$$0 < \int_X \left(\liminf_{n \rightarrow \infty} \widehat{T}^n f \right) d\mu \leq \liminf_{n \rightarrow \infty} \left\| \widehat{T}^n f \right\|_1$$

which only implies that (X, \mathcal{A}, μ, T) is not exact.

9. ANOTHER NORM AND L^1 -CHARACTERIZATION OF ERGODICITY

It is possible to define yet another norm, this time on $L^1(\mu)$, closely related to $\|\cdot\|_*$:

Definition 54. The map

$$f \mapsto \mathbb{E}_{\mu^*} [|N(f)|] = \|N(f)\|_{L^1(\mu^*)}$$

defines a norm on $L^1(\mu)$ that we denote by $\|\cdot\|_{**}$.

To derive some of its properties, we need further distributional properties of the random measure N .

9.1. Reduced moment measure of order 2. By taking the direct product of N , $N \otimes N$, we get a random measure on $(X \times X, \mathcal{A} \otimes \mathcal{A})$ and by restricting it to the complement of the diagonal $\Delta := \{(x, x), x \in X\}$, we get the random measure $N \otimes N_{|\Delta^c}$ whose intensity is called the *reduced moment measure of order 2*. In our case, it is the product measure $\mu \otimes \mu$, indeed, if A and B are finite μ -measure sets in \mathcal{A} :

$$\begin{aligned} \mathbb{E}_{\mu^*} [N \otimes N_{|\Delta^c} (A \times B)] &= \mathbb{E}_{\mu^*} [N(A)N(B) - N(A \cap B)] \\ &= \mathbb{E}_{\mu^*} [(N(A) - \mu(A))(N(B) - \mu(B))] \\ &\quad + \mu(A)\mathbb{E}_{\mu^*} [(N(B) - \mu(B))] + \mu(B)\mathbb{E}_{\mu^*} [(N(A) - \mu(A))] \\ &\quad + \mu(A)\mu(B) - \mathbb{E}_{\mu^*} [N(A \cap B)] \\ &= \mu(A \cap B) + \mu(A)\mu(B) - \mu(A \cap B) \\ &= \mu \otimes \mu (A \times B). \end{aligned}$$

In particular, for any g and h in $L^1(\mu)$, $N \otimes N_{|\Delta^c}(g \otimes h) \in L^1(\mu^*)$ and we have:

$$\mathbb{E}_{\mu^*} [N \otimes N_{|\Delta^c}(g \otimes h)] = \mu(g)\mu(h).$$

Proposition 55. *For any g and h in $L^1(\mu)$, gh is in $L^1(\omega)$ for μ^* -a.e. $\omega \in X^*$ and the formula*

$$N \otimes N_{|\Delta^c}(g \otimes h) = N(g)N(h) - N(gh)$$

holds μ^ -a.e..*

Proof. Let g and h in $L^1(\mu)$. As $N(|g|)$ and $N(|h|)$ are integrable and thus finite μ^* -a.e., we get:

$$\begin{aligned} N(|g|)N(|h|) &= N \otimes N_{|\Delta^c}(|g| \otimes |h|) + N \otimes N_{|\Delta}(|g| \otimes |h|) \\ &= N \otimes N_{|\Delta^c}(|g| \otimes |h|) + N(|gh|) \end{aligned}$$

And since $N(|g|)$, $N(|h|)$ and $N \otimes N_{|\Delta^c}(|g| \otimes |h|)$ are all in $L^1(\mu^*)$, they are also finite μ^* -a.e. and thus $N(|gh|)$ as well, that is, gh is in $L^1(\omega)$ for μ^* -a.e. $\omega \in X^*$.

Therefore the formula

$$N(g)N(h) = N \otimes N_{|\Delta^c}(g \otimes h) + N \otimes N_{|\Delta}(g \otimes h)$$

holds true and we obtain

$$N \otimes N_{|\Delta^c}(g \otimes h) = N(g)N(h) - N(gh).$$

□

9.2. Properties of $\|\cdot\|_{**}$.

Proposition 56. *The following holds:*

- For every $f \in L^1(\mu)$, $\|f\|_{**} \leq \| |f| \|_{**} = \|f\|_1$.
- $\|f\|_* = \|f\|_1$ if and only if $f = \lambda|f|$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$.
- For every $f \in L_0^1(\mu)$, $\|f\|_{**} = \|f\|_*$.

- If T is an endomorphism then $\|f \circ T\|_{**} = \|f\|_{**}$.

Proof. For every $f \in L^1(\mu)$,

$$\|f\|_{**} = \mathbb{E}_{\mu^*} [|N(f)|] \leq \mathbb{E}_{\mu^*} [N(|f|)] = \|f\|_1$$

If $\|f\|_{**} = \|f\|_1$, then $\mathbb{E}_{\mu^*} [|N(f)|] = \mathbb{E}_{\mu^*} [N(|f|)]$ and as $|N(f)| \leq N(|f|)$ we have

$$\mathbb{E}_{\mu^*} [N(|f|) - |N(f)|] = 0,$$

therefore $N(|f|) = |N(f)|$ μ^* -a.e..

Thanks to Proposition 55:

$$\begin{aligned} N \otimes N_{|\Delta^c} (|f| \otimes |f|) &= N(|f|) N(|f|) - N(|f||f|) \\ &= N(|f|)^2 - N(|f|^2) \end{aligned}$$

and

$$\begin{aligned} N \otimes N_{|\Delta^c} (f \otimes \bar{f}) &= N(f) N(\bar{f}) - N(f\bar{f}) \\ &= |N(f)|^2 - N(|f|^2) \end{aligned}$$

Therefore

$$N \otimes N_{|\Delta^c} (|f| \otimes |f|) = N \otimes N_{|\Delta^c} (f \otimes \bar{f})$$

and by integrating, we get

$$\left(\int_X |f| d\mu \right)^2 = \left| \int_X f d\mu \right|^2$$

thus $|\int_X f d\mu| = \int_X |f| d\mu$ which is the equality case in the triangular inequality for integrals. In other words, there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that

$$f = \lambda |f|.$$

If $f \in L_0^1(\mu)$, then $I_0(f) = N(f) - \mu(f) = N(f)$, therefore

$$\|f\|_{**} = \|f\|_*.$$

For the last assertion, observe that if T preserves μ , then T_* preserves μ^* and

$$\begin{aligned} \|f \circ T\|_{**} &= \mathbb{E}_{\mu^*} [|N(f \circ T)|] \\ &= \mathbb{E}_{\mu^*} [|N(f) \circ T_*|] \\ &= \mathbb{E}_{\mu^*} [|N(f)| \circ T_*] \\ &= \mathbb{E}_{\mu^*} [|N(f)|] \\ &= \|f\|_{**}. \end{aligned}$$

□

9.3. The space $(\overline{L^1(\mu)}, \|\cdot\|_{})$ and Birkhoff ergodic theorem.** By construction $(\overline{L^1(\mu)}, \|\cdot\|_{**})$ is isomorphic and isometric to $\overline{\text{Vect}\{N(f), f \in L^1(\mu)\}} \subset L^1(\mu^*)$.

Proposition 57. *We have $\overline{\text{Vect}\{N(f), f \in L^1(\mu)\}} = \mathcal{I}_1 + \mathbb{C}$.*

Proof. Set $V := \overline{\text{Vect}\{N(f), f \in L^1(\mu)\}}$. The first step consists in showing the constants belong to V and this amounts to use Birkhoff's ergodic theorem to the ergodic transformation T_* if T is chosen so that (X, \mathcal{A}, μ, T) has no absolutely continuous T -invariant probability measure. Let $f \in L^1(\mu)$, $\mu(f) = 1$:

$$N(f) \circ T_* = N(f \circ T) \in V$$

and the convergence holds in $L^1(\mu^*)$:

$$\frac{1}{n} \sum_{k=1}^n N(f) \circ T_*^k \rightarrow \mathbb{E}_{\mu^*}[N(f)] = 1,$$

as n tends to infinity.

Therefore the constants belong to V .

Now if f is in $L^\Phi(\mu)$, setting $f_n := f1_{|f| > \frac{1}{n}}$, then $f_n \in L^1(\mu)$, $f1_{|f| \leq \frac{1}{n}} \in L^2(\mu)$ and thus $I_1(f_n) := N(f_n) - \mu(f_n) \in V$ and we have

$$\begin{aligned} \|I_1(f) - I_1(f_n)\|_{L^1(\mu^*)} &= \left\| I_1\left(f1_{|f| \leq \frac{1}{n}}\right) \right\|_{L^1(\mu^*)} \\ &\leq \left\| I_1\left(f1_{|f| \leq \frac{1}{n}}\right) \right\|_{L^2(\mu^*)} \\ &= \left\| f1_{|f| \leq \frac{1}{n}} \right\|_2 \\ &\rightarrow 0, \end{aligned}$$

as n tends to infinity by dominated convergence. Therefore $\{I_1(f_n)\}_{n \in \mathbb{N}}$ tends to $I_1(f)$ in $L^1(\mu^*)$. That is V contains both the constants and \mathcal{I}_1 , hence the result. \square

Definition 58. We denote by $\bar{1} \in (\overline{L^1(\mu)}, \|\cdot\|_{**})$ the unit element that corresponds to 1_{X^*} through the isometry between $(\overline{L^1(\mu)}, \|\cdot\|_{**})$ and $\mathcal{I}_1 + \mathbb{C}$.

In the course of the proof of the last proposition, we have used Birkhoff's ergodic theorem that translates into the following:

Theorem 59. *Let T be an endomorphism of the infinite measure Lebesgue space (X, \mathcal{A}, μ) without absolutely continuous T -invariant probability measure. Then for any $f \in L^1(\mu)$ (or any $f \in (\overline{L^1(\mu)}, \|\cdot\|_{**})$), the following convergence holds:*

$$\frac{1}{n} \sum_{k=1}^n f \circ T^k \rightarrow_{\|\cdot\|_{**}} \left(\int_X f d\mu \right) \cdot \bar{1},$$

as n tends to infinity.

9.4. L^1 -characterization of ergodicity. Theorem 59 invites us to find a way to tell if a system is ergodic or not, using $L^1(\mu)$ endowed with different norms. For this purpose, a measure-preserving transformation T being given on (X, \mathcal{A}, μ) , we will consider the following norms:

Definition 60. A T -norm on $L^1(\mu)$ is a norm $\|\cdot\|$ on $L^1(\mu)$ such that

- T is a mean-ergodic contraction on $(L^1(\mu), \|\cdot\|)$
- There exists $K \geq 0$ such that for all $f \in L^1(\mu)$, $\|f\| \leq K \|f\|_1$.

If in addition T is an isometry for $\|\cdot\|$, then it is said to be a T -isometric norm.

Obviously, $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are T -(isometric) norms, $\|\cdot\|_1$ is a T -(isometric) norm if $\mu(X) < \infty$.

It is worth mentioning that we could have replaced the first requirement by

- T is a mean-ergodic contraction on $(\overline{L^1(\mu)}, \|\cdot\|)$

Indeed, it is easily seen that a contraction is mean-ergodic as soon as the convergence of the Birkhoff sums holds on a dense subspace.

We can now formulate our theorem:

Theorem 61. *Let T be a measure-preserving automorphism (resp. endomorphism) of (X, \mathcal{A}, μ) . Then (X, \mathcal{A}, μ, T) is ergodic if and only if*

$$\sup \left(\dim \left(\text{fix} \left(T, \left(\overline{L^1(\mu)}, \|\cdot\| \right) \right) \right) \right) = 1,$$

the supremum being taken over the T -norms (resp. T -isometric norms).

Proof. Assume first (X, \mathcal{A}, μ, T) is not ergodic and consider the partition (see [1], page 47) $X = K_\infty \cup K_{<\infty}$ into T -invariant sets where K_∞ is the maximal subset on which there is no T -invariant probability measure absolutely continuous with respect to μ .

If $\mu(K_{<\infty}) > 0$ and $\mu(K_\infty) = 0$ then one finds two disjoint T -invariant finite μ -measure sets A and B . Then 1_A and 1_B are linearly independent T -invariant vectors of $L^1(\mu)$ and thus $\dim(\text{fix}(T, (L^1(\mu), \|\cdot\|_1))) \geq 2$.

If $\mu(K_{<\infty}) > 0$ and $\mu(K_\infty) > 0$, then we define the T -isometric norm $\|\cdot\|$, for any $f \in L^1(\mu)$ by:

$$\|f\| := \|f|_{K_{<\infty}}\|_1 + \|f|_{K_\infty}\|_{**}.$$

We can find a T -invariant set $A \in \mathcal{A}_f$ inside $K_{<\infty}$ so that 1_A is a T -invariant vector of $L^1(\mu)$. Observe also that the restriction of the Poisson suspensions to K_∞ ($(K_\infty^*, (\mathcal{A}|_{K_\infty})^*, \mu|_{K_\infty}, (T|_{K_\infty})_*$), is ergodic and thus, from Theorem 59 there exists a T -invariant vector, say $\bar{1}$, in $(\overline{L^1(\mu)}, \|\cdot\|)$. Of course 1_A and $\bar{1}$ are linearly independent, since, for instance, $\bar{1}$ doesn't belong to $L^1(\mu)$.

If now $\mu(K_{<\infty}) = 0$ and $\mu(K_\infty) > 0$, we partition X into two non-trivial T -invariant sets Y_1 and Y_2 and define again the following T -isometric norm $\|\cdot\|$, for any $f \in L^1(\mu)$:

$$\|f\| := \|f|_{Y_1}\|_{**} + \|f|_{Y_2}\|_{**}.$$

Again, the restriction of the Poisson suspensions $(Y_1^*, (\mathcal{A}|_{Y_1})^*, \mu|_{Y_1}, (T|_{Y_1})_*)$ and $(Y_2^*, (\mathcal{A}|_{Y_2})^*, \mu|_{Y_2}, (T|_{Y_2})_*)$ are both ergodic and, thanks to Theorem 59 again we can find two non-trivial T -invariant vectors in $(\overline{L^1(\mu|_{Y_1})}, \|\cdot\|_{**}) \subset (\overline{L^1(\mu)}, \|\cdot\|)$ and $(\overline{L^1(\mu|_{Y_2})}, \|\cdot\|_{**}) \subset (\overline{L^1(\mu)}, \|\cdot\|)$ respectively, and they are linearly independent since $\overline{L^1(\mu)} = \overline{L^1(\mu|_{Y_1})} \oplus \overline{L^1(\mu|_{Y_2})}$.

For the converse, assume that (X, \mathcal{A}, μ, T) is ergodic.

If $\mu(X) < \infty$, then $\dim(\text{fix}(T, (L^1(\mu), \|\cdot\|_1))) = 1$ and if $\mu(X) = +\infty$, then $\dim(\text{fix}(T, (\overline{L^1(\mu)}, \|\cdot\|_{**}))) = 1$ from Theorem 59 once again.

Assume now that T is invertible and there exists a T -norm $\|\cdot\|$ such that

$$\dim(\text{fix}(T, (\overline{L^1(\mu)}, \|\cdot\|))) \geq 2.$$

Let ϕ be a continuous linear form on $(L^1(\mu), \|\cdot\|)$, denote by H its dual and by $\|\cdot\|'$ its dual norm. We have, for all $f \in L^1(\mu)$:

$$(9.1) \quad |\phi(f)| \leq \|\phi\|' \|f\| \leq K \|\phi\|' \|f\|_1,$$

therefore ϕ is continuous with respect to $\|\cdot\|_1$ and in particular, $H \hookrightarrow L^\infty(\mu)$:

For any $\phi \in H$, there exists a unique $g \in L^\infty(\mu)$ such that for any $f \in L^1(\mu)$,

$$\phi(f) = \int_X g f d\mu,$$

We thus identify H to a subspace of $L^\infty(\mu)$.

We now want to identify the adjoint operator of T acting on H as T^{-1} . First observe that T^{-1} actually acts on H : If $g \in H$, then

$$\begin{aligned} \left| \int_X g \circ T^{-1} f d\mu \right| &= \left| \int_X g f \circ T d\mu \right| \\ &\leq \|g\|' \|f \circ T\| \\ &\leq \|g\|' \|f\|. \end{aligned}$$

Therefore $g \circ T^{-1}$ belongs to H and the map $g \mapsto g \circ T^{-1}$ is continuous on it. Now from

$$\int_X g \circ T^{-1} f d\mu = \int_X g f \circ T d\mu,$$

for any $f \in L^1(\mu)$ and $g \in H$, we deduce that the adjoint operator of T is T^{-1} as announced. Now we refer to [6] (Exercise 5 page 149) to get that,

as the set of fixed points of T on $\left(\overline{L^1(\mu)}, \|\cdot\|\right)$ is not trivial, it is also the case for T^{-1} on H . But since T is ergodic, T^{-1} is ergodic as well and the only T^{-1} -invariant functions on $L^\infty(\mu)$ are the constants. This implies that $1_X \in H$ or, in other words, the integral $\mu : f \mapsto \mu(f)$ extends continuously to $\left(\overline{L^1(\mu)}, \|\cdot\|\right)$ and thus $\text{fix}(T^{-1}, H) = \mathbb{C} \cdot \mu$.

We refer to [6] (Theorem 8.20 page 138) again for the following result translated in our context: for any $f \in \text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)$, $f \neq 0$, there exists $\phi \in \text{fix}(T^{-1}, H)$ such that $\phi(f) \neq 0$ and therefore, for any $f \in \text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)$, $f \neq 0$, we get $\mu(f) \neq 0$.

Take f_1 and f_2 in $\text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)$, linearly independent. From the above result, $\mu(f_1) \neq 0$ and $\mu(f_2) \neq 0$ and if we consider the vector $f_1 - \frac{\mu(f_1)}{\mu(f_2)}f_2$, it is a non-zero element of $\text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)$ such that

$$\mu\left(f_1 - \frac{\mu(f_1)}{\mu(f_2)}f_2\right) = 0$$

which contradicts the above result. Therefore, for any T -norm,

$$\dim\left(\text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)\right) \leq 1.$$

We finish by the non-invertible case. Assume (X, \mathcal{A}, μ, T) is ergodic, and let $\|\cdot\|$ be a T -isometric norm such that

$$\dim\left(\text{fix}\left(T, \left(\overline{L^1(\mu)}, \|\cdot\|\right)\right)\right) \geq 2.$$

We will show how we can connect to the invertible case already treated.

For this purpose, consider the natural extension $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{T})$ of (X, \mathcal{A}, μ, T) where now \tilde{T} is invertible and ergodic as well. We can identify \mathcal{A} as a sub- σ -algebra of $\tilde{\mathcal{A}}$ such that $\sigma\left(\tilde{T}^n \mathcal{A}, n \geq 0\right) = \tilde{\mathcal{A}}$ and $L^1(\mu)$ as the subspace of $L^1(\tilde{\mu})$ of \mathcal{A} -measurable integrable functions and T as the restriction of \tilde{T} to $L^1(\mu)$.

In particular $\cup_{n \geq 0} \tilde{T}^{-n} L^1(\mu)$ is a $\|\cdot\|_1$ -dense linear subspace of $L^1(\tilde{\mu})$. We extend the norm $\|\cdot\|$ by $\|\cdot\|_{ext}$ from $L^1(\mu)$ to $L^1(\tilde{\mu})$ in the following way:

If $f \in \tilde{T}^{-n} L^1(\mu)$ for some n , then $\tilde{T}^n f \in L^1(\mu)$ and we define

$$\|f\|_{ext} := \left\| \tilde{T}^n f \right\|.$$

Recall that by definition, there exists $K > 0$ such that for all $f \in L^1(\mu)$

$$\|f\| \leq K \|f\|_1,$$

and by construction, this inequality holds with $\|\cdot\|_{ext}$ for f in $\cup_{n \geq 0} \tilde{T}^{-n} L^1(\mu)$. In particular, a Cauchy sequence with respect to $\|\cdot\|_1$ is Cauchy with respect

to $\|\cdot\|_{ext}$ as well. Thus $\left(\overline{\cup_{n \geq 0} \tilde{T}^{-n} L^1(\mu)}, \|\cdot\|_{ext}\right)$ contains $L^1(\tilde{\mu})$. $\|\cdot\|_{ext}$ is in particular well defined on $L^1(\tilde{\mu})$.

\tilde{T} is by construction an isometry, we want to check it is a mean-ergodic operator. It is sufficient to check it on $\cup_{n \geq 0} \tilde{T}^{-n} L^1(\mu)$ which is $\|\cdot\|_{ext}$ -dense in $\left(\overline{L^1(\tilde{\mu})}, \|\cdot\|_{ext}\right)$.

Let $f \in \tilde{T}^{-m} L^1(\mu)$ for some $m \geq 0$, we have $\tilde{T}^n f \in L^1(\mu)$ and for $n \geq m$

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \tilde{T}^k f &= \frac{1}{n} \sum_{k=1}^m \tilde{T}^k f + \frac{1}{n} \sum_{k=m+1}^n \tilde{T}^k f \\ (9.2) \quad &= \frac{1}{n} \sum_{k=1}^m \tilde{T}^k f + \frac{n-m}{n} \left(\frac{1}{n-m} \sum_{k=1}^{n-m} T^k (\tilde{T}^m f) \right) \end{aligned}$$

which converges since T is mean ergodic for $\|\cdot\|$ and $\left\| \frac{1}{n} \sum_{k=1}^m \tilde{T}^k f \right\|_{ext} \leq \frac{m}{n} \|f\|_{ext} \rightarrow 0$ as n tends to infinity. $\|\cdot\|_{ext}$ is thus a \tilde{T} -isometric norm.

Of course, $\dim \left(\text{fix} \left(\tilde{T}, \left(\overline{L^1(\tilde{\mu})}, \|\cdot\|_{ext} \right) \right) \right) \geq 2$ and we can discard this situation with the arguments previously given in the invertible case. \square

10. CONCLUSION

This paper aims to illustrate once more the fact that the Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is a canonical theoretical tools to study the dynamical system (X, \mathcal{A}, μ, T) when the measure μ is infinite. The inherent complexity of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ helps to reveal features of (X, \mathcal{A}, μ, T) that might be not obvious at first: here, the Orlicz space $L^\Phi(\mu)$ appears as a natural companion space to $L^1(\mu)$. Although the treatment given here is almost exclusively theoretical, we believe the Poisson-Orlicz norms could potentially have applications in limit theorems where L^1 -norms cannot be used. The choice of the most appropriate equivalent version of the norm and its evaluation for a particular function in $L^\Phi(\mu)$ is in itself a challenging mathematical problem that will probably require new techniques.

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