An adaptive finite element method for viscoplastic flows in a pipe with stick-slip at the wall

Nicolas Roquet, Pierre Saramito

An adaptive finite element method for viscoplastic flows in a pipe with stick-slip at the wall

Nicolas Roquet

LCPC
Centre de Nantes, route de Bouaye, BP 4129, 44341 Bouguenais cedex, France
mél : Nicolas.Roquet@lcpc.fr

Pierre Saramito

CNRS
Laboratoire Jean Kuntzmann, B.P. 53, 38041 Grenoble cedex 9, France
mél : Pierre.Saramito@imag.fr
http ://www-ljk.imag.fr/membres/Pierre.Saramito

Abstract :

This communication presents the numerical resolution of non-linear yield stress phenomena by using a new mixed anisotropic auto-adaptive finite element method. The Poiseuille flow of a Bingham fluid with slip yield boundary condition at the wall is considered. Despite its practical interest, for instance for pipeline flows of yield stress fluids such as concrete and cements, this problem was not yet addressed to our knowledge. The case of a pipe with a square section has been investigated into details. The computations cover the full range of the two main dimensionless numbers and exhibit complex flow patterns: all the different flow regimes are completely identified.

Résumé :

Dans cette communication, nous présentons la résolution numérique de phénomènes non-linéaires de seuil de contrainte par une nouvelle méthode d'éléments finis mixtes auto-adaptatifs. On considère l'écoulement de Poiseuille d'un fluide de Bingham avec une condition aux bords de glissement à seuil. Malgré son intérêt pratique, par exemple pour les écoulement en pipeline *de ciments et bétons liquides, ce problème n'avait pas encore été abordé à notre connaissance. Le cas d'un tuyau de section carrée est étudié en détails. Les calculs couvrent la plage entière des deux nombres sans dimension principaux du problème et mettent en évidence des situations d'écoulements complexes : tous les différents régimes d'écoulement sont complètement identifiés.*

Mots-clefs :

viscoplasticity ; slip at the wall ; finite element method

1 Introduction

The flow of a viscoplastic fluid in a straight pipe with constant cross-section and *with non-slip* condition at the wall as been considered several times in the literature. In the 60's, an extensive mathematical study was presented by Mosolov and Miasnikov [1965]. These authors have presented impressive results on the existence and shape of rigid zones in the flow. In particular, they were the first to characterize the critical value of the yield stress above which the flow stops. Next, Duvaut and Lions [1972] clarified the problem of existence and uniqueness of a solution and renewed the mathematical study by using the powerful tools of variational inequalities. They recovered some properties already established by Mossolov and Miasnikov, and found new interesting properties. Saramito and Roquet [2001] revisited the classical fully developed Poiseuille flow of a Bingham yield stress fluid in pipe with non-circular cross-section.

Addressing the case of a square cross-section, they pointed out the lack of precision of the previous numerical computations, that were not able to compute accurately the yield surfaces that separates the shear region from the central plug and the dead zones. They proposed a new mixed anisotropic auto-adaptive finite element method coupled to the augmented Lagrangian algorithm proposed by Fortin and Glowinski [1983]. The mesh refinement is expected to capture accurately the free boundaries of the rigid zones. Based on *a priori* error estimate on adapted meshes, Roquet et al. [2000] performed the numerical analysis of the method and showed that it converges with an optimal global order of accuracy. Finally, the extension of this approach to more general flows of a Bingham fluid is addressed in Roquet and Saramito [2003] where the authors considered the flow around a cylinder. Nevertheless, in practical viscoplastic flow problems such as concrete pumping, it appears that a non-slip boundary condition is not stisfactory. The fluid slips when the tangential strength exceeds a critical value, and, otherwise the fluid sticks at the wall. This critical value may be considered as an intrinsic characteristic of the material : in the following, it will be called the *yield-force* of the fluid. The model defined by Weber and Amerongen [1969] describes this yield-force slip phenomenon. It has already been used for the flow of a Newtonian fluid with the Weber slip law by Roquet and Saramito [2004] for the straight pipe flow with a square cross-section. The aim of this paper is to extend the technique presented in Saramito and Roquet [2001] and Roquet and Saramito [2004] in order to apply it to the flow of a Bingham fluid in a straight pipe with constant cross-section with the Weber slip law at the wall.

2 Constitutive equation and conservation laws

The total Cauchy stress tensor is expressed by $\sigma_{\rm tot} = -p.I + \sigma$, where σ denote its deviatoric part, and p the pressure. In this paper, the fluid is supposed to be viscoplastic, and the relation between σ and $D(\mathbf{u})$ is given by the Bingham model (see Oldroyd [1947]) :

$$\left\{ \begin{array}{rcll} \sigma & = & 2\eta D(\mathbf{u}) + \sigma_0 \frac{D(\mathbf{u})}{|D(\mathbf{u})|} & \text{when } D(\mathbf{u}) \neq 0 \\ |\sigma| & \leq & \sigma_0 & \text{when } D(\mathbf{u}) = 0 \end{array} \right. \tag{1}$$

here $\sigma_0 \geq 0$ is the yield stress, $\eta > 0$ is the constant viscosity, $\mathbf{u}$ is the velocity field and $D(\mathbf{u}) = \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)/2$. For any tensor $\tau = (\tau_{ij})$, the notation $|\tau|$ represents the matrix norm : $|\tau|^2 = \left(\sum_{i,j} \tau_{ij}\right)/2$. The constitutive equation (1) could be written equivalently :

$$D(\mathbf{u}) = \left\{ \begin{array}{cl} \left(1 - \frac{\sigma_0}{|\sigma|}\right) \frac{\sigma}{2\eta} & \text{when } |\sigma| > \sigma_0 \\ 0 & \text{otherwise} \end{array} \right. \tag{2}$$

The slip boundary condition reads :

$$\mathbf{u_t} = \left\{ \begin{array}{cl} -\left(1 - \frac{s_0}{|\sigma_{\boldsymbol{\nu t}}|}\right) \frac{\sigma_{\boldsymbol{\nu t}}}{c_f}, & \text{when } |\sigma_{\boldsymbol{\nu t}}| > s_0, \\ 0, & \text{otherwise}, \end{array} \right. \tag{3}$$

where $s_0 \geq 0$ is the slip yield stress and $c_f > 0$ the friction dissipation coefficient. The notations $\mathbf{u_t}$ and $\sigma_{\boldsymbol{\nu t}}$ are defined by $\mathbf{u_t} = \mathbf{u} - (\mathbf{u}.\boldsymbol{\nu})\,\boldsymbol{\nu}$ and $\sigma_{\boldsymbol{\nu t}} = \sigma.\boldsymbol{\nu} - (\sigma_{\boldsymbol{\nu\nu}})\,\boldsymbol{\nu}$ where $\sigma_{\boldsymbol{\nu\nu}} = (\sigma.\boldsymbol{\nu}).\boldsymbol{\nu}$ and $\boldsymbol{\nu}$ is the unit outward normal vector. For any vector field $\mathbf{v}$, the notation $|.|$ represents the vector norm $|\mathbf{v}| = (\mathbf{v}.\mathbf{v})^{1/2}$. Notice that the vector field $\sigma_{\boldsymbol{\nu t}}$ is tangent to the boundary and that $\sigma_{\boldsymbol{\nu\nu}}$ is a scalar field defined on the boundary. Observe the analogy of structure between the slip law (3) and the Bingham constitutive equation (2). The slip relation can be also written as :

$$\left\{ \begin{array}{ll} \sigma_{\boldsymbol{\nu t}} = -c_f \mathbf{u_t} - s_0 \frac{\mathbf{u_t}}{|\mathbf{u_t}|}, & \text{when } |\mathbf{u_t}| \neq 0, \\ |\sigma_{\boldsymbol{\nu t}}| \leq s_0, & \text{when } |\mathbf{u_t}| = 0. \end{array} \right. \tag{4}$$

Also, observe the analogy between (4) and (1). The boundary condition is completed by a condition expressing that the fluid does not cross the boundary : $\mathbf{u}.\boldsymbol{\nu} = 0$. Remark that for $s_0 = 0$, one obtains the classical linear slip boundary condition : the fluid slips for any non-vanishing shear stress $\sigma_{\boldsymbol{\nu t}}$. For $s_0 > 0$, boundary parts where the fluid sticks can be observed. As s_0 becomes larger, these stick regions develop. The system of equations is closed by conservation laws. The conservation of momentum is :

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} \right) - \mathbf{div}\, \sigma + \nabla p = 0,$$

where ρ is the constant density. Since the fluid is supposed to be incompressible, the mass conservation leads to : $\operatorname{div} \mathbf{u} = 0$.

3 The pipe flow problem

We consider the fully developed flow in a prismatic tube. Let (Oz) be the axis of the tube and (Oxy) the plane of the bounded cross-section $\Omega \subset I\!R^2$. The pressure gradient is written as $\nabla p = (0, 0, -f)$ in Ω, where $f > 0$ is the constant applied force density. The velocity is written as $\mathbf{u} = (0, 0, u)$, where the third component u along the (Oz) axis depends only upon x and y, and is independent of t and z. The problem can be written as a two-dimensional one, and the stress tensor σ is equivalent to a two shear stress component vector : $\boldsymbol{\sigma} = (\sigma_{xz}, \sigma_{yz})$. We also use the following notations :

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \qquad \operatorname{div} \boldsymbol{\sigma} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \quad \text{and} \quad |\boldsymbol{\sigma}| = \left(\sigma_{xz}^2 + \sigma_{yz}^2 \right)^{1/2}.$$

Finally, the problem of the flow of a Bingham fluid in a pipe with slip at the wall can be summarized as :

(P) : find $\boldsymbol{\sigma}$ and u defined in Ω such that

$$\operatorname{div} \boldsymbol{\sigma} = -f \text{ in } \Omega, \tag{5}$$

$$\max \left(0;\ 1 - \frac{\sigma_0}{|\boldsymbol{\sigma}|} \right) \boldsymbol{\sigma} - \eta \nabla u = 0 \text{ in } \Omega, \tag{6}$$

$$\max \left(0;\ 1 - \frac{s_0}{|\boldsymbol{\sigma}.\mathbf{n}|} \right) \boldsymbol{\sigma}.\mathbf{n} + c_f u = 0 \text{ on } \partial\Omega, \tag{7}$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$ of the cross-section Ω. Here (5) expresses the conservation of momentum, (6) the constitutive equation and (7) the slip boundary condition. Let L be a characteristic length of the cross-section Ω, e.g. the half-length of an edge of a square cross-section. A characteristic velocity is given by $U = L^2 f/\eta$. Let :

$$Bi = \frac{\sigma_0}{L\, f}, \qquad S = \frac{s_0}{L\, f} \quad \text{and} \quad C_f = \frac{c_f U}{\Sigma} = \frac{c_f L}{\eta}. \tag{8}$$

The Bingham dimensionless number Bi is defined by the ratio of yield stress σ_0 by the representative stress $\Sigma = \eta U/L = Lf$. The slip yield dimensionless number S is defined as the ratio of the slip yield stress s_0 to Σ. The three dimensionless numbers Bi, S and C_f characterize the problem. The C_f coefficient is chosen equal to the unity for all numerical experiments. In this paper, we explore the problem related to the variation of both Bi and S.

4 Numerical experiments and identification of the flow regimes

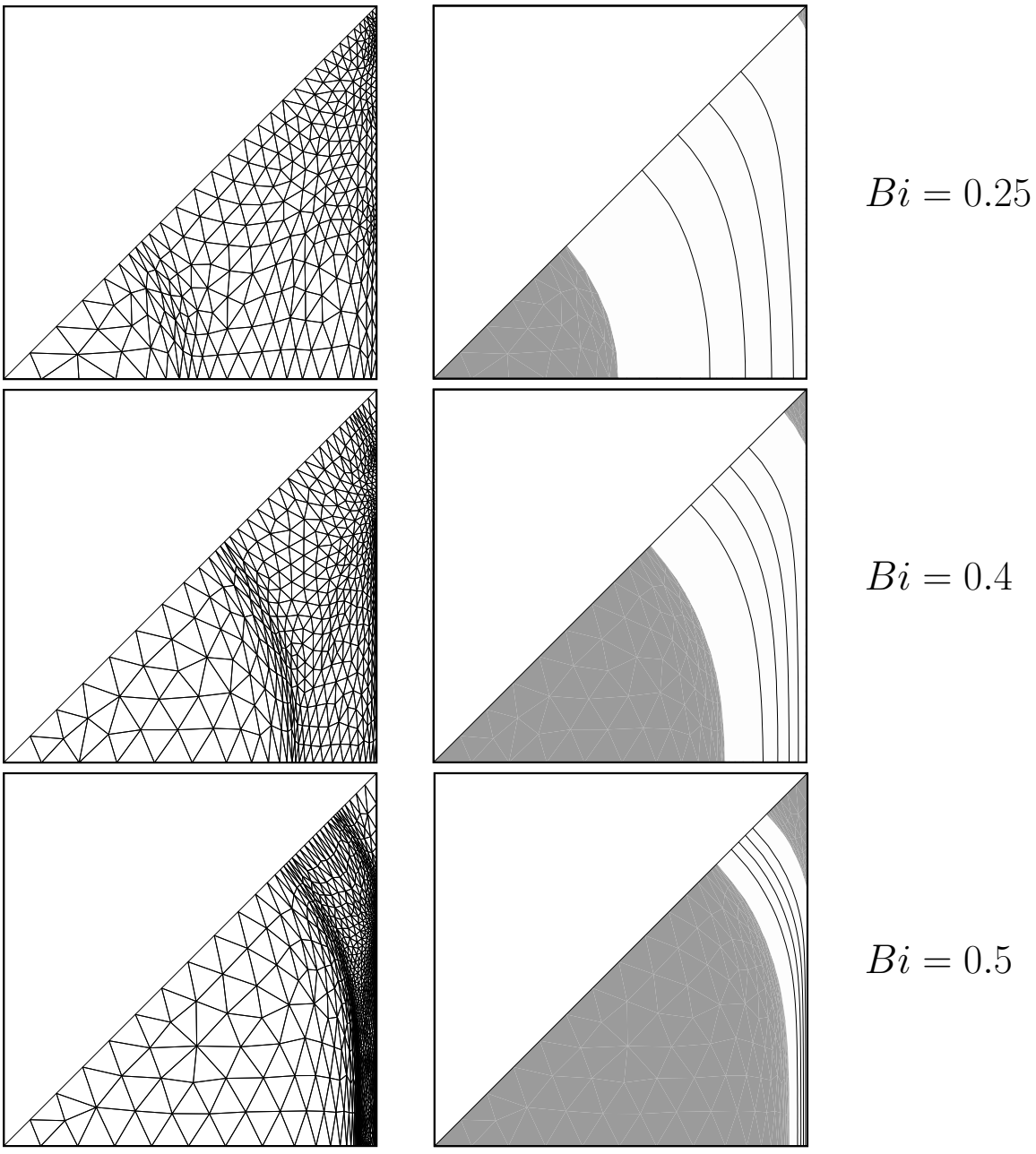

FIG. 1 – Adapted meshes and their associated solutions for $S = 0.6$: rigid zones in light gray, deforming zones in dark gray, and isovalues of the velocity.

The numerical procedure for solving the problem has been previously detailed by Roquet and Saramito [2007] and is not recalled here. Fig. 1 represents the development of rigid zones for $S = 0.6$ versus Bi and the associated adapted meshes. The area of rigid zones increases with Bi and its boundary flattens when approaching the wall. Simultaneously, concave dead zones appear and develop in the corners of the square cross-section. In this situation, the width of the deforming zone decreases and progressively reduces to a thin band arround the central plug. Finaly, the flow stops completely when the central plug simultaneously merges with the dead zones and reaches the wall.

Our computations cover the full range of the two main dimensionless numbers (Bi, S). Fig. 2 shows the existence of five flow regimes that appear in the (Bi, S) plane as areas delimted by some critical curves :

A : full adhesion,

G : full slip,

A+G : adhesion in the corners vicinity and slip elsewhere,

B stop flow ($u = 0$),

T block translation ($u = constant$).

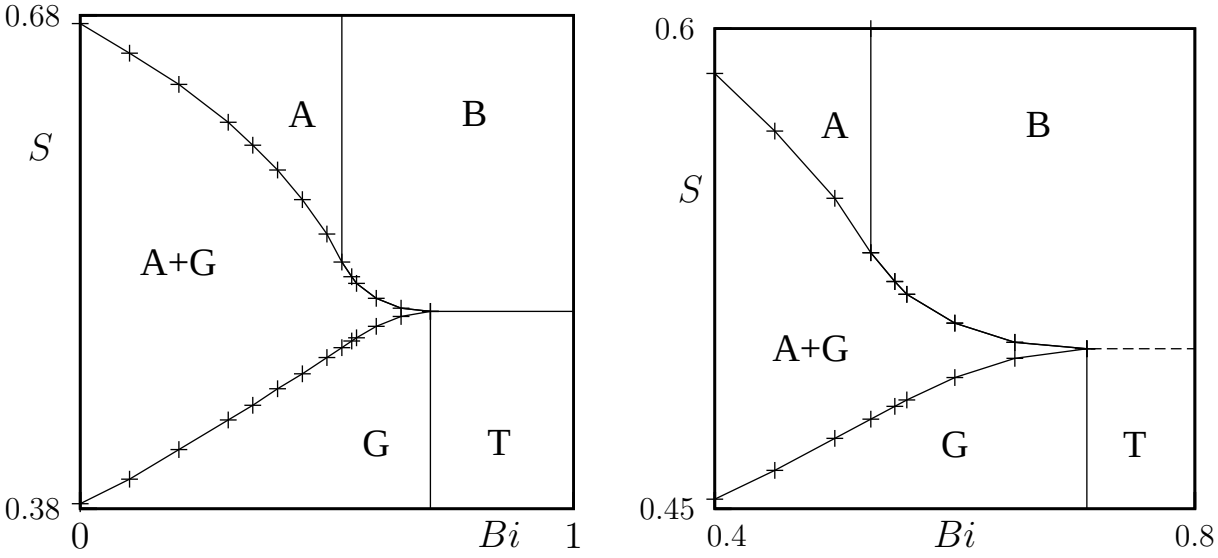

FIG. 2 – The main flow regimes for a square cross-section : (left) curves separating the regimes ; (right) zoom in the central zone.

5 Conclusion

The communication presents a combination of the two previous non-linear yield stress phenomena : the Poiseuille flow of a Bingham fluid with slip yield boundary condition at the wall. This problem is of practical interest, for instance for pipeline flows of yield stress fluids such as concrete and cements, and has not been addressed to the best of our knowledge from a computational point of view. This generalizes the works previously achieved for two particular cases : a viscoplastic fluid with non-slip at the wall, and a Newtonian fluid with the yield-force slip law. The case of a pipe with a square cross-section has been investigated in detail. The computations cover the full range of the two main dimensionless numbers and exhibit complex flow patterns. Considering the two main parameters S and Bi of the material, five flow regimes and three sub-regimes have been identified. More precisely :

- the limiting values of Bi and S separating the regimes have been obtained ;
- the evolution of the rigid zones stick-slip transition points have been established, with respect to Bi and S in each of the height regimes.

The simulations results have evidenced complex flow patterns, which have been caught thanks to the use of an auto-adaptive mesh process.

Références

G. Duvaut and J. L. Lions. *Les inéquations en mécanique et en physique*. Dunod, 1972.

M. Fortin and R. Glowinski. *The augmented Lagrangian method*. Elsevier, North-Holland, 1983.

P. P. Mosolov and V. P. Miasnikov. Variational methods in the theory of viscous-plastic medium. *J. Mech. and Appl. Math. (P.M.M.)*, 29 :468–492, 1965.

J. G. Oldroyd. A rational formulation of the equations of plastic flow for a Bingham solid. *Proc. Camb. Phil. Soc.*, 43 :100–105, 1947.

N. Roquet and P. Saramito. An adaptive finite element method for Bingham fluid flows around a cylinder. *Comput. Appl. Meth. Mech. Engrg.*, 192(31-32) :3317–3341, 2003.

N. Roquet and P. Saramito. Stick-slip transition capturing by using an adaptive finite element method. *Mathematical Modelling and Numerical Analysis*, 38(2) :249–260, 2004.

N. Roquet and P. Saramito. An adaptive finite element method for viscoplastic flows in a pipe with stick-slip at the wall. submitted, 2007. `http://hal.archives-ouvertes.fr/hal-00133116`.

N. Roquet, R. Michel, and P. Saramito. Errors estimate for a viscoplastic fluid by using Pk finite elements and adaptive meshes. *C. R. Acad. Sci. Paris, Série I*, 331(7) :563–568, 2000.

P. Saramito and N. Roquet. An adaptive finite element method for viscoplastic fluid flows in pipes. *Comput. Meth. Applied Mech. Engng*, 190(40-41) :5391–5412, 2001.

R. Weber and C. V. Amerongen. *The transport of concrete by pipeline*. C. and C. A. library translation, no. 129, London : Cement and Concrete Association, 1969.