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A Geometrical Deduction of the Kinematics of Some Plate's Models

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Abstract :

We present a deduction of Kirchhoff-Love and Reissner-Mindlin kinematics of a simply-connected plate by using the formal asymptotic developments method applied to the compatibility conditions of Saint Venant and the formula of Cesàro-Volterra, without the use of any information coming from the loading or the constitutive behavior.

Résumé :

Nous donnons une déduction purement géométrique des équations cinématiques de Kirchhoff-Love et de celles de Reissner-Mindlin dans le cas d'une plaque simplement connexe. Cette déduction est obtenue uniquement à partir des relations de compatibilité de Saint Venant et de la représentation intégrale de Cesàro-Volterra. Aucune information concernant le matériau constitutif de la plaque ou le chargement n'est utilisée.

Key-words :

Saint Venant relations; Cesàro-Volterra representation; plate models

1 Introduction

The justification of the usual models of thin plates has been a subject largely discussed in the last twenty years. In particular the Kirchhoff-Love's *kinematical assumptions* have been obtained by formal asymptotic developments from the 3D equilibrium equations taking into account the loading and the peculiar geometry of the structure: indeed, the reference configuration of the plate is $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$ where ω is a domain of \mathbf{R}^2 and the thickness is $2\varepsilon > 0$. The *small* parameter ε describes the thinness of the plate and will tend to zero. Let us recall that, under suitable assumptions on the loads, the formal asymptotic expansions have been justified by different arguments (see e.g. ref.(1) and (2))

In this paper we give, when ω is simply-connected, a different *purely geometrical* formal deduction of the Kirchhoff-Love's kinematical assumptions and we apply the same method to obtain the Reissner-Mindlin kinematics.

2 The Saint Venant Compatibility Conditions and the Formula of Cesàro-Volterra for a 3D Plate-like cylindrical domain

Let be given a deformation matrix field $\mathbf{e}^\varepsilon(\mathbf{x}^\varepsilon) = (e_{ij}^\varepsilon)(\mathbf{x}^\varepsilon) \in M_{sym}^2$, defined into the variable (simply-connected) domain Ω^ε and satisfying the Saint Venant compatibility conditions:

$$\partial_{lj}^\varepsilon e_{ik}^\varepsilon(\mathbf{x}^\varepsilon) + \partial_{ki}^\varepsilon e_{jl}^\varepsilon(\mathbf{x}^\varepsilon) - \partial_{li}^\varepsilon e_{jk}^\varepsilon(\mathbf{x}^\varepsilon) - \partial_{kj}^\varepsilon e_{il}^\varepsilon(\mathbf{x}^\varepsilon) = 0 \text{ in } \Omega^\varepsilon. \quad (1)$$

Then, there exists a displacement field $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ whose (linearized) strain tensor field is $\mathbf{e}^\varepsilon(\mathbf{x}^\varepsilon)$ (Saint Venant stated this result in 1864, but it was rigorously proved by Beltrami only in 1886).

V. Volterra in 1906 provided a formula giving *explicitly* the displacement as a line integral on a path contained in the domain of a certain function of the symmetric matrix field. In 1907 Cesàro re-wrote the formula of Volterra in a more refined way. Precisely, let $\gamma^\varepsilon(\mathbf{x}_0^\varepsilon, \mathbf{x}^\varepsilon)$ denote a (regular) path from a fixed point \mathbf{x}_0^ε to \mathbf{x}^ε completely contained in Ω^ε , then the representation of Cesàro-Volterra of the displacement field $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ takes the following form:

$$u_i^\varepsilon(\mathbf{x}^\varepsilon) = \int_{\gamma^\varepsilon(\mathbf{x}_0^\varepsilon, \mathbf{x}^\varepsilon)} [e_{ij}^\varepsilon(\mathbf{y}^\varepsilon) + (x_k^\varepsilon - y_k^\varepsilon)(\partial_k^\varepsilon e_{ij}^\varepsilon(\mathbf{y}^\varepsilon) - \partial_i^\varepsilon e_{kj}^\varepsilon(\mathbf{y}^\varepsilon))] dy_j^\varepsilon \quad (2)$$

In order to study the asymptotic behavior of the deformation field \mathbf{e}^ε and of the displacement field \mathbf{u}^ε when the thickness ε goes to zero, we introduce the usual change of variables that allows to transform the problem posed on a variable domain (dependent of ε) onto a problem on a fixed domain (independent of ε), (see ref.(1)). Let $\Omega := \omega \times]-1, 1[$ and let $\mathbf{x} = (x_i)$ denote the generic point in the set $\bar{\Omega}$; at each point $\mathbf{x} \in \bar{\Omega}$, we associate the point $\mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon$ through the bijection (Fig. 1):

$$\Pi^\varepsilon : \mathbf{x} = (x_i) \in \bar{\Omega} \rightarrow \mathbf{x}^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon. \quad (3)$$

Consequently,

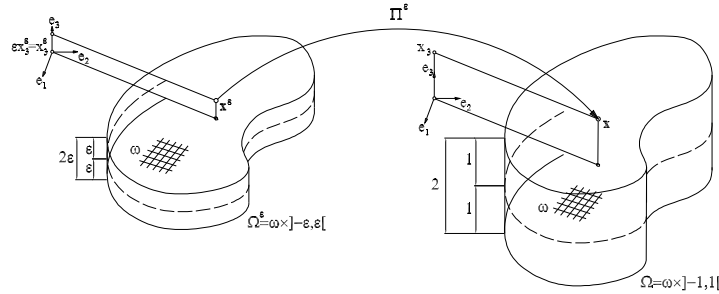


Figure 1: The reference configuration

$$\partial_\alpha^\varepsilon = \partial_\alpha \text{ and } \partial_3^\varepsilon = \frac{1}{\varepsilon} \partial_3.$$

and we denote $e_{ij}(\varepsilon)(\mathbf{x}) = e_{ij}^\varepsilon(\mathbf{x}^\varepsilon)$, resp. $u_i(\varepsilon)(\mathbf{x}) = u_i^\varepsilon(\mathbf{x}^\varepsilon)$, the deformation, resp. the displacement, transformed by Π^ε . In the sequel, we will omit the explicit dependence of $e_{ij}(\varepsilon)$ from \mathbf{x} . By applying the change of coordinates (3) to the Cesàro-Volterra formula (2) we obtain for the in-plane displacements u_α and the transverse displacement u_3 :

$$\begin{aligned} u_\alpha(\varepsilon)(\mathbf{x}) &= \int_{\gamma(\mathbf{x}_0, \mathbf{x})} [e_{\alpha\beta}(\varepsilon)(\mathbf{y}) + (x_\sigma - y_\sigma)(\partial_\sigma e_{\alpha\beta}(\varepsilon)(\mathbf{y}) - \partial_\alpha e_{\sigma\beta}(\varepsilon)(\mathbf{y})) + \\ &\quad + \varepsilon(x_3 - y_3)(\frac{1}{\varepsilon} \partial_3 e_{\alpha\beta}(\varepsilon)(\mathbf{y}) - \partial_\alpha e_{3\beta}(\varepsilon)(\mathbf{y}))] dy_\beta \\ &\quad + \varepsilon[e_{\alpha 3}(\varepsilon)(\mathbf{y}) + (x_\sigma - y_\sigma)(\partial_\sigma e_{\alpha 3}(\varepsilon)(\mathbf{y}) - \partial_\alpha e_{\sigma 3}(\varepsilon)(\mathbf{y})) + \\ &\quad + \varepsilon(x_3 - y_3)(\frac{1}{\varepsilon} \partial_3 e_{\alpha 3}(\varepsilon)(\mathbf{y}) - \partial_\alpha e_{33}(\varepsilon)(\mathbf{y}))] dy_3, \\ u_3(\varepsilon)(\mathbf{x}) &= \int_{\gamma(\mathbf{x}_0, \mathbf{x})} [e_{3\beta}(\varepsilon)(\mathbf{y}) + (x_\sigma - y_\sigma)(\partial_\sigma e_{3\beta}(\varepsilon)(\mathbf{y}) - \frac{1}{\varepsilon} \partial_3 e_{\sigma\beta}(\varepsilon)(\mathbf{y}))] dy_\beta \\ &\quad + \varepsilon[e_{33}(\varepsilon)(\mathbf{y}) + (x_\sigma - y_\sigma)(\partial_\sigma e_{33}(\varepsilon)(\mathbf{y}) - \frac{1}{\varepsilon} \partial_3 e_{\sigma 3}(\varepsilon)(\mathbf{y}))] dy_3. \end{aligned} \quad (4)$$

In the sequel, we will omit the dependence on \mathbf{y} of the integrand function and the dependence of γ from the starting and ending points \mathbf{x}_0 and \mathbf{x} .

3 Deduction of the Kirchhoff-Love kinematics

By applying the previous coordinates change in (1) we obtain the following equations on the fixed domain Ω :

$$\begin{aligned}
& \partial_{11}e_{22}(\varepsilon) + \partial_{22}e_{11}(\varepsilon) - 2\partial_{12}e_{12}(\varepsilon) = 0, \\
& \frac{1}{\varepsilon^2}\partial_{33}e_{22}(\varepsilon) + \partial_{22}e_{33}(\varepsilon) = \frac{2}{\varepsilon}\partial_{23}e_{23}(\varepsilon), \\
& \frac{1}{\varepsilon^2}\partial_{33}e_{11}(\varepsilon) + \partial_{11}e_{33}(\varepsilon) = \frac{2}{\varepsilon}\partial_{13}e_{13}(\varepsilon), \\
& \frac{1}{\varepsilon}\partial_{23}e_{11}(\varepsilon) = \partial_1(-\partial_1e_{23}(\varepsilon) + \partial_2e_{13}(\varepsilon) + \frac{1}{\varepsilon}\partial_3e_{12}(\varepsilon)), \\
& \frac{1}{\varepsilon}\partial_{13}e_{22}(\varepsilon) = \partial_2(\partial_1e_{23}(\varepsilon) - \partial_2e_{13}(\varepsilon) + \frac{1}{\varepsilon}\partial_3e_{12}(\varepsilon)), \\
& \frac{1}{\varepsilon}\partial_{12}e_{33}(\varepsilon) = \frac{1}{\varepsilon}\partial_3(\partial_1e_{23}(\varepsilon) + \partial_2e_{13}(\varepsilon) - \frac{1}{\varepsilon}\partial_3e_{12}(\varepsilon)).
\end{aligned} \tag{5}$$

Since in the re-scaled equations (5) the parameter of smallness ε appears explicitly in a polynomial form, we assume that there exists a formal asymptotic expansion of the deformations $e_{ij}(\varepsilon)$:

$$e_{ij}(\varepsilon) = e_{ij}^0 + \varepsilon e_{ij}^1 + \varepsilon^2 e_{ij}^2 + \varepsilon^3 e_{ij}^3 + \dots \tag{6}$$

By substituting the asymptotic development (6) in (5) and by isolating the zeroth order term e_{ij}^0 , we find:

$$\begin{aligned}
& \partial_{33}e_{\alpha\beta}^0 = 0, \\
& \partial_{23}e_{11}^0 - \partial_{13}e_{12}^0 = 0, \\
& \partial_{23}e_{12}^0 - \partial_{13}e_{22}^0 = 0, \\
& \partial_{11}e_{22}^0 + \partial_{22}e_{11}^0 - 2\partial_{12}e_{12}^0 = 0.
\end{aligned} \tag{7}$$

From equation (7)₁ we derive immediately the characterization of the components of the leading term $e_{\alpha\beta}^{KL} := e_{\alpha\beta}^0$:

$$e_{\alpha\beta}^{KL} = c_{\alpha\beta}(x_1, x_2) + x_3 r_{\alpha\beta}(x_1, x_2). \tag{8}$$

Thus, we can notice that the deformation field is decomposed in two parts, which are well-known in theory of plates: the former $c_{\alpha\beta}$, the so-called *in-plane strain tensor*, governs the in-plane deformations; the latter $r_{\alpha\beta}$, the so-called *curvature tensor*, characterizes the deformations outside the middle-plane of the plate.

By substituting (8) in (7)_{2,3,4}, we obtain the compatibility conditions for the fields $c_{\alpha\beta}$ and $r_{\alpha\beta}$ in the case of a plate:

$$\begin{aligned}
& \partial_{22}c_{11} + \partial_{11}c_{22} - 2\partial_{12}c_{12} = 0, \\
& \partial_1 r_{12} - \partial_2 r_{11} = 0, \\
& \partial_1 r_{22} - \partial_2 r_{12} = 0.
\end{aligned} \tag{9}$$

No conditions have been found for the remaining components e_{i3}^{KL} of the deformation. They follow from the formula of Cesàro-Volterra (2). Indeed inserting the asymptotic development (6) in (4) we obtain the two formal asymptotic expansions:

$$u_\alpha(\varepsilon) = u_\alpha^0 + \varepsilon u_\alpha^1 + \varepsilon^2 u_\alpha^2 + \varepsilon^3 u_\alpha^3 + \dots \tag{10}$$

$$u_3(\varepsilon) = \frac{1}{\varepsilon} u_3^{-1} + u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \varepsilon^3 u_3^3 + \dots \tag{11}$$

whose leading terms are:

$$\begin{aligned} u_\alpha^0(\mathbf{x}) &= \int_\gamma [e_{\alpha\beta}^0 + (x_\sigma - y_\sigma)(\partial_\sigma e_{\alpha\beta}^0 - \partial_\alpha e_{\sigma\beta}^0) + (x_3 - y_3)\partial_3 e_{\alpha\beta}^0] dy_\beta, \\ u_3^{-1}(\mathbf{x}) &= - \int_\gamma (x_\sigma - y_\sigma)\partial_3 e_{\sigma\beta}^0 dy_\beta, \end{aligned} \quad (12)$$

and represent the formula of Cesàro-Volterra in the case of a plate. Naturally, by applying the independence from the path to the line integral (12), we obtain once more the equations (7) and hence (8)-(9). The substitution of (8) into (12) suggests the following definitions $u_\alpha^{KL} := u_\alpha^0$, $u_3^{KL} := u_3^{-1}$ and hence $e_{\alpha 3}^{KL} := \frac{1}{2}(\partial_\alpha u_3^{KL} + \partial_3 u_\alpha^{KL})$ and $e_{33}^{KL} := \partial_3 u_3^{KL}$. Moreover, by deriving the formulae (12) along the through-the-thickness coordinate x_3 , we obtain that $e_{i3}^{KL} = 0$.

Let us stress explicitly the difference between e_{i3}^{KL} and e_{i3}^0 :

- e_{i3}^{KL} are calculated starting from the formula of Cesàro-Volterra and they represent the strain functions associated to the displacements u_α^{KL} and u_3^{KL} ;
- e_{i3}^0 are only the zeroth order terms of the asymptotic development of the matrix field $\mathbf{e}(\varepsilon)$.

In order to recover from (12) the classical Kirchhoff-Love kinematics of a plate:

$$\begin{cases} u_\alpha^{KL}(x_1, x_2, x_3) = \bar{u}_\alpha(x_1, x_2) - x_3 \partial_\alpha w, \\ u_3^{KL}(x_1, x_2) = w(x_1, x_2), \end{cases} \quad (13)$$

one has only to consider the Airy function $w(x_1, x_2) \in C^3(\omega)$ (i.e. such that $r_{\alpha\beta} := -\partial_\alpha \partial_\beta w$) and to choose a path $\hat{\gamma}$ all contained in the middle plane of the plate (i.e. $\hat{\gamma} \subset \omega$). One then obtains (13) with $\bar{u}_\alpha(x_1, x_2) := \int_{\hat{\gamma}} [c_{\alpha\beta} + (x_\sigma - y_\sigma)(\partial_\sigma c_{\alpha\beta} - \partial_\alpha c_{\sigma\beta})] dy_\beta$.

It is important to remark once more that in order to obtain (13) we did not make any assumption on the constitutive behavior of the material or on the equilibrium conditions of the system.

4 Deduction of the Reissner-Mindlin kinematics

All plate theories obtained by asymptotic analysis and based on a limit process when the thickness tends to zero give rise to non-shearable kinematics, as in Kirchhoff-Love plate model. It has been remarked (see e.g. ref. (3)) that the the engineering derivation of equations of shearable plates, like Reissner-Mindlin plate model, is based on the following set of internal constraints:

$$\partial_{33} u_\alpha = 0 \text{ and } \partial_3 u_3 = 0. \quad (14)$$

Let now be given a deformation matrix \mathbf{e}^ε satisfying the compatibility conditions (1) and the constraints

$$\begin{aligned} e_{\alpha 3}^\varepsilon(\mathbf{x}^\varepsilon) &= e_{\alpha 3}^\varepsilon(x_1, x_2) \\ e_{33}^\varepsilon(\mathbf{x}^\varepsilon) &= 0 \end{aligned} \quad (15)$$

which are the analogous of (14). It then follows that $e_{\alpha\beta}^\varepsilon$ is a linear function of the transverse coordinate x_3^ε :

$$e_{\alpha\beta}^\varepsilon(\mathbf{x}^\varepsilon) = c_{\alpha\beta}^\varepsilon(x_1, x_2) + x_3^\varepsilon r_{\alpha\beta}^\varepsilon(x_1, x_2). \quad (16)$$

and that $c_{\alpha\beta}^\varepsilon$ and $r_{\alpha\beta}^\varepsilon$ satisfy the following set of Reissner-Mindlin compatibility conditions in Ω^ε :

$$\begin{aligned} \partial_{11}^\varepsilon c_{22}^\varepsilon + \partial_{22}^\varepsilon c_{11}^\varepsilon - 2\partial_{12}^\varepsilon c_{12}^\varepsilon &= 0, \\ \partial_{11}^\varepsilon r_{22}^\varepsilon + \partial_{22}^\varepsilon r_{11}^\varepsilon - 2\partial_{12}^\varepsilon r_{12}^\varepsilon &= 0, \\ \partial_2^\varepsilon r_{11}^\varepsilon - \partial_1^\varepsilon r_{12}^\varepsilon &= \partial_{12}^\varepsilon e_{13}^\varepsilon - \partial_{11}^\varepsilon e_{23}^\varepsilon, \\ \partial_1^\varepsilon r_{22}^\varepsilon - \partial_2^\varepsilon r_{12}^\varepsilon &= \partial_{12}^\varepsilon e_{23}^\varepsilon - \partial_{22}^\varepsilon e_{13}^\varepsilon. \end{aligned} \quad (17)$$

By applying the change of coordinates (3), $c_{\alpha\beta}^\varepsilon$, resp. $r_{\alpha\beta}^\varepsilon$, resp. $e_{\alpha 3}^\varepsilon$ turns into $c_{\alpha\beta}(\varepsilon)$, resp. $r_{\alpha\beta}(\varepsilon)$, resp. $e_{\alpha 3}(\varepsilon)$ and one can define the corresponding asymptotic expansions.

The representation of Cesàro-Volterra of the displacement field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ takes the form:

$$\begin{aligned} u_\alpha(\varepsilon)(\mathbf{x}) &= \int_\gamma [c_{\alpha\beta}(\varepsilon) + (x_\sigma - y_\sigma)(\partial_\sigma c_{\alpha\beta}(\varepsilon) - \partial_\alpha c_{\sigma\beta}(\varepsilon)) + \\ &\quad + \varepsilon y_3(r_{\alpha\beta}(\varepsilon) + (x_\sigma - y_\sigma)(\partial_\sigma r_{\alpha\beta}(\varepsilon) - \partial_\alpha r_{\sigma\beta}(\varepsilon))) + \\ &\quad + \varepsilon(x_3 - y_3)(r_{\alpha\beta}(\varepsilon) - \partial_\alpha e_{3\beta}(\varepsilon))] dy_\beta + \\ &\quad + \varepsilon[e_{\alpha 3}(\varepsilon) + (x_\sigma - y_\sigma)(\partial_\sigma e_{3\beta}(\varepsilon) - \partial_\alpha e_{3\sigma}(\varepsilon))] dy_3, \\ u_3(\varepsilon)(\mathbf{x}) &= \int_\gamma [e_{3\beta}(\varepsilon) + (x_\sigma - y_\sigma)(\partial_\sigma e_{3\beta}(\varepsilon) - r_{\sigma\beta}(\varepsilon))] dy_\beta. \end{aligned} \quad (18)$$

The asymptotic expansions of $c_{\alpha\beta}(\varepsilon)$, $r_{\alpha\beta}(\varepsilon)$ and $e_{\alpha 3}(\varepsilon)$ induce an asymptotic development for the displacement functions $u_i(\varepsilon)$:

$$u_i(\varepsilon) = u_i^0 + \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \varepsilon^3 u_i^3 + \dots \quad (19)$$

with :

$$\begin{aligned} u_\alpha^0(\mathbf{x}) &= \int_\gamma [c_{\alpha\beta}^0 + (x_\sigma - y_\sigma)(\partial_\sigma c_{\alpha\beta}^0 - \partial_\alpha c_{\sigma\beta}^0)] dy_\beta, \\ u_\alpha^1(\mathbf{x}) &= \int_\gamma [c_{\alpha\beta}^1 + (x_\sigma - y_\sigma)(\partial_\sigma c_{\alpha\beta}^1 - \partial_\alpha c_{\sigma\beta}^1) + y_3(r_{\alpha\beta}^0 + (x_\sigma - y_\sigma)(\partial_\sigma r_{\alpha\beta}^0 - \partial_\alpha r_{\sigma\beta}^0)) + \\ &\quad + (x_3 - y_3)(r_{\alpha\beta}^0 - \partial_\alpha e_{3\beta}^0)] dy_\beta + [e_{\alpha 3}^0 + (x_\sigma - y_\sigma)(\partial_\sigma e_{3\beta}^0 - \partial_\alpha e_{3\sigma}^0)] dy_3, \\ u_3^0(\mathbf{x}) &= \int_\gamma [e_{3\beta}^0 + (x_\sigma - y_\sigma)(\partial_\sigma e_{3\beta}^0 - r_{\sigma\beta}^0)] dy_\beta, \end{aligned} \quad (20)$$

We can easily prove that u_α^0 is independent from the transverse coordinate x_3 . Thus,

$$u_\alpha^0(\mathbf{x}) = \int_\gamma [c_{\alpha\beta}^0 + (x_\sigma - y_\sigma)(\partial_\sigma c_{\alpha\beta}^0 - \partial_\alpha c_{\sigma\beta}^0)] dy_\beta := \bar{u}_\alpha^0(x_1, x_2). \quad (21)$$

We notice that u_α^0 governs only the in-plane deformation state, while we are looking for a Reissner-Mindlin kinematics which takes into account not only membranal and flexural but also shear behaviors. In order to obtain the missing information we intensively use (20).

At first, we derive (20)₃ with respect to x_3 and we have $\partial_3 u_3^0 = 0$. Hence:

$$u_3^0 := w(x_1, x_2). \quad (22)$$

Secondly, from (20)₂ and from (20)₃, by deriving u_α^1 with respect to x_3 and u_3^0 with respect to x_α and by summing up these two derivatives, we find:

$$\partial_3 u_\alpha^1 + \partial_\alpha u_3^0 = 2e_{\alpha 3}^0(x_1, x_2). \quad (23)$$

and hence one has the complete characterization of the displacement functions u_α^1 and u_3^0 :

$$\begin{cases} u_\alpha^1(x_1, x_2, x_3) = \bar{u}_\alpha^1(x_1, x_2) + x_3(2e_{\alpha 3}^0 - \partial_\alpha w), \\ u_3^0(x_1, x_2) = w(x_1, x_2). \end{cases} \quad (24)$$

Thirdly, from (20)₂, if we derive u_α^1 with respect to x_β and u_β^1 with respect to x_α , we have:

$$c_{\alpha\beta}^1(x_1, x_2) + x_3 r_{\alpha\beta}^0(x_1, x_2) = \frac{1}{2}(\partial_\beta u_\alpha^1 + \partial_\alpha u_\beta^1), \quad (25)$$

and hence using (24)

$$c_{\alpha\beta}^1 = \frac{1}{2}(\partial_\beta \bar{u}_\alpha^1 + \partial_\alpha \bar{u}_\beta^1) \text{ and } r_{\alpha\beta}^0 = \frac{1}{2}(2\partial_\beta e_{\alpha 3}^0 + 2\partial_\alpha e_{\beta 3}^0 - \partial_{\alpha\beta} w). \quad (26)$$

We can now define the Reissner-Mindlin kinematics:

$$\begin{cases} u_\alpha^{RM}(x_1, x_2, x_3) := \bar{u}_\alpha^0(x_1, x_2) + \bar{u}_\alpha^1(x_1, x_2) + x_3(2e_{\alpha 3}^0 - \partial_\alpha w), \\ u_3^{RM}(x_1, x_2) := w(x_1, x_2), \end{cases} \quad (27)$$

and consequently, we can calculate the deformation field as follows:

$$\begin{aligned} e_{\alpha\beta}^{RM} &:= c_{\alpha\beta}^0(x_1, x_2) + c_{\alpha\beta}^1(x_1, x_2) + x_3 r_{\alpha\beta}^0(x_1, x_2), \\ e_{\alpha 3}^{RM} &:= e_{\alpha 3}^0(x_1, x_2), \\ e_{33}^{RM} &= 0. \end{aligned} \quad (28)$$

As we can notice from the definition (27) of the Reissner-Mindlin displacements, the quantity $2e_{\alpha 3}^0 - \partial_\alpha w$ clearly represents the rotation of the transverse fiber of the plate. This fiber does not remain perpendicular to the middle plane after the deformation, as in the Kirchhoff-Love plate: hence, this is an example of a plate which can shear. The function $e_{\alpha 3}^0$, which appears in the rotation of the transverse segment, measures precisely the deviation from a Kirchhoff-Love strain state and identifies a new deformation state of a shearable plate, namely the Reissner-Mindlin strain state.

5 Conclusions

The previous *formal* deduction of the Kirchhoff-Love kinematics (12) or (13) and of the Reissner-Mindlin kinematics (27) for simply-connected plates is *purely geometrical* since we have used no information about the material of the plate or about the loading. We have *only* used the Saint Venant compatibility conditions and the Cesàro-Volterra integral representation. Both are deeply connected with the *metric* structure of the deformation and of the manifold to whom this deformation is applied. Hence one can develop an analogous approach in other situations : beams, shells, ...

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