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Optimal Boundary ISS Controller for Heterogeneous and Congested Traffic [★]

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Abstract

This paper studies an optimal boundary control law for a heterogeneous traffic flow model with disturbances in order to remove the traffic congestion. The macroscopic first-order N -class Aw-Rascle (AR) traffic model consists of $2N$ hyperbolic partial differential equations (PDEs). The vehicle size and driver's behavior characterize the type of vehicle. There are m positive characteristic velocities and $2N - m$ negative velocities in the congested traffic after linearizing the model equations around the steady-states depending on the spacial variable. By using the backstepping method, a controller implemented by a ramp metering at the inlet boundary is designed for rejecting the disturbances to stabilize the $2N \times 2N$ hetero-directional traffic system. The developed controller in terms of proportional integral (PI) control is derived from mapping the original system to a target system with a PI boundary control rejecting the disturbances. The integral input-to-state stability (iISS) of the target system is proved by using the Lyapunov method. Finally, an optimization problem is established and solved for seeking the optimal controller.

Key words: Multi-type AR traffic model, Traffic congestion, Disturbances, Backstepping, Optimal controller

1 Introduction

Traffic congestion is a pervasive problem that leads to the increased fuel consumption and risky driving conditions. It is natural to use boundary control on available control signals as ramp metering or variable speed limits to stabilize the highway traffic systems. Paper [15] contributes to the boundary control design for multi-directional congested traffic evolving on large-scale urban networks represented by a continuum two-dimensional plane. In [18], the reinforcement learning boundary controllers is designed to mitigate stop-and-go congested traffic for 2×2 quasilinear Aw-Rascle-Zhang (ARZ) partial differential equations (PDEs) model by

using the proximal policy optimization that is an algorithm based on neural networks. In [3], a delay-robust stabilizing state feedback boundary control law is developed for an underactuated network of two subsystems of heterodirectional linear first-order $n + m$ hyperbolic PDE systems.

Usually, macroscopic models typically described by PDEs are more suitable to study the congested traffic and disturbances in the traffic flow. The exact boundary controllability of a class of nonlocal conservation laws modeling traffic flow is studied in [5]. In [10], the authors propose a new continuum model with an additional anisotropic term which ensures the characteristic velocities can be less than or equal to the macroscopic flow speed. An extension of speed gradient (SG) model is introduced to study the mixed traffic flow system in [11]. Paper [12] extends the Aw-Rascle (AR) model for heterogeneous traffic by using area occupancy and analyzes the properties of the extended model. A new car-following model for heterogeneous traffic flow is presented in [14]. In [12], the macroscopic N -class AR traffic model with the consideration of vehicle size is used because of the validation of simulation. A contin-

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uum multi-class traffic model is proposed on the basis of a three-dimensional flow–concentration surface in [13]. Paper [9] studies a two-type vehicle heterogeneous traffic model to acquire overtaking and creeping traffic flows.

The backstepping method is used to derive a boundary controller in some papers. In [16], a boundary observer for nonlinear ARZ traffic flow model is designed to estimate the information of traffic states using the backstepping method. A controller is designed for the underactuated cascade network of interconnected PDEs systems using backstepping in [2]. In consideration of the limits of technology and cost, there have been works inspired by [8], designing a control law for the linearized ARZ traffic flow model by using backstepping transformations (see also [17]). Paper [7] uses the backstepping method to design an output feedback boundary control for the stop and go traffic problem of linearized two-class AR traffic flow system. Paper [8] uses a backstepping transformation to design a control law and derives \mathbf{H}^2 exponential stability for a quasilinear 2×2 system of first-order hyperbolic PDEs. Paper [4] studies the sufficient conditions for local Input-to-State Stability in sup norm of general quasilinear hyperbolic systems with boundary input disturbances.

Contribution: This paper states a new result on controller design by using backstepping method for the linearized multi-type traffic flow system around a nonuniform steady-state to reject disturbances and then to remove the traffic congestion. Firstly, this work presents the derivation of an extended multi-type AR traffic flow model in the characteristic form. Secondly, we prove the iISS of the target system which has a source term of integral form and a proportional-integral (PI) boundary control for rejecting disturbances. Moreover, a controller implemented by ramp metering is designed to robustly stabilize the heterogeneous traffic system by applying the backstepping method to the multi-type vehicle traffic model.

This paper is organized as follows: Section 2 introduces the multi-type AR traffic flow model with the parameters characterizing different vehicle types and the formulation of the control problem to be solved. In Section 3, the iISS of the target system is proved by the Lyapunov method and a controller is designed by using the backstepping approach. In Section 4, the optimization problem is presented and the numerical results are provided for verifying the existence of the optimal controller. The paper ends with concluding remarks in Section 5.

Notation. $\mathcal{M}_{n,n}(\mathbb{R})$ denotes the set of $n \times n$ real matrices. $[A]_{i,j}$ denotes the entry of matrix A in the i -th row and the j -th column. For a function $\varphi = [\varphi_1, \dots, \varphi_n]^\top : [0, L] \times [0, \infty) \rightarrow \mathbb{R}^n$, $\|\varphi\|_{\mathbf{L}^\infty((0,L);\mathbb{R}^n)} = \max(\|\varphi_1\|_{\mathbf{L}^\infty((0,L);\mathbb{R}^n)}, \dots, \|\varphi_n\|_{\mathbf{L}^\infty((0,L);\mathbb{R}^n)})$.

$$\begin{aligned} \|\varphi\|_{\mathbf{L}^2((0,L);\mathbb{R}^n)} &= \sqrt{\int_0^L (\varphi_1^2(\xi, t) + \dots + \varphi_n^2(\xi, t)) d\xi}, \\ \|\varphi\|_{\mathbf{H}^1((0,L);\mathbb{R}^n)} &= \|\varphi\|_{\mathbf{L}^2((0,L);\mathbb{R}^n)} + \|\varphi_x\|_{\mathbf{L}^2((0,L);\mathbb{R}^n)}. \end{aligned}$$

The symbol $*$ stands for a symmetric block in a matrix.

2 Traffic Flow System and Control Problem

In this section, the multi-type AR traffic flow model and the interpretation of crucial parameters are presented. The preparations for designing controller are also done including the transformations of states and the linearization around a nonuniform steady-state. On the basis of the control problem to be solved, the corresponding boundary conditions are derived.

2.1 Multi-type AR traffic flow model

We investigate the multi-type AR traffic flow model in [12] that describes the dynamics of a heterogeneous traffic consisting of N vehicle types on a road segment with the length L ,

$$\begin{aligned} \partial_t \rho_i(x, t) + \partial_x (\rho_i(x, t) v_i(x, t)) &= 0, \quad (1) \\ \partial_t (v_i(x, t) + p_i(Ao)) + v_i(x, t) \partial_x (v_i(x, t) + p_i(Ao)) &= \frac{V_{e,i}(Ao) - v_i(x, t)}{\tau_i}, \quad (2) \end{aligned}$$

with the independent space variable $x \in (0, L)$ and the independent time variable $t \in [0, \infty)$, where i is the index of vehicle type with $i = 1, 2, \dots, N$, $\rho_i(x, t)$ and $v_i(x, t)$ are respectively the density and velocity of vehicle type i . Additionally, the density $\rho_i(x, t)$ is defined as the number of vehicles passing road section per unit length and the velocity $v_i(x, t)$ is defined as the average speed of vehicles passing location x in unit time. The relaxation time τ_i of vehicle type i is subject to the driving behavior, the variable $Ao(\rho)$ is the area occupancy, and

$$Ao(\rho) = \frac{\sum_{i=1}^N a_i \rho_i}{W}, \quad (3)$$

where $\rho = (\rho_1, \rho_2, \dots, \rho_N)^\top$, a_i is the occupied surface per vehicle for type i and W is the width of the road segment. Area occupancy Ao describes the percentage of road space that is occupied by all the vehicle classes on the considered road section. In the physical sense, $0 < Ao \leq 1$.

For the heterogeneous traffic, the traffic pressure function $p_i(Ao)$ of vehicle type i is an increasing function of the area occupancy Ao as follows (see [7]),

$$p_i(Ao) = v_i^M \left(\frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i}, \quad i = 1, 2, \dots, N, \quad (4)$$

where the free-flow velocity v_i^M and the maximum area occupancy $0 < Ao_i^M \leq 1$ of vehicle type i respectively describe the maximal velocity in the free regime and the maximum percentage of occupied surface in the congested regime, if there is only vehicle class i on the road. The constant $\gamma_i > 1$ is the pressure exponent of type i and it makes formulating the experienced traffic pressure $p_i(Ao)$ correct and flexible.

The steady-state speed-Ao relationship of vehicle class i ($= 1, 2, \dots, N$) is given by Greenshields model in [6] as

$$V_{e,i}(Ao) = v_i^M - p_i(Ao) = v_i^M \left(1 - \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i} \right). \quad (5)$$

There is a negative connection from the decreasing function $V_{e,i}(Ao)$ describing the desired velocity of the drivers to the crowded degree.

2.2 Linearization of multi-type AR traffic flow model

Inspired by the case (2 vehicle classes) in [7], the multi-type AR traffic model (1)-(2) is linearized around a nonuniform steady-state

$$u^*(x) = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x), \dots, \rho_N^*(x), v_N^*(x))^T,$$

for $x \in (0, L)$, where $\rho_i^*, v_i^* \in \mathbb{C}^2([0, L]; \mathbb{R})$ satisfy, for $i = 1, 2, \dots, N$,

$$v_i^*(x) \frac{d\rho_i^*(x)}{dx} + \rho_i^*(x) \frac{dv_i^*(x)}{dx} = 0, \quad (6)$$

$$v_i^*(x) \frac{dv_i^*(x)}{dx} + v_i^*(x) \frac{dp_i(x)}{dx} = \frac{V_{e,i}(Ao(\rho^*(x))) - v_i^*(x)}{\tau_i}. \quad (7)$$

From (6), note that $\rho_i^*(x)v_i^*(x) = d_i$ with constants $d_i, i = 1, 2, \dots, N$.

Denote $(\tilde{\rho}_1, \tilde{v}_1, \tilde{\rho}_2, \tilde{v}_2, \dots, \tilde{\rho}_N, \tilde{v}_N)^T$ with $\tilde{\rho}_i = \rho_i - \rho_i^*(x), \tilde{v}_i = v_i - v_i^*(x), i = 1, 2, \dots, N, x \in (0, L)$ by $\tilde{u} \in \mathbf{H}^1([0, L] \times [0, \infty); \mathbb{R}^{2N})$, the system (1)-(2) is transformed to the following equation,

$$A(\tilde{u})\tilde{u}_t(x, t) + B(\tilde{u})\tilde{u}_x(x, t) + C(\tilde{u})\tilde{u}(x, t) = 0, \quad (8)$$

where, for $i, j = 1, 2, \dots, N$,

$$A(\tilde{u}) = \begin{bmatrix} A_{11}(\tilde{u}) & A_{12}(\tilde{u}) & \cdots & A_{1N}(\tilde{u}) \\ A_{21}(\tilde{u}) & A_{22}(\tilde{u}) & \cdots & A_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}(\tilde{u}) & A_{N2}(\tilde{u}) & \cdots & A_{NN}(\tilde{u}) \end{bmatrix}, \quad (9)$$

with

$$A_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \delta_{ii}(\rho) & 1 \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (10)$$

$$B(\tilde{u}) = \begin{bmatrix} B_{11}(\tilde{u}) & B_{12}(\tilde{u}) & \cdots & B_{1N}(\tilde{u}) \\ B_{21}(\tilde{u}) & B_{22}(\tilde{u}) & \cdots & B_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1}(\tilde{u}) & B_{N2}(\tilde{u}) & \cdots & B_{NN}(\tilde{u}) \end{bmatrix}, \quad (11)$$

with

$$B_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} \tilde{v}_i + v_i^*(x) & \tilde{\rho}_i + \rho_i^*(x) \\ (\tilde{v}_i + v_i^*(x))\delta_{ii}(\rho) & \tilde{v}_i + v_i^*(x) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (\tilde{v}_i + v_i^*(x))\delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (12)$$

and

$$C(\tilde{u}) = \begin{bmatrix} C_{11}(\tilde{u}) & C_{12}(\tilde{u}) & \cdots & C_{1N}(\tilde{u}) \\ C_{21}(\tilde{u}) & C_{22}(\tilde{u}) & \cdots & C_{2N}(\tilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1}(\tilde{u}) & C_{N2}(\tilde{u}) & \cdots & C_{NN}(\tilde{u}) \end{bmatrix}, \quad (13)$$

with

$$C_{ij}(\tilde{u}) = \begin{cases} \begin{bmatrix} \frac{dv_i^*(x)}{dx} & \frac{d\rho_i^*(x)}{dx} \\ \left[\frac{1}{\tau_i} \delta_{ii}(\rho) + v_i^*(x) \sigma_{ii}(\rho) \frac{d\rho_i^*(x)}{dx} \right] \frac{1}{\tau_i} + \frac{dv_i^*(x)}{dx} + \sum_{j=1}^N \delta_{ij}(\rho) \frac{d\rho_j^*(x)}{dx} & \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \left[\frac{1}{\tau_i} \delta_{ij}(\rho) + v_i^*(x) \sigma_{ij}(\rho) \frac{d\rho_j^*(x)}{dx} \right] & 0 \end{bmatrix}, & \text{if } j \neq i. \end{cases} \quad (14)$$

Therein,

$$\delta_{ij}(\rho) = \frac{\partial p_i(Ao)}{\partial \rho_j} = \frac{v_i^M \gamma_i a_j}{Ao_i^M W} \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i - 1},$$

$$\sigma_{ij}(\rho) = \frac{\partial \delta_{ij}(\rho)}{\partial \rho_j} = \frac{v_i^M \gamma_i (\gamma_i - 1) a_j^2}{(Ao_i^M W)^2} \left(\frac{Ao}{Ao_i^M} \right)^{\gamma_i - 2}.$$

Because of the invertibility of $A(\tilde{u})$, i.e., $|A(\tilde{u})| \neq 0$, we transform and linearize the system (8) around the nonuniform steady-state u^* , then for all $t \in [0, +\infty)$,

$x \in (0, L)$, the linearized system is derived with $\rho^*(x) = (\rho_1^*(x), \rho_2^*(x), \dots, \rho_N^*(x))^T$ as follows,

$$\tilde{u}_t(x, t) + F(u^*(x))\tilde{u}_x(x, t) = G(u^*(x))\tilde{u}(x, t), \quad (15)$$

where, for $i, j = 1, 2, \dots, N$,

$$F(u^*(x)) = \begin{bmatrix} F_{11}(u^*(x)) & F_{12}(u^*(x)) & \cdots & F_{1N}(u^*(x)) \\ F_{21}(u^*(x)) & F_{22}(u^*(x)) & \cdots & F_{2N}(u^*(x)) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(u^*(x)) & F_{N2}(u^*(x)) & \cdots & F_{NN}(u^*(x)) \end{bmatrix}, \quad (16)$$

with

$$F_{ij}(u^*(x)) = \begin{cases} \begin{bmatrix} v_i^*(x) & \rho_i^*(x) \\ 0 & v_i^*(x) - \rho_i^*(x)\delta_{ii}(\rho^*(x)) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (v_i^*(x) - v_j^*(x))\delta_{ij}(\rho^*(x)) & -\rho_j^*(x)\delta_{ij}(\rho^*(x)) \end{bmatrix}, & \text{if } j \neq i, \end{cases} \quad (17)$$

and

$$G(u^*(x)) = \begin{bmatrix} G_{11}(u^*(x)) & G_{12}(u^*(x)) & \cdots & G_{1N}(u^*(x)) \\ G_{21}(u^*(x)) & G_{22}(u^*(x)) & \cdots & G_{2N}(u^*(x)) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}(u^*(x)) & G_{N2}(u^*(x)) & \cdots & G_{NN}(u^*(x)) \end{bmatrix}, \quad (18)$$

with (19).

Assume that the system (15) is hyperbolic, for all $u^* \in \mathbb{R}^{2N}([0, L])$, the matrix $F(u^*)$ has $2N$ real distinct eigenvalues different to zero. Given an invertible transformation matrix $T(x) \in \mathcal{M}_{2N, 2N}(\mathbb{R})$ and $2N$ eigenvalues

$$\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_m(x) > 0 > -\lambda_{m+1}(x) > \cdots > -\lambda_{2N}(x), \quad \forall x \in [0, L], \quad (20)$$

of $F(u^*)$ ($\lambda_i \in \mathcal{C}^2([0, L])$, $i = 1, \dots, 2N$, m is the number of positive eigenvalues and $1 \leq m < 2N$), by using the transformation $\omega = T^{-1}(x)\tilde{u}$, the linearized system (15) is rewritten as follows, for $\forall t \in [0, +\infty)$, $x \in (0, L)$,

$$\partial_t \omega(x, t) + \Lambda(x)\partial_x \omega(x, t) = M(x)\omega(x, t), \quad (21)$$

where

$$\Lambda(x) = \text{diag}\{\Lambda^+(x), -\Lambda^-(x)\} \in \mathcal{D}_{2N}, \\ \Lambda^+(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \dots, \lambda_m(x)\} \in \mathcal{D}_m^+,$$

$$\Lambda^-(x) = \text{diag}\{\lambda_{m+1}(x), \lambda_{m+2}(x), \dots, \lambda_{2N}(x)\} \in \mathcal{D}_{2N-m}^+, \\ M(x) = T^{-1}(x)G(u^*(x))T(x) \in \mathcal{M}_{2N, 2N}(\mathbb{R}).$$

Note that the columns of matrix $T(x)$ are the corresponding right eigenvectors of $2N$ eigenvalues, and for $x \in [0, L]$,

$$|\Lambda(x)| = \text{diag}\{\Lambda^+(x), \Lambda^-(x)\},$$

$$\Lambda'(x) = \text{diag}\{\lambda_1'(x), \dots, \lambda_m'(x), -\lambda_{m+1}'(x), \dots, -\lambda_{2N}'(x)\}.$$

In (20), $1 \leq m < 2N$ means that the traffic waves moves against the traffic flow (upstream) in congested traffic due to the reaction of the drivers to their respective leading vehicles, and the value of the derivative of Ao with respect to the spacial variable $x \in (0, L)$ is very high. Therefore, we will study the traffic congested regime in the \mathbf{H}^1 sense in this paper.

2.3 Problem statement

The control problem is motivated by the dissolution of traffic congestion on a road segment with the disturbances at the inlet boundary and the flow restriction at the downstream boundary. For example, the occurrence of traffic congestion is attributed to the excess of the capacity of bottleneck downstream outlet and the high traffic demand (serve as disturbances) upstream inlet on the considered road section.

In order to control the traffic system, we develop a boundary control law to reject disturbances by ramp metering installed at the inlet $x = 0$ of a investigated road segment with a constant density $\rho_i^*(L)$ and a flow restriction at the downstream boundary. The diagram of the control model is illustrated in Figure 1.

We can derive the following equation on the basis of the flow conservation at the upstream inlet $x = 0$,

$$Q_{in}^* + \bar{p}(t) + Q_{rmp}^* + \Theta U(t) = \begin{bmatrix} \rho_1(0, t)v_1(0, t) \\ \rho_2(0, t)v_2(0, t) \\ \vdots \\ \rho_N(0, t)v_N(0, t) \end{bmatrix}, \quad (22)$$

where $Q_{in}^* \in \mathbb{R}^N$ is a constant inflow, and $\bar{p} \in \mathcal{C}^1([0, \infty); \mathbb{R}^N)$ is the unknown disturbances of flow rate serving as an exogenous variable depending on time t . The actuation signal vector $U \in \mathcal{C}^0([0, \infty); \mathbb{R}^{2N-m})$ with a coefficient matrix $\Theta \in \mathcal{M}_{N, 2N-m}(\mathbb{R})$ is implemented by the on-ramp metering at the upstream boundary of the considered road segment. The nominal on-ramp flux rate at the inlet $x = 0$ of road section

$$G_{ij}(u^*(x)) = \begin{cases} \left[\begin{array}{c} \frac{dv_i^*(x)}{dx} \\ \frac{1}{\tau_i} \delta_{ii}(\rho^*(x)) + v_i^*(x) \sigma_{ii}(\rho^*(x)) \frac{d\rho_i^*(x)}{dx} - \delta_{ii}(\rho^*(x)) \frac{dv_i^*(x)}{dx} \frac{1}{\tau_i} + \frac{dv_i^*(x)}{dx} + \sum_{j=1, j \neq i}^N \delta_{ij}(\rho^*(x)) \frac{d\rho_j^*(x)}{dx} \end{array} \right], & \text{if } j = i, \\ \left[\begin{array}{c} 0 \\ \frac{1}{\tau_i} \delta_{ij}(\rho^*(x)) + v_i^*(x) \sigma_{ij}(\rho^*(x)) \frac{d\rho_j^*(x)}{dx} - \delta_{ij}(\rho^*(x)) \frac{dv_j^*(x)}{dx} - \delta_{ij}(\rho^*(x)) \frac{d\rho_j^*(x)}{dx} \end{array} \right], & \text{if } j \neq i. \end{cases} \quad (19)$$

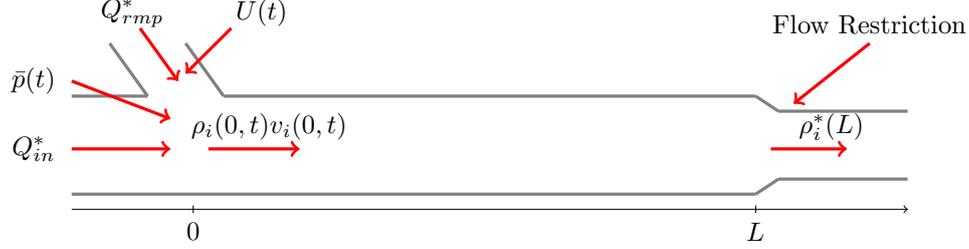


Fig. 1. Multi-type vehicles traffic on a road with disturbances and flow restriction.

$Q_{rmp}^* \in \mathbb{R}^N$ satisfying the relation

$$Q_{in}^* + Q_{rmp}^* = \begin{bmatrix} \rho_1^*(0)v_1^*(0) \\ \rho_2^*(0)v_2^*(0) \\ \vdots \\ \rho_N^*(0)v_N^*(0) \end{bmatrix}. \quad (23)$$

From the boundary condition at $x = L$, by combining control laws (22) with (23) and linearizing, the boundary conditions are derived, for $t \geq 0$,

$$A_1 \tilde{u}(0, t) = \bar{p}(t) + \Theta U(t), \quad (24)$$

$$B_1 \tilde{u}(L, t) = 0, \quad (25)$$

with

$$A_1 = \text{diag}\{[v_1^*(0), \rho_1^*(0)], \dots, [v_N^*(0), \rho_N^*(0)]\} \in \mathcal{M}_{N,2N},$$

$$B_1 = \text{diag}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \in \mathcal{M}_{2N,2N}.$$

For the sake of reducing the congestion and preventing the capacity drop, a controller is designed by using the backstepping approach in this paper. In the next subsection, the Riemann coordinate transformation of the state ω is dealt with in order to make the development and analysis of controller easier.

2.4 Riemann Coordinates Transformation

By the transformation

$$R = \begin{bmatrix} R^+ \\ R^- \end{bmatrix} = \Psi(x)\omega, \quad x \in [0, L], \quad (26)$$

with $\Psi(x) = \text{diag}\{\Psi^+(x), \Psi^-(x)\}$,

$$\Psi^+(x) = \text{diag} \left\{ e^{-\int_0^x \frac{[M(s)]_{1,1}}{\lambda_1(s)} ds}, e^{-\int_0^x \frac{[M(s)]_{2,2}}{\lambda_2(s)} ds}, \dots, e^{-\int_0^x \frac{[M(s)]_{m,m}}{\lambda_m(s)} ds} \right\},$$

$$\Psi^-(x) = \text{diag} \left\{ e^{\int_0^x \frac{[M(s)]_{m+1,m+1}}{\lambda_{m+1}(s)} ds}, e^{\int_0^x \frac{[M(s)]_{m+2,m+2}}{\lambda_{m+2}(s)} ds}, \dots, e^{\int_0^x \frac{[M(s)]_{2N,2N}}{\lambda_{2N}(s)} ds} \right\},$$

from ω to the new variable R with $R^+ : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^m$, $R^- : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^{2N-m}$, we can derive the following system with a simpler source term in which all the diagonal entries of the coefficient matrix are zero,

$$R_t(x, t) + \Lambda(x)R_x(x, t) = \Sigma(x)R(x, t), \quad (27)$$

$$R_{in}(t) = K_P R_{out}(t) + \Gamma_0(\bar{p}(t) + \Theta U(t)), \quad (28)$$

where

$$\begin{aligned} \Sigma(x) &= \begin{bmatrix} \Sigma^{++}(x) & \Sigma^{+-}(x) \\ \Sigma^{-+}(x) & \Sigma^{--}(x) \end{bmatrix}, \\ R_{in}(t) &= \begin{bmatrix} R^+(0, t) \\ R^-(L, t) \end{bmatrix}, \quad R_{out}(t) = \begin{bmatrix} R^+(L, t) \\ R^-(0, t) \end{bmatrix}, \\ K_P &= \begin{bmatrix} 0_{m \times m} & \Gamma_1 \\ \Gamma_3 & 0_{(2N-m) \times (2N-m)} \end{bmatrix}, \\ \Gamma_0 &= \begin{bmatrix} \Gamma_2 \\ 0_{(2N-m) \times N} \end{bmatrix}, \end{aligned}$$

with

$$\begin{aligned} \Sigma^{++}(x) &= \{\epsilon_{ij}(x)\}_{1 \leq i, j \leq m}, \\ \Sigma^{+-}(x) &= \{\epsilon_{ij}(x)\}_{1 \leq i \leq m, m+1 \leq j \leq 2N}, \\ \Sigma^{-+}(x) &= \{\epsilon_{ij}(x)\}_{m+1 \leq i \leq 2N, 1 \leq j \leq m}, \\ \Sigma^{--}(x) &= \{\epsilon_{ij}(x)\}_{m+1 \leq i \leq 2N, m+1 \leq j \leq 2N}, \end{aligned}$$

and $\epsilon_{ij} \in \mathcal{C}^0([0, L])$,

$$\epsilon_{ij}(x) = \begin{cases} 0, & \text{if } j = i, \\ [\Psi(x)]_{i,i} \cdot [M(x)]_{i,j} \cdot [\Psi(x)]_{j,j}^{-1}, & \text{if } j \neq i. \end{cases}$$

Under the assumption that there are matrices $\Upsilon_1 \in \mathcal{M}_{m,N}(\mathbb{R})$ and $\Upsilon_2 \in \mathcal{M}_{2N-m,N}(\mathbb{R})$ such that $\Upsilon_1 A_1 T^+(0) \in \mathcal{M}_{m,m}(\mathbb{R})$ and $\Upsilon_2 A_2 T^-(L) \in \mathcal{M}_{2N-m,2N-m}(\mathbb{R})$ are invertible, we obtain

$$\begin{aligned} \Gamma_1 &= -(\Upsilon_1 A_1 T^+(0))^{-1} \Upsilon_1 A_1 T^-(0), \\ \Gamma_2 &= (\Upsilon_1 A_1 T^+(0))^{-1} \Upsilon_1, \\ \Gamma_3 &= -\Psi^-(L) (\Upsilon_2 A_2 T^-(L))^{-1} \Upsilon_2 A_2 T^+(L) (\Psi^+(L))^{-1}, \end{aligned}$$

with $A_2 = \text{diag}\{[1, 0], \dots, [1, 0]\} \in \mathcal{M}_{N,2N}$,

$$\begin{aligned} T^+(0) &= \{T_{ij}(0)\}_{1 \leq i \leq 2N, 1 \leq j \leq m}, \\ T^-(0) &= \{T_{ij}(0)\}_{1 \leq i \leq 2N, m+1 \leq j \leq 2N}, \\ T^+(L) &= \{T_{ij}(L)\}_{1 \leq i \leq 2N, 1 \leq j \leq m}, \\ T^-(L) &= \{T_{ij}(L)\}_{1 \leq i \leq 2N, m+1 \leq j \leq 2N}. \end{aligned}$$

Since the transformation (26) is invertible, the linearized system in density and velocity has the same stability property as the system R . Inspired by [1], we are now in position to design the controller.

3 Controller Design

3.1 Target system

Consider the backstepping transformations

$$\begin{aligned} Z^+(x, t) &= R^+(x, t), \\ Z^-(x, t) &= R^-(x, t) - \int_x^L G^1(x, \xi) R^+(\xi, t) d\xi \\ &\quad - \int_x^L G^2(x, \xi) R^-(\xi, t) d\xi, \end{aligned} \quad (29)$$

where

$$\begin{aligned} G^1 &= \{G_{ij}^1\}_{1 \leq i \leq 2N-m, 1 \leq j \leq m} \in \mathcal{C}^2(\mathbb{T}_1), \\ G^2 &= \{G_{ij}^2\}_{1 \leq i, j \leq 2N-m} \in \mathcal{C}^2(\mathbb{T}_1), \end{aligned}$$

are kernels defined on the triangular domain $\mathbb{T}_1 = \{(x, \xi) \in \mathbb{R}^2 \mid 0 \leq x \leq \xi \leq L\}$.

The following target system is introduced,

$$\begin{aligned} Z_t(x, t) + \Lambda(x) Z_x(x, t) &= \Sigma_1(x) Z(x, t) \\ &\quad + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi, \end{aligned} \quad (31)$$

$$\dot{X}(t) = K_I Z_{out}(t) + \Gamma_0 \dot{\bar{p}}(t), \quad (32)$$

$$Z_{in}(t) = K_P Z_{out}(t) + X(t), \quad (33)$$

$$X(t) = K_I \int_0^t Z_{out}(\sigma) d\sigma + \Gamma_0 \bar{p}(t), \quad (34)$$

where

$$\begin{aligned} Z(x, t) &= \begin{bmatrix} Z^+(x, t) \\ Z^-(x, t) \end{bmatrix}, \quad \Sigma_1(x) = \begin{bmatrix} \Sigma^{++}(x) & \Sigma^{+-}(x) \\ 0 & 0 \end{bmatrix}, \\ C_1(x, \xi) &= \begin{bmatrix} C^+(x, \xi) & C^-(x, \xi) \\ 0 & 0 \end{bmatrix}, \\ Z_{in}(t) &= \begin{bmatrix} Z^+(0, t) \\ Z^-(L, t) \end{bmatrix}, \quad Z_{out}(t) = \begin{bmatrix} Z^+(L, t) \\ Z^-(0, t) \end{bmatrix}, \\ K_I &= \begin{bmatrix} K_I^{11} & K_I^{12} \\ 0_{(2N-m) \times m} & 0_{(2N-m) \times (2N-m)} \end{bmatrix}, \end{aligned}$$

with $K_I^{11} = \{K_I\}_{1 \leq i, j \leq m}$, $K_I^{12} = \{K_I\}_{1 \leq i \leq m, m+1 \leq j \leq 2N}$. Here C^+ , C^- are given as the solutions to the Volterra integral equations, for all (x, ξ) in \mathbb{T}_1 ,

$$C^+(x, \xi) = \Sigma^{+-}(x) G^1(x, \xi) + \int_x^\xi C^-(x, s) G^1(s, \xi) ds, \quad (35)$$

$$C^-(x, \xi) = \Sigma^{+-}(x)G^2(x, \xi) + \int_x^\xi C^-(x, s)G^2(s, \xi) ds. \quad (36)$$

The system (31)-(34) is considered under the initial conditions,

$$Z(\cdot, 0) = Z_0(\cdot) = \begin{bmatrix} Z_0^+(\cdot) \\ Z_0^-(\cdot) \end{bmatrix}, \quad (37)$$

$$X(0) = X_0 = \Gamma_0 \bar{p}(0) \in \mathbb{R}^{2N}. \quad (38)$$

The exponential stability for the \mathbf{H}^1 -norm of target system (31)-(34) is as follows.

Theorem 1 *The steady-state $Z(x, t) \equiv \mathbf{0}$ of the system (31)-(34) is integral input-to-state stable for the \mathbf{H}^1 -norm if there exist positive constants α, q_1, q_2, q_3 , diagonal positive-definite matrices $P_1, P_4 \in \mathbb{R}^{2N \times 2N}$, a symmetric positive-definite matrix $P_2 \in \mathbb{R}^{2N \times 2N}$ and a matrix $P_3 \in \mathbb{R}^{2N \times 2N}$ such that the following matrix inequalities hold, for all $x \in [0, L]$,*

(i)

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12} & \Omega_{13}(x) & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix} \geq 0, \quad (39)$$

where

$$\begin{aligned} \Omega_{11}(x) &= -\Lambda'(x)P_1 - \alpha P_1 - \left[\Sigma_1^\top(x)P_1 + P_1\Sigma_1(x) \right. \\ &\quad \left. + q_1 L\nu_1^2 I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2} \right) C_1^\top(0, x)C_1(0, x) \right], \\ \Omega_{12} &= -P_3 K_I, \\ \Omega_{13}(x) &= -\Lambda'(x)P_3 - \alpha P_3 - \Sigma_1^\top(x)P_3, \\ \Omega_{14} &= 0, \\ \Omega_{22} &= \frac{1}{L}E_2 P_1 - \frac{1}{L}K_P^\top E_1 P_1 K_P - \frac{1}{L}K_I^\top E_1 P_4 K_I, \\ \Omega_{23} &= -\frac{1}{L}K_P^\top E_1 P_1 - \frac{1}{L}(K_P^\top M_1 + M_2) - K_I^\top P_2, \\ \Omega_{24} &= -\frac{1}{L}K_I^\top E_1 P_4 K_P, \\ \Omega_{33} &= -\frac{1}{L}E_1 P_1 - \frac{1}{L}(M_1 + M_1^\top) - \alpha P_3 \\ &\quad - q_2 L\nu_2^2 I_{2N}, \\ \Omega_{34} &= 0, \\ \Omega_{44} &= \frac{1}{L}E_2 P_4 - \frac{1}{L}K_P^\top E_1 P_4 K_P, \end{aligned}$$

with

$$M_1 = \begin{bmatrix} \Lambda^+(0)P_3^{++} & \Lambda^+(0)P_3^{+-} \\ -\Lambda^-(L)P_3^{-+} & -\Lambda^-(L)P_3^{--} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -\Lambda^+(L)P_3^{++} & -\Lambda^+(L)P_3^{+-} \\ \Lambda^-(0)P_3^{-+} & \Lambda^-(0)P_3^{--} \end{bmatrix},$$

$$P_3^{++} = \{P_3\}_{1 \leq i, j \leq m},$$

$$P_3^{+-} = \{P_3\}_{1 \leq i \leq m, m+1 \leq j \leq 2N},$$

$$P_3^{-+} = \{P_3\}_{m+1 \leq i \leq 2N, 1 \leq j \leq m},$$

$$P_3^{--} = \{P_3\}_{m+1 \leq i \leq 2N, m+1 \leq j \leq 2N},$$

$$E_1 = \text{diag}\{\Lambda^+(0), \Lambda^-(L)\}, E_2 = \text{diag}\{\Lambda^+(L), \Lambda^-(0)\},$$

$$\nu_1 = \max(\lambda(P_1)), \nu_2 = \max(|\lambda(P_3)|),$$

(ii)

$$\begin{aligned} M(x) &= (-\Lambda'(x) - \alpha I_{2N})P_4 - \left[\Sigma_1^\top(x)P_4 + P_4\Sigma_1(x) \right. \\ &\quad \left. + q_3 L\nu_3^2 I_{2N} + \frac{L}{q_3} C_1^\top(0, x)C_1(0, x) \right] \geq 0, \quad (40) \end{aligned}$$

with $\nu_3 = \max(\lambda(P_4))$.

In other words, there exist positive constants b_1, c_1 such that, for every $Z_0 \in \mathbf{H}^1((0, L); \mathbb{R}^{2N})$, $X_0 \in \mathbb{R}^{2N}$, and for any \bar{p} such that $\dot{\bar{p}} \in \mathbf{L}^1[0, \infty)$, the solution $Z \in C^0([0, +\infty); \mathbf{H}^1((0, L); \mathbb{R}^{2N}))$, $X \in C^0([0, +\infty); \mathbb{R}^{2N})$ to the Cauchy problem (31)-(34), (37)-(38) is defined on $[0, +\infty) \times [0, L]$ and satisfies

$$\begin{aligned} &\|Z(\cdot, t)\|_{\mathbf{H}^1((0, L); \mathbb{R}^{2N})}^2 + \|X(t)\|^2 \\ &\leq c_1 e^{-\alpha t} \left(\|Z_0\|_{\mathbf{H}^1((0, L); \mathbb{R}^{2N})}^2 + \|X_0\|^2 \right) \\ &\quad + b_1 \int_0^t \dot{\bar{p}}^\top(s)\dot{\bar{p}}(s) ds, \quad \forall t \in [0, +\infty). \quad (41) \end{aligned}$$

Proof. The following \mathbf{H}^1 Lyapunov function candidate is introduced for the stability analysis of system (31)-(34), for $\forall t \in [0, +\infty)$,

$$V(Z(x, t), X(t), Z_t(x, t)) = V_1 + V_2 + V_3 + V_4, \quad (42)$$

where

$$V_1 = \int_0^L [Z^\top(x, \cdot)\mathcal{P}_1(x)Z(x, \cdot)] dx, \quad (43)$$

$$V_2 = \int_0^L [Z^\top(x, \cdot)\mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x)Z(x, \cdot)] dx, \quad (44)$$

$$V_3 = LX^\top(\cdot)P_2X(\cdot), \quad (45)$$

$$V_4 = \int_0^L [Z_t^\top(x, \cdot) \mathcal{P}_4(x) Z_t(x, \cdot)] dx, \quad (46)$$

and

$$\begin{aligned} \mathcal{P}_1(x) &\triangleq P_1 \text{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\}, \\ \mathcal{P}_3(x) &\triangleq P_3 \text{diag} \left\{ e^{-\frac{\mu}{2} x} I_m, e^{\frac{\mu}{2} x} I_{2N-m} \right\}, \\ \mathcal{P}_4(x) &\triangleq P_4 \text{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\}, \end{aligned}$$

where, by definition, the notation Z_t must be understood as

$$\begin{aligned} Z_t(x, t) &\triangleq -\Lambda(x) Z_x(x, t) + \Sigma_1(x) Z(x, t) \\ &\quad + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi. \end{aligned}$$

Under the definition of V and straightforward estimations, there exists a positive real constant β such that, for every Z , we can obtain the following inequality,

$$\begin{aligned} &\frac{1}{\beta} \int_0^L (|Z(x, \cdot)|^2 + |X(\cdot)|^2 + |Z_x(x, \cdot)|^2) dx \\ &\leq V \\ &\leq \beta \int_0^L (|Z(x, \cdot)|^2 + |X(\cdot)|^2 + |Z_x(x, \cdot)|^2) dx. \quad (47) \end{aligned}$$

By time differentiation of (31) and (33), Z_t can be shown to satisfy the following equations, for all $x \in [0, L]$, $t \in [0, +\infty)$,

$$\begin{aligned} Z_{tt}(x, t) &= -\Lambda(x) Z_{tx}(x, t) + \Sigma_1(x) Z_t(x, t) \\ &\quad + \int_x^L C_1(x, \xi) Z_t(\xi, t) d\xi, \quad (48) \end{aligned}$$

$$\dot{Z}_{in}(t) = K_P \dot{Z}_{out}(t) + \dot{X}(t). \quad (49)$$

Taking time derivative of V_1 along the solutions to (31)-(34) and using integrations by parts, the following result is achieved,

$$\begin{aligned} \dot{V}_1 &\leq Z_{out}^\top(t) [K_P^\top \bar{E}_1 P_1 K_P - e^{-\mu L} \bar{E}_2 P_1] Z_{out}(t) \\ &\quad + Z_{out}^\top(t) K_P^\top \bar{E}_1 P_1 X(t) + X^\top(t) P_1 \bar{E}_1 K_P Z_{out}(t) \\ &\quad + X^\top(t) \bar{E}_1 P_1 X(t) \\ &\quad + \int_0^L Z^\top(x, t) [\Lambda'(x) \mathcal{P}_1(x) - \mu |\Lambda(x)| \mathcal{P}_1(x)] Z(x, t) dx. \\ &\quad + \int_0^L \left[\left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right)^\top \right. \\ &\quad \left. P_1(x) Z(x, t) + Z^\top(x, t) \mathcal{P}_1(x) \right. \\ &\quad \left. \left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right) \right] dx. \quad (50) \end{aligned}$$

with

$$\begin{aligned} \bar{E}_1 &= \text{diag} \{ \Lambda^+(0), e^{\mu L} \Lambda^-(L) \}, \\ \bar{E}_2 &= \text{diag} \{ \Lambda^+(L), e^{\mu L} \Lambda^-(0) \}. \end{aligned}$$

By taking time derivative of V_2 along the solutions to (31)-(34) and using integrations by parts, we get

$$\begin{aligned} \dot{V}_2 &\leq Z_{out}^\top(t) [K_P^\top \bar{M}_1 + \bar{M}_2] X(t) + X^\top(t) \bar{M}_1 X(t) \\ &\quad + X^\top(t) [\bar{M}_1^\top K_P + \bar{M}_2^\top] Z_{out}(t) + X^\top(t) \bar{M}_1^\top X(t) \\ &\quad + \int_0^L Z^\top(x, t) \left[\Lambda'(x) \mathcal{P}_3(x) - \frac{\mu}{2} |\Lambda(x)| \mathcal{P}_3(x) \right] X(t) dx \\ &\quad + \int_0^L X^\top(t) \left[-\frac{\mu}{2} \mathcal{P}_3^\top(x) |\Lambda(x)| + \mathcal{P}_3^\top(x) \Lambda'(x) \right] Z(x, t) dx \\ &\quad + \int_0^L [Z^\top(x, t) \mathcal{P}_3(x) K_I Z_{out}(t) \\ &\quad + Z_{out}^\top(t) K_I^\top \mathcal{P}_3^\top(x) Z(x, t)] dx \\ &\quad + \kappa_1 \int_0^L Z^\top(x, t) \mathcal{P}_3(x) \Gamma_0 (Z^\top(x, t) \mathcal{P}_3(x) \Gamma_0)^\top dx \\ &\quad + \frac{L}{\kappa_1} \dot{\bar{p}}^\top(t) \dot{\bar{p}}(t) \\ &\quad + \int_0^L \left[\left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right)^\top \right. \\ &\quad \left. \mathcal{P}_3(x) X(t) + X^\top(t) \mathcal{P}_3^\top(x) \right. \\ &\quad \left. \left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right) \right] dx, \quad (51) \end{aligned}$$

with a positive constant κ_1 and

$$\begin{aligned} \bar{M}_1 &= \begin{bmatrix} \Lambda^+(0) P_3^{++} & \Lambda^+(0) P_3^{+-} \\ -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_3^{-+} & -e^{-\frac{\mu}{2} L} \Lambda^-(L) P_3^{--} \end{bmatrix}, \\ \bar{M}_2 &= \begin{bmatrix} -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_3^{++} & -e^{-\frac{\mu}{2} L} \Lambda^+(L) P_3^{+-} \\ \Lambda^-(0) P_3^{-+} & \Lambda^-(0) P_3^{--} \end{bmatrix}. \end{aligned}$$

By taking time derivative of V_3 along the solutions to (32), we can derive the following result with a positive constant κ_2 ,

$$\begin{aligned} \dot{V}_3 &\leq L Z_{out}^\top(t) K_I^\top P_2 X(t) + L X^\top(t) P_2 K_I Z_{out}(t) \\ &\quad + L \kappa_2 X^\top(t) P_2 \Gamma_0 (X^\top(t) P_2 \Gamma_0)^\top + \frac{L}{\kappa_2} \dot{\bar{p}}^\top(t) \dot{\bar{p}}(t). \quad (52) \end{aligned}$$

Taking time derivative of V_4 along the solutions to (31)-(34), (48) and using integrations by parts, we get

$$\begin{aligned} \dot{V}_4 &\leq \dot{Z}_{out}^\top(t) [K_P^\top \bar{E}_1 P_4 K_P - e^{-\mu L} \bar{E}_2 P_4] \dot{Z}_{out}(t) \\ &\quad + \dot{Z}_{out}^\top(t) K_P^\top P_4 \bar{E}_1 K_I Z_{out}(t) \\ &\quad + Z_{out}^\top(t) K_I^\top \bar{E}_1 P_4 K_P \dot{Z}_{out}(t) \end{aligned}$$

$$\begin{aligned}
& + Z_{out}^\top(t) K_I^\top \bar{E}_1 P_4 K_I Z_{out}(t) + \frac{1}{\kappa_3} \dot{\bar{p}}^\top(t) \dot{\bar{p}}(t) \\
& + \kappa_3 \dot{Z}_{out}^\top(t) K_P^\top \bar{E}_1 P_4 \Gamma_0 (K_P^\top \bar{E}_1 P_4 \Gamma_0)^\top \dot{Z}_{out}(t) \\
& + \kappa_4 Z_{out}^\top(t) K_I^\top \bar{E}_1 P_4 \Gamma_0 (K_I^\top \bar{E}_1 P_4 \Gamma_0)^\top Z_{out}(t) \\
& + \frac{1}{\kappa_4} \dot{\bar{p}}^\top(t) \dot{\bar{p}}(t) + \dot{\bar{p}}(t)^\top \Gamma_0^\top \bar{E}_1 P_4 \Gamma_0 \dot{\bar{p}}(t) \\
& + \int_0^L Z_t^\top(x, t) [\Lambda'(x) \mathcal{P}_4(x) - \mu |\Lambda(x)| \mathcal{P}_4(x)] Z_t(x, t) dx \\
& + \int_0^L \left[\left(\Sigma_1(x) Z_t(x, t) + \int_x^L C_1(x, \xi) Z_t(\xi, t) d\xi \right)^\top \right. \\
& \mathcal{P}_4(x) Z_t(x, t) + Z_t^\top(x, t) \mathcal{P}_4(x) \left. \left(\Sigma_1(x) Z_t(x, t) \right. \right. \\
& \left. \left. + \int_x^L C_1(x, \xi) Z_t(\xi, t) d\xi \right) \right] dx, \tag{53}
\end{aligned}$$

with positive constants κ_3 and κ_4 .

The three rightmost integrals in (50), (51) and (53) are considered individually,

$$\begin{aligned}
& \int_0^L \left[\left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right)^\top \right. \\
& \mathcal{P}_1(x) Z(x, t) + Z^\top(x, t) \mathcal{P}_1(x) \\
& \left. \left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right) \right] dx \\
& \leq \int_0^L \left[\left(\Sigma_1(x) Z(x, t) \right)^\top \mathcal{P}_1(x) Z(x, t) \right. \\
& \left. + Z^\top(x, t) \mathcal{P}_1(x) \left(\Sigma_1(x) Z(x, t) \right) \right] dx \\
& + q_1 L e^{2\mu L} \nu_1^2 \int_0^L Z^\top(x, t) Z(x, t) dx \\
& + \frac{L}{q_1} \int_0^L (C_1(0, x) Z(x, t))^\top (C_1(0, x) Z(x, t)) dx. \tag{54}
\end{aligned}$$

Similarly, we derive the inequalities for the other two integrals,

$$\begin{aligned}
& \int_0^L \left[\left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right)^\top \right. \\
& \mathcal{P}_3(x) X(t) + X^\top(t) \mathcal{P}_3^\top(x) \\
& \left. \left(\Sigma_1(x) Z(x, t) + \int_x^L C_1(x, \xi) Z(\xi, t) d\xi \right) \right] dx \\
& \leq \int_0^L \left[\left(\Sigma_1(x) Z(x, t) \right)^\top \mathcal{P}_3(x) X(t) \right. \\
& \left. + X^\top(t) \mathcal{P}_3^\top(x) \left(\Sigma_1(x) Z(x, t) \right) \right] dx \\
& + q_2 L e^{\mu L} \nu_2^2 \int_0^L X^\top(t) X(t) dx
\end{aligned}$$

$$+ \frac{L}{q_2} \int_0^L (C_1(0, x) Z(x, t))^\top (C_1(0, x) Z(x, t)) dx, \tag{55}$$

$$\begin{aligned}
& \int_0^L \left[\left(\Sigma_1(x) Z_t(x, t) + \int_x^L C_1(x, \xi) Z_t(\xi, t) d\xi \right)^\top \right. \\
& \mathcal{P}_4(x) Z_t(x, t) + Z_t^\top(x, t) \mathcal{P}_4(x) \\
& \left. \left(\Sigma_1(x) Z_t(x, t) + \int_x^L C_1(x, \xi) Z_t(\xi, t) d\xi \right) \right] dx \\
& \leq \int_0^L \left[\left(\Sigma_1(x) Z_t(x, t) \right)^\top \mathcal{P}_4(x) Z_t(x, t) \right. \\
& \left. + Z_t^\top(x, t) \mathcal{P}_4(x) \left(\Sigma_1(x) Z_t(x, t) \right) \right] dx \\
& + q_3 L e^{2\mu L} \nu_3^2 \int_0^L Z_t^\top(x, t) Z_t(x, t) dx \\
& + \frac{L}{q_3} \int_0^L (C_1(0, x) Z_t(x, t))^\top (C_1(0, x) Z_t(x, t)) dx. \tag{56}
\end{aligned}$$

Using (50)-(56), there exists a constant $\alpha > 0$ such that, for all $t \geq 0$,

$$\begin{aligned}
\dot{V} & = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 \\
& \leq -\alpha V - \int_0^L \begin{bmatrix} Z(x, \cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix}^\top \bar{\Omega}(x) \begin{bmatrix} Z(x, \cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix} dx \\
& - \int_0^L Z_t^\top(x, t) \bar{M}(x) Z_t(x, t) dx \\
& + \dot{\bar{p}}^\top(t) \left[\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \Gamma_0^\top \bar{E}_1 P_4 \Gamma_0 \right] \dot{\bar{p}}(t), \tag{57}
\end{aligned}$$

where

$$\bar{\Omega}(x) = \begin{bmatrix} \bar{\Omega}_{11}(x) & \bar{\Omega}_{12}(x) & \bar{\Omega}_{13}(x) & \bar{\Omega}_{14} \\ * & \bar{\Omega}_{22} & \bar{\Omega}_{23} & \bar{\Omega}_{24} \\ * & * & \bar{\Omega}_{33}(x) & \bar{\Omega}_{34} \\ * & * & * & \bar{\Omega}_{44} \end{bmatrix} \tag{58}$$

with

$$\begin{aligned}
\bar{\Omega}_{11}(x) & = \mu |\Lambda(x)| \mathcal{P}_1(x) - \Lambda'(x) \mathcal{P}_1(x) - \alpha \mathcal{P}_1(x) \\
& - \kappa_1 \mathcal{P}_3(x) \Gamma_0 (\mathcal{P}_3(x) \Gamma_0)^\top - \left[\Sigma_1^\top(x) \mathcal{P}_1(x) \right. \\
& \left. + \mathcal{P}_1(x) \Sigma_1(x) + q_1 L e^{2\mu L} \nu_1^2 I_{2N} \right. \\
& \left. + \left(\frac{L}{q_1} + \frac{L}{q_2} \right) C_1^\top(0, x) C_1(0, x) \right], \\
\bar{\Omega}_{12}(x) & = -\mathcal{P}_3(x) K_I,
\end{aligned}$$

$$\begin{aligned}
\bar{\Omega}_{13}(x) &= \frac{\mu}{2} |\Lambda(x)| \mathcal{P}_3(x) - \Lambda'(x) \mathcal{P}_3(x) - \alpha \mathcal{P}_3(x) \\
&\quad - \Sigma_1^\top(x) \mathcal{P}_3(x), \\
\bar{\Omega}_{14} &= 0, \\
\bar{\Omega}_{22} &= \frac{e^{-\mu L}}{L} \bar{E}_2 P_1 - \frac{1}{L} K_P^\top \bar{E}_1 P_1 K_P - \frac{1}{L} K_I^\top \bar{E}_1 P_4 K_I \\
&\quad - \frac{\kappa_4}{L} K_I^\top \bar{E}_1 P_4 \Gamma_0 (K_I^\top \bar{E}_1 P_4 \Gamma_0)^\top, \\
\bar{\Omega}_{23} &= -\frac{1}{L} K_P^\top \bar{E}_1 P_1 - \frac{1}{L} (K_P^\top \bar{M}_1 + \bar{M}_2) - K_I^\top P_2, \\
\bar{\Omega}_{24} &= -\frac{1}{L} K_I^\top \bar{E}_1 P_4 K_P, \\
\bar{\Omega}_{33}(x) &= -\frac{1}{L} \bar{E}_1 P_1 - \frac{1}{L} (\bar{M}_1 + \bar{M}_1^\top) \\
&\quad - \kappa_2 P_2 \Gamma_0 (P_2 \Gamma_0)^\top - \alpha P_2 - q_2 L e^{\mu L} \nu_2^2 I_{2N}, \\
\bar{\Omega}_{34} &= 0, \\
\bar{\Omega}_{44} &= \frac{e^{-\mu L}}{L} \bar{E}_2 P_4 - \frac{1}{L} K_P^\top \bar{E}_1 P_4 K_P \\
&\quad - \frac{\kappa_3}{L} K_P^\top \bar{E}_1 P_4 \Gamma_0 (K_P^\top \bar{E}_1 P_4 \Gamma_0)^\top,
\end{aligned}$$

and

$$\begin{aligned}
\bar{M}(x) &= (-\Lambda'(x) + \mu |\Lambda(x)| - \alpha I_{2N}) \mathcal{P}_4(x) \\
&\quad - \left[\Sigma_1^\top(x) \mathcal{P}_4(x) + \mathcal{P}_4(x) \Sigma_1(x) \right. \\
&\quad \left. + q_3 L e^{2\mu L} \nu_3^2 I_{2N} + \frac{L}{q_3} C_1^\top(0, x) C_1(0, x) \right]. \quad (59)
\end{aligned}$$

Under the conditions (39), (40), $\exists \mu, \kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$ small enough, such that $\bar{\Omega}(x) \geq 0$ and $\bar{M}(x) \geq 0$, thus

$$\dot{V} \leq -\alpha V + \alpha_1 \dot{\bar{p}}^\top(t) \dot{\bar{p}}(t), \quad (60)$$

with $\alpha_1 = \max \left(\lambda \left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4} \right) I_{2N} + \Gamma_0^\top P_4 \bar{E}_1 \Gamma_0 \right) \right)$. Thus along the solutions to the system (31)-(34),

$$V \leq V(0) e^{-\alpha t} + \alpha_1 \int_0^t \dot{\bar{p}}^\top(s) \dot{\bar{p}}(s) ds. \quad (61)$$

Combining this relation with (47), there exist positive constants $c_1 = \beta^2$, $b_1 = \beta \alpha_1$ such that, for all $t \geq 0$,

$$\begin{aligned}
&\int_0^L (|Z(x, t)|^2 + |X(t)|^2 + |Z_x(x, t)|^2) dx \\
&\leq c_1 e^{-\alpha t} \left(\int_0^L (|Z_0(x)|^2 + |X_0|^2 + |Z_x(x, 0)|^2) dx \right) \\
&\quad + b_1 \int_0^t \dot{\bar{p}}^\top(s) \dot{\bar{p}}(s) ds, \quad (62)
\end{aligned}$$

completing the proof of Theorem 1. \square

By applying [1] (Theorem D.6), the transformations defined in (29) and (30) are in $\mathcal{C}^2(\mathbb{T}_1)$. Moreover, differentiating these transformations with respect to x , and applying Theorem 1.2 in [1], it can be shown that the \mathbf{H}^1 norm of Z is equivalent to the \mathbf{H}^1 norm of R . Thus, the exponential stability of the \mathbf{H}^1 norm of Z system implies the corresponding one for the \mathbf{H}^1 norm of R system.

3.2 Control law

Take time derivative and spatial derivative on (29) and (30), and substitute them into (31)-(34) to get the following equations of the kernels $G^1(x, \xi)$ and $G^2(x, \xi)$,

$$\begin{aligned}
&\Lambda^-(x) G_x^1(x, \xi) - G_\xi^1(x, \xi) \Lambda^+(\xi) \\
&= G^1(x, \xi) [(\Lambda^+) '(\xi) + \Sigma^{++}(\xi)] + G^2(x, \xi) \Sigma^{-+}(\xi), \quad (63)
\end{aligned}$$

$$\begin{aligned}
&\Lambda^-(x) G_x^2(x, \xi) + G_\xi^2(x, \xi) \Lambda^-(\xi) \\
&= G^2(x, \xi) [-(\Lambda^-)'(\xi) + \Sigma^{--}(\xi)] + G^1(x, \xi) \Sigma^{+-}(\xi), \quad (64)
\end{aligned}$$

with the boundary conditions

$$G^1(x, x) \Lambda^+(x) + \Lambda^-(x) G^1(x, x) = \Sigma^{-+}(x), \quad (65)$$

$$G^2(x, x) \Lambda^-(x) - \Lambda^-(x) G^2(x, x) = -\Sigma^{--}(x), \quad (66)$$

$$G^1(x, L) \Lambda^+(L) - G^2(x, L) \Lambda^-(L) \Gamma_3 = K_1(x), \quad (67)$$

and $K_1(x)$ is a vector in the form $K_1(x) = \{k_1^j(x)\}_{1 \leq j \leq m}$. These equations are under-determined, and to ensure the well-posedness, the additional boundary conditions are added,

$$G_{ij}^2(0, \xi) = g_{ij}^2(\xi), \quad 1 \leq j < i \leq 2N - m, \quad (68)$$

for some arbitrary functions $g_{ij}^2, 1 \leq j < i \leq 2N - m$.

The wellposedness of solution to the kernel equations (63)-(68) follows from a coordinate change $(x, \xi) \mapsto (L - x, L - \xi)$ and an application of Theorem D.6 in [1] in the triangular domain $\mathbb{T}_0 = \{(L - x, L - \xi) \in \mathbb{R}^2 \mid 0 \leq L - \xi \leq L - x \leq L\}$.

Assume that there is a matrix $\bar{\Theta} \in \mathcal{M}_{2N-m, m}$ such that $\bar{\Theta} \Gamma_2 \bar{\Theta}$ is invertible, then we deduce, from (28), (29), (30), (33), (34), the following controller defined as, $\forall t \in [0, +\infty)$,

$$\begin{aligned}
U(t) &= (\bar{\Theta} \Gamma_2 \bar{\Theta})^{-1} \bar{\Theta} \int_0^t (K_I^{11} R^+(L, \sigma) + K_I^{12} R^-(0, \sigma)) d\sigma \\
&\quad - (\bar{\Theta} \Gamma_2 \bar{\Theta})^{-1} \bar{\Theta} K_I^{12} \int_0^t \int_0^L [G^1(0, \xi) R^+(\xi, \sigma) \\
&\quad \quad \quad + G^2(0, \xi) R^-(\xi, \sigma)] d\xi d\sigma
\end{aligned}$$

$$\begin{aligned}
& - (\bar{\Theta}\Gamma_2\Theta)^{-1}\bar{\Theta}\Gamma_1 \int_0^L \left[G^1(0, \xi)R^+(\xi, t) \right. \\
& \quad \left. + G^2(0, \xi)R^-(\xi, t) \right] d\xi. \tag{69}
\end{aligned}$$

Under the assumptions of Theorem 1, the target system (31)-(34) is integral input-to-state stable. Thus, using the invertibility of backstepping transformation, the original system (27), (28) is integral input-to-state stable in the \mathbf{H}^1 -norm with the control law (69).

4 Optimal controller and numerical studies

In this section, an optimization problem is presented to obtain the optimal values of parameters of the designed controller. According to the analysis of the parameters of controller, conclude that the initial values of density of all the vehicle classes on the considered road segment are the key parameters to determine the values of controller parameters and to compute the area occupancy. The experiment is set and the results of computation are presented and discussed.

4.1 Optimal controller

From (3), we note that Ao depends on the density vector ρ . The higher value of $\|\rho\|_{\mathbf{L}^\infty((0,L);\mathbb{R}^N)}$ is, the higher value of $\|Ao\|_{\mathbf{L}^\infty((0,L);\mathbb{R})}$ which implies the highest road occupancy on the road segment under consideration is, then the traffic congestion is more possible to happen. In order to minimize the likelihood of congested traffic, we set the following optimization problem to derive the optimal control law $U(t)$,

$$\begin{aligned}
& \min_{\Gamma_1, \Gamma_2, K_I^{11}, K_I^{12}} \|Ao\|_{\mathbf{L}^\infty((0,L);\mathbb{R})} \\
& \text{subject to (39) and (40)}. \tag{70}
\end{aligned}$$

From (69), the value of U depends on the parameters $\Gamma_1, \Gamma_2, K_I^{11}, K_I^{12}$, and the kernels $G^1(0, \xi), G^2(0, \xi), \xi \in [0, L]$. The values of them for the optimal controller are obtained by solving the optimization problem (70).

The parameters $\Gamma_1, K_I^{11}, K_I^{12}$ are respectively included in the proportional and integral tuning matrices K_P and K_I , which have effects on the iISS of the target system Z in the \mathbf{H}^1 sense in terms of the matrix inequalities. The parameter Γ_2 is contained in the proportional coefficient matrix Γ_0 of the disturbances, moreover, Γ_0 has effects on the coefficient of integral term of iISS. Notice that the value of $u^*(0)$ determines Γ_1 and Γ_2 , and due to $\rho_i^*(x)v_i^*(x) = d_i, i = 1, 2, \dots, N, x \in (0, L), \rho_i^*(0)$ is key to the values of Γ_1 and Γ_2 . Similarly, $\rho_i^*(x)$ is key to the values of kernels $G^1(0, \xi), G^2(0, \xi), \xi \in [0, L]$.

The spatial variable x is discretized on the domain $[0, L]$. The values of $\rho_i^*(x), v_i^*(x), (i = 1, 2, \dots, N)$ are de-

rived by solving the ODEs (6)-(7), given the initial values $\rho_i^*(0), v_i^*(0)$ through line search methods. The values of K_I^{11}, K_I^{12} are given and tuned, the values of Γ_1 and Γ_2 are computed and the variables P_1, P_2, P_3, P_4 are derived by solving linear matrix inequalities (LMIs).

4.2 Numerical studies

For numerical computation, the traffic parameters of two vehicle classes on a considered road section in the congested regime are chosen as in [7], see Table 1. The steady-state velocities are determined by

Name	Symbol	Value	Unit
Number of vehicle class	N	2	1
Relaxation time	τ_1	30	s
	τ_2	60	s
Pressure exponent	γ_1	2.5	1
	γ_2	2	1
Free-flow velocity	v_1^M	80	$\frac{km}{h}$
	v_2^M	60	$\frac{km}{h}$
Maximum AO	Ao_1^M	0.9	1
	Ao_2^M	0.85	1
Occupied surface per vehicle	a_1	10	m^2
	a_2	42	m^2
steady-state density at the inlet	$\rho_1^*(0)$	110	$\frac{veh}{km}$
	$\rho_2^*(0)$	70	$\frac{veh}{km}$
steady-state velocity at the inlet	$v_1^*(0)$	50	$\frac{km}{h}$
	$v_2^*(0)$	25	$\frac{km}{h}$
Road width	W	6.5	m
Road length	L	1	km
Number of grid points	N_x	40	1

Table 1

Selected values of parameters.

the choice of the steady-state density at the same location on the considered road. By solving the optimization problem (70), the values of steady-state density and steady-state velocity of two classes are derived as shown in Table 1. Then the steady-state $u^*(x) = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x))^T, x \in [0, 1]$ is derived by solving the ordinary differential equations (6)-(7) with the initial value $\rho^*(0) = (\rho_1^*(0), \rho_2^*(0))^T$, see Figure 2. The relationships $a_1 < a_2, \tau_1 < \tau_2$ in Table 1 and $v_1^* > v_2^*$ in Figure 2 illustrate that, class 1 represents small and fast vehicles, and class 2 describes big and slow vehicles. Given

$$K_I^{11} = \begin{bmatrix} -20 & 30 & 30 \\ -24 & -7 & 26 \\ -10 & 20 & -30 \end{bmatrix} \times 10^{-5}, \quad K_I^{12} = \begin{bmatrix} 60 \\ 30 \\ 20 \end{bmatrix} \times 10^{-5},$$

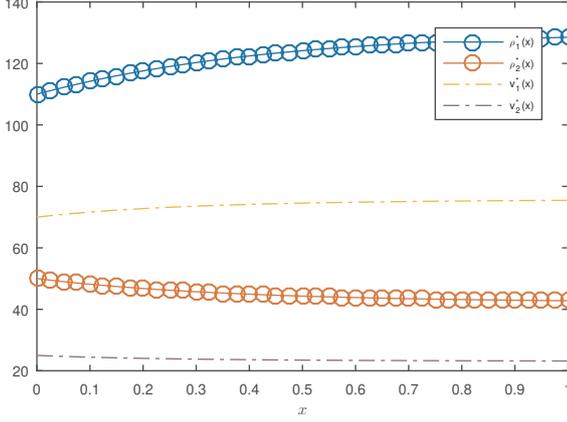


Fig. 2. Relation between spatial variable x and the nonuniform steady-state $u^*(x) = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x))^T$.

we get

$$\Gamma_1 = \begin{bmatrix} -0.785 \\ 1.0467 \\ -4.2039 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} 0 & 0.0469 \\ 0 & -0.0625 \\ 0.0332 & 0.2051 \end{bmatrix},$$

$$P_1 = \text{diag} \{2.1347, 2.6029, 4.8043, 2.5084\} \times 10^3,$$

$$P_2 = \begin{bmatrix} 8.8861 & 0 & 0 & 0 \\ * & 8.8862 & 0 & 0 \\ * & * & 8.8861 & 0 \\ * & * & * & 8.8861 \end{bmatrix} \times 10^3,$$

$$P_3 = \begin{bmatrix} -13.2307 & -0.0486 & -0.0275 & -0.0458 \\ 0.0555 & -15.9336 & 0.0023 & -0.2843 \\ 0.0550 & -0.0041 & -29.1363 & 1.0618 \\ -0.0411 & -0.2236 & 0.4766 & 14.8374 \end{bmatrix},$$

$$P_4 = \text{diag} \{2.3839, 2.7027, 4.2496, 1.5339\} \times 10^3.$$

The conditions of Theorem 1 are satisfied, thus the iISS of the target system and the original system are valid. In addition, the optimal controller exists.

5 Conclusion

The robust control problem was studied to stabilize the multi-type linearized AR traffic flow system. A controller was designed by using backstepping and the existence of the optimal controller was validated by solving the optimization problem.

Inspired by [8], the \mathbf{H}^1 locally iISS and state estimation problem will be studied for the quasilinear system in the future research. It would be of interest to solve this analogous problem by using a more complicated backstepping transformations to simplify the target system.

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