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1                   **EXAMPLES OF FINITE TIME BLOW UP IN MASS**  
2                   **DISSIPATIVE REACTION-DIFFUSION SYSTEMS WITH**  
3                   **SUPERQUADRATIC GROWTH**

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ABSTRACT. We provide explicit examples of finite time  $L^\infty$ -blow up for the solutions of  $2 \times 2$  reaction-diffusion systems for which three main properties hold: positivity is preserved for all time, the total mass is uniformly controlled and the growth of the nonlinear reaction terms is superquadratic. They are obtained by choosing the space dimension large enough. This is to be compared with recent global existence results of uniformly bounded solutions for the same kind of systems with quadratic or even slightly superquadratic growth depending on the dimension. Such blow up may occur even with homogeneous Neumann boundary conditions. All these  $L^\infty$ -blowing up solutions may be extended as weak global solutions. Blow up examples are also provided in space dimensions one, two and three with various growths.

4   **1. Introduction.** The main goal of this paper is to provide examples of blow up  
5 in finite time for  $2 \times 2$  reaction-diffusion systems of the form

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \text{ on } (0, T) \times B_N, \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \text{ on } (0, T) \times B_N, \\ \text{"good" boundary conditions on } (0, T) \times \partial B_N, \\ u_1(0, \cdot) = u_1^0 \geq 0, u_2(0, \cdot) = u_2^0 \geq 0, \end{cases} \quad (1)$$

6 where  $B_N$  is the open unit ball in  $\mathbb{R}^N$ ,  $d_1, d_2 \in (0, +\infty)$ ,  $f_1, f_2 : [0, +\infty)^2 \rightarrow \mathbb{R}$  are  
7 regular nonlinearities such that the nonnegativity of the solutions to (1) is preserved  
8 for all time which, as it is well-known, is equivalent to

$$\text{(P) quasipositivity: } f_1(0, s_2) \geq 0, f_2(s_1, 0) \geq 0, \forall s_1, s_2 \in [0, +\infty), \quad (2)$$

9 and such that the following 'mass dissipativity' condition holds

$$\text{(M) dissipativity: } f_1(s_1, s_2) + f_2(s_1, s_2) \leq 0, \forall s_1, s_2 \in [0, +\infty). \quad (3)$$

10 This last property implies that the  $L^1(B_N)$ -norm of the solutions  $u_1(t), u_2(t)$  of (1)  
11 does not blow up in finite time (= control of the total mass for all time).

12

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1 It is well-known that, for  $u_1^0, u_2^0 \in L^\infty(B_N)^+$ , there exist  $T \in (0, +\infty]$  and a  
 2 classical solution to (1) on  $[0, T)$ . If all the  $d_i$  are equal, then  $T = +\infty$  so that  
 3 global classical existence holds. However, if the  $d_i$  are different from each other,  
 4 it is known that the two conditions (P), (M) are in general not sufficient to pro-  
 5 vide global classical solutions on  $[0, +\infty)$  as proved in [11], [12] where examples of  
 6 solutions blowing up in  $L^\infty(B_N)$  in finite time are described for this class of systems.

7  
 8 A main new result proved here (see Theorem 2.1) is that actually, for all  $d =$   
 9  $2 + \eta, \eta > 0$ , there exists a choice of  $d_1, d_2 \in (0, +\infty)$ , of the space dimension  $N$  and  
 10 of the nonlinearities  $f_1, f_2$  satisfying (P), (M) and the growth condition (G) below,  
 11 for which the solution to (1) **blows up in  $L^\infty(B_N)$  in finite time  $T < +\infty$**  :

$$(G) \text{ d-growth: } |f_i(s_1, s_2)| \leq C_0 + C_1(s_1 + s_2)^d, \quad \forall s_1, s_2 \in [0, +\infty), i = 1, 2, \quad (4)$$

12 where  $C_0, C_1 \in [0, +\infty)$ . The point is that finite time blow up may occur *for any*  
 13 *superquadratic growth*.

14 This result has to be analyzed in parallel with the recent results in [4], [5] saying  
 15 that, for any dimension  $N$ , there exists  $\eta > 0$ , depending on the dimension  $N$ , such  
 16 that, if the growth of  $f_1, f_2$  at infinity is at most  $d = 2 + \eta$ , then all solutions of (1)  
 17 are **global in time**: in particular, the  $L^\infty(B_N)$ -norm of  $u_1(t), u_2(t)$  is bounded on  
 18 all intervals  $[0, T]$ . Note that these global existence results are proved in [4], [5] even  
 19 for all  $m \times m$  systems,  $m \geq 2$ , with the properties (P), (M), (G). Theorem 2.1 shows  
 20 that these results are in some sense optimal. Besides [4], [5], several papers had  
 21 provided global existence results of classical solutions for close to quadratic systems  
 22 like [7], [6], [2], [14]. More results may also be found in [3], [15], [8], [1], [13]. We re-  
 23 fer to the introduction in [4] where a nice state of art on these questions is described.

24  
 25 Obviously, given  $\eta > 0$  and  $f_1, f_2$  with a  $2 + \eta$  growth, to obtain blow up examples,  
 26 it is necessary to work in space dimensions  $N$  which are higher and higher when  $\eta$   
 27 is smaller and smaller.

28 Four more facts are also interesting about these blow up examples.

29 1) First it turns out that, not only the  $L^\infty$ -norm blows up at time  $t = T$ , but  
 30 also the  $L^m$ -norm for all  $m \geq N(1 + \eta)/2, N$  large.

31 2) Next these  $T$ -blowing up solutions can nevertheless be extended as *global weak*  
 32 *solutions on  $[0, +\infty)$*  of the corresponding system (1) (see Section 7).

33 3) We also prove (see Theorem 2.2) ) that the same kind of solutions provide  
 34 blow up examples for the following kind of systems:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = c_1(t, x) u_1^\alpha u_2^\beta & \text{on } (0, T) \times B_N, \\ \partial_t u_2 - d_2 \Delta u_2 = c_2(t, x) u_1^\alpha u_2^\beta & \text{on } (0, T) \times B_N, \\ \text{"good" boundary conditions on } (0, T) \times \partial B_N, \\ u_1(0, \cdot) = u_1^0 \geq 0, \quad u_2(0, \cdot) = u_2^0 \geq 0, \end{cases} \quad (5)$$

35 where  $c_1 + c_2 \leq 0, c_1, c_2 \in L^\infty(Q_T)$  and  $\alpha + \beta = 2 + \eta$  with  $\eta > 0$  as small as we  
 36 want.

37 4) The boundary conditions of the examples provided in Theorem 2.1 are nonho-  
 38 mogeneous boundary conditions. We also describe blowing up examples for systems  
 39 with *homogeneous Neumann boundary conditions*, up to replacing  $f_i, i = 1, 2$  by  
 40 more general (still regular) nonlinearities  $g_i = g_i(t, x, s_1, s_2)$ . This is indicated in  
 41 Theorem 2.3.

An optimal dependence of  $N$  in terms of  $\eta$  or  $d$  is not explicitly known in the main result of Theorem 2.1. What comes out in the proof of this theorem [see (28)], is that one should choose  $N$  large enough so that

$$\frac{2N - \theta}{N - \theta} \leq d = 2 + \eta \Leftrightarrow N \geq \theta(1 + \eta^{-1}),$$

1 where  $\theta \in (4/3, +\infty)$ . This means that we should have  $N > 4(1 + \eta^{-1})/3$ .

2 Another natural question is, given the dimension  $N$ , to find the optimal growth  
3 which provides global existence (or blow up in finite time) for nonlinearities satis-  
4 fying the main properties **(P)**, **(M)**. As proved in [11], [12], examples of blow up  
5 may occur even in dimension  $N = 1$  by choosing a large enough growth for the  
6 nonlinearity. It is known that in dimension  $N = 1$ , global existence is proved for at  
7 most cubic growth [see [6], [15]], but the optimal growth is not known.

To progress in this understanding, we give in Section 8 explicit blow up examples in the following cases :

$$N = 1, d = 6; \quad N = 2, d = 7/2; \quad N = 3, d = 3.$$

8 **2. Statement of the main results. Some notation.** Here  $B_N$  denotes the open  
9 euclidian unit ball in  $\mathbb{R}^N$ ,  $Q_T = (0, T) \times B_N$ ,  $\Sigma_T = (0, T) \times \partial B_N$ . We also denote by  
10  $C^\infty(A_1, A_2)$  the family of  $C^\infty$ -mappings from  $A_1$  to  $A_2$  where  $A_i \subset \mathbb{R}^{N_i}$ ,  $i = 1, 2$ ,  
11 and  $N_1, N_2$  are positive integers. If  $A_2 = \mathbb{R}$ , we simply denote  $C^\infty(A_1)$ .

**Theorem 2.1.** *Let  $d := 2 + \eta, \eta \in (0, +\infty)$  and  $T \in (0, +\infty)$ . Then there exist  $f_1, f_2 \in C^\infty([0, +\infty)^2)$  satisfying **(P)** + **(M)** + **(G)**,  $d_1, d_2 \in (0, +\infty)$ , a dimension  $N$  large enough,  $u_1^0, u_2^0 \in C^\infty(\overline{B_N})^+$ ,  $\alpha_1, \alpha_2 \in C^\infty([0, T])^+$  and  $u_1, u_2$  nonnegative  $C^\infty$ -solutions of (1) with*

$$u_1(t, x) = \alpha_1(t), \quad u_2(t, x) = \alpha_2(t) \text{ on } \Sigma_T,$$

and such that

$$\lim_{t \rightarrow T^-} \|u_1(t)\|_{L^m(B_N)} = \lim_{t \rightarrow T^-} \|u_2(t)\|_{L^m(B_N)} = +\infty,$$

12 for  $m \geq N(d - 1)/2$ . Moreover, for  $\lambda$  close to 1

$$(M_\lambda) \quad f_1(s_1, s_2) + \lambda f_2(s_1, s_2) \leq 0, \quad \forall s_1, s_2 \in [0, +\infty). \quad (6)$$

13 **Remark 2.1.** Despite the blow up of  $u_1(t), u_2(t)$  at time  $t = T$ , it turns out that the  
14 solutions  $u_i$  provided by Theorem 2.1 can be extended to the whole interval  $[0, +\infty)$   
15 as *weak global solutions* of the system (1) on the whole interval  $[0, +\infty)$ . By weak  
16 solution, we mean that the nonlinear terms  $f_i(u_1, u_2), i = 1, 2$  are in  $L^1((0, \tau) \times B_N)$   
17 for all  $\tau \in (0, +\infty)$  and  $u_1, u_2$  are solutions in the sense of distributions or in the  
18 sense of the 'variation of constants' formula. This is a consequence of the fact that,  
19 in the blow up examples, not only  $f_1 + f_2 \leq 0$  holds, but also  $f_1 + \lambda f_2 \leq 0$  for some  
20  $\lambda \neq 1$ . This provides the  $L^1$ -bound on the nonlinear terms and this is sufficient for  
21 the existence of weak global solutions. All this is made more precise in Section 7.

**Theorem 2.2.** *Let  $d := 2 + \eta, \eta \in (0, +\infty), \alpha, \beta \in (1, +\infty)$  with  $\alpha + \beta = d$  and  $T \in (0, +\infty)$ . Then there exist a dimension  $N$  large enough,  $d_1, d_2 \in (0, +\infty)$  and*

$$\begin{cases} c_1, c_2 \in C^\infty([0, T) \times \overline{B_N}) \cap L^\infty(Q_T), \\ u_1^0, u_2^0 \in C^\infty(\overline{B_N})^+, \quad \alpha_1, \alpha_2 \in C^\infty([0, T])^+, \end{cases}$$

and  $u_1, u_2$  nonnegative  $C^\infty$ -solutions of the system (5) with  $u_i = \alpha_i, i = 1, 2$  on  $\Sigma_T$  and such that

$$\begin{aligned} c_1(t, x) + c_2(t, x) &\leq 0, \quad \forall (t, x) \in \overline{Q_T}, \\ \lim_{t \rightarrow T^-} \|u_1(t)\|_{L^m(B_N)} &= \lim_{t \rightarrow T^-} \|u_2(t)\|_{L^m(B_N)} = +\infty, \end{aligned}$$

1 for  $m \geq N(d-1)/2$ .

2 **Remark 2.2.** The solutions  $u_1, u_2$  appearing in this result are the same as those in  
3 Theorem 2.1. Only the interpretation of the nonlinear part changes. In particular,  
4 they blow up in  $L^m(B_N)$  as  $t \rightarrow T^-$  for the same values of  $m$ .

5 It turns out that we can also use the same kind of solutions to construct new  
6 blowing up solutions for similar systems, but with *homogeneous Neumann boundary*  
7 *conditions*. Here the nonlinearities depend also on  $(t, x)$ . This is the purpose of the  
8 next theorem.

9 **Theorem 2.3.** Let  $d := 2 + \eta, \eta \in (0, +\infty)$  and  $T \in (0, +\infty)$ . Then there exist  $\tau_0 \in$   
10  $[0, T), g_1, g_2 \in C^\infty([\tau_0, T) \times \overline{B_N} \times [0, +\infty)^2)$  satisfying  $(\mathbf{P}')$  +  $(\mathbf{M}'_\lambda)$  +  $(\mathbf{G}')$  below,  
11  $d_1, d_2 \in (0, +\infty)$ , a dimension  $N$  large,  $v_1^0, v_2^0 \in C^\infty(\overline{B_N})^+$  and  $v_1, v_2$  nonnegative  
12  $C^\infty$ -solutions of

$$\begin{cases} \partial_t v_1 - d_1 \Delta v_1 = g_1(t, x, v_1, v_2) \text{ in } (\tau_0, T) \times B_N, \\ \partial_t v_2 - d_2 \Delta v_2 = g_2(t, x, v_1, v_2) \text{ in } (\tau_0, T) \times B_N, \\ \partial_\nu v_1 = 0 = \partial_\nu v_2 \text{ on } (\tau_0, T) \times B_N, \\ v_i(\tau_0, \cdot) = v_i^0 \geq 0, \end{cases} \quad (7)$$

and such that

$$\lim_{t \rightarrow T^-} \|v_1(t)\|_{L^m(B_N)} = \lim_{t \rightarrow T^-} \|v_2(t)\|_{L^m(B_N)} = +\infty,$$

13 for  $m \geq N(d-1)/2$ .

14 Here the conditions  $(\mathbf{P}')$  +  $(\mathbf{M}'_\lambda)$  +  $(\mathbf{G}')$  are the same as  $(\mathbf{P})$  +  $(\mathbf{M}_\lambda)$  +  $(\mathbf{G})$ , but  
15 with a  $(t, x)$ -dependence, namely,

$$\begin{aligned} \forall (t, x) \in [\tau_0, T] \times \overline{B_N}, \forall (s_1, s_2) \in [0, +\infty)^2, \\ (\mathbf{P}') \quad g_1(t, x, 0, s_2) \geq 0, \quad g_2(t, x, s_1, 0) \geq 0, \\ (\mathbf{M}'_\lambda) \quad g_1(t, x, s_1, s_2) + \lambda g_2(t, x, s_1, s_2) \leq 0, \quad \forall \lambda \text{ close to } 1, \\ (\mathbf{G}') \quad |g_1(t, x, s_1, s_2)| + |g_2(t, x, s_1, s_2)| \leq C_0 + C_1(s_1 + s_2)^d. \end{aligned} \quad (8)$$

16

17 **3. Steps of the proof of Theorem 2.1.** The main idea in the proof of Theorem 2.1  
18 is similar to the approach in [11], [12]. It consists in working with functions  $u_1, u_2$  of the  
19 form

$$u_i(t, x) = \frac{a_i(T-t) + b_i r^2}{[(T-t) + r^2]^\gamma}, \quad r = |x|, i = 1, 2, \quad (9)$$

20 where  $a_1, a_2, b_1, b_2$  will be well-chosen in  $(0, +\infty)$ , as well as  $\gamma \in (1, 2)$  and the dimension  
21  $N$  (large enough), in such a way that there exist  $d_1, d_2 \in (0, +\infty)$  and two functions  $f_1, f_2$   
22 as described in Theorem 2.1 for which

$$\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2), \quad \text{in } (0, T) \times B_N, \quad i = 1, 2. \quad (10)$$

23 The values of  $u_1, u_2$  on  $\partial B_N$  are obviously  $C^\infty$ -functions. Moreover the  $L^\infty(B_N)$ -norm of  
24  $u_1(t), u_2(t)$  blows up as  $t \rightarrow T^-$  since  $\gamma > 1$ . More precisely, we directly check (see Lemma  
25 3.1) that

$$\lim_{t \rightarrow T^-} \|u_i(t)\|_{L^m(B_N)} = +\infty, \quad \forall m \geq \frac{N}{2(\gamma-1)}, \quad i = 1, 2. \quad (11)$$

26 Thus Theorem 2.1 will be proved if we can achieve (10) with regular functions  $f_1, f_2$  sat-  
27 isfying  $(\mathbf{P}), (\mathbf{M}), (\mathbf{G})$  with  $d = 2 + \eta \geq \gamma/(\gamma-1)$ . It will be a consequence of the three

1 following lemmas.

2

3 First note that  $u_i, i = 1, 2$ , is of the following form:

$$u_i(t, x) = (T - t)^{1-\gamma} \rho_i \left( \frac{r^2}{T - t} \right), \quad \rho_i(\sigma) = \frac{a_i + b_i \sigma}{(1 + \sigma)^\gamma}, \quad i = 1, 2. \quad (12)$$

4 We have the following technical lemma.

5 **Lemma 3.1.** *Let  $u_1, u_2$  be given by (9). For  $i = 1, 2$ , we have in  $(0, T) \times B_N$*

$$\partial_t u_i - d_i \Delta u_i = (T - t)^{-\gamma} (1 + \sigma)^{-(\gamma+2)} [A_i + B_i \sigma + C_i \sigma^2], \quad \sigma = r^2 / (T - t), \quad (13)$$

6 where  $A_i, B_i, C_i$  are real numbers given by

$$\begin{cases} A_i = (\gamma - 1)a_i - 2d_i N(b_i - \gamma a_i), \\ B_i = (\gamma - 1)[2a_i + b_i(1 + 2d_i N + 4d_i)] + (b_i - \gamma a_i)[4d_i(\gamma + 1) - 2d_i N + 1], \\ C_i = (\gamma - 1)[a_i + b_i(1 + 2d_i N - 4\gamma d_i)] + b_i - \gamma a_i. \end{cases} \quad (14)$$

7 As a consequence

$$\partial_t u_1 - d_1 \Delta u_1 + \partial_t u_2 - d_2 \Delta u_2 = (T - t)^{-\gamma} (1 + \sigma)^{-(\gamma+2)} [A + B\sigma + C\sigma^2], \quad (15)$$

8

$$\begin{cases} A = (\gamma - 1)a - 2NE + 2N\gamma G, \\ B = (\gamma - 2)a + \gamma b + [2N(\gamma - 2) + 8\gamma]E - 2\gamma[2(\gamma + 1) - N]G, \\ C = -a + \gamma b + 2(\gamma - 1)(N - 2\gamma)E, \end{cases} \quad (16)$$

9 where only the four following quantities are involved

$$a := a_1 + a_2, \quad b := b_1 + b_2, \quad E := b_1 d_1 + b_2 d_2, \quad G := d_1 a_1 + d_2 a_2. \quad (17)$$

10 Moreover,  $\lim_{t \rightarrow T^-} \|u_i(t)\|_{L^m(B_N)} = +\infty$  if  $m \geq N/2(\gamma - 1)$ .

11 The proof of this lemma is elementary (see below). Next we check that  $\partial_t u_i - d_i \Delta u_i$   
12 may be written as a function of  $u_1, u_2$ . This is the purpose of the following lemma. Here  
13  $\mathbb{Q}$  denotes the set of rational numbers.

14 **Lemma 3.2.** *Let  $u_1, u_2$  be given by (9). Assume*

$$\gamma \in (1, 2) \cap \mathbb{Q}, \quad a_1 b_2 - a_2 b_1 \neq 0. \quad (18)$$

15 Then for  $i = 1, 2$ , there exists a  $C^\infty$ -function  $P_i : (0, +\infty)^2 \rightarrow \mathbb{R}^2$  which is homogeneous  
16 of degree  $\gamma' = \gamma / (\gamma - 1)$  and such that

$$\partial_t u_i - d_i \Delta u_i = P_i(u_1, u_2) \quad \text{in } (0, T) \times B_N, \quad (19)$$

17

$$|P_i(s_1, s_2)| \leq K(s_1 + s_2)^{\gamma'}, \quad \forall (s_1, s_2) \in (0, +\infty)^2, \quad \text{for some } K \in (0, +\infty), \quad (20)$$

18

$$P_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma) = (1 + \sigma)^{\frac{2-\gamma}{\gamma-1}} [A_i + B_i \sigma + C_i \sigma^2], \quad \forall \sigma \in (0, +\infty). \quad (21)$$

19 **Remark 3.1.** By homogeneous of degree  $\gamma'$ , we mean that

$$P_i(\lambda s_1, \lambda s_2) = \lambda^{\gamma'} P_i(s_1, s_2), \quad \forall \lambda \in (0, +\infty), (s_1, s_2) \in (0, +\infty)^2. \quad (22)$$

20 This lemma will be proved below. But let us continue describing the scheme of the proof  
21 of Theorem 2.1. A main step is now to prove that we can choose the various parameters  
22  $a_i, b_i, d_i, i = 1, 2, d, \gamma$  and the dimension  $N$  such that the functions  $P_1, P_2$  appearing in  
23 Lemma 3.2 satisfy

$$P_1(s_1, s_2) + P_2(s_1, s_2) \leq 0, \quad \forall (s_1, s_2) \in (0, +\infty)^2. \quad (23)$$

By (21), this inequality will be satisfied for  $(s_1, s_2) = (a_1 + b_1 \sigma, a_2 + b_2 \sigma), \sigma \in (0, +\infty)$  if  
and only if

$$A + B\sigma + C\sigma^2 \leq 0, \quad \forall \sigma \in (0, +\infty),$$

24 where  $A, B, C$  are defined in (15) and (16) of Lemma 3.1. It will be the case if

$$A < 0, \quad C < 0, \quad B^2 < 4AC. \quad (24)$$

25 And we then construct the  $P_i$  so that this inequality extends to all  $(s_1, s_2) \in (0, +\infty)^2$ .

1 The main point in the proof of Theorem 2.1 is that we can choose the various param-  
 2 eters in the definition (9) so that (24) holds for  $A, B, C$  as defined in Lemma 3.1. This is  
 3 the purpose of the following main lemma where we use the following notation.

4

**Notation.** Writing a function  $D : \mathbb{R} \mapsto \mathbb{R}$  as  $D(N) = O(N^{-\alpha})$ ,  $\alpha \in \mathbb{R}$ , means that

$$\limsup_{N \rightarrow +\infty} |D(N)|N^\alpha < +\infty.$$

5 **Lemma 3.3.** *In the definition (9) of  $u_1, u_2$ , let us choose*

$$\begin{cases} a_1 = 1/N^2, a_2 = 2N - N^{-2}, b_1 = b_2 = 1/\sqrt{N}, \\ d_1 = \sqrt{N}, d_2 = 1/N^3, \gamma = 2 - \theta/N, \theta \in (4/3, +\infty) \cap \mathbb{Q}. \end{cases} \quad (25)$$

6 Then as  $N \rightarrow +\infty$ , the values  $A, B, C$ , as defined in Lemma 3.1, satisfy

$$\begin{cases} A = -2\theta + O(N^{-\frac{1}{2}}) < 0, & C = -(8 + 2\theta) + O(N^{-\frac{1}{2}}) < 0, \\ B^2 - 4AC = 64(4 - 3\theta) + O(N^{-\frac{1}{2}}) < 0, \\ A + B\sigma + C\sigma^2 \leq A - B^2/4C = \frac{8(4-3\theta)}{4+\theta} + O(N^{-\frac{1}{2}}) < 0, \forall \sigma \in [0, +\infty). \end{cases} \quad (26)$$

7 Moreover (18) is satisfied and the functions  $P_1, P_2$  can be chosen in Lemma 3.2 so that  
 8 they satisfy (23) and even

$$P_1(s_1, s_2) + \lambda P_2(s_1, s_2) \leq 0, \quad \forall (s_1, s_2) \in (0, +\infty)^2, \quad (27)$$

9 for  $\lambda$  close to 1.

10 **4. Proof of Theorem 2.1.** Theorem 2.1 is a consequence of Lemmas 3.1, 3.2, 3.3. We  
 11 will prove these lemmas below. But let us first show how these lemmas imply Theorem  
 12 2.1.

13 We consider the functions  $u_1, u_2$  as defined in (9) with the choice of parameters  
 14  $a_1, a_2, b_1, b_2, d_1, d_2, \gamma$  given in Lemma 3.3 with  $\theta \in (4/3, +\infty) \cap \mathbb{Q}$  and  $N$  large enough so  
 15 that

$$\gamma' = \frac{2N - \theta}{N - \theta} \leq d = 2 + \eta. \quad (28)$$

16 Since by assumption  $\theta \in \mathbb{Q}$ , we have  $\gamma \in \mathbb{Q}$ . Moreover  $\gamma \in (1, 2)$  for  $N > \theta$ . For this choice  
 17 of the parameters,  $a_2 b_1 - a_1 b_2 = 2N^{1/2} + O(N^{-5/2}) \neq 0$  for  $N$  large.

Thus by Lemma 3.2, there exist two  $C^\infty$ -functions  $P_1, P_2 : (0, +\infty)^2 \rightarrow \mathbb{R}$  such that

$$\partial_t u_i - d_i \Delta u_i = P_i(u_1, u_2), \quad \text{in } (0, T) \times B_N,$$

$$|P_i(s_1, s_2)| \leq K(s_1 + s_2)^{\gamma'}, \quad \forall (s_1, s_2) \in (0, +\infty)^2, i = 1, 2, \text{ for some } K \in (0, +\infty).$$

18 By Lemma 3.3, we can even choose  $P_1, P_2$  so that (27) holds for  $\lambda$  close to 1.

19

Note that there exists  $m_N \in (0, +\infty)$  such that

$$u_i(t, x) \geq m_N, \quad \forall (t, x) \in [0, T] \times \bar{B}_N, \quad i = 1, 2.$$

Indeed, we may choose  $m_N = (T + 1)^{1-\gamma} \min\{a_1, a_2, b_1, b_2\}$  as it can be seen by writing

$$u_i(t, x) = (T - t + r^2)^{1-\gamma} \frac{a_i + b_i \sigma}{1 + \sigma}.$$

Let us introduce  $\varphi \in C^\infty([0, +\infty), [0, +\infty))$  such that

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 0 \text{ on } [0, m_N/2], \quad \varphi \equiv 1 \text{ on } [m_N, +\infty).$$

Let us denote  $f_i(s_1, s_2) := \varphi(s_1)\varphi(s_2)P_i(s_1, s_2)$ ,  $i = 1, 2$  on  $[0, +\infty)^2$ . Then

$$\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2), \quad \text{in } (0, T) \times B_N,$$

20 and  $f_1, f_2$  are exactly as described in Theorem 2.1. More precisely, and for future reference,  
 21 we even have

$$f_i(s_1, s_2) = 0, \quad \forall (s_1, s_2) \in [0, +\infty)^2 \setminus [m_N/2, +\infty)^2. \quad (29)$$

Next we have  $u_i^0 \in C^\infty(\overline{B_N})$  and  $\alpha_i \in C^\infty([0, T])$  for  $i = 1, 2$  since

$$\begin{cases} u_i^0(x) = (T + r^2)^{-\gamma}(a_i T + b_i r^2), \\ \alpha_i(t) = (T - t + 1)^{-\gamma}(a_i(T - t) + b_i), \quad i = 1, 2. \end{cases}$$

1 Now, we know by Lemma 3.1, that the  $L^m(B_N)$ -norm of  $u_i(t)$ ,  $i = 1, 2$  blows up if  
 2  $m \geq N[2(\gamma - 1)]^{-1}$ . Here  $\gamma$  is chosen [see (28)] so that  $\gamma' \leq d$ , that is  $(\gamma - 1)^{-1} \leq d - 1$ . If  
 3  $m \geq N(d - 1)/2$ , then  $m \geq N[2(\gamma - 1)]^{-1}$  and the  $L^m(B_N)$ -norm of  $u_i(t)$  blows up. This  
 4 ends the proof of the main Theorem 2.1, assuming Lemmas 3.1, 3.2, 3.3.

5  
 6 Let us now prove these three lemmas.

7  
**Proof of Lemma 3.1.** This result is derived by technical but easy computations. We may use (see (12))  $u_i = (T - t)^{1-\gamma}\rho_i(\sigma)$ ,  $\sigma = r^2/(T - t)$  so that

$$\begin{aligned} \partial_t u_i &= (\gamma - 1)(T - t)^{-\gamma}\rho_i(\sigma) + \frac{r^2}{(T - t)^2}(T - t)^{1-\gamma}\rho_i'(\sigma), \\ \partial_r u_i &= (T - t)^{1-\gamma} \left( \frac{2r}{T - t} \right) \rho_i' \left( \frac{r^2}{T - t} \right), \\ \partial_{rr} u_i &= (T - t)^{1-\gamma} \left[ \frac{4r^2}{(T - t)^2} \rho_i'' \left( \frac{r^2}{T - t} \right) + \frac{2}{T - t} \rho_i' \left( \frac{r^2}{T - t} \right) \right], \\ \frac{N - 1}{r} \partial_r u_i + \partial_{rr} u_i &= (T - t)^{-\gamma} [2N\rho_i'(\sigma) + 4\sigma\rho_i''(\sigma)]. \\ \partial_t u_i - d_i \Delta u_i &= (T - t)^{-\gamma} [(\gamma - 1)\rho_i(\sigma) + (\sigma - 2d_i N)\rho_i'(\sigma) - 4d_i \sigma \rho_i''(\sigma)]. \end{aligned}$$

With the choice of  $\rho_i(\sigma) = (a_i + b_i \sigma)/(1 + \sigma)^\gamma$ , we have

$$\begin{aligned} \rho_i'(\sigma) &= \frac{b_i - \gamma a_i + b_i(1 - \gamma)\sigma}{(1 + \sigma)^{\gamma+1}}, \\ \rho_i''(\sigma) &= \frac{b_i(1 - \gamma)}{(1 + \sigma)^{\gamma+1}} - \frac{(\gamma + 1)[b_i - \gamma a_i + b_i(1 - \gamma)\sigma]}{(1 + \sigma)^{\gamma+2}}. \end{aligned}$$

8 This leads to

$$\partial_t u_i - d_i \Delta u_i = (T - t)^{-\gamma} (1 + \sigma)^{-(\gamma+2)} [A_i + B_i \sigma + C_i \sigma^2], \quad (30)$$

9 where  $A_i, B_i, C_i$  are given as stated in Lemma 3.1.

10 By summing for  $i = 1, 2$  these two expressions, we obtain the claims (15)-(16)-(17) of  
 11 Lemma 3.1. It is important to notice that this sum can be written only in terms of the  
 12 four parameters  $a, b, E, G$ .

Finally we compute the  $L^m(B_N)$ -norm of  $u_i(t)$ . For some  $C_N \in (0, +\infty)$

$$\int_{B_N} u_i^m(t, x) dx = C_N (T - t)^{m(1-\gamma)} \int_0^1 r^{N-1} \rho_i^m(r^2/(T - t)) dr.$$

By setting  $\sigma = r^2/(T - t) \Leftrightarrow r = \sqrt{T - t} \sqrt{\sigma}$ , we have

$$\int_{B_N} u_i^m(t, x) dx = \frac{C_N}{2} (T - t)^{\frac{N}{2} + m(1-\gamma)} \int_0^{(T-t)^{-1}} \sigma^{\frac{N}{2}-1} \rho_i^m(\sigma) d\sigma.$$

As  $\sigma^{\frac{N}{2}-1} \rho_i^m(\sigma)$  is equivalent to  $\sigma^{\frac{N}{2}-1+m(1-\gamma)}$ , up to a positive constant, as  $\sigma \rightarrow +\infty$ , we have

$$\frac{N}{2} < m(\gamma - 1) \Rightarrow \int_0^{+\infty} \sigma^{\frac{N}{2}-1} \rho_i^m(\sigma) d\sigma < +\infty.$$

13 As a consequence, in this case  $\int_{B_N} u_i^m(t, x) dx$  behaves like  $(T - t)^{\frac{N}{2} + (1-\gamma)m}$ , up to a con-  
 14 stant, as  $t \rightarrow T^-$  and therefore tends to  $+\infty$ . The conclusion is the same in the equality  
 15 case  $N = 2m(\gamma - 1)$  since then  $\int_{B_N} u_i^m(t, x) dx$  behaves like  $|\log(T - t)|$  as  $t \rightarrow T^-$ .

16



1 **Proof of Lemma 3.2.** We are looking for two functions  $P_i(\cdot, \cdot), i = 1, 2$  such that

$$\partial_t u_i - d_i \Delta u_i = P_i(u_1, u_2) = P_i((T-t)^{1-\gamma} \rho_1(\sigma), (T-t)^{1-\gamma} \rho_2(\sigma)), \quad \sigma = r^2/(T-t). \quad (31)$$

According to (13) in Lemma 3.1, and by the expected  $\gamma'$ -homogeneity of  $P_i$ , this is equivalent to

$$\begin{aligned} & (1+\sigma)^{-(\gamma+2)} [A_i + B_i \sigma + C_i \sigma^2] = (1+\sigma)^{\gamma^2/(1-\gamma)} P_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma), \\ & \Leftrightarrow (1+\sigma)^{\frac{2-\gamma}{\gamma-1}} [A_i + B_i \sigma + C_i \sigma^2] = P_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma), \quad \forall \sigma \in [0, +\infty). \end{aligned} \quad (32)$$

Let us write  $\gamma' = p/q \in (2, +\infty) \cap \mathbb{Q}$  with  $p, q$  two coprime positive integers. Then

$$\gamma = p/(p-q), \quad (2-\gamma)/(\gamma-1) = (p-2q)/q > 0.$$

3 Thus (32) may be rewritten

$$(1+\sigma)^{\frac{p-2q}{q}} [A_i + B_i \sigma + C_i \sigma^2] = P_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma), \quad (33)$$

4 which also implies

$$(1+\sigma)^{p-2q} [A_i + B_i \sigma + C_i \sigma^2]^q = \{P_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma)\}^q. \quad (34)$$

Here  $p > 2q$  and the left-hand side is a polynomial of degree at most  $p$  in  $\sigma$ . It is easy to see that, since  $a_1 b_2 - a_2 b_1 \neq 0$ , the family of  $p+1$  polynomials

$$\{(a_1 + b_1 \sigma)^k (a_2 + b_2 \sigma)^{p-k}, \quad k = 0, \dots, p\},$$

is a *linear basis of the linear space of polynomials of degree at most  $p$* . Indeed, let us analyze the relation

$$\sum_{k=0}^p \mu_k (a_1 + b_1 \sigma)^k (a_2 + b_2 \sigma)^{p-k} \equiv 0, \quad \mu_k \in \mathbb{R}, \quad \forall k = 0, \dots, p.$$

Assume by contradiction that there exists  $r \in \{0, \dots, p\}$  such that

$$\mu_k = 0, \quad \forall k = 0, \dots, r-1, \quad \mu_r \neq 0.$$

Then after dividing by  $(a_1 + b_1 \sigma)^r$ , the above relation may be rewritten

$$\sum_{k=r}^p \mu_k (a_1 + b_1 \sigma)^{k-r} (a_2 + b_2 \sigma)^{p-k} = 0.$$

5 If  $r = p$ , this obviously implies  $\mu_r = 0$  whence a contradiction. If  $r < p$ , choosing  $\sigma =$   
6  $-a_1/b_1$  leads to  $\mu_r (a_2 - b_2 a_1/b_1)^{p-r} = 0$ , whence again  $\mu_r = 0$  thanks to the assumption  
7 (18) on  $a_i, b_i, i = 1, 2$ , and this is a contradiction.

8 We are now going to define the convenient function  $P_i$ . Going back to the polynomial  
9 of degree at most  $p$  involved in (34), we can claim that there exist  $\lambda_k^i \in \mathbb{R}, k = 0, \dots, p$  such  
10 that for all  $\sigma \in \mathbb{R}$

$$(1+\sigma)^{p-2q} [A_i + B_i \sigma + C_i \sigma^2]^q = \sum_{k=0}^p \lambda_k^i (a_1 + b_1 \sigma)^k (a_2 + b_2 \sigma)^{p-k}. \quad (35)$$

11 *Assume first that  $q$  is odd.* Let us define

$$\tilde{P}_i(s_1, s_2) := \left\{ \sum_{k=0}^p \lambda_k^i s_1^k s_2^{p-k} \right\}^{1/q}, \quad (s_1, s_2) \in [0, +\infty)^2. \quad (36)$$

12 Note that  $\tilde{P}_i$  is homogeneous of degree  $p/q$  and

$$(1+\sigma)^{\frac{p-2q}{q}} [A_i + B_i \sigma + C_i \sigma^2] = \tilde{P}_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma). \quad (37)$$

13 According to (30), this implies [see also (31), (32)]

$$\partial_t u_i - d_i \Delta u_i = \tilde{P}_i(u_1, u_2) \text{ in } Q_T. \quad (38)$$

Moreover, we directly see on the definition (36) of  $\tilde{P}_i$  that for  $(s_1, s_2) \in [0, +\infty)$

$$|\tilde{P}_i(s_1, s_2)| \leq \left\{ \sum_{k=0}^p |\lambda_k| \right\}^{1/q} (s_1 + s_2)^{p/q} = C(s_1 + s_2)^{\gamma'}.$$

Now, for the regularity of  $\tilde{P}_i$ , it is obvious from its definition (36) that  $\tilde{P}_i$  is  $C^\infty$  around each point  $(s_1, s_2)$  for which  $\tilde{P}_i(s_1, s_2) \neq 0$ . We are going to modify it into a new function  $P_i$  which will be in  $C^\infty((0, +\infty)^2)$  with the same main properties. For this, let us assume without loss of generality in (18) that  $a_1 b_2 - a_2 b_1 > 0$  or  $b_2/b_1 > a_2/a_1$  and let us consider the function

$$s \in (0, +\infty) \rightarrow X := \frac{a_2 + b_2 s}{a_1 + b_1 s} \in I := \left( \frac{a_2}{a_1}, \frac{b_2}{b_1} \right).$$

- 1 It is  $C^\infty$  with derivative  $(a_1 b_2 - a_2 b_1)/(a_1 + b_1 s)^2 > 0$  on  $(0, +\infty)$ . Thus its inverse  
 2  $\psi : I \mapsto (0, +\infty)$  is also  $C^\infty$ .

From (36), we may write

$$\tilde{P}_i(s_1, s_2) = s_1^{p/q} \tilde{P}_i(1, s_2/s_1),$$

- 3 and thanks to the relation (37) and the  $p/q$  homogeneity of  $\tilde{P}_i$ , we have

$$\tilde{P}_i(1, X) = [a_1 + b_1 \psi(X)]^{-p/q} [1 + \psi(X)]^{(p-2q)/q} [A_i + B_i \psi(X) + C_i \psi(X)^2]. \quad (39)$$

Whence the  $C^\infty$ -property of  $[X \in I \mapsto \tilde{P}_i(1, X)]$  and therefore of  $\tilde{P}_i(s_1, s_2)$  on  $\mathcal{I}$  where we set

$$\mathcal{I} := \{(s_1, s_2) \in (0, +\infty)^2; s_2/s_1 \in I\}.$$

Actually,  $X \in \mathbb{R} \mapsto \tilde{P}_i(1, X)$  is continuous and does not vanish in a neighborhood of the extremities of  $I$ , namely

$$X = a_2/a_1 \Leftrightarrow \psi(X) = 0, \quad \tilde{P}_i(1, X) = A_i a_1^{-p/q},$$

$$X = b_2/b_1 \Leftrightarrow \psi(X) = +\infty, \quad \tilde{P}_i(1, X) = C_i b_1^{-p/q}.$$

- 4 Thus, as noticed before thanks to the definition (36),  $X \mapsto \tilde{P}_i(1, X)$  is  $C^\infty$  even on some  
 5 open interval  $J$  containing  $\bar{I}$ . We now introduce a function  $\chi \in C^\infty([0, +\infty))$  such that:

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } I, \quad \chi \equiv 0 \text{ on } [0, +\infty) \setminus J. \quad (40)$$

And we set

$$P_i(s_1, s_2) := s_1^{p/q} \chi(s_2/s_1) \tilde{P}_i(1, s_2/s_1) = \chi(s_2/s_1) \tilde{P}_i(s_1, s_2).$$

- 6 Thus  $P_i$  is  $C^\infty$  on all of  $(0, +\infty)^2$  and coincides with  $\tilde{P}_i$  on  $\mathcal{I}$ . It follows that we may  
 7 replace  $\tilde{P}_i$  by  $P_i$  in the relations (37) and (38). This ends the proof of Lemma 3.2 when  $q$   
 8 is odd.

- 9  
 10 Assume now that  $q$  is even. We start again from the relation (35) from which we deduce

$$(1 + \sigma)^{(p-2q)/q} [A_i + B_i \sigma + C_i \sigma^2] = \tilde{P}_i(a_1 + b_1 \sigma, a_2 + b_2 \sigma) \text{sign}\{A_i + B_i \sigma + C_i \sigma^2\}, \quad (41)$$

- 11 where we again set

$$\tilde{P}_i(s_1, s_2) := \left\{ \sum_{k=0}^p \lambda_k^i s_1^k s_2^{p-k} \right\}^{1/q} = s_1^{p/q} \tilde{P}_i(1, s_2/s_1), \quad \forall (s_1, s_2) \in (0, +\infty)^2, \quad (42)$$

and where

$$\text{sign}(r) = 1, \forall r \in (0, +\infty), \quad \text{sign}(0) = 0, \quad \text{sign}(r) = -1, \forall r \in (-\infty, 0).$$

- 12 We deduce from (41) and the  $p/q$  homogeneity of  $\tilde{P}_i$  that, for all  $X \in I$ ,

$$\begin{cases} \tilde{P}_i(1, X) \text{sign}(A_i + B_i \psi(X) + C_i \psi(X)^2) \\ = (a_1 + b_1 \psi(X))^{-p/q} (1 + \psi(X))^{(p-2q)/q} [A_i + B_i \psi(X) + C_i \psi(X)^2] \end{cases} \quad (43)$$

and this function of  $X$  is  $C^\infty$  on  $I$ . Since  $\tilde{P}_i(1, X)$  does not vanish near the extremities of the interval  $I$ , it is also locally  $C^\infty$  there. Moreover  $\text{sign}(A_i + B_i\psi(X) + C_i\psi(X)^2)$  is constant as  $X$  tends to these extremities from inside  $I$ . It follows that the function of  $X$  appearing in (43) may be extended in a  $C^\infty$ -way to an open interval  $J$  containing  $\bar{I}$ . Using the same function  $\chi$  as above (see (40)), by setting

$$P_i(s_1, s_2) := s_1^{p/q} \chi(s_2/s_1) \tilde{P}_i(1, s_2/s_1) \text{sign}(A_i + B_i\psi(s_2/s_1) + C_i\psi^2(s_2/s_1))$$

we define a  $C^\infty$ -function on  $(0, +\infty)^2$  which is  $p/q$ -homogeneous and satisfies

$$P_i(a_1 + b_1\sigma, a_2 + b_2\sigma) = (1 + \sigma)^{(p-2q)/q} [A_i + B_i\sigma + C_i\sigma^2], \forall \sigma \in [0, +\infty)$$

and therefore (see (31), (32), (33) )

$$\partial_i u_i - d_i \Delta u_i = P_i(u_1, u_2) \text{ on } Q_T.$$

1 This ends the proof of Lemma 3.2.

2

**Proof of Lemma 3.3.** As a consequence of the choice of the parameters in Lemma 3.3, we obtain

$$\begin{aligned} a &= a_1 + a_2 = 2N, \quad b = b_1 + b_2 = 2/\sqrt{N}, \\ E &= d_1 b_1 + d_2 b_2 = \sqrt{N}(1/\sqrt{N}) + N^{-3}(1/\sqrt{N}) = 1 + O(N^{-7/2}), \\ G &= d_1 a_1 + d_2 a_2 = \sqrt{N}N^{-2} + N^{-3}[2N - (1/N^2)] = O(N^{-3/2}). \end{aligned}$$

It follows from the formulas (16) in Lemma 3.1 that for  $N$  large

$$\begin{aligned} A &= [1 - \theta/N]2N - 2N[1 + O(N^{-7/2})] + 2N[2 - \theta/N]O(N^{-3/2}) = -2\theta + O(N^{-1/2}), \\ C &= -2N + [2 - \theta/N]2/\sqrt{N} + 2[1 - \theta/N][N - 4 + 2\theta/N][1 + O(N^{-7/2})] = -(8 + 2\theta) + O(N^{-1/2}), \\ B &= \begin{cases} [-\theta/N]2N + [2 - \theta/N][2/\sqrt{N}] + [2N(-\theta/N) + 8(2 - \theta/N)][1 + O(N^{-7/2})] \\ -2[2 - \theta/N][2(3 - \theta/N) - N][O(N^{-3/2})] = 16 - 4\theta + O(N^{-1/2}). \end{cases} \\ B^2 - 4AC &= 16[4 - \theta]^2 - 8\theta(8 + 2\theta) + O(N^{-1/2}) = 64[4 - 3\theta] + O(N^{-1/2}). \end{aligned}$$

3 With the choice of  $\theta > 4/3$ , we obtain  $A < 0, C < 0, B^2 - 4AC < 0$  for  $N$  large as claimed  
4 in Lemma 3.3.

5 Since the maximum of  $\sigma \in \mathbb{R} \mapsto A + B\sigma + C\sigma^2$  is reached for  $\sigma = -B/2C$ , we deduce

$$A + B\sigma + C\sigma^2 \leq A - B^2/4C = -K_\theta + O(N^{-\frac{1}{2}}), \forall \sigma \in [0, +\infty) \quad (44)$$

6 with  $K_\theta := 8(3\theta - 4)/(4 + \theta) > 0$ .

On the other hand,  $a_2 b_1 - a_1 b_2 = 2N^{1/2} + O(N^{-5/2}) \neq 0$  and  $\gamma \in (1, 2) \cap \mathbb{Q}$  so that (18) is satisfied and we may apply Lemma 3.2. Using now the relation (21) in this Lemma 3.2 together with (44), we obtain that

$$(P_1 + P_2)(a_1 + b_1\sigma, a_2 + b_2\sigma) \leq -K_\theta + O(N^{-\frac{1}{2}}), \forall \sigma \in [0, +\infty).$$

By the  $p/q$ -homogeneity of  $P_1 + P_2$ , we deduce that

$$(P_1 + P_2)(1, X) \leq -K_\theta a_1^{-p/q} + O(N^{-\frac{1}{2}}), \forall X \in I,$$

where  $I$  is defined as in the previous proof of Lemma 3.2. Since  $I$  is bounded, so is  $P_2(1, X)$  for  $X \in I$ . It follows that for  $\lambda$  close to 1 and  $N$  large enough, we may claim that

$$(P_1 + \lambda P_2)(1, X) \leq -\frac{K_\theta a_1^{-p/q}}{2}, \forall X \in I.$$

Up to reducing  $J$  and the support of the function  $\chi$  in (40) of the proof of Lemma 3.2, we may claim that

$$(P_1 + \lambda P_2)(1, X) \leq -\frac{K_\theta a_1^{-p/q}}{4}, \forall X \in J.$$

By homogeneity, this implies that

$$(P_1 + \lambda P_2)(s_1, s_2) \leq 0, \forall (s_1, s_2) \in (0, +\infty)^2.$$

7 This ends the proof of Lemma 3.3.

1 **5. Proof of Theorem 2.2.** We use the same functions  $u_1, u_2$  as those introduced in the  
 2 proof of Theorem 2.1 and defined in (9), and we choose  $\gamma$  as in (28), namely such that

$$\gamma' \leq d \Leftrightarrow \gamma \geq d/(d-1). \quad (45)$$

We deduce from (30) that

$$\partial_t u_i - d_i \Delta u_i = \frac{A_i(T-t)^2 + B_i(T-t)r^2 + C_i r^4}{(T-t+r^2)^{\gamma+2}}, \quad i = 1, 2.$$

Given  $\alpha, \beta$  as indicated in the statement of Theorem 2.2, we may understand this expression as being of the form

$$\partial_t u_i - d_i \Delta u_i = c_i(t, x) u_1^\alpha u_2^\beta,$$

where we define

$$\begin{aligned} c_i(t, x) &:= \frac{A_i(T-t)^2 + B_i(T-t)r^2 + C_i r^4}{(T-t+r^2)^{\gamma+2}} u_1^{-\alpha} u_2^{-\beta}, \\ \Rightarrow c_i(t, x) &= \frac{(T-t+r^2)^{\gamma(d-1)-2} [A_i(T-t)^2 + B_i(T-t)r^2 + C_i r^4]}{[a_1(T-t) + b_1 r^2]^\alpha [a_2(T-t) + b_2 r^2]^\beta}. \end{aligned}$$

Obviously  $c_i$  is  $C^\infty$  on  $[0, T] \times \overline{B_N}$ . Let us check that it is in  $L^\infty(Q_T)$  as follows. Let  $H := T - t + r^2$ . We use

$$T - t \leq H, r^2 \leq H, a_i(T-t) + b_i r^2 \geq \min\{a_i, b_i\}H.$$

Thus for some  $K_1, K_2 \in (0, +\infty)$ , and using that  $\gamma(d-1) - 2 \geq \gamma(d-1) - d \geq 0$ , we have

$$|c_i(t, x)| \leq \frac{H^{\gamma(d-1)-2} [|A_i| + |B_i| + |C_i|] H^2}{K_1 H^{\alpha+\beta}} \leq K_2 H^{\gamma(d-1)-d} \leq K_2 (T+1)^{\gamma(d-1)-d}.$$

3 Finally, we see that the sign of  $c_1 + c_2$  is the same as the sign of  $A(T-t)^2 + B(T-t)r^2 + Cr^4$ ,  
 4 that is  $c_1 + c_2 \leq 0$  since  $A < 0, C < 0, B^2 - 4AC < 0$  by Lemma 3.3. This ends the proof  
 5 of Theorem 2.2.

6 **6. Proof of Theorem 2.3.** Let  $u_1, u_2$  be the solutions obtained in Theorem 2.1. Their  
 7 normal derivatives on  $\partial B_N$  are given as follows

$$\begin{aligned} \partial_r u_1(t, 1) &= \frac{2[(b_1 - \gamma a_1)(T-t) - (\gamma-1)b_1]}{(T-t+1)^{\gamma+1}}, \quad \partial_r u_2(t, 1) = \frac{2[(b_2 - \gamma a_2)(T-t) - (\gamma-1)b_2]}{(T-t+1)^{\gamma+1}}, \\ b_1 &= N^{-1/2} = b_2, \quad b_1 - \gamma a_1 = N^{-1/2} - 2N^{-2} + O(N^{-3}), \\ b_2 - \gamma a_2 &= -4N + O(1). \end{aligned} \quad (46)$$

8 Obviously for  $N$  large

$$\begin{cases} \partial_r u_1(t, 1) \leq 0, \quad \forall t \in [\tau_0, T], \quad \tau_0 := (T - \gamma + 1)^+ (< T), \\ \partial_r u_2(t, 1) \leq 0, \quad \forall t \in [0, T]. \end{cases} \quad (47)$$

9 We now introduce the solutions  $\beta_i, i = 1, 2$  of

$$\begin{cases} \partial_t \beta_i - d_i \Delta \beta_i = 0 \text{ in } (\tau_0, T) \times B_N, \\ \partial_\nu \beta_i = -\partial_r u_i(t, 1) \text{ on } (\tau_0, T) \times \partial B_N, \\ \beta_i(\tau_0, \cdot) \equiv 0. \end{cases} \quad (48)$$

By the  $C^\infty$ -property of  $\partial_r u_i(t, 1)$ , these solutions are also in  $C^\infty(\overline{Q_T})$ . Moreover, thanks to (47), and by maximum principle, we have  $\beta_i(t, x) \geq 0$  on  $[\tau_0, T] \times B_N$ . We now introduce for  $i = 1, 2$

$$v_i(t, x) := u_i(t, x) + \beta_i(t, x) \geq 0 \text{ so that } \partial_\nu v_i = \partial_r v_i(t, 1) = 0 \text{ on } \Sigma_T.$$

And according to Theorem 2.1 and the definition of  $\beta_i$ , we have

$$\partial_t v_i - d_i \Delta v_i = \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2) = g_i(t, x, v_1, v_2) \text{ in } Q_T, \quad i = 1, 2,$$

where we set

$$g_i(t, x, s_1, s_2) := f_i(s_1 - \beta_1(t, x), s_2 - \beta_2(t, x)).$$

Here the functions  $f_i$  are extended by 0 outside  $[0, +\infty)^2$  and with the help of (29), we see that they are  $C^\infty$  on  $\mathbb{R}^2$ . The functions  $g_i$  defined in this way satisfy  $(\mathbf{P}')$ ,  $(\mathbf{M}'_\lambda)$ ,  $(\mathbf{G}')$  since, by nonnegativity of  $\beta_i$ :

$$\begin{aligned} g_1(t, x, 0, s_2) &= f_1(-\beta_1(t, x), s_2 - \beta_2(t, x)) = 0, \\ g_2(t, x, s_1, 0) &= f_2(s_1 - \beta_1(t, x), -\beta_2(t, x)) = 0, \\ (g_1 + \lambda g_2)(t, x, s_1, s_2) &= (f_1 + \lambda f_2)(s_1 - \beta_1(t, x), s_2 - \beta_2(t, x)) \leq 0 \text{ [using (6)]}, \\ |g_i(t, x, s_1, s_2)| &= |f_i(s_1 - \beta_1(t, x), s_2 - \beta_2(t, x))| \\ &\leq C_0 + C_1[(s_1 - \beta_1(t, x))^+ + (s_2 - \beta_2(t, x))^+]^d \\ &\Rightarrow |g_i(t, x, s_1, s_2)| \leq C'_0 + C'_1|s_1 + s_2|^d, \end{aligned}$$

1 where  $C'_0, C'_1$  depend on  $C_0, C_1, d, \|\beta_i\|_{L^\infty(Q_T)}$ . Finally, the  $L^m(B_N)$ -norm of  $v_i(t)$  blows  
2 up as  $t \rightarrow T^-$  together with the  $L^m(B_N)$ -norm of  $u_i(t)$ . This ends the proof of Theorem  
3 **2.3**.

4 **7. About global weak solutions.** Let us prove here that the solutions  $u_1, u_2$  obtained  
5 in Theorem 2.1 may be extended to global weak solutions of the same system. For this we  
6 will use the approach of the previous section and, as a first step, we extend the functions  
7  $v_1, v_2$  to weak solutions on  $[0, +\infty)$ . This will provide extensions of  $u_1, u_2$  as well.

8 Let us start by extending the functions  $\beta_i$  as the solutions on  $[0, +\infty)$  of

$$\begin{cases} \partial_t \beta_i - d_i \Delta \beta_i = 0 \text{ in } (\tau_0, +\infty) \times B_N, \\ \partial_\nu \beta_i = \begin{cases} -\partial_r u_i(t, 1) \text{ on } (\tau_0, T) \times \partial B_N \\ -\partial_r u_i(T, 1) \text{ on } (T, +\infty) \times \partial B_N, \end{cases} \\ \beta_i(\tau_0, \cdot) \equiv 0. \end{cases} \quad (49)$$

These extensions are  $C^\infty$  on  $[\tau_0, +\infty) \times \overline{B_N}$ . We also extend the nonlinear function  $g_i$  as follows for  $(t, x) \in (\tau_0, +\infty) \times \overline{B_N}$

$$g_i(t, x, s_1, s_2) := f_i(s_1 - \beta_1(t, x), s_2 - \beta_2(t, x)), \quad \forall (s_1, s_2) \in [0, +\infty)^2.$$

9 They satisfy the same properties  $(\mathbf{P}')$ ,  $(\mathbf{M}'_\lambda)$ ,  $(\mathbf{G}')$  on  $[\tau_0, +\infty)$ . We now apply Theorem  
10 5.9 in [10] to the system

$$\begin{cases} \partial_t v_1 - d_1 \Delta v_1 = g_1(t, x, v_1, v_2) \text{ in } (\tau_0, +\infty) \times B_N, \\ \partial_t v_2 - d_2 \Delta v_2 = g_2(t, x, v_1, v_2) \text{ in } (\tau_0, +\infty) \times B_N, \\ \partial_\nu v_1 = 0 = \partial_\nu v_2 \text{ on } (\tau_0, +\infty) \times B_N, \\ v_i(\tau_0, \cdot) = v_i^0 = u_i(\tau_0, \cdot) + \beta(\tau_0, \cdot). \end{cases} \quad (50)$$

Indeed, thanks to the property  $(\mathbf{M}'_\lambda)$ , any good approximation of this system with bounded nonlinearities  $g_i^n$  in place of  $g_i$  and the same regular data at  $t = \tau_0$ , will provide approximate regular solutions  $v_i^n$  such that, for all  $S \in [\tau_0, +\infty)$

$$\sup_{n \geq 1} \int_{(\tau_0, S) \times B_N} |g_i^n(t, x, v_1^n(t, x), v_2^n(t, x))| < +\infty, \quad i = 1, 2,$$

We may for instance choose

$$g_i^n(t, x, s_1, s_2) := \frac{g_i(t, x, s_1, s_2)}{1 + n^{-1}[(|g_1| + |g_2|)(t, x, s_1, s_2)]},$$

so that  $|g_i^n| \leq n$ . These approximate functions  $g_i^n$  satisfy also  $(\mathbf{P}')$ ,  $(\mathbf{M}'_\lambda)$  and it is easy to prove that these properties imply the above  $L^1((\tau_0, S) \times B_N)$ -bounds (see e.g. Proposition 5.1 in [10]). As a consequence, and as proved in [10], these approximate solutions  $v_i^n$  converge to a *weak solution* of (50) which means that

$$v_i(t) = S_i(t - \tau_0)v_i^0 + \int_{\tau_0}^t S_i(t - s)g_i(s, \cdot, v_1(s, \cdot), v_2(s, \cdot)) ds, \quad i = 1, 2,$$

11 where  $S_i(t)$  denotes the semigroup generated by the operator  $-d_i \Delta$  on  $B_N$  with homoge-  
12 neous Neumann boundary conditions.

These weak solution coincides on the interval  $[\tau_0, T)$  with the classical solution of (7) as found in Theorem 2.3. Indeed, since  $v_i^0 \in L^\infty(B_N)$ , it is classical to prove that the approximate solution  $(v_1^n, v_2^n)$  stays uniformly bounded, independently of  $n$ , on some interval  $[\tau_0, \tau_1] \subset [\tau_0, T]$ . To see it, we might for instance use the inequalities

$$\partial_t v_i^n - d_i \Delta v_i^n \leq |g_i(t, x, v_1^n, v_2^n)| \leq C'_0 + C'_1 |v_1^n + v_2^n|^d.$$

1 Thus the limit  $(v_1, v_2)$  is therefore a classical solution on  $[\tau_0, \tau_1]$ . By uniqueness of classical  
 2 solutions for the reaction-diffusion system (7), both solutions coincide at least on  $[\tau_0, \tau_1]$ ,  
 3 and subsequently on the whole interval  $[\tau_0, T]$  by a classical continuity argument.

Finally, if we now set  $u_i(t, x) := v_i(t, x) - \beta_i(t, x), i = 1, 2$  on the whole domain  $[\tau_0, +\infty) \times \overline{B_N}$ , we find a *global weak solution* of the initial system (1) on  $[0, +\infty)$  which extends the (classical) solution obtained on  $[0, T)$  in Theorem 2.1. It remains to get convinced that  $u_i \geq 0$  for all  $t$ . This can be seen by noticing that  $u_i^n := v_i^n - \beta_i$  is a classical solution on  $[\tau_0, +\infty)$  of

$$\partial_t u_i^n - d_i \Delta u_i^n = g_i^n(t, x, v_1^n, v_2^n) = f_i^n(u_1^n, u_2^n),$$

4 with boundary conditions  $\partial_\nu u_i^n = -\partial_r \beta_i \geq 0$  with a quasi-positive nonlinearity  $(f_1^n, f_2^n)$ .  
 5 Thus  $u_i^n \geq 0$  and this is preserved at the limit.

**8. Some more explicit examples in small dimensions.** We use here Lemma 3.1 and Lemma 3.2. All blow up examples we are going to describe here are defined as in (9) so that they blow up as  $t \rightarrow T^-$ . Moreover by Lemma 3.2, we know that we can then find  $P_1, P_2$  so that (19) holds, namely

$$\partial_t u_i - d_i \Delta u_i = P_i(u_1, u_2), \text{ in } Q_T, \quad i = 1, 2.$$

6 The goal is then to choose all parameters involved in the definition (9) well enough so that  
 7 (24) holds, namely

$$A < 0, \quad C < 0, \quad B^2 - 4AC < 0, \tag{51}$$

8 where  $A, B, C$  are defined in Lemma 3.1. Then, exactly as in the proof of Theorem 2.1,  
 9 we prove that the previous inequalities implies that  $P_1 + P_2 \leq 0$  and that  $P_1, P_2$  can then  
 10 slightly be modified into functions  $f_1, f_2$  satisfying (P), (M) and (1). Moreover the growth  
 11 at infinity of  $f_1, f_2$  is at most  $\gamma'$ .

12 The following technical proposition is very useful to help finding examples of blow up.  
 13 Notations are as in Lemma 3.1.

14 **Proposition 8.1.** *Let  $a, b, E, G \in (0, +\infty)$ . Then we can find  $a_i, b_i, d_i, i = 1, 2 \in (0, +\infty)$   
 15 such that*

$$a_1 + a_2 = a, \quad b_1 + b_2 = b, \quad b_1 d_1 + b_2 d_2 = E, \quad a_1 d_1 + a_2 d_2 = G, \tag{52}$$

16

$$a_1 b_2 - a_2 b_1 = \frac{aE - bG}{d_2 - d_1}. \tag{53}$$

**Proof of Proposition 8.1.** Let us choose  $d_1, d_2$  such that

$$0 < d_1 < \min\{Eb^{-1}, Ga^{-1}\} \leq \max\{Eb^{-1}, Ga^{-1}\} < d_2.$$

Then, given the four parameters  $a, b, E, G$ , a solution to (52) is given by

$$a_1 = \frac{ad_2 - G}{d_2 - d_1}, \quad a_2 = \frac{G - ad_1}{d_2 - d_1}, \quad b_1 = \frac{bd_2 - E}{d_2 - d_1}, \quad b_2 = \frac{E - bd_1}{d_2 - d_1}.$$

17 We easily check that, with this choice, the relations (52), (53) are satisfied and all param-  
 18 eters are positive. The relation (53) will be used to check (18).

19 **Remark 8.1.** In what follows, we give several explicit choices of parameters which lead  
 20 to blow up examples according to the analysis of the previous sections. Thanks to the  
 21 Proposition 8.1, it is sufficient to provide the values of  $a, b, E, G$  such that (51) holds for  
 22 the corresponding values of  $A, B, C$ .

8.1. **Blow up with a cubic growth in dimension  $N = 3$ .** Here we choose with  $\epsilon \in (0, 1)$  small enough,

$$\gamma = 3/2, a = 10, b = 1, E = 1, G = \epsilon.$$

Then,  $aE - bG = 10 - \epsilon \neq 0$  and

$$A = (\gamma - 1)a - 6E + 6\gamma G = 5 - 6 + 9\epsilon = -1 + 9\epsilon < 0,$$

$$C = -a + \gamma b + 2(\gamma - 1)(3 - 2\gamma)E = -10 + 3/2 + 0 = -17/2 < 0,$$

$$B = (\gamma - 2)a + \gamma b + [6(\gamma - 2) + 8\gamma]E - 2\gamma[2(\gamma + 1) - 3]G = 11/2 - 6\epsilon.$$

$$B^2 - 4AC = \frac{11^2}{4} - 4\frac{17}{2} + 0(\epsilon) = -15/4 + O(\epsilon) < 0.$$

- 1 Thus (51) holds so that this provides a blowing up example with growth  $\gamma' = 3$ . By  
 2 Lemma 3.1, for  $m \geq 3$ , the  $L^m(B_N)$ -norm of the solution blows up as  $t \rightarrow T^-$ . Note  
 3 that since all inequalities are strict in this example, we could slightly modify it to obtain  
 4 a blowing up example with growth  $3 - \sigma$  for some small  $\sigma > 0$ .

8.2. **Blow up with a 7/2-growth in dimension  $N = 2$ .** An example with such a growth was already mentioned in [12]. We give here another one, easier to compute. Let us choose the following parameters with  $\epsilon > 0$  small enough.

$$\gamma = 7/5, a = 4, b = \epsilon, E = 1, G = \epsilon.$$

then  $aE - bG = 4 - \epsilon^2 \neq 0$  and

$$A = \frac{2}{5}a - 4E + \frac{28}{5}G = \frac{8}{5} - 4 + O(\epsilon) = -\frac{12}{5} + O(\epsilon) < 0,$$

$$C = -a + \frac{7}{5}b - \frac{16}{25}E = -4 - \frac{16}{25} + O(\epsilon) = -\frac{116}{25} + O(\epsilon) < 0,$$

$$B = -\frac{3}{5}a + \frac{7}{5}b + \frac{44}{5}E - \frac{14^2}{25}G = -\frac{12}{5} + \frac{44}{5} + O(\epsilon) = \frac{32}{5} + O(\epsilon).$$

$$B^2 - 4AC = \frac{32^2}{25} - 4\frac{12}{5}\frac{116}{25} + O(\epsilon) = \frac{1}{125}[5120 - 5568] + O(\epsilon) = -\frac{448}{125} + O(\epsilon) < 0.$$

- 5 Thus this provides a blow up example with growth  $\gamma' = 7/2$  and an  $L^m(B_N)$ -norm  
 6 blowing up for  $m \geq 5/2$ . Again, since all inequalities are strict, we could find another  
 7 example with growth  $7/2 - \sigma$  for some  $\sigma > 0$ .

8.3. **Blow up with a 6-growth in dimension  $N = 1$ .** Here we choose

$$\gamma = 6/5, a = 5, b = 1/2, E = 1, G = 1/10.$$

then  $aE - bG = 5 - 1/20 \neq 0$  and

$$A = 1 - 2 + 6/25 = -19/25 < 0,$$

$$C = -5 + 3/5 + 2(1/5)(1 - 12/5) = -22/5 - 14/25 = -124/25 < 0,$$

$$B = -4 + 3/5 + 8 - 12.17/250 = 473/125.$$

$$125^2[B^2 - 4AC] = 473^2 - 4.95.620 = 223729 - 235600 = -11871 < 0.$$

- 8 This provides a blow up example with growth  $\gamma' = 6$  and with an  $L^m(B_N)$ -norm  
 9 blowing up for  $m \geq 5/2$ . As above again, we could improve it to a lower growth  $6 - \sigma$   
 10 for some  $\sigma > 0$ .

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