



HAL
open science

A Quasi-Normal Scale Elimination theory of turbulent flows with stable stratification

Semion Sukoriansky, Boris Galperin

► **To cite this version:**

Semion Sukoriansky, Boris Galperin. A Quasi-Normal Scale Elimination theory of turbulent flows with stable stratification. CFM 2007 - 18ème Congrès Français de Mécanique, Aug 2007, Grenoble, France. <hal-03359845>

HAL Id: hal-03359845

<https://hal.science/hal-03359845v1>

Submitted on 30 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

A Quasi-Normal Scale Elimination (QNSE) theory of turbulent flows with stable stratification

Sukoriansky Semion & Galperin Boris

*Ben-Gurion University of the Negev
Department of Mechanical Engineering
Beer-Sheva 84105, Israel
and
University of South Florida
College of Marine Science
St. Petersburg, Florida, USA
semion@bgu.ac.il*

Abstract :

A new spectral model of turbulent flows with stable stratification is presented. The theory is based upon a mapping of the actual velocity field to a quasi-Gaussian field using the Langevin equation. The parameters of the mapping are calculated using a systematic process of successive averaging over small shells of velocity and temperature modes that eliminates them from the equations of motion. This procedure does not differentiate between turbulence and internal waves and accounts for their combined effect. This approach offers a powerful mathematical tool for dealing with previously nearly intractable aspects of anisotropic turbulence; among these aspects are the threshold criterion for generation of internal waves and the modification of their dispersion relation by turbulence. The process of successive small scales elimination results in a model describing the largest scales of a flow. Partial scale elimination yields subgrid-scale viscosities and diffusivities that can be used in large eddy simulations. The elimination of all fluctuating scales results in RANS models. The model predicts various important characteristics of stably stratified flows, such as the dependence of the vertical turbulent Prandtl number on Froude and Richardson numbers, anisotropization of the flow field, and decay of vertical diffusivity under strong stratification, all in good agreement with computational and observational data. The theory also yields analytical expressions for various 1D and 3D kinetic and potential energy spectra that reflect the effects of waves and anisotropy. The model's results are suitable for immediate use in practical applications.

Key-words :

Stable stratification; anisotropy; quasi-normal closure

1 Introduction

Vertical-horizontal anisotropy and turbulence-internal wave interaction are among the most fundamental problems one encounters in modeling of turbulent flows with stable stratification. Governed by strong nonlinearity, these two phenomena have been beyond reach for analytical theories. Putting aside direct simulations, which at present are feasible only under idealized conditions and for simple domains, one generally stipulates “closure” assumptions that relate unknown correlations to the mean and known correlations, the approach known as the Reynolds-Averaged Navier-Stokes, or RANS modeling. Although a great deal of progress has been made with RANS models, they fail in strongly anisotropic flows. Among the deficiencies of RANS models is that the models’ coefficients are neither constant nor universal. Attempts have been made to derive equations for these coefficients in terms of the Reynolds stress tensor invariants. However, the complexity of these equations makes this approach impractical. Another deficiency of the RANS models is poor representation of anisotropic flows with waves. Scale-dependent physics of such flows cannot be accommodated in the procedure of ensemble-

averaging that lumps together processes on all scales. This weakness hampers the performance of RANS models and ensuing subgrid-scale (SGS) representations in geophysical applications where flows are usually strongly anisotropic and affected by various types of waves.

Sukoriansky et al. (2005) developed an alternative approach of studying the large-scale properties of turbulent flows with stable stratification using the spectral space representation and employing a self-consistent algorithm of successive elimination of small subsets of fast modes in the assumption of quasi-normality. This process gradually “shrinks” the domain of explicitly resolved scales and produces corrections to the “eddy” viscosity and diffusivity resulting from that shrinking. Partial scale elimination yields a SGS representation for large-eddy simulation (LES) models while the elimination of all fluctuating scales results in an equivalence of a RANS model. In both LES and RANS formats, the Quasi-Normal Scale Elimination (QNSE)-based SGS viscosities and diffusivities account for the effects of anisotropy and waves. In stably stratified flows, a QNSE method predicts turbulence anisotropization with increased strength of stratification and shows how the dispersion relationships for linear internal waves are modified by turbulence. The method yields the internal wave frequency shift, the threshold criterion of internal wave generation in the presence of turbulence, and various one-dimensional and three-dimensional spectra that reflect the effects of waves and anisotropy. All this information is unavailable in conventional RANS models.

The use of the quasi-normality assumption in the QNSE method is not new; it has been utilized in other spectral theories of turbulence [e.g. Orszag (1977), McComb (1991)]. Although higher-order statistical moments in neutral turbulence point to a deviation from the Gaussianity, this assumption works reasonably well for the second moments which are most important for the calculation of eddy viscosity and diffusivity. Recent direct numerical simulations (DNS) of stably stratified flows indicate that in these flows, the hypothesis of Gaussianity holds even better than in neutral flows [Kimura & Herring (1996)].

One may argue that spectral models have not gotten much “traction” in practical applications over the years. Usually, these models consider all Fourier modes at once, and when the flow field is strongly anisotropic and embeds waves, the analytical derivations become nearly intractable. For this reason, such models can only deal with weak anisotropy. By contrast, in the QNSE model, we successively eliminate small shells of wave numbers thus allowing the effects of anisotropy to gradually accumulate towards the large scales. The mathematical burden is still significant but the analytical derivations can be carried out in full yielding closed expressions for horizontal and vertical eddy viscosities and eddy diffusivities for strongly anisotropic turbulent flows with waves [Sukoriansky et al. (2005)]. The final product of the QNSE theory in a RANS format is a set of equations for the horizontal and vertical eddy viscosities and eddy diffusivities as functions of the gradient Richardson number that can be readily used in various atmospheric and ocean models.

2 Basics of the QNSE theory of turbulent flows with stable stratification

The theory is developed for a fully three-dimensional turbulent flow field with imposed vertical, stabilizing temperature gradient. The flow is governed by the momentum, temperature and continuity equations in the Boussinesq approximation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} - \alpha g T \hat{e}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathbf{f}^0 \quad (1)$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \nabla) T + \frac{d\Theta}{dz} u_3 = \kappa_0 \nabla^2 T \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

where \mathbf{u} and T are the fluctuating velocity and potential temperature, respectively; P is the pressure, ρ is the constant density, ν_0 and κ_0 are the molecular viscosity and diffusivity, respectively, α is the thermal expansion coefficient, \mathbf{g} is the gravity acceleration directed downwards, $d\Theta/dz$ is the mean potential temperature gradient, and \mathbf{f}^0 represents a large-scale external energy source customarily used in spectral theories of turbulence; it maintains turbulence in statistically steady state and may originate from large-scale shear instabilities. According to Kolmogorov theory of turbulence, the net effect of this forcing is communicated to the fluid via the rate of the energy injection at large scales. Through nonlinear interactions, the external forcing excites all Fourier modes down to the dissipation scale $k_d = (\varepsilon/\nu_0^3)^{1/4}$ (ε is the dissipation rate). The modes exert indiscriminant, random agitation upon each other, which is accompanied by simultaneous random damping. In statistically steady state, the processes of forcing and damping balance each other; in other words, every Fourier mode $u_i(\mathbf{k}, \omega)$ (\mathbf{k} and ω being Fourier space wave number and frequency, respectively) receives and loses equal (in a statistical sense) amounts of energy. Reflecting the duality between nonlinear interactions and stochastic forcing and damping, the nonlinear term in Eq. (1) can be replaced by a random, *modal* forcing f_i and damping represented by turbulent viscosity. The resulting linear equation with stochastic forcing and damping is known as the Langevin equation,

$$u_i(\mathbf{k}, \omega) = G_{ij}(\mathbf{k}, \omega) f_j(\mathbf{k}, \omega) \quad (4)$$

Here, $G_{ij}(\mathbf{k}, \omega)$ is the velocity response (or Green function). The Langevin equation thus replaces the original nonlinear Navier-Stokes equation by a linear, forced, stochastic equation in which the energy budget is systematically adjusted for every Fourier mode [Kraichnan (1987)]. For the temperature, a similar Langevin-type equation can be derived; the random forcing in this case is due to the vertical velocity fluctuation [Sukoriansky et al. 2005a],

$$T(\mathbf{k}, \omega) = -\frac{d\Theta}{dz} G_T(\mathbf{k}, \omega) u_3(\mathbf{k}, \omega) \quad (5)$$

Here, $G_T(\mathbf{k}, \omega)$ is the temperature response function. The major closure assumption is that the forcing \mathbf{f} is quasi-Gaussian, as in Kraichnan's (1959) theory of direct interaction approximation (DIA) or Orszag's (1977) eddy-damped, quasi-normal, Markovian approximation (EDQNM). The assumption of quasi-normality enables one to derive expressions for eddy viscosity and eddy diffusivity. The fluctuating velocity field is zero-mean and incompressible imposing the same restraints upon the forcing $\mathbf{f}(\mathbf{k}, \omega)$ in (4); in the assumption of quasi-Gaussianity, this forcing is fully determined by its two-point, two-time correlation function. The choice of this correlation function is guided by general considerations of spatial and temporal homogeneity and incompressibility, which almost completely determine its functional form [Sukoriansky et al. (2003)]. The correlation function of the forcing $\mathbf{f}(\mathbf{k}, \omega)$ accounts for the statistically mean energy input to a given mode \mathbf{k} via its interaction with all other modes such that its amplitude, D , is proportional to the mean rate ε of the energy transfer through that mode. The balance between the energy gain due to eddy forcing and the energy loss due to eddy damping enables one to relate the forcing amplitude D to ε and the buoyancy destruction. In the neutral case, this approach recovers some basic features of isotropic homogeneous turbulence including the Kolmogorov spectrum and the Kolmogorov constant [Sukoriansky et al. (2003)].

3 Some results from the QNSE model

Rigorous derivations of $G_{ij}(\mathbf{k}, \omega)$ and $G_T(\mathbf{k}, \omega)$ from the original system (1-3) are given in Sukoriansky et al. (2005). From these functions, we obtain the following dispersion relation for internal waves in the presence of turbulence:

$$\omega^2 = N^2 \sin^2 \theta \left[1 - \left(\frac{k}{k_o} \right)^{4/3} \left(\frac{\frac{\kappa_z - \nu_z}{\nu_n} \cos^2 \theta + \frac{\kappa_h - \nu_h}{\nu_n} \sin^2 \theta}{4 \sin \theta} \right)^2 \right] \quad (6)$$

Here, k_h and k_z are the horizontal and vertical wave numbers; ν_h , κ_h and ν_z , κ_z are the horizontal and vertical eddy viscosities and diffusivities, respectively; $N=[\alpha g(d\Theta/dz)]^{1/2}$ is the Brunt-Väisälä frequency; θ is the angle between the vector \mathbf{k} and the vertical; $k_o = (N^3/\varepsilon)^{1/2}$ is the Ozmidov wave number, and ν_n is the eddy viscosity in the case of neutral stratification. In the limit $k/k_o \rightarrow 0$ (large scales; strong stratification), (6) reduces to the classical dispersion relation. Otherwise, Eq. (6) describes internal wave frequency shift due to turbulence. This shift is expressed in terms of the non-dimensional eddy viscosities and diffusivities, ν_h/ν_n , ν_z/ν_n , κ_h/ν_n and κ_z/ν_n (see Fig. 1 below). Eq. (6) shows that gravity waves exist only at wave numbers k smaller than a threshold value $k_t \equiv k_o |10 \sin \theta|^{3/2} \approx 32 k_o |\sin \theta|^{3/2}$. Physically, with decreasing scales, the turbulent overturn exerts increasingly disorganizing action on the wave motion that eventually overwhelms waves at smallest turbulent scales.

The algorithm of successive averaging over small shells of velocity and temperature modes utilizing the Langevin equations (4) and (5) yields a system of coupled non-linear ODEs for eddy viscosities and diffusivities [Sukoriansky et al. (2005)]. The computation starts at the Kolmogorov scale k_d where the initial values of ν_z and κ_z are equal to their respective molecular values, ν_0 and κ_0 , and is continued to an arbitrary wave number $k < k_d$. Figure 1 presents the solutions. At large k/k_o (small scales), all nondimensional parameters approach values typical of the neutral case. At small k/k_o , approximately at the threshold of internal wave generation, ν_h , κ_h , ν_z and κ_z depart markedly from their neutral values and from each other. Horizontal viscosity increases by a factor of 1.3 compared to the neutral case, while vertical viscosity decreases to approximately 1/6 of the horizontal viscosity value. The vertical viscosity ν_z preserves a finite asymptotic value even for very strong stable stratification. This “residual” momentum mixing is due to internal waves, which are an integral part of the QNSE model.

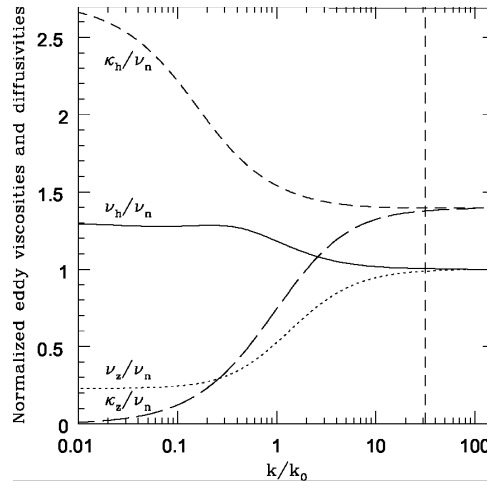


FIG. 1 – Normalized horizontal and vertical eddy viscosities and diffusivities as functions of k/k_o . The vertical dashed line shows the threshold of internal wave generation in the presence of turbulence, Sukoriansky et al. (2005)

In the limit of weak stratification, the QNSE model yields asymptotic expressions for ν_h , κ_h , ν_z and κ_z and allows one to analytically calculate various one-dimensional spectra that

quantify deviations from the Kolmogorov regime and characterize turbulence anisotropy. In particular, Sukoriansky et al. (2005) show that in the vicinity of k_0 , the vertical spectrum of the horizontal velocity undergoes transition from the Kolmogorov $-5/3$ slope to the -3 slope. The numerical coefficients in the transition equation agree well with LES by Carnevale et al. (2001) and with the measurements of the vertical spectrum of the horizontal shear by Gargett et al. (1981) [see Sukoriansky et al. (2006)]. Note that the analytical derivation of this transition has never before been obtained from a basic theory.

The scale-dependent viscosities and diffusivities are used to derive the SGS parameters in large-eddy simulations (LES). Let k_c be the wave number of the LES grid. Figure 1 indicates that the effect of stable stratification is small for $k_c \gg k_0$ but cannot be neglected for $k_c < k_0$. The process of small scales elimination can be extended to the largest scale of turbulence identified with the integral length scale, k_L^{-1} . One can then calculate ν_h/ν_n , ν_z/ν_n , κ_h/ν_n and κ_z/ν_n either as functions of the Froude number, $Fr = \varepsilon/KN$, where K is the turbulence kinetic energy, or, as functions of the gradient Richardson number $Ri = N^2/S^2$ (S is magnitude of the shear). Figure 2 shows that major features of the dependence on Fr and on Ri replicate those of the dependence on k/k_0 shown in Fig. 1.

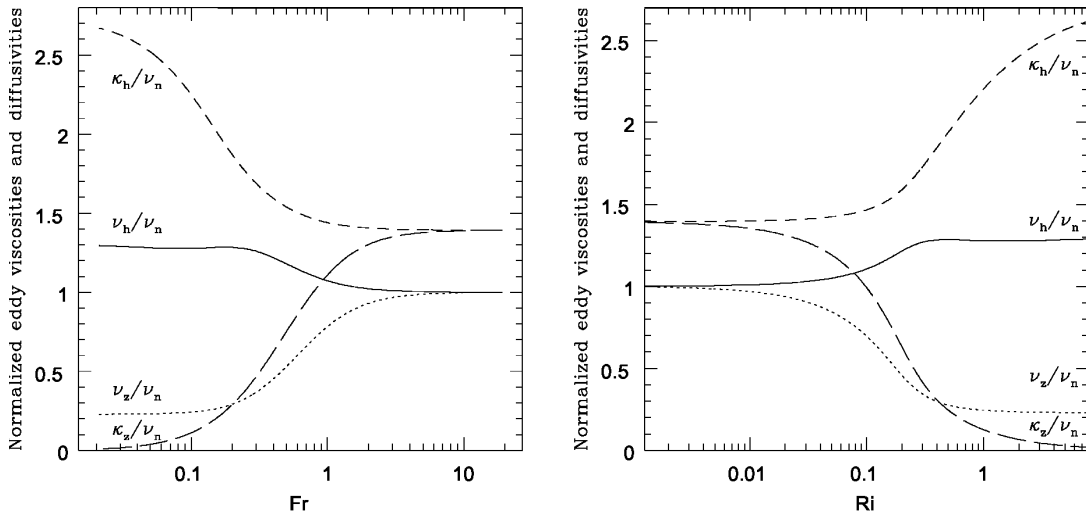


FIG. 2 – Normalized turbulent exchange coefficients as functions of the Froude number Fr (left panel) and the gradient Richardson number Ri (right panel).

Eddy viscosities and eddy diffusivities undergo the most significant change for Fr and Ri varying between 0.1 and 1. For $Fr < 1$ and $Ri > 0.1$, both vertical eddy viscosity and eddy diffusivity decrease, while the diffusivity decreases faster. Eventually, stable stratification may totally suppress vertical scalar mixing. The vertical momentum mixing continues even at relatively high values of Ri such that the “vertical” turbulent Prandtl number, $Pr_t = \nu_z/\kappa_z$, increases for $Ri > 0.1$, in agreement with extensive laboratory and observational data.

4 Conclusions

The basic assumption of the new spectral model of turbulence presented in this paper, the QNSE model, is the hypothesis of quasi-normality which leads to a self-contained procedure of consecutive coarse-graining. The model explicitly resolves the stable stratification-induced disparity between the transport processes in the horizontal and vertical directions and accounts for the combined effect of turbulence and waves. Anisotropic turbulent viscosities and diffusivities obtained from the model are in good agreement with experimental data. Various anisotropic spectra can be calculated analytically in the limit of weak stratification; one of the

results is the demonstration that the energy of the horizontal flow components increases at the expense of their vertical counterpart. For the first time, the transition from the Kolmogorov to the N^2k^{-3} spectral laws for the one-dimensional spectrum of the horizontal velocity, $E_1(k_3)$, has been derived analytically. The QNSE model yields modification of the classical dispersion relation for internal waves that accounts for turbulence. The QNSE theory provides SGS parameterizations that can be used in both LES and RANS models. The QNSE approach has been implemented in a $K-l$ and $K-\varepsilon$ models of stratified atmospheric boundary layers (ABL) yielding good agreement with CASES99, BASE and SHEBA datasets for cases of moderate and strong stable stratification. When implemented in the numerical weather forecast system HIRLAM, the QNSE-based vertical eddy viscosity and eddy diffusivity substantially improve the system's predictive skills (Sukoriansky et al. (2005)).

The QNSE model indicates that there is no single Richardson number at which turbulence is completely suppressed. Instead, there exists a range of Ri where the vertical eddy viscosity and eddy diffusivity undergo dramatic decrease but turbulence survives even at very large values of Ri. The QNSE-based transport models can be applied to engineering flows and ABL with and without effects of stratification using invariant set of constants. In this approach, the problem of the model realizability does not arise. Thus, the QNSE-based RANS models present a viable alternative to the Reynolds stress closure models widely used in meteorological, oceanographic and engineering applications.

Acknowledgements. This research has been partially supported by the US Army Research Office under Grant Nos. DAAD19-01-1-0816 and W911NF-05-1-0055 and Israel Science Foundation under Grant No. 134/03.

References

- Gargett, A., P., Hendricks, T., Sanford, T., Osborn & Williams, A. III, 1981 A composite spectrum of vertical shear in the upper ocean. *J. Phys. Oceanogr.*, **11**, 1258–1271.
- Kimura, Y. & J. R. Herring 1996 Diffusion in stably stratified turbulence. *J. Fluid Mech.*, **328**, 253-263.
- Kraichnan, R., 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.*, **5**, 497–543.
- Kraichnan, R. 1987 An interpretation of the Yaghot-Orszag turbulence theory. *Phys. Fluids*, **30**, 2400–2405.
- McComb, W. 1991 *The Physics of Fluid Turbulence*. Oxford University Press.
- Orszag, S. 1977 Statistical theory of turbulence. *Les Houches Summer School in Physics*, (ed. R. Balian and J.-L. Peabe), pp. 237–374. Gordon and Breach.
- Sukoriansky, S., Galperin, B. & Staroselsky, I. 2003 Cross-terms and ε -expansion in RNG theory of turbulence. *Fluid Dyn. Res.*, **33**, 319–331.
- Sukoriansky, S., Galperin, B. & Staroselsky, I. 2005 A quasinormal scale elimination model of turbulent flows with stable stratification. *Phys. Fluids*, **17**, 085107.
- Sukoriansky, S., Galperin, B. & Perov, V. 2006 A quasi-normal scale elimination model of turbulence and its application to stably stratified flows. *Nonlin. Proc. Geophys.*, **13**, 9-22.
- Sukoriansky, S., Galperin, B. & Perov, V. 2005 Application of a new spectral theory of stably stratified turbulence to the atmospheric boundary layer over sea ice. *Boundary- Layer Meteorology*, **117**, 231-257.