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# Application of variational calculus to parameter estimation in a real hydrological system <sup>\*</sup>

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## Abstract

This paper presents an estimation study based on calculus of variations for parameters in a real hydrological system, namely Tondi-Kiboro catchment, in Niger. The considered dynamical representation for this system is first given, from the well-known Saint-Venant equations under some simplifying assumptions. A cost function is then defined based on the gap between measured and predicted water discharges at the real sensor position, and the calculus of variations along with the first order optimality condition are used to find the gradients of Lagrangian objective functional with respect to the parameters to be estimated. Depending on the infiltration model used in the system model, two applications of this adjoint-based approach are finally presented: the first one only estimates the friction coefficient, with the assumption that infiltration is described by the physical so-called Green-Ampt model, while the second one additionally estimates the parameters of an alternative infiltration representation called Horton model. Some simulation-based validation and robustness discussion concludes the paper.

*Keywords:* Parameter estimation; Adjoint method; Overland flow; Green-Ampt infiltration model; Horton infiltration model; Saint-Venant model

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## 1. Introduction

### 1.1. Environmental challenges at Tondi-Kiboro catchment

In recent decades, scientists have observed a serious problem of hydrological changes, with strong rainfall decrements and significant modifications in the land

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use noted all around the world (see [1]). This is in particular what happens in the *Tondi-Kiboro* catchment of the Sahelian desert area in Niger. The water cycle is indeed strongly modified there, and the related natural changes, like vegetation clearing and topsoil debasement [2] at this location create a lot of difficulties for the population, especially in agricultural productivity and crops. One of the main components in those phenomena is the *overland flow* which is generated by the rainfall and plays an essential role in natural water cycle. Mathematically, its dynamics can be modeled by hyperbolic partial differential equations of Saint-Venant type [3]. More details about considered water dynamics will be discussed in section 2.1.

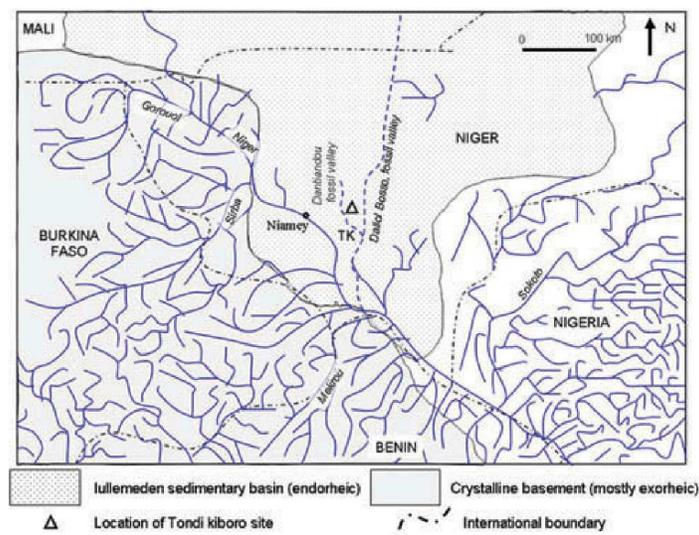


Figure 1: Location of Tondi Kiboro catchment [4].

As for geometric and hydrological characteristics of Tondi-Kiboro catchment, they can be briefly described as follows [4]: the catchment is located near Tondi Kiboro village, 70 km east of the Niamey region, southwest Niger (as shown in Fig. 1). It consists of three small basins. Two of them are nested as illustrated in Figure 2 with surfaces of  $46\,800\text{ m}^2$  and  $63\,720\text{ m}^2$  for the upper and lower basins, respectively. Only the upper basin is taken into account in this paper. Its length is about 600 m approximately, between upstream and downstream ends, and its bed slope  $S_0$  is about 3.5%. The components of the top soil are approximately 10% of silt and clay, 90% of sands. They play an important role on vegetation, crop growth, infiltration and overland processes. They also contribute significantly to fix the value of the so-called Manning coefficient in friction.

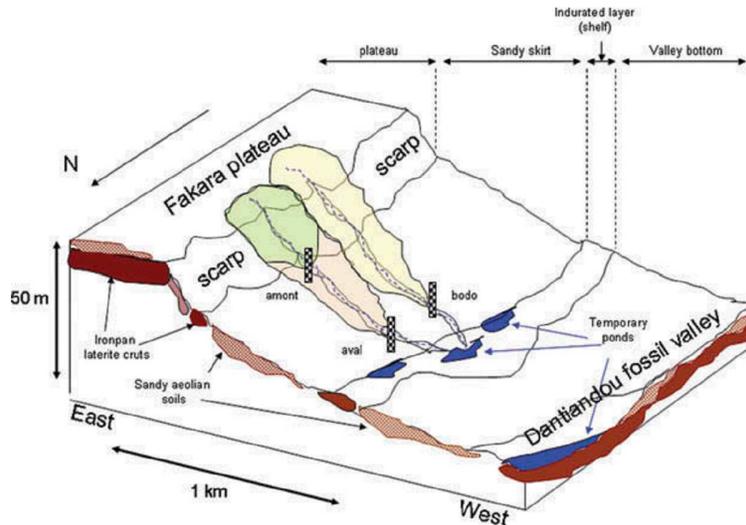


Figure 2: Overview of Tondi Kiboro basins [4].

### 1.2. State-of-the-art and paper contribution

Tondi Kiboro catchment has been attracting attention of scientists in hydrology for several years, with studies like [5] (investigating land use change and its hydrogeomorphological consequences) or [2] (providing analysis about the effect of water discharge on land use change). In those references, there is no dynamical model of water flow, and some of the parameters which are used - including Manning roughness coefficient of the soil, characterizing the friction of water flow on the soil surface, or coefficients of infiltration process - are in general not easily, nor accurately, known. Roughness is for instance usually chosen empirically (from Chow's Manning table in [3]) and adjusted to adapt to the specific context if necessary, or obtained from some empirical calculations applied to measured data. Its value normally depends on a lot of factors such as soil surface characteristics, water topology, vegetation and crop. On the other hand, infiltration process in a real catchment is usually complicated, and can be seen as an empirical process. Some quantities of infiltration can be measured by sensor (e.g. infiltration rate, cumulative infiltration rate) and from these measurements, the dynamics of infiltration may be recovered by using fitting techniques. The aim of the present paper is to propose a *direct estimation method, on the basis of calculus of variations, to identify Manning coefficient and infiltration parameters.*

The application of calculus of variations technique, or adjoint state method, for both state and estimation has been addressed by various studies in the domain of hydrology. One can cite here for instance: [6] (estimating the bottom friction coefficient and the water depth in the tidal flow model), [7] (providing some aspects of stability to validate the estimated result and a survey on

the use of calculus of variations for inverse problems in oceanography and meteorology), [8] (proposing a gradient-based approach to identify the spatially distributed Manning coefficient in the shallow water equations), [9] and [10] (using the same approach for estimating also the Manning roughness but on the Asahi River in Japan during flooding in 2011 and for flood hydrograph modeled by shallow-water equations). Work [11] applied the adjoint method on the three-dimensional cohesive sediment transport model of Hangzhou Bay to estimate both parameters and states of the system and [12] shared the same approach but using the data from the Lower Yellow River in China. In [13], the adjoint method was also applied on a two-dimensional tidal model to estimate the spatially varying bottom friction coefficient of the Bohai Sea, the Yellow Sea and the East China Sea (with data provided by the TOPEX/Poseidon altimeter).

More specially, the work of Castaings et al. in [14], where the applicability of variational calculus for parameter estimation in distributed overland flows is analyzed and illustrated by a numerical example, shares the same idea as the one of parameter estimation in an overland flow investigated in the present study. However, in addition to be limited to a simulation study, the estimation method proposed in [14] addresses the different Thoré catchment, and uses Saint-Venant equation only considering Green-Ampt model for infiltration.

In the present paper, the adjoint method is considered again, but in such a way that it allows to deal with parameter estimation in different cases of infiltration model, for the Tondi-Kiboro catchment: physical Green-Ampt model and empirical Horton model. The estimated results in those two cases help to compare the applicability of these models on the considered catchment. Notice that the adjoint approach for state and parameter estimation was already investigated in some of our former works [15, 16, 17, 18], but with synthetic measurements taken from simulations only. Notice also that some inverse problems on this basin were previously considered in two publications of ours as well: [19], dealing with Manning roughness estimation only, and [20] additionally handling parameters of an infiltration model, but considering Horton case only. The present study thus recasts those former ones in a single frame, and extends them, with: 1) an appropriate general development of adjoint-based parameter estimation method for a class of 1-D hyperbolic system, allowing to handle identification of Manning coefficient for two cases of infiltration model - Green-Ampt and Horton ones, in overland flow dynamics; 2) an application on experimental data from a real catchment, considering and comparing the two cases of infiltration models; 3) a discussion on robustness issues.

The paper goes on as follows: Section 2 provides the mathematical modeling of the system, and states the related estimation problem. Section 3 presents the full development of variational calculus for the considered estimation problem, and section 4 gives corresponding application results with real data and some robustness analysis. Section 5 finally concludes the paper.

## 2. System dynamics and estimation problem

### 2.1. System dynamics

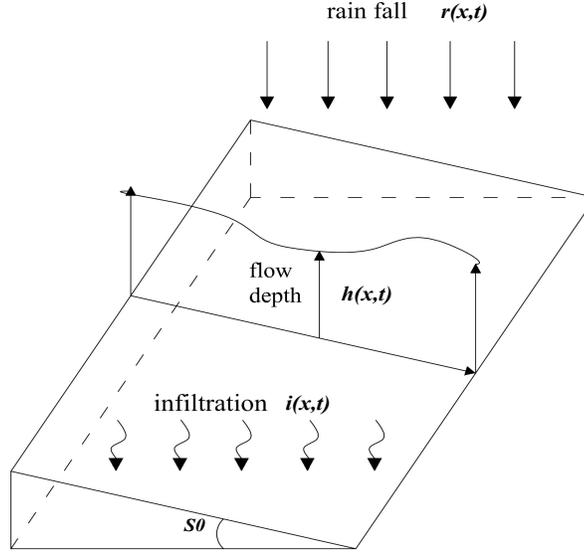


Figure 3: Illustration of overland flow model [17].

The water flow caused by rainfall in Tondi-Kiboro catchment can be illustrated by Figure 3 and its dynamics can be characterized by so-called Saint Venant equations [3], which belong to a class of distributed flow routing. They are classically made of two partial differential equations, a continuity one, and a momentum one, but owing to characteristics of the basin under consideration, the momentum equation will here be omitted (see [20, 16] for more details about this approximation). The resulting model is recalled as follows:

$$\begin{cases} \frac{\partial h(x,t)}{\partial t} + \frac{\partial f(h(x,t),x)}{\partial x} = r(t) - i(t) \\ h(x,0) = h_0^i(x) \\ h(0,t) = h_0^b(t) \end{cases} \quad (1)$$

where  $h$  is the water flow depth, ( $m$ );  $x$  is the space variable, ( $m$ );  $t$  is the time, ( $s$ );  $r$  is the rainfall intensity, ( $m/s$ );  $i$  is the infiltration intensity, ( $m/s$ );  $f$  is the flow of water per unit width  $m^2/s$ . It is a function of water depth  $h(x,t)$  and spatial variable  $x$  due to the distributed Manning coefficient  $n(x)$  in some cases. It is calculated by equation  $f(h(x,t),x) = S_0^{1/2} h(x,t)^{5/3} / n(x)$  where  $S_0$

denotes the bed slope, ( $m/m$ ). Two terms  $h_0^i(x)$  and  $h_0^b(t)$  are respectively the initial condition and boundary condition. Due to the small size of the considered Tondi-Kiboro basin, the rain fall rate and infiltration rate can be considered to be uniform all over the area. Manning coefficient  $n$  is also constant because the soil characteristics do not vary spatially.

Depending on empirical or physical points of view, there are different ways to characterize the infiltration process, such as Horton model or Green-Ampt model. Due to its simplicity and good physical meaning preservation, Green-Ampt model is used in a lot of studies. In the case of *unsteady rain*, as in Tondi Kiboro, a modified version has to be considered, the full details of which can be found in [21] and [22], together with entire formula and a calculation validation. Based on those formulas, and some knowledge on soil characteristics, the infiltration rate is calculated offline.

An alternative is to adopt the more empirical representation called Horton model. Being empirical, this model is used to describe the dynamics of infiltration process after the rainfall intensity exceeds the infiltration capacity of soil. With the meaning of fitting equation to data, Horton model does not depend on rainfall or soil properties and is described by a function of time rather than of rainfall rate, as:

$$i(t) = i_c + (i_0 - i_c)e^{-kt} \quad (2)$$

where  $i(t)$  is the infiltration rate at time  $t$ , ( $m/s$ );  $k$  some exponential decay, ( $1/s$ );  $i_c$  the equilibrium infiltration rate, ( $m/s$ );  $i_0$  the initial infiltration capacity, ( $m/s$ ).

## 2.2. Estimation problem

In dynamical equation (1) of Tondi-Kiboro's water flow, the initial state function  $h_0^i(x)$  and boundary function  $h_0^b(t)$  are supposed to be equal to zero at the beginning of rainfall process. This assumption is reasonable because this region is generally dry between two rain events.

In the case of using Green-Ampt model, only Manning coefficient is considered to be unknown, and to be estimated. The infiltration rate is calculated offline from rainfall data and some soil knowledge.

In Horton infiltration case, the estimation problem is more complicated, with 4 parameters to be estimated, including Manning coefficient  $n$ , equilibrium infiltration rate  $i_c$ , initial infiltration rate  $i_0$  and exponential rate  $k$ .

The only available measurement for parameter estimation is the water discharge during all rainfall events at the downstream end of the basin, provided by stream flow sensor on this basin, with locations presented in Figure 4, in the measurement campaign of 2012. Each observed water discharge is associated to the corresponding measurements of rainfall rate provided by 1 day rain gauge (see again Figure 4 for corresponding locations). Notice that only the lower basin is taken into account in this paper, and thus that the water discharge data which are used only come from measurements at observation station called 'bodo' in Figure 4.

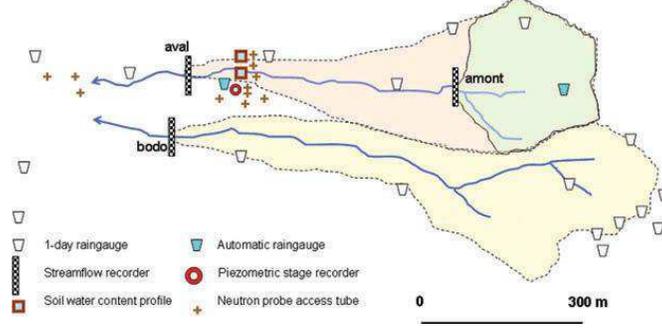


Figure 4: Observation stations on Tondi-Kiboro catchment [4].

This measurement allows to define the problem of parameters estimation as an optimization one related to some cost function  $\mathcal{J}$ . This function is defined as a least square error between real data and model-predicted value of water discharge as follows:

$$\mathcal{J} = \frac{1}{2} \int_0^T \left\{ \int_0^L \Delta(x - x_{meas}) Q(x, t) dx - Q^{meas}(x_{meas}, t) \right\}^2 dt \quad (3)$$

where  $Q(x, t)$  is the calculated value of water discharge at position  $x$  and time  $t$ , obtained as

$$Q(x, t) = Wf(h, x) = W \frac{S_0^{1/2}}{n} h^{5/3}; \quad (4)$$

$T$  is the optimization horizon (*hours*);  $L$  is the considered spatial length where the system takes place (*m*);  $Q^{meas}(x_{meas}, t)$  is the measured value of water discharge at observation position  $x_{meas}$ ;  $\Delta(x - x_{meas})$  is a continuous approximation of Delta-Dirac function representing the sensor spatial accuracy. It is described by a Gaussian function with a very small variance  $\sigma^2$  with equation

$$\Delta(x - x_{meas}) = \frac{1}{\Lambda(x_{meas})} e^{-(x-x_{meas})^2/\sigma^2}$$

where the coefficient  $\Lambda(x_{meas})$  is defined at sensor location  $x_{meas}$  by equation:

$$\Lambda(x_{meas}) = \int_0^L e^{-(x-x_{meas})^2/\sigma^2} dx \quad (5)$$

Playing the role of space observation window, the value of  $\sigma$  must be chosen to keep the balance between good accuracy of observation operator (small enough

but not too small) and stiffness of the gradients which will be presented in the next section. The value of  $\sigma$  is selected manually after several trials.

### 3. Estimation methodology

The estimation method is here presented for a case general enough to cover at least the two infiltration models discussed before. Equation (1) is thus rewritten under the form of a general 1-D hyperbolic system of variable  $h(x, t)$ , flow  $f$  function of  $h(x, t)$ , and source term  $g$  as follows:

$$\frac{\partial h(x, t)}{\partial t} + \frac{\partial f(h(x, t), \alpha)}{\partial x} = g(x, t, p) \quad (6)$$

where the flow function is supposed to have the form  $f(h, \alpha) = \sum_{i=1}^N \alpha_i \varphi_i(h)$  for some known functions  $\varphi_i(h)$  and some *unknown parameters*  $\alpha_i$  gathered in a vector  $\alpha$ . Similarly, the source term  $g$  also contains unknown parameters in a vector called  $p$ , and of size  $M$ . This means that in this formulation,  $N + M$  parameters are to be estimated.

In the case of Tondi Kiboro model, since  $f(h(x, t), x) = S_0^{1/2} h(x, t)^{5/3} / n$  where bed slope  $S_0$  is known, vector  $\alpha$  reduces to a single parameter representing unknown Manning coefficient, of the form  $S_0^{1/2} / n$ . Notice that with Green Ampt model,  $p$  reduces to an empty vector, while with Horton model, it contains 3 parameters:  $k, i_0, ic$ . Notice also that from equation (4), in this problem statement the measured variable is also a function of some unknown parameter, in addition to state variable  $h$ :

$$Q = Q(h, \alpha) \quad (7)$$

At this point, the parameter estimation considered hereby, or inverse problem, is to estimate the value of some unknown parameters of Tondi-Kiboro catchment, whose dynamical equation is generalized as equation (6). With cost function  $\mathcal{J}$  of equation (3), this estimation problem becomes a minimization issue to be solved under equality constraints (6)-(7). This is a popular problem in variational calculus, and the Lagrangian formulation can then be a powerful tool to combine the system equation and the cost function.

Even though some variants of this problem have been formerly presented in previous papers, for instance in [15], but without unknown parameter in source term  $g$ , or for more general switched dynamics in [18], but with no unknown parameter in measurement function, let us, for the sake of completeness, recall here the main lines of this methodology for the specific problem under study.

#### 3.1. Calculus of variations

In the Lagrangian formulation, a new variable  $\lambda(x, t)$  is first introduced, the so-called Lagrangian multiplier, giving rise to a modified objective functional  $\mathcal{L}$

as follows:

$$\mathcal{L}(h(x, t), \alpha, p) = \mathcal{J} + \int_0^T \int_0^L \lambda \left[ \frac{\partial h(x, t)}{\partial t} + \frac{\partial f(h(x, t), \alpha)}{\partial x} - g(x, t, p) \right] dx dt \quad (8)$$

The first order necessary condition for functional optimality is the basic idea to solve this optimization problem. This condition and the calculus of variations (see [23] for more details) are used to cancel the first order variation of  $\mathcal{L}(h(x, t), \alpha, p)$ . The candidates for optimal solutions of the cost functional are found at the point where its variations with respect to the variations of parameters vanish. Firstly, it is necessary to take the first variation of the left part of equation (8) with respect to  $h(x, t)$ ,  $\alpha$  and  $p$ . These variations exist because of the continuity of the first partial derivative of cost function  $\mathcal{J}$  and system equations. The variation of cost functional  $\mathcal{L}$  with respect to all possible variation directions can be written under the form of Gâteaux derivative of  $\mathcal{L}$ . The Gâteaux derivative of  $\mathcal{J}$  in all variation directions  $\pi = [\delta h \ \delta \alpha \ \delta p]$  reads:

$$\delta \mathcal{L} = \langle \nabla \mathcal{L}, \pi \rangle \quad (9)$$

where  $\nabla \mathcal{L}$  is the weak form of gradient of  $\mathcal{L}$  with respect to  $h(x, t)$ ,  $\alpha$  and  $p$  and notation  $\langle \cdot, \cdot \rangle$  stands for the inner product defined on the space  $L_2([0, T], [0, L]) \times \mathbb{R}^M \times \mathbb{R}^N \rightarrow \mathbb{R}$  as

$$\langle \nabla \mathcal{L}, \pi \rangle = \int_0^T \int_0^L \nabla \mathcal{L}_{h(x, t)} \delta h(x, t) dx dt + \nabla \mathcal{L}_\alpha \delta \alpha + \nabla \mathcal{L}_p \delta p \quad (10)$$

where  $\nabla \mathcal{L}$  is a vector of 3 elements  $\nabla \mathcal{L} = [\nabla \mathcal{L}_{h(x, t)} \ \nabla \mathcal{L}_\alpha \ \nabla \mathcal{L}_p]$ . The first-order necessary condition for optimality means that the first variation  $\delta \mathcal{L}$  must be equal to zero for all admissible variation vectors  $\pi$ . As a result the weak form of gradients  $\nabla \mathcal{L}$  must satisfy:

$$\nabla \mathcal{L} = 0 \quad (11)$$

For the sake of simplicity, notations  $h(x, t)$ ,  $f(h(x, t), \alpha)$ ,  $g(x, t, p)$  and  $Q(h(x, t), \alpha)$  are shortly denoted by  $h$ ,  $f$ ,  $g$  and  $Q$ . By gathering all variation terms of each element of  $\pi$  appearing in  $\delta \mathcal{L}$  and applying the optimization condition of the weak gradient in all directions of  $\pi$ , one can obtain the adjoint equation of variable  $\lambda(x, t)$  (equation (12)), and weak forms of gradient of  $\mathcal{L}$  with respect to  $\alpha$  (equation (13)) and  $p$  (equation (14)).

$$\begin{cases} -\frac{\partial \lambda}{\partial t} - \frac{\partial f}{\partial h} \frac{\partial \lambda}{\partial x} + \frac{\partial Q}{\partial h} \Delta (x - x_{meas}) \left[ \int_0^L \Delta (x - x_{meas}) Q dx - Q^{meas} \right] = 0 \\ \lambda(x, T) = 0 \quad \text{and} \quad \lambda(L, t) = 0 \end{cases} \quad (12)$$

$$\begin{aligned} \nabla \mathcal{L}_\alpha = & \int_0^T \left[ \lambda \frac{\partial f}{\partial \alpha} \right] \Big|_0^L dt - \int_0^T \int_0^L \frac{\partial f}{\partial \alpha} \frac{\partial \lambda}{\partial x} dx dt + \int_0^T \int_0^L \Delta(x - x_{meas}) \frac{\partial Q}{\partial \alpha} \\ & \times \left[ \int_0^L \Delta(x - x_{meas}) Q dx - Q^{meas} \right] dx dt \end{aligned} \quad (13)$$

$$\nabla \mathcal{L}_p = \int_0^T \int_0^L \lambda \frac{\partial g}{\partial p} dx dt \quad (14)$$

By applying these general calculations on the considered estimation problem of overland flow on Tondi-Kiboro, the adjoint system has the same form as equation (12) with the partial derivatives  $\partial f/\partial h = 5\alpha h^{2/3}/3$  and  $\partial Q/\partial h = 5W\alpha h^{2/3}/3$ . The gradient of parameter  $\alpha$  is obtained by replacing the derivatives  $\partial f/\partial \alpha = h^{5/3}$  and  $\partial Q/\partial \alpha = Wh^{5/3}$  in equation (13). In the case of estimation of Horton parameters, the gradient for  $p$  are is obtained as follows:

$$\begin{aligned} \nabla \mathcal{L}_k = & - \int_0^T \int_0^L \lambda t (i_0 - i_c) e^{-kt} dx dt; & \nabla \mathcal{L}_{i_0} = & \int_0^T \int_0^L \lambda e^{-kt} dx dt \\ \nabla \mathcal{L}_{i_c} = & \int_0^T \int_0^L \lambda (1 - e^{-kt}) dx dt \end{aligned} \quad (15)$$

### 3.2. Numerical implementation

#### 3.2.1. Numerical schemes

To solve the equations of direct system (1) and adjoint system (12), a discretization is applied here based on the explicit Lax-Wendroff scheme. Because the direct system is defined by specifying the initial condition at time  $t = 0$  and boundary condition at position  $x = 0$ , it will be solved forward in time and space. The development of the forward Lax-Wendroff scheme is already presented in our previous works (details of this scheme can be found in [15] for instance). Due to its conditions specified at the boundary of  $x - t$  plane, the adjoint equation is to be simulated backward in time-space plane.

#### 3.2.2. Optimization algorithm

The gradient vector  $\nabla \mathcal{L}$  is the input of the optimization algorithm which is a gradient-based one. More clearly, the *interior point* method is used as the optimization algorithm, through *fmincon* function of *Matlab* [24]. The gradients obtained by calculus of variations are fed to this function. Their role is to give the directions to update the current estimated values and to be used to approximate the Hessian matrix (by Broyden-Fletcher-Goldfarb-Shanno algorithm [25]). Some bounds are also specified in the optimization tool in order to preserve the meaning of the estimated parameters. Physics-preserving bounds

of the Manning coefficient are loosely chosen according to the top soil characteristics of Tondi-Kiboro catchment (see Table 1 of [26]). Numerically, lower and upper bounds are [0.005, 0.04].

## 4. Estimation results

### 4.1. Real data

The approach is applied to real measurements available for the Tondi-Kiboro catchment. Two sets of data are used here: one for optimization and one for validation (verification of estimated model consistency). Each data set includes the measurement of rainfall rate  $r(t)$  and corresponding water discharge  $Q^{meas}(x_{meas}, t)$  at sensor position  $x_{meas}$  for a rainfall event over  $t \in [0, T]$ . The first data set, called DS1, is taken on 2012/06/21 from 08:12 to 09:17 universal time, and the second one, called DS2, is taken on 2012/08/03 from 20:33 to 21:09. *The rainfall rate is recorded at 1/10 Hz by the rain gauge. The temporal resolution  $\Delta t$  and spatial resolution  $\Delta x$  of the discretization scheme should be chosen to respect the CFL condition [15].  $\Delta t$  is set to 10s, being equal to the temporal resolution of the rainfall rate measurement. The original sampling frequency of the water discharge  $Q^{meas}(x_{meas}, t)$  is 1/60 Hz. Its values are then linearly interpolated to 1/10 Hz in order to be consistent with the discretization scheme.*

The estimation process is firstly carried out in the case of estimation of Manning coefficient, while infiltration is modeled by Green-Ampt representation. Then, Horton model is used in the second estimation problem, where Manning roughness is again estimated, along with the parameters of infiltration equation. In those two applications, geometric characteristics of Tondi-Kiboro and numerical parameters are set to the same values, and summarized in Table 1.

Parameters	Value	Unit
Length of basin $L$	600	$m$
Simulation time $T$	3960	$second$
Space step $\Delta x$	60	$m$
Time step $\Delta t$	10	$second$
Bed slope $S_0$	3.5%	$m/m$
Surface flow width at position $x_{meas}$ , $W$	78	$m$

Table 1: Geometric characteristics and discretization parameters for Tondi-Kiboro catchment.

### 4.2. Case of Green-Ampt infiltration model

In this first case, as mentioned in subsection 2.1, the infiltration is obtained offline. In fact, as also mentioned, rainfall rate in Tondi Kiboro is unsteady, meaning that infiltration cannot be represented by classical Green Ampt model (as we considered it for instance in our simulated example of [18]): in a steady rain case, infiltration process begins with an unponded surface, and when becoming ponded, it remains so till the end of rainfall event. But in the case

of unsteady rain, there are more than one ponding times, because the rainfall rate exceeds the infiltration rate at several moments, and the infiltration process can shift back to original stage from the current one. To overcome this difficulty, S.T. Chu [21] proposed modifications to Green Ampt model: in short, the infiltration rate is no longer determined by an explicit formula (as in the case of steady rainfall), but by an iteration technique. This is why it is here established offline using the procedure described in [21], and independently of the optimization process.

The soil parameters which are needed in this process are presented in Table A.3 of the appendix.

After estimation of parameter  $\alpha$  by the methodology of section 3, Manning coefficient is recalculated by simple equation  $n = S_0^{1/2}/\alpha$ . Its value converges to the value of 0.0151, as illustrated by Figure 5. The corresponding minimal cost function value is reached after less than 10 iterations of optimization algorithm, and is equal to  $\mathcal{J}^* = 0.3288$ , as it can also be seen on Figure 5.

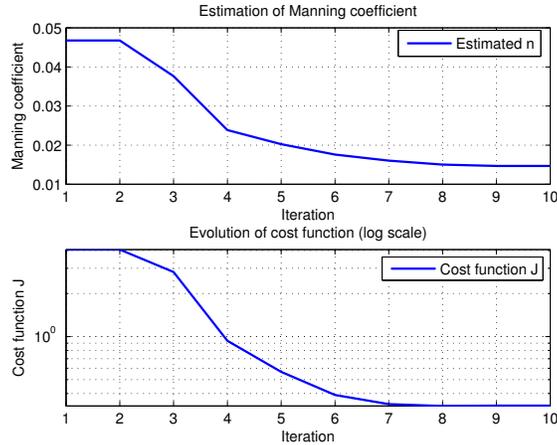


Figure 5: Estimation of Manning coefficient and cost function  $\mathcal{J}$ .

Notice that the obtained Manning coefficient estimate clearly falls within the reference range of  $[0.012, 0.026]$  for natural channels whose bed material is sand (see Table 1 of [26] for more details), which is pretty consistent with the considered soil (containing 90% of sand).

After the optimization procedure, the estimated value of  $n$  is checked by simulating again system equation (1) with this value of  $n$  and comparing the discharge at the end of watercourse with the measured data (already used for optimization). The results are shown in Figure 6a, where the standard deviation (std) value of estimation error (between simulated and measured discharge) is  $0.0512(m^3/s)$ . This allows to conclude that the model simulates pretty well the characteristics of overland flow on Tondi-Kiboro catchment.

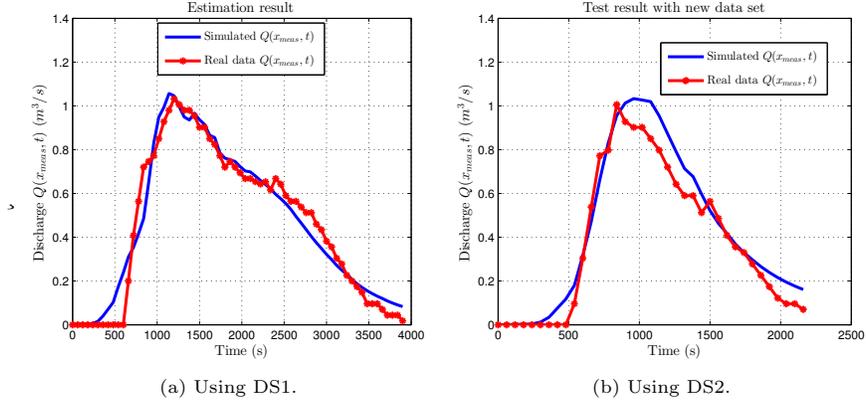


Figure 6: Simulated discharge and measured discharge with two data sets.

The consistency of the numerical model and effectiveness of the optimization approach is further checked by simulating again system equation (1) with this value of  $n$  under new rainfall condition of data set DS2, and comparing the simulated discharge with that of DS2. The simulated water discharge at position  $x_{meas}$  again follows the measured value fairly well, with an error of  $0.067(m^3/s)$ , as shown in Figure 6b. The small bias which can be seen may come from measurement errors or model simplicity. Moreover, the changes of soil surface and cultivated plant over the time can also be a part of the reason. This test allows to consider that equation (1) with mentioned parameter and estimated Manning coefficient as a validated numerical model for overland flow taking place on Tondi-Kiboro basin.

#### 4.3. Case of Horton infiltration model

The data used in this second case are the same ones as in the preceding example, with the single observation value of water discharge, provided by stream flow sensor at position  $x_{meas} = L$ . Parameter estimation is achieved on the first data set DS1, whose rainfall intensity is depicted in the upper part of Figure 7a, and 4 parameters are estimated here, providing Manning coefficient together with the 3 parameters of Horton infiltration model.

The proposed optimization approach is again used, with initial guesses for those 4 unknown parameters chosen in a physically admissible range (various trials give rise to similar estimation results), and for instance, starting with  $[n^0; i_c^0; i_0^0; k^0] = [0.006; 1 \times 10^{-8}; 4 \times 10^{-7}; 3 \times 10^{-3}]$ , parameter estimates converge after 44 iterations to the optimal values  $[0.0119; 9.3 \times 10^{-8}; 9.995 \times 10^{-6}; 0.0017]$ . The estimated friction coefficient in this example (equal to 0.0119) is slightly different from the one obtained in the former approach (0.0151). But both of these two values are consistent with the characteristics of the basin soil, and their difference may come from differences in infiltration models. The cost function  $\mathcal{J}$  decreases from 13.0643 and reaches the minimum value of 0.9833.

Similarly to the previous case, it is interesting to compare the water discharge measurement with the simulated one obtained with estimated parameters. This comparison is shown in Figure 7b, where it can be seen that the difference is very small (std of estimation error is  $0.042(m^3/s)$ ), confirming the correctness of estimation result in this case again. Moreover, the obtained result is very similar to the one already obtained in subsection 4.2.

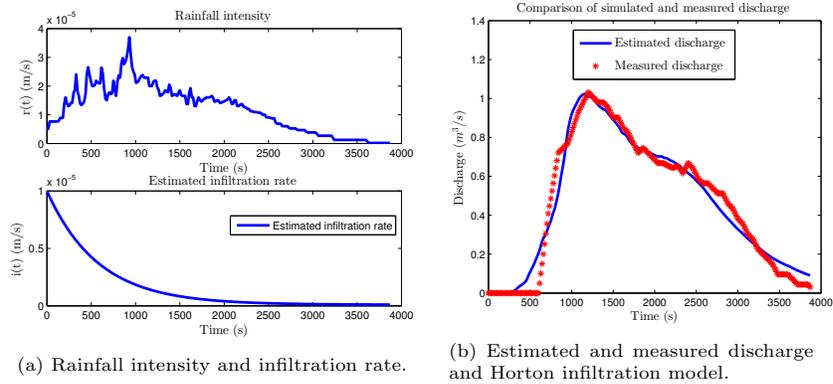


Figure 7: Estimation result of Horton model using DS1.

The role of the second rainfall data set DS2 (taken on 2012/08/03 from 20:33 to 21:09 universal time) is again to verify the consistency of estimated values. The rainfall and infiltration rate (simulated with estimated parameters) are given in Figure 8a. As observed in Figure 8b, the simulated discharged again fits pretty well the measurements provided by the sensors. By comparing it with Figure 6b, one can notice that the simulation of water discharge using estimated values of Horton model follows the measured data better than in the case when using Green-Ampt model (std of estimation error in this case is  $0.046(m^3/s)$ ). Those results tend to show that estimation of infiltration parameters can provide a more accurate dynamical model of Tondi-Kiboro's water flow. This comparison is discussed further in next subsection.

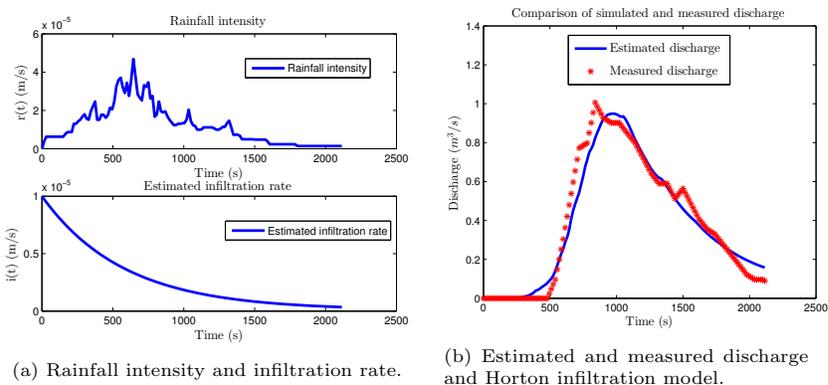


Figure 8: Validation results for Horton model using DS2.

#### 4.4. Robustness discussion

- Green Ampt sensitivity to soil parameters. From results of the previous subsections, using Horton empirical model may give better results in Manning coefficient estimation. This can be further confirmed by the fact that Green Ampt approach is very sensitive to the prior knowledge used to generate infiltration data offline: in this approach indeed, various soil parameters must be known beforehand (Table A.3), and it appears that small changes in such values can turn into large deviations in Manning coefficient estimates. In order to illustrate it, uncertainties on saturated water content (SWC) are considered: the optimization process presented in subsection 4.2 is carried out with different sets of soil parameters. In each set, only the value of SWC is modified, considering variations of 1%, 3%, 5%, 7% with respect to its reference value of Table A.3.

The resulting Manning coefficient estimates are then compared to the value obtained with the reference set of soil parameters, namely 0.0151 (see subsection 4.2), and Table 2 illustrates how a small variation of SWC value, e.g. 1%, can cause a large variation of the friction coefficient, here 32.2%.

SWC variation (%)	Manning coeff. estimate variation (%)
1	32.2
3	53.4
5	59.2
7	63.2

Table 2: Variation of estimated Manning coefficient versus variation of SWC.

The use of Green-Ampt offline model to describe infiltration rate can be efficient, but is also very sensitive to the knowledge on soil parameters.

- *Robustness to measurement conditions.* The objective of this part is to complete estimation results of previous subsections with longer measurements in time, and to test the robustness of proposed estimation algorithm against measurement noise, and in front of arbitrary (admissible) values of Manning coefficient, or rainfall rates. To that end, Monte-Carlo simulation method is used. For the Green-Ampt model, 450 rainfall rate and Manning coefficient profiles are generated. Similarly, the same number of random sets of three infiltration parameters and Manning coefficient are generated in the case of Horton model. All those parameters are considered as white Gaussian noises. The random values of the Manning coefficient are in the range  $[0.01 \ 0.1]$  corresponding to typical characteristics of natural streams (from the Chow's Manning table in [3]). The observed water discharged is accordingly obtained by simulating the Tondi-Kiboro dynamical model with given randomly generated rainfall rate. The duration of rainfall event in this simulation is 5 hours. To represent the additive measurement noise in the simulation, a white Gaussian noise is added to the observed water discharged with 40 dB signal-to-noise ratio per sample. Other parameters in the simulations are identical to the ones of Tondi-Kiboro catchment.

Figure 9 presents the relative estimation error between the estimated Manning coefficient and the ones generated in the 450 random profiles with (exact) Green Ampt infiltration model. The mean value of estimation error is 0.79% and the variance is 0.74%, showing a pretty good robustness of the approach.

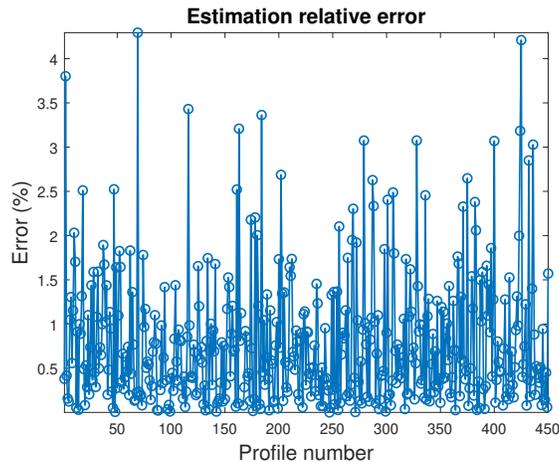


Figure 9: Estimation relative errors of 450 scenarios with Green-Ampt model.

As for the second case, the average and variance of average estimated errors over the 450 profiles of Manning coefficient and Horton infiltration parameters are respectively 0.7% and 0.71%, as shown in Figure 10, which are similar to

the previous case of Green-Ampt model. Regarding the noise level added to the simulated measurement, the estimation errors are pretty low. It is worth recalling that all three infiltration parameters of Horton model are estimated together with friction coefficient  $n$ , with no need of further prior knowledge on soil characteristics, making Horton model a good candidate for a robust and complete way to represent infiltration process in Tondi-Kiboro catchment.

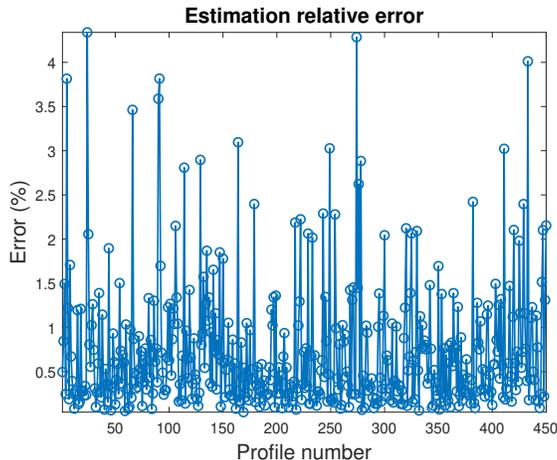


Figure 10: Estimation relative errors of 450 random scenarios with Horton model.

## 5. Conclusion

In this paper, an optimal estimation approach, based on calculus of variations, have been proposed, and validated, to estimate constant Manning coefficient, and even some infiltration parameters, in an overland flow dynamical description. The whole methodology has been presented from mathematical developments to numerical implementation, and it has been successfully applied to real data taken from Tondi-Kiboro catchment in Sahel region.

In fact two data sets have been used: a first one to solve for the identification problem and a second one to validate the results, while two infiltration models have been tested and compared. A robustness simulation study has finally been proposed to confirm the effectiveness of the approach, and if the dynamical model with estimated parameters finally seems to give a realistic representation of the real behavior of the Tondi-Kiboro catchment in both cases of infiltration models, Horton one appears to be more robust in this study.

Applications to more general cases or problems, such as optimal control of overland flows will be part of future works.

## Appendix A. Infiltration parameter of modified Green-Ampt

Parameters	Value	Unit
Effective hydraulic conductivity $K_i$	0.0479	$cm/s$
Average suction at the wetting front $\Psi$	28.5	$cm$
Saturated water content	0.1	$cm^3/cm^3$
Initial water content at start of the rainfall event	0.01	$cm$
Soil surface storage	10	$cm$

Table A.3: Infiltration parameter of modified Green-Ampt model on Tondi Kiboro catchment.

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