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NAKAI-MOISHEZON CRITERION FOR ADELIC \mathbb{R} -CARTIER DIVISORS

FRANÇOIS BALLAÏ

ABSTRACT. We prove a Nakai-Moishezon criterion for adelic \mathbb{R} -Cartier divisors, which is an arithmetic analogue of a theorem of Campana and Peternell. Our main result answers a question of Burgos Gil, Philippon, Moriwaki and Sombra. We deduce it from the case of adelic Cartier divisors (due to Zhang) by continuity arguments and reductions involving a generalization of Zhang's theorem on successive minima.

1. INTRODUCTION

In algebraic geometry, the Nakai-Moishezon criterion asserts that a Cartier divisor $D \in \text{Div}(X)$ on a projective variety X over an algebraically closed field is ample if and only if $D^{\dim Y} \cdot Y > 0$ for every subvariety $Y \subseteq X$. By a theorem of Campana and Peternell [CP90], this statement remains valid when $D \in \text{Div}(X)_{\mathbb{R}} = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is a \mathbb{R} -Cartier divisor. In [Zha95a], Zhang started the study of arithmetic ampleness in the context of Arakelov geometry, and proved an arithmetic Nakai-Moishezon criterion for adelic metrized line bundles ([Zha95a, Theorem 4.2]). Our purpose is to extend this result to adelic \mathbb{R} -Cartier divisors (in the sense of Moriwaki [Mor16]), thus proving an arithmetic analogue of Campana and Peternell's theorem.

Let X be a normal and geometrically integral projective scheme over a number field K . An adelic \mathbb{R} -Cartier divisor $\overline{D} = (D, (g_v)_v)$ on X is a pair consisting of a \mathbb{R} -Cartier divisor $D \in \text{Div}(X)_{\mathbb{R}}$ and a suitable collection of Green functions $(g_v)_v$ on the analytifications X_v^{an} of X , where v runs over the set of places of K (see Definition 3.1). The set $\widehat{\text{Div}}(X)_{\mathbb{R}}$ of adelic \mathbb{R} -Cartier divisors is a \mathbb{R} -vector space; it contains the set of adelic Cartier divisors $\widehat{\text{Div}}(X)$, defined by

$$\widehat{\text{Div}}(X) = \{(D, (g_v)_v) \in \widehat{\text{Div}}(X)_{\mathbb{R}} \mid D \in \text{Div}(X)\} \subseteq \widehat{\text{Div}}(X)_{\mathbb{R}}.$$

To any adelic Cartier divisor $\overline{D} \in \widehat{\text{Div}}(X)$ we can associate an adelic metrized line bundle $(\mathcal{O}_X(D), (\|\cdot\|_v^{\overline{D}})_v)$ in the sense of Zhang [Zha95b], and a global section $s \in H^0(X, D)$ of $\mathcal{O}_X(D)$ is called strictly small if $\sup_{x \in X_v^{\text{an}}} \|s\|_v^{\overline{D}}(x) \leq 1$ for every place v , with strict inequality at archimedean places. We say that an adelic \mathbb{R} -Cartier divisor \overline{D} is ample if it is semi-positive (see Definition 3.4) and if it can be written as a finite sum

$$\overline{D} = \sum_{i=1}^{\ell} \lambda_i \overline{A}_i$$

where for each $i \in \{1, \dots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and $\overline{A}_i = (A_i, (g_{i,v})_v) \in \widehat{\text{Div}}(X)$ is an adelic Cartier divisor such that $A_i \in \text{Div}(X)$ is ample and $H^0(X, mA_i)$ has a K -basis consisting of strictly small sections for every $m \gg 1$. This definition of ampleness for adelic \mathbb{R} -Cartier divisors coincides with the one used in [BGMPS16] (see Remark 6.5). For any semi-positive $\overline{D} = (D, (g_v)_v) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ and for any

subvariety $Y \subseteq X$, we denote by $h_{\overline{D}}(Y)$ the height of Y with respect to \overline{D} (see section 3.2). The main result in this paper is the following (see Corollary 6.4).

Theorem 1.1. *Let $\overline{D} = (D, (g_v)_v)$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . Then \overline{D} is ample if and only if $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$.*

This theorem gives an affirmative answer to a question of Burgos Gil, Moriwaki, Philippon and Sombra [BGMP16, Remark 3.21]. To our knowledge, it was known only under one of the following additional assumptions up to now:

- \overline{D} is an adelic Cartier divisor (Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2], [Mor15, Corollary 5.1], [CM18, Theorem 1.2]);
- \overline{D} is a toric metrized \mathbb{R} -Cartier divisor ([BGMP16, Corollary 6.3]);
- X has dimension one ([Iko21, Corollary A.4]).

Given a semi-positive adelic \mathbb{R} -Cartier divisor $\overline{D} = (D, (g_v)_v)$ on X and a subvariety $Y \subseteq X$ with $\deg_D(Y) := D^{\dim Y} \cdot Y \neq 0$, the normalized height of Y with respect to \overline{D} is defined by

$$\widehat{h}_{\overline{D}}(Y) = \frac{h_{\overline{D}}(Y)}{(\dim Y + 1) \deg_D(Y)}.$$

We also let $\zeta_{\text{abs}}(\overline{D}) = \inf_{x \in X(\overline{K})} \widehat{h}_{\overline{D}}(x)$. The following theorem is our second main result, which plays an important role in this paper and might be of independent interest.

Theorem 1.2. *Let $\overline{D} = (D, (g_v)_v)$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . If D is ample, there exists a subvariety $Y \subseteq X$ such that*

$$\zeta_{\text{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) = \min_{Z \subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the minimum is over the subvarieties $Z \subseteq X$.

In other words, the infimum of the normalized heights of subvarieties $Z \subseteq X$ is attained at a subvariety Y , which moreover satisfies $\widehat{h}_{\overline{D}}(Y) = \zeta_{\text{abs}}(\overline{D})$. Our proof of Theorem 1.2 is based on Zhang's theorem on successive minima [Zha95a, Theorem 5.2]. Although the latter does not appear in the literature for adelic \mathbb{R} -Cartier divisors, we shall prove that it remains valid in this context thanks to a continuity property for successive minima (see Lemma 4.1 and Theorem 4.3). This approach also provides additional information on the subvariety $Y \subseteq X$ of Theorem 1.2 (see Theorem 5.1). Our proof of Theorem 1.1 is very direct, and goes roughly as follows. Let $\overline{D} = (D, (g_v)_v) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be semi-positive, with D ample. Given a real number $t \in \mathbb{R}$, we define an adelic \mathbb{R} -Cartier divisor $\overline{D}(t)$ by rescaling the metrics at archimedean places to have $\widehat{h}_{\overline{D}(t)}(Y) = \widehat{h}_{\overline{D}}(Y) - t$ for every subvariety $Y \subseteq X$ (see Definition 3.3 and Lemma 3.7). In view of Theorem 1.2, it suffices to prove that

$$\sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is ample}\} = \zeta_{\text{abs}}(\overline{D}).$$

We denote by $\theta(\overline{D})$ the supremum on the left hand side. We first observe that Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2] implies that $\theta(\overline{D}) = \zeta_{\text{abs}}(\overline{D})$ provided that \overline{D} is an adelic Cartier divisor. We simply deduce the general case (Theorem 6.1) by slightly perturbing \overline{D} and by applying a continuity property for the invariants $\zeta_{\text{abs}}(\overline{D})$ and $\theta(\overline{D})$ (see Lemmas 4.1 and 6.2).

Organization of the paper. We fix some notation in section 2. In section 3 we recall the definition of adelic \mathbb{R} -Cartier divisors and of height of subvarieties. We also study some basic properties of ample adelic \mathbb{R} -Cartier divisors. We define successive minima in section 4, and we establish a continuity property allowing us to extend Zhang's theorem on minima to adelic \mathbb{R} -Cartier divisors (Lemma 4.1 and

Theorem 4.3). We prove Theorem 1.2 in section 5 (Theorem 5.1) and Theorem 1.1 in section 6 (Corollary 6.4).

2. CONVENTIONS AND TERMINOLOGY

2.1. We say that a scheme is integral if it is reduced and irreducible. Given a Noetherian integral scheme X , we denote by $\text{Div}(X)$ the group of Cartier divisors on X and by $\text{Rat}(X)$ the field of rational functions on X . If \mathbb{K} denotes \mathbb{Z} , \mathbb{Q} or \mathbb{R} , we let $\text{Div}(X)_{\mathbb{K}} = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$. The elements of $\text{Div}(X)_{\mathbb{K}}$ are called \mathbb{K} -Cartier divisors on X . If X is normal, we denote by $\text{Supp } D$ the support of a \mathbb{K} -Cartier divisor D (see [Mor16, section 1.2] for details). It is a Zariski-closed subset of X . We let (ϕ) be the Cartier divisor associated to a rational function $\phi \in \text{Rat}(X)^{\times}$.

2.2. Let X be a projective variety over a field K , i.e. X is an integral projective scheme on $\text{Spec } K$. A subvariety $Y \subseteq X$ is an integral closed subscheme of X . Given an integer $r \in \{0, \dots, \dim X\}$, a r -cycle is a formal linear combination with integer coefficients of r -dimensional subvarieties in X . Given a \mathbb{K} -Cartier divisor D on X , we define the degree of a r -cycle Z with respect to D by $\deg_D(Z) = D^{\dim Z} \cdot Z$. In particular, if $x \in X(\overline{K})$ is a closed point (considered as a subvariety of X), then $\deg_D(\{x\}) = [K(x) : K]$ is the degree over K of the residue field $K(x)$ of $x \in X$.

2.3. Throughout this text, we fix a number field K and an algebraic closure \overline{K} of K . We denote by Σ_K the set of places of K and by $\Sigma_{K,\infty} \subset \Sigma_K$ the set of archimedean places. For each $v \in \Sigma_K$, we let K_v be the completion of K with respect to v and we denote by $|\cdot|_v$ the unique absolute value on K_v extending the usual absolute value $|\cdot|_v$ on $\mathbb{Q}_v : |p|_v = p^{-1}$ if v is a non-archimedean place over a prime number p , and $|\cdot|_v = |\cdot|$ is the usual absolute value on \mathbb{R} if v is archimedean.

2.4. Let X be a scheme on $\text{Spec } K$. For each $v \in \Sigma_K$, we let $X_v = X \times_K \text{Spec } K_v$ be the base change of X to K_v , and we denote by X_v^{an} the analytification of X_v in the sense of Berkovich (see [Mor16, section 1.3] for a short introduction). Given a closed point $x \in X_v$, we let $x^{\text{an}} \in X_v^{\text{an}}$ be the point corresponding to the unique absolute value on $K_v(x)$ extending $|\cdot|_v$.

2.5. Let X be a normal projective variety on $\text{Spec } K$. Let $D \in \text{Div}(X)_{\mathbb{R}}$, $v \in \Sigma_K$ and let $D_v \in \text{Div}(X_v)_{\mathbb{R}}$ be the pullback of D to X_v . We consider an open covering $X_v = \cup_{i=1}^{\ell} U_i$ such that D_v is defined by $f_i \in \text{Rat}(X_v)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ on U_i for each $i \in \{1, \dots, \ell\}$. A continuous D -Green function on X_v^{an} is a function

$$g_v : X_v^{\text{an}} \setminus (\text{Supp } D_v)^{\text{an}} \rightarrow \mathbb{R}$$

such that $g_v + \ln |f_i|_v^2$ extends to a continuous function on the analytification U_i^{an} of U_i for each $i \in \{1, \dots, \ell\}$. When v is archimedean, we say that g_v is smooth (respectively plurisubharmonic) if the extension of $g_v + \ln |f_i|_v^2$ to U_i^{an} is smooth (respectively plurisubharmonic) for each $i \in \{1, \dots, \ell\}$. We refer the reader to [Mor16, sections 1.4 and 2.1] for more details on Green functions.

2.6. Let X be a normal projective variety on $\text{Spec } K$. Let $D \in \text{Div}(X)_{\mathbb{K}}$ and let $U \subseteq \text{Spec } \mathcal{O}_K$ be a non-empty open subset, where \mathcal{O}_K is the ring of integers of K . A normal model \mathcal{X} of X over U is an integral normal scheme \mathcal{X} together with a projective dominant morphism $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow U$ with generic fiber X . If \mathcal{D} is a \mathbb{K} -Cartier divisor on \mathcal{X} such that the restriction of \mathcal{D} to X is equal to D , we say that $(\mathcal{X}, \mathcal{D})$ is a normal model of (X, D) over U . For each non-archimedean place $v \in U$, we denote by $g_{\mathcal{D},v}$ the D -Green function on X_v^{an} induced by \mathcal{D} (see [Mor16, section 0.2] for details on this construction).

3. ADELIC \mathbb{R} -CARTIER DIVISORS

In the rest of the text, we consider a normal and geometrically integral projective variety X over the number field K . We define adelic \mathbb{R} -Cartier divisors in subsection 3.1. We then recall the notion of semi-positive adelic \mathbb{R} -Cartier divisors and we define heights of subvarieties in subsection 3.2. Subsection 3.3 contains basic facts concerning ample adelic \mathbb{R} -Cartier divisors.

3.1. Definitions. In this paragraph, \mathbb{K} denotes either \mathbb{Z} , \mathbb{Q} or \mathbb{R} .

Definition 3.1. An adelic \mathbb{K} -Cartier divisor on X is a pair $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ consisting of a \mathbb{K} -Cartier divisor D on X and of a continuous D -Green function g_v on X_v^{an} for each $v \in \Sigma_K$, satisfying the following condition: there exist a dense open subset U of $\text{Spec } \mathcal{O}_K$ and a normal model $(\mathcal{X}, \mathcal{D})$ of (X, D) over U such that $g_v = g_{\mathcal{D}, v}$ for all $v \in U$.

The set of adelic \mathbb{K} -Cartier divisors on X is a \mathbb{K} -module, denoted by $\widehat{\text{Div}}(X)_{\mathbb{K}}$. Since X is normal, the natural map $\text{Div}(X) \rightarrow \text{Div}(X)_{\mathbb{K}}$ is injective. It follows that $\widehat{\text{Div}}(X)_{\mathbb{Z}} \subset \widehat{\text{Div}}(X)_{\mathbb{Q}} \subset \widehat{\text{Div}}(X)_{\mathbb{R}}$. In the sequel, the elements of $\widehat{\text{Div}}(X) := \widehat{\text{Div}}(X)_{\mathbb{Z}}$ will be called adelic Cartier divisors for simplicity.

Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be an adelic \mathbb{R} -Cartier divisor on X . We consider the K -vector space

$$H^0(X, D) := \{\phi \in \text{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}.$$

For any $\phi \in (\text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}) \cup \{0\}$ and any $v \in \Sigma_K$, we let ϕ_v be the pullback of ϕ on X_v^{an} and we consider the function $\|\phi\|_v^{\overline{D}} := |\phi_v|_v \exp(-g_v/2)$, defined on an open subset of X_v^{an} . If $\phi \in H^0(X, D)$, the function $\|\phi\|_v^{\overline{D}}$ extends to a continuous function on X_v^{an} (see [Mor16, Propositions 1.4.2 and 2.1.3]). In that case, we let $\|\phi\|_{v, \text{sup}}^{\overline{D}} := \sup_{x \in X_v^{\text{an}}} \|\phi\|_v^{\overline{D}}(x)$. We also define the set of strictly small sections of \overline{D} by

$$\widehat{H}^0(X, \overline{D}) := \{\phi \in H^0(X, D) \mid \|\phi\|_{v, \text{sup}}^{\overline{D}} \leq 1 \ \forall v \in \Sigma_K, \ \|\phi\|_{v, \text{sup}}^{\overline{D}} < 1 \ \forall v \in \Sigma_{K, \infty}\}.$$

Remark 3.2. Let $\overline{D} \in \widehat{\text{Div}}(X)$ be an adelic Cartier divisor. With the above notation, the pair $(\mathcal{O}_X(D), (\|\cdot\|_v^{\overline{D}})_{v \in \Sigma_K})$ is an adelic metrized line bundle in the sense of Zhang [Zha95b, (1.2)]. One can see that every adelic metrized line bundle $\overline{L} = (L, (\|\cdot\|_v)_{v \in \Sigma_K})$ on X can be obtained in this way by considering the Cartier divisor $D = \text{div}(s)$ associated to a trivialization s of L and the D -Green functions $g_v = -\ln \|s_v\|_v^2$ for every $v \in \Sigma_K$, where s_v is the pullback of s to X_v^{an} .

We end this paragraph with the definition of twists of adelic \mathbb{R} -Cartier divisors, which we shall use frequently in the rest of the text.

Definition 3.3. Let $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. For any real number $t \in \mathbb{R}$, we define the t -twist of \overline{D} by

$$\overline{D}(t) = \overline{D} - t\overline{\xi}_\infty \in \widehat{\text{Div}}(X)_{\mathbb{R}},$$

where $\overline{\xi}_\infty = (0, (\xi_v)_{v \in \Sigma_K})$ is the adelic Cartier divisor on X given by $\xi_v = 2$ if v is archimedean, and $\xi_v = 0$ otherwise.

It follows from the definitions that for any $\phi \in H^0(X, D)$, we have $\|\phi\|_v^{\overline{D}(t)} = e^t \|\phi\|_v^{\overline{D}}$ for every $v \in \Sigma_{K, \infty}$ and $\|\phi\|_v^{\overline{D}(t)} = \|\phi\|_v^{\overline{D}}$ for every $v \in \Sigma_K \setminus \Sigma_{K, \infty}$.

3.2. Semi-positivity and heights of subvarieties. Let us first define the height of a point $x \in X(\overline{K})$ with respect to an adelic \mathbb{R} -Cartier divisor \overline{D} on X . Let $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ be a function with $x \notin \text{Supp}(D + (\phi))$ and let $K(x)$ be the function field of $x \in X$. For each place $w \in \Sigma_{K(x)}$, we fix a K -embedding $\sigma_w : K(x) \hookrightarrow \overline{K}_v$, where v denotes the restriction of w to K (note that there are exactly $[K(x)_w : K_v]$ such embeddings). The pair (x, σ_w) determines uniquely a point $x_w \in X_v$, and the quantity $\|\phi\|_{\overline{D}}^w(x) := \|\phi\|_v^{\overline{D}}(x_w^{\text{an}})$ does not depend on the choice of σ_w . The normalized height of x with respect to \overline{D} is the real number

$$\widehat{h}_{\overline{D}}(x) = - \sum_{w \in \Sigma_{K(x)}} \frac{[K(x)_w : \mathbb{Q}_w]}{[K(x) : \mathbb{Q}]} \ln \|\phi\|_{\overline{D}}^w(x).$$

This definition does not depend on the choice of ϕ by [Mor16, (4.2.1)]. Moreover, if $\phi \in H^0(X, D) \setminus \{0\}$ then it follows from the definitions that

$$(3.1) \quad \widehat{h}_{\overline{D}}(x) \geq - \sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \ln \|\phi\|_{v, \text{sup}}^{\overline{D}}.$$

In order to define the height of higher dimensional subvarieties, we need the notion of semi-positive adelic \mathbb{R} -Cartier divisors which we recall below.

Definition 3.4. Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. We say that \overline{D} is semi-positive if there exists a sequence $(\mathcal{X}_n, \mathcal{D}_n, (g_{n,v})_{v \in \Sigma_K})_{n \in \mathbb{N}}$ such that :

- for all $n \in \mathbb{N}$, $(\mathcal{X}_n, \mathcal{D}_n)$ is a normal Spec \mathcal{O}_K -model for (X, D) with \mathcal{D}_n relatively nef,
- for all $n \in \mathbb{N}$, $g_{n,v}$ is a smooth plurisubharmonic D -Green function if $v \in \Sigma_{K, \infty}$ and $g_{n,v} = g_{\mathcal{D}_n, v}$ for every non-archimedean $v \in \Sigma_K$,
- for every $v \in \Sigma_K$, $(g_{n,v})_{n \in \mathbb{N}}$ converges uniformly to g_v .

Remark 3.5.

- (1) It follows from the definition that the sum of semi-positive adelic \mathbb{R} -Cartier divisors is semi-positive. Moreover, if $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ is semi-positive then $\overline{D}(t)$ is semi-positive for any $t \in \mathbb{R}$.
- (2) An adelic Cartier divisor $\overline{D} \in \widehat{\text{Div}}(X)$ is semi-positive if and only if the associated line bundle $(\mathcal{O}_X(D), (\|\cdot\|_v^{\overline{D}})_{v \in \Sigma_K})$ of Remark 3.2 is semi-positive in the sense of Zhang [Zha95b, (1.3)] (see [BGMPS16], (1) page 229).

Following [BGMPS16], we say that an adelic \mathbb{R} -Cartier divisor \overline{D} on X is DSP if $\overline{D} = \overline{D}_1 - \overline{D}_2$ is the difference of two semi-positive $\overline{D}_1, \overline{D}_2 \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. Let \overline{D} be a DSP adelic \mathbb{R} -Cartier divisor on X and let $Y \subseteq X$ be a r -dimensional subvariety, where $0 \leq r \leq \dim X$ is an integer. For any place $v \in \Sigma_K$, we define a measure $c_1(\overline{D})^{\wedge \dim Y} \wedge \delta_{Y_v^{\text{an}}}$ on X_v^{an} as in [BGMPS16, page 225]. It is obtained by multilinearity from the corresponding measures associated to semi-positive adelic Cartier divisors defined in [BGPS14, Definition 1.4.6]. The measure $c_1(\overline{D})^{\wedge \dim Y} \wedge \delta_{Y_v^{\text{an}}}$ is supported on $Y_v^{\text{an}} \subseteq X_v^{\text{an}}$ and has total mass $\deg_D(Y)$.

Let $\Phi = (\phi_0, \dots, \phi_r) \in (\text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R})^{\oplus r}$ be a family intersecting Y properly in the following sense: for every $I \subseteq \{0, \dots, r\}$,

$$Y \cap \left(\bigcap_{i \in I} \text{Supp}((\phi_i) + D) \right)$$

is of pure dimension $r - \#I$. The local height $h_{\overline{D}, \Phi, v}(Y)$ of Z at v with respect to (\overline{D}, Φ) is defined inductively as follows. We put $h_{\overline{D}, \Phi, v}(\emptyset) = 0$, and

$$(3.2) \quad h_{\overline{D}, \Phi, v}(Z) = h_{\overline{D}, (\phi_1, \dots, \phi_r), v}(Y \cdot (D + (\phi_0))) - \int_{X_v^{\text{an}}} \ln \|\phi_0\|_v^{\overline{D}} c_1(\overline{D})^{\wedge \dim Y} \wedge \delta_{Y_v^{\text{an}}}.$$

It follows from [BGPS14, Proposition 1.5.14] that $h_{\overline{D}, \Phi, v}(Y) = 0$ for all except finitely many places $v \in \Sigma_K$. The height of Y with respect to \overline{D} is the real number

$$h_{\overline{D}}(Y) = \sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} h_{\overline{D}, \Phi, v}(Y);$$

it does not depend on the choice of Φ . If $Y \subseteq X$ is a subvariety with $\deg_D(Y) \neq 0$, the normalized height of Y with respect to \overline{D} is the real number

$$\widehat{h}_{\overline{D}}(Y) = \frac{h_{\overline{D}}(Y)}{(\dim Y + 1) \deg_D(Y)}.$$

Remark 3.6.

- (1) If $Y = \{x\}$ is a closed point in X , then $\widehat{h}_{\overline{D}}(Y)$ coincides with the normalized height $\widehat{h}_{\overline{D}}(x)$ of x .
- (2) The height function is continuous in the following sense: for any DSP adelic \mathbb{R} -Cartier divisor \overline{D}' on X , we have

$$\lim_{t \rightarrow 0} h_{\overline{D} + t\overline{D}'}(Y) = h_{\overline{D}}(Y).$$

If moreover $\deg_D(Y) \neq 0$, then $\deg_{\overline{D} + t\overline{D}'}(Y) \neq 0$ for any sufficiently small $t \in \mathbb{R}$ and we have $\lim_{t \rightarrow 0} \widehat{h}_{\overline{D} + t\overline{D}'}(Y) = \widehat{h}_{\overline{D}}(Y)$.

- (3) Assume that $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ is semi-positive, and let

$$(\mathcal{X}_n, \mathcal{D}_n, (g_{n,v})_{v \in \Sigma_K})_{n \in \mathbb{N}}$$

be a sequence as in Definition 3.4. Given $n \in \mathbb{N}$, let $\overline{D}_n = (D, (g_{n,v})_{v \in \Sigma_K})$. Then we have $\lim_{n \rightarrow \infty} h_{\overline{D}_n}(Y) = h_{\overline{D}}(Y)$, and moreover $\lim_{n \rightarrow \infty} \widehat{h}_{\overline{D}_n}(Y) = \widehat{h}_{\overline{D}}(Y)$ if $\deg_D(Y) \neq 0$.

- (4) Assume that $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)$ is a semi-positive adelic Cartier divisor such that there exists a Spec \mathcal{O}_K -model $(\mathcal{X}, \mathcal{D})$ of (X, D) with $g_v = g_{\mathcal{D}, v}$ for every non-archimedean place $v \in \Sigma_K$. Then

$$\overline{\mathcal{L}} = (\mathcal{O}_{\mathcal{X}}(\mathcal{D}), (\|\cdot\|_v^{\overline{D}})_{v \in \Sigma_K, \infty})$$

is a semi-positive hermitian line bundle in the sense of [Zha95a] and we have $h_{\overline{D}}(Y) = c_1(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}}$, where \mathcal{Y} is the Zariski-closure of Y in \mathcal{X} (see [Zha95a, (1.2)] for the definition of $c_1(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}}$).

We have the following lemma concerning the behaviour of heights with respect to twists of adelic \mathbb{R} -Cartier divisors (see Definition 3.3).

Lemma 3.7. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a DSP adelic \mathbb{R} -Cartier divisor on X and let $Y \subseteq X$ be a subvariety. For any $t \in \mathbb{R}$, we have*

$$h_{\overline{D}(t)}(Y) = h_{\overline{D}}(Y) - t(\dim Y + 1) \deg_D(Y).$$

In particular, if $\deg_D(Y) \neq 0$ then $\widehat{h}_{\overline{D}(t)}(Y) = \widehat{h}_{\overline{D}}(Y) - t$.

Proof. The result follows from (3.2) by induction on $\dim Y$. □

Let $r \in \{0, \dots, \dim X\}$ and let Z be a r -cycle in $X_{\overline{K}} = X \times_K \text{Spec } \overline{K}$. There exists a finite extension K' of K such that Z is defined over K' , i.e. Z is a r -cycle in $X_{K'} = X \times_K \text{Spec } K'$: there exists integers a_1, \dots, a_ℓ and subvarieties Y_1, \dots, Y_ℓ of $X_{K'}$ such that $Z = \sum_{i=1}^{\ell} a_i Y_{i, \overline{K}}$. Given a DSP $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$, we define a DSP adelic \mathbb{R} -Cartier divisor $\overline{D}_{K'}$ by pulling back \overline{D} to $X_{K'}$. The height of Z with respect to \overline{D} is then defined by $h_{\overline{D}}(Z) = \sum_{i=1}^{\ell} a_i h_{\overline{D}_{K'}}(Y_i)$. This definition does not depend on the choice of K' by [BGPS14, Proposition 1.5.10].

Lemma 3.8. *Let \overline{D} be a DSP adelic \mathbb{R} -Cartier divisor on X . The following conditions are equivalent:*

- (1) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$;
- (2) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X_{\overline{K}}$.

Proof. The implication (2) \implies (1) is clear. Assume that (1) holds and let $Y \subseteq X_{\overline{K}}$ be a subvariety. Let $\text{Gal}(\overline{K}/K)$ be the set of K -automorphisms $\sigma: \overline{K} \rightarrow \overline{K}$. For any $\sigma \in \text{Gal}(\overline{K}/K)$, we denote by Y^σ the pullback of Y by the automorphism of $X_{\overline{K}}$ induced by σ ; it is a subvariety of $X_{\overline{K}}$. We consider the set

$$O(Y) = \{Y^\sigma \mid \sigma \in \text{Gal}(\overline{K}/K)\}.$$

It follows easily from the definitions that $h_{\overline{D}}(Y') = h_{\overline{D}}(Y)$ for any $Y' \in O(Y)$ (alternatively, this fact is a direct consequence of [BGPS14, Theorem 1.5.11]). By [BG06, A.4.13],

$$Z_Y = \bigcup_{Y' \in O(Y)} Y'$$

is a subvariety of X (i.e. its image in X is an irreducible Zariski closed subset of X , which we still denote by Z_Y). Therefore $h_{\overline{D}}(Z_Y) > 0$ by assumption. Let K' be a finite extension such that every $Y' \in O(Y)$ is a subvariety of $X_{K'}$. Let $(Z_Y)_{K'}$ be the cycle in $X_{K'}$ associated to Z_Y : we have

$$(Z_Y)_{K'} = \sum_{Y' \in O(Y)} n_{Y'} Y',$$

where $n_{Y'}$ is a positive integer for every $Y' \in O(Y)$. By [BGPS14, Proposition 1.5.10], we have $h_{\overline{D}}((Z_Y)_{K'}) = h_{\overline{D}}(Z_Y) > 0$. On the other hand, we have

$$h_{\overline{D}}((Z_Y)_{K'}) = \sum_{Y' \in O(Y)} n_{Y'} h_{\overline{D}}(Y') = h_{\overline{D}}(Y) \times \sum_{Y' \in O(Y)} n_{Y'},$$

and therefore $h_{\overline{D}}(Y) > 0$. \square

We end this paragraph with a sufficient condition for the ampleness of the underlying divisor of an adelic \mathbb{R} -Cartier divisor.

Lemma 3.9. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be semi-positive. Assume that $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$. Then D is ample.*

We want to combine Campana and Peternell's Nakai-Moishezon criterion for \mathbb{R} -Cartier divisors [CP90] with Moriwaki's generalized Hodge index theorem [Mor16, Theorem 5.3.2] applied to subvarieties of X . We must pay attention to the fact that [Mor16, Theorem 5.3.2] applies only to normal and geometrically integral subvarieties.

Proof. Let $Y \subseteq X_{\overline{K}}$ be a subvariety and let K' be a finite extension of K such that Y is defined over K' . We consider the adelic \mathbb{R} -Cartier divisor $\overline{D}_{K'} = (D_{K'}, (g_w)_{w \in \Sigma_{K'}})$ defined by pulling back \overline{D} to $X_{K'}$. Let $f: Y' \rightarrow Y$ be the normalization of Y and let $\phi \in \text{Rat}(X_{K'})^\times \otimes_{\mathbb{Z}} \mathbb{R}$ be such that $Y \not\subseteq \text{Supp}(D_{K'} + (\phi))$. Note that Y' is normal and geometrically integral. We define a semi-positive adelic \mathbb{R} -Cartier divisor $\overline{D}_{Y'} = (D_{Y'}, (g_{Y',w})_{w \in \Sigma_{K'}})$ on Y' as follows: $D_{Y'} = f^*(D_{K'} + (\phi))|_Y$ and for each $w \in \Sigma_{K'}$, the $D_{Y'}$ -Green function $g_{Y',w}$ is the pullback of $(g_w - 2 \ln |\phi|_w)|_{Y_w^{\text{an}}}$ to $(Y'_w)^{\text{an}}$. By [BGPS14, Theorem 1.5.11 (2)], we have $h_{\overline{D}_{Y'}}(Y') = h_{\overline{D}_{K'}}(Y)$. Therefore our assumption together with Lemma 3.8 implies that $h_{\overline{D}_{Y'}}(Y') = h_{\overline{D}_{K'}}(Y) > 0$. It follows from [Mor16, Theorem 5.3.2] that $\overline{D}_{Y'}$ is big in the sense of [Mor16, Definition 4.4.1]. In particular, $D_{Y'}$ is big. Since $D_{Y'}$

is also nef by semi-positivity, we have $D_{K'}^{\dim Y} \cdot Y = D_{Y'}^{\dim Y'} \cdot Y' > 0$. Therefore D is ample by [CP90, Theorem 1.3]. \square

3.3. Ample adelic \mathbb{R} -Cartier divisors. We now define ample adelic \mathbb{R} -Cartier divisors and study some of their properties.

Definition 3.10. Let $\bar{D} = (D, (g_v)_{v \in \Sigma_K})$ be an adelic \mathbb{R} -Cartier divisor. We say that \bar{D} is

- weakly ample (w-ample for short) if $\bar{D} = \sum_{i=1}^{\ell} \lambda_i \bar{A}_i$ is a \mathbb{R} -linear combination of adelic Cartier divisors $\bar{A}_i \in \widehat{\text{Div}}(X)$ such that for each $i \in \{1, \dots, \ell\}$, $\lambda_i > 0$, A_i is ample and for every $m \gg 1$, $H^0(X, mA_i)$ has a K -basis consisting of strictly small sections;
- ample if it is w-ample and semi-positive.

The terminology of weakly ample adelic \mathbb{R} -Cartier divisors is due to Ikoma [Iko21]. We end this section with three lemmas concerning basic properties of w-ample adelic \mathbb{R} -Cartier divisors.

Lemma 3.11. *Let $\bar{D}, \bar{D}' \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. If \bar{D} is w-ample, there exists a real number $\varepsilon > 0$ such that $\bar{D} + t\bar{D}'$ is ample for any $t \in \mathbb{R}$ with $|t| \leq \varepsilon$.*

Proof. Without loss of generality, we only consider the case where $\bar{D}' \in \widehat{\text{Div}}(X)$ and $t \geq 0$. If \bar{D} is w-ample, $\bar{D} = \sum_{i=1}^{\ell} \lambda_i \bar{A}_i$ is a \mathbb{R} -linear combination with positive coefficients of adelic Cartier divisors $\bar{A}_i \in \widehat{\text{Div}}(X)$ such that for each $i \in \{1, \dots, \ell\}$, A_i is ample and $H^0(X, mA_i)$ has a K -basis consisting of strictly small sections for $m \gg 1$. By [Iko16, Proposition 5.4 (5)], there exists a $\delta > 0$ such that $\bar{A}_1 + \delta\bar{D}'$ is w-ample. Let $\varepsilon = \delta\lambda_1$. Then for every real number $t \in [0, \varepsilon]$,

$$\bar{D} + t\bar{D}' = \frac{t}{\delta}(\bar{A}_1 + \delta\bar{D}') + (\lambda_1 - \frac{t}{\delta})\bar{A}_1 + \sum_{i=2}^{\ell} \lambda_i \bar{A}_i$$

is w-ample. \square

Remark 3.12. By Lemma 3.11 and [Mor16, Lemma 1.1.1], an adelic Cartier divisor $\bar{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)$ on X is w-ample if and only if D is ample and $H^0(X, mD)$ has a K -basis consisting of strictly small sections for every $m \gg 1$.

Lemma 3.13. *Let \bar{D} be a w-ample adelic \mathbb{R} -Cartier divisor on X . Then*

$$\inf_{x \in X(\bar{K})} \widehat{h}_{\bar{D}}(x) > 0.$$

Proof. By definition, we can write $\bar{D} = \sum_{i=1}^{\ell} \lambda_i \bar{A}_i$ where for each $i \in \{1, \dots, \ell\}$, λ_i is a positive real number, \bar{A}_i is an adelic Cartier divisor such that A_i is ample, and $H^0(X, mA_i)$ has a K -basis consisting of strictly small sections for every $m \gg 1$. Let $m \geq 1$ be an integer such that for each $i \in \{1, \dots, \ell\}$, there exists a set of functions $\phi_{i,1}, \dots, \phi_{i,k_i} \in \widehat{H}^0(X, mA_i)$ with

$$\bigcap_{j=1}^{k_i} \text{Supp}(mA_i + (\phi_{i,j})) = \emptyset.$$

Letting

$$\Lambda_i := - \max_{1 \leq j \leq k_i} \sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \ln \|\phi_{i,j}\|_{v, \text{sup}}^{m\bar{A}_i} > 0,$$

we have $\widehat{h}_{\overline{A}_i}(x) \geq \Lambda_i/m$ for every $x \in X(\overline{K})$ (see (3.1)). Therefore we have

$$\inf_{x \in X(\overline{K})} \widehat{h}_{\overline{D}}(x) \geq \sum_{i=1}^{\ell} \lambda_i \inf_{x \in X(\overline{K})} \widehat{h}_{\overline{A}_i}(x) \geq \sum_{i=1}^{\ell} \lambda_i \Lambda_i/m > 0.$$

□

Lemma 3.14. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be an adelic \mathbb{R} -Cartier divisor. If D is ample, there exists a real number $t \in \mathbb{R}$ such that $\overline{D}(t)$ is w -ample.*

Proof. Since D is ample, there exists an ample $\overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ such that $\overline{D} - \overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$ and $D - A$ is ample. For a sufficiently large and divisible integer m , $m(D - A)$ is a very ample Cartier divisor on X . Let $(\phi_1, \dots, \phi_{\ell})$ be basis of $H^0(X, m(D - A))$ such that $\|\phi_i\|_{v, \text{sup}}^{m(\overline{D} - \overline{A})} \leq 1$ for every $i \in \{1, \dots, \ell\}$ and every non-archimedean place $v \in \Sigma_K$. Let $t \in \mathbb{R}$ be a real number such that

$$t < -\max_{1 \leq i \leq \ell} \max_{v \in \Sigma_{K, \infty}} \ln \|\phi_i\|_{v, \text{sup}}^{m(\overline{D} - \overline{A})}.$$

Then $\phi_i \in \widehat{H}^0(X, m(\overline{D} - \overline{A})(t))$ for every i , and it follows that $\overline{A}'_t := (\overline{D} - \overline{A})(t) = \overline{D}(t) - \overline{A}$ is ample. Therefore $\overline{D}(t) = \overline{A} + \overline{A}'_t$ is ample. □

4. ZHANG'S THEOREM ON SUCCESSIVE MINIMA

In this section we recall the notion of successive minima for adelic \mathbb{R} -Cartier divisors, which was first introduced by Zhang for hermitian line bundles [Zha95a, section 5]. We then prove a continuity property which allows to extend Zhang's theorem on minima [Zha95a, Theorem 5.2] to the case of adelic \mathbb{R} -Cartier divisors (see Lemma 4.1 and Theorem 4.3 below).

Let $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ and let $Z \subseteq X$ be a subvariety. For any $i \in \{1, \dots, \dim Z + 1\}$, we define the i -th successive minimum of \overline{D} on Z by

$$\zeta_i(\overline{D}, Z) = \sup_{\substack{Y \subseteq Z \\ \dim Y < i-1}} \inf_{x \in Z(\overline{K}) \setminus Y} \widehat{h}_{\overline{D}}(x) \in \mathbb{R} \cup \{-\infty\},$$

where the supremum is over all the Zariski-closed subsets $Y \subseteq Z$ of dimension $\dim Y < i - 1$. We obtain a chain of real numbers

$$\zeta_{\dim Z + 1}(\overline{D}, Z) \geq \zeta_{\dim Z}(\overline{D}, Z) \geq \dots \geq \zeta_1(\overline{D}, Z).$$

Successive minima satisfy the following properties.

Lemma 4.1. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. Let $Z \subseteq X$ be a subvariety and let $1 \leq i \leq \dim Z + 1$ be an integer.*

(1) *For any $\overline{D}' \in \widehat{\text{Div}}(X)_{\mathbb{R}}$, we have*

$$\zeta_i(\overline{D} + \overline{D}', Z) \geq \zeta_i(\overline{D}, Z) + \zeta_i(\overline{D}', Z).$$

(2) *Let $\overline{D}_1, \dots, \overline{D}_{\ell} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. If D is ample, then*

$$\lim_{\max\{|t_1|, \dots, |t_{\ell}|\} \rightarrow 0} \zeta_i(\overline{D} + t_1 \overline{D}_1 + \dots + t_{\ell} \overline{D}_{\ell}, Z) = \zeta_i(\overline{D}, Z).$$

Proof. (1) We may assume that $\zeta_i(\overline{D}, Z) > -\infty$ and $\zeta_i(\overline{D}', Z) > -\infty$. Let $t < \zeta_i(\overline{D}, Z)$ and $t' < \zeta_i(\overline{D}', Z)$ be real numbers. By definition, there exist two closed subsets $Y, Y' \subseteq Z$ of dimension $< i - 1$ such that for any $x \in Z(\overline{K}) \setminus (Y \cup Y')$, we have

$$\widehat{h}_{\overline{D} + \overline{D}'}(x) = \widehat{h}_{\overline{D}}(x) + \widehat{h}_{\overline{D}'}(x) \geq t + t'.$$

Since $\dim(Y \cup Y') < i - 1$, we have

$$\zeta_i(\overline{D} + \overline{D}', Z) \geq \inf_{x \in Z(\overline{K}) \setminus (Y \cup Y')} \widehat{h}_{\overline{D}}(x) \geq t + t',$$

and we conclude by letting t and t' tend to $\zeta_i(\overline{D}, Z)$ and $\zeta_i(\overline{D}', Z)$.

(2) If we replace \overline{D} by $\overline{D}(t)$ for some real number t , both sides of the equality differ by $-t$. By Lemma 3.14, we may therefore assume that \overline{D} is w-ample. Let $\varepsilon > 0$ be a real number. For $t_1, \dots, t_\ell \in \mathbb{R}$ small enough, the adelic \mathbb{R} -Cartier divisors

$$(1 + \varepsilon)\overline{D} - (\overline{D} + t_1\overline{D}_1 + \dots + t_\ell\overline{D}_\ell) = \varepsilon\overline{D} - (t_1\overline{D}_1 + \dots + t_\ell\overline{D}_\ell)$$

and

$$\overline{D} + t_1\overline{D}_1 + \dots + t_\ell\overline{D}_\ell - (1 - \varepsilon)\overline{D} = \varepsilon\overline{D} + (t_1\overline{D}_1 + \dots + t_\ell\overline{D}_\ell)$$

are w-ample by Lemma 3.11. Combining (1) and Lemma 3.13, we have

$$(1 + \varepsilon)\zeta_i(\overline{D}, Z) \geq \zeta_i(\overline{D} + t_1\overline{D}_1 + \dots + t_\ell\overline{D}_\ell, Z) \geq (1 - \varepsilon)\zeta_i(\overline{D}, Z)$$

and the result follows. \square

Remark 4.2. Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be semi-positive. We consider a sequence $(\mathcal{X}_n, \mathcal{D}_n, (g_{n,v})_{v \in \Sigma_K})_{n \in \mathbb{N}}$ associated to \overline{D} as in Definition 3.4. For each $n \in \mathbb{N}$, let $\overline{D}_n = (D, (g_{n,v})_{v \in \Sigma_K})$. Then we have

$$\lim_{n \rightarrow \infty} \zeta_i(\overline{D}_n, Z) = \zeta_i(\overline{D}, Z)$$

for any subvariety $Z \subseteq X$ and any $i \in \{1, \dots, \dim Z + 1\}$. Indeed, the sum

$$\varepsilon_n := 2 \sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \sup_{z \in X_v^{\text{an}}} |g_v(z) - g_{n,v}(z)|$$

is finite for every $n \in \mathbb{N}$, and the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converges to zero. By construction, we have

$$\widehat{h}_{\overline{D}_n}(x) - \varepsilon_n \leq \widehat{h}_{\overline{D}}(x) \leq \widehat{h}_{\overline{D}_n}(x) + \varepsilon_n.$$

for any $n \in \mathbb{N}$ and $x \in X(\overline{K})$. It follows that

$$\zeta_i(\overline{D}_n, Z) - \varepsilon_n \leq \zeta_i(\overline{D}, Z) \leq \zeta_i(\overline{D}_n, Z) + \varepsilon_n$$

as in the proof of Lemma 4.1 (1), and we conclude by letting n tend to infinity.

The following theorem was originally proved by Zhang for adelic Cartier divisors equipped with Green functions induced by a fixed model [Zha95a, Theorem 5.2]. Thanks to the continuity property of Lemma 4.1, it remains valid for adelic \mathbb{R} -Cartier divisors.

Theorem 4.3. *Assume that $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ is semi-positive and that D is ample. For any subvariety $Z \subseteq X$, we have*

$$\zeta_{\dim Z + 1}(\overline{D}, Z) \geq \widehat{h}_{\overline{D}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z + 1} \zeta_i(\overline{D}, Z).$$

Proof. Since D is ample, we can write $D = \sum_{i=1}^{\ell} \lambda_i A_i$ where for each $i \in \{1, \dots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and $A_i \in \text{Div}(X)$ is an ample Cartier divisor on X . Let $(g_{i,v})_{v \in \Sigma_K}$ be a collection of A_i -Green functions such that $\overline{A}_i = (A_i, (g_{i,v})_{v \in \Sigma_K})$ is a semi-positive adelic Cartier divisor on X . Given a ℓ -tuple of real numbers $\mathbf{t} = (t_1, \dots, t_\ell) \in \mathbb{R}^\ell$, we denote by $\overline{D}_{\mathbf{t}} = (D_{\mathbf{t}}, (g_{\mathbf{t},v})_{v \in \Sigma_K})$ the adelic \mathbb{R} -Cartier divisor

$$\overline{D}_{\mathbf{t}} = \overline{D} + \sum_{i=1}^{\ell} t_i \overline{A}_i = \left(\sum_{i=1}^{\ell} (\lambda_i + t_i) A_i, (g_v + \sum_{i=1}^{\ell} t_i g_{i,v})_{v \in \Sigma_K} \right).$$

Let $\varepsilon > 0$ be a real number. We can choose $\mathbf{t} \in [0, \varepsilon]^\ell$ such that $\overline{D}_{\mathbf{t}} \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$. Note that $\overline{D}_{\mathbf{t}} \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$ is semi-positive. We consider a sequence

$$(\mathcal{X}_{\mathbf{t},n}, \mathcal{D}_{\mathbf{t},n}, (g_{\mathbf{t},n,v})_{v \in \Sigma_K})_{n \in \mathbb{N}}$$

associated to $\overline{D}_{\mathbf{t}}$ as in Definition 3.4, and we let $\overline{D}_{\mathbf{t},n} = (D_{\mathbf{t},n}, (g_{\mathbf{t},n,v})_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$. Let m be a positive integer such that $mD_{\mathbf{t},n} \in \text{Div}(X)$. By [Mor15, Theorem 0.2], the hermitian metrized line bundle $\overline{\mathcal{L}}_{m,\mathbf{t},n}$ associated to $m\overline{D}_{\mathbf{t},n}$ in Remark 3.6 (4) is semiample metrized in the sense of [Zha95a, section 5]. Therefore we can apply [Zha95a, Theorem 5.2] to the restriction of $\overline{\mathcal{L}}_{m,\mathbf{t},n}$ to the closure of Z in $\mathcal{X}_{\mathbf{t},n}$. We obtain

$$(4.1) \quad \zeta_{\dim Z+1}(m\overline{D}_{\mathbf{t},n}, Z) \geq \widehat{h}_{m\overline{D}_{\mathbf{t},n}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z+1} \zeta_i(m\overline{D}_{\mathbf{t},n}, Z)$$

for any $n \in \mathbb{N}$ (see Remark 3.6 (4)). On the other hand we have $\widehat{h}_{m\overline{D}_{\mathbf{t},n}}(Z) = m\widehat{h}_{\overline{D}_{\mathbf{t},n}}(Z)$ and $\zeta_i(m\overline{D}_{\mathbf{t},n}, Z) = m\zeta_i(\overline{D}_{\mathbf{t},n}, Z)$ for any $i \in \{1, \dots, \dim Z + 1\}$, and therefore (4.1) remains true for $m = 1$. Letting n tend to infinity, we obtain

$$\zeta_{\dim Z+1}(\overline{D}_{\mathbf{t}}, Z) \geq \widehat{h}_{\overline{D}_{\mathbf{t}}}(Z) \geq \frac{1}{\dim Z + 1} \sum_{i=1}^{\dim Z+1} \zeta_i(\overline{D}_{\mathbf{t}}, Z)$$

by Remarks 3.6 (3) and 4.2. Letting ε tend to zero, the result follows from the continuity of normalized heights and successive minima given by Remark 3.6 (2) and Lemma 4.1 (2). \square

5. ABSOLUTE MINIMUM AND HEIGHT OF SUBVARIETIES

For any $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$, we call $\zeta_{\text{abs}}(\overline{D}) := \zeta_1(\overline{D}, X) = \inf_{x \in X(\overline{K})} \widehat{h}_{\overline{D}}(x)$ the absolute minimum of \overline{D} . The goal of this section is to prove the following statement, which refines Theorem 1.2 in the introduction.

Theorem 5.1. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . If D is ample, there exists a subvariety $Y \subseteq X$ such that*

$$\zeta_{\text{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) = \min_{Z \subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the minimum is over the subvarieties $Z \subseteq X$. Moreover, $\zeta_{\text{abs}}(\overline{D}) = \zeta_i(\overline{D}, Y)$ for any $i \in \{1, \dots, \dim Y + 1\}$.

We begin with two preliminary lemmas.

Lemma 5.2. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . Assume that D is ample. Then for any subvariety $Z \subseteq X$, the following conditions are equivalent:*

- (1) $\widehat{h}_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq Z$;
- (2) $\zeta_1(\overline{D}, Z) > 0$.

Proof. (1) \implies (2): If $Z = \{x\}$ is a point, then $\zeta_1(\overline{D}, Z) = \widehat{h}_{\overline{D}}(x) > 0$. We assume by induction that $\dim Z > 0$ and that $\zeta_1(\overline{D}, Y) > 0$ for every subvariety $Y \subsetneq Z$. Since $\widehat{h}_{\overline{D}}(Z) > 0$, it follows from Theorem 4.3 that there exists a closed subset $Y \subsetneq Z$ such that $\inf_{x \in Z(\overline{K}) \setminus Y} \widehat{h}_{\overline{D}}(x) > 0$. On the other hand, if Y_1, \dots, Y_ℓ are the irreducible components of Y then

$$\inf_{x \in Z(\overline{K}) \cap Y} \widehat{h}_{\overline{D}}(x) = \min_{1 \leq i \leq \ell} \zeta_1(\overline{D}, Y_i) > 0$$

by the induction hypothesis. Therefore we have

$$\zeta_1(\overline{D}, Z) = \min\left\{\inf_{x \in Z(\overline{K}) \setminus Y} \widehat{h}_{\overline{D}}(x), \inf_{x \in Z(\overline{K}) \cap Y} \widehat{h}_{\overline{D}}(x)\right\} > 0.$$

(2) \implies (1): Let $\zeta = \zeta_1(\overline{D}, Z) > 0$. Note that $\overline{D}(\zeta) = \overline{D} - \zeta \overline{\xi}_\infty$ is semi-positive and $\zeta_1(\overline{D}(\zeta), Z) = \zeta_1(\overline{D}, Z) - \zeta = 0$. For any subvariety $Y \subseteq Z$ we have

$$\widehat{h}_{\overline{D}(\zeta)}(Y) \geq \zeta_1(\overline{D}(\zeta), Y) \geq \zeta_1(\overline{D}(\zeta), Z) = 0,$$

where the first inequality is given by Theorem 4.3 and the second one follows from the definitions. By Lemma 3.7, we have

$$\widehat{h}_{\overline{D}}(Y) = \widehat{h}_{\overline{D}(\zeta)}(Y) + \zeta > \widehat{h}_{\overline{D}(\zeta)}(Y) \geq 0.$$

□

Lemma 5.3. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X with D ample. Then*

$$\zeta_{\text{abs}}(\overline{D}) = \inf_{Z \subseteq X} \widehat{h}_{\overline{D}}(Z),$$

where the infimum is over the subvarieties $Z \subseteq X$.

Proof. By Zhang's Theorem 4.3, we have

$$\widehat{h}_{\overline{D}}(Z) \geq \zeta_1(\overline{D}, Z) \geq \zeta_{\text{abs}}(\overline{D})$$

for any subvariety $Z \subseteq X$, and we deduce one inequality of the lemma by taking the infimum on Z . The converse inequality follows directly from the definition of $\zeta_{\text{abs}}(\overline{D})$.

□

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\zeta = \zeta_{\text{abs}}(\overline{D}) \in \mathbb{R}$. Note that $\overline{D}(\zeta)$ is semi-positive and $\zeta_{\text{abs}}(\overline{D}(\zeta)) = \zeta_{\text{abs}}(\overline{D}) - \zeta = 0$. By Theorem 4.3, we have

$$\widehat{h}_{\overline{D}(\zeta)}(Y) \geq \zeta_1(\overline{D}(\zeta), Y) \geq \zeta_{\text{abs}}(\overline{D}(\zeta)) = 0$$

for every subvariety $Y \subseteq X$. By Lemma 5.2 applied to $Z = X$, there exists a subvariety $Y \subseteq X$ such that $\widehat{h}_{\overline{D}(\zeta)}(Y) = 0$. Therefore Lemma 3.7 gives

$$\zeta_{\text{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y) - \widehat{h}_{\overline{D}(\zeta)}(Y) = \widehat{h}_{\overline{D}}(Y).$$

The fact that $\zeta_{\text{abs}}(\overline{D})$ coincides with the minimum in the theorem follows from Lemma 5.3. Finally, we observe that $\zeta_1(\overline{D}, Y) \geq \zeta_{\text{abs}}(\overline{D}) = \widehat{h}_{\overline{D}}(Y)$. Therefore Zhang's Theorem 4.3 implies that $\zeta_{\text{abs}}(\overline{D}) = \zeta_i(\overline{D}, Y)$ for every integer $1 \leq i \leq \dim Y + 1$.

□

6. PROOF OF THEOREM 1.1

Given an adelic \mathbb{R} -Cartier divisor \overline{D} on X , we introduce the invariant

$$\theta(\overline{D}) := \sup\{t \in \mathbb{R} \mid \overline{D}(t) \text{ is w-ample}\} \in \mathbb{R} \cup \{-\infty\}$$

(with the convention that $\sup \emptyset = -\infty$). The main result of this section is the following theorem, from which we shall deduce Theorem 1.1 (see Corollary 6.4 below).

Theorem 6.1. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . If D is ample, then $\zeta_{\text{abs}}(\overline{D}) = \theta(\overline{D})$.*

Before proving this theorem, we gather some basic properties satisfied by the invariant $\theta(\overline{D})$ in the following lemma.

Lemma 6.2. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$.*

(1) *For any $\overline{D}' \in \widehat{\text{Div}}(X)_{\mathbb{R}}$, we have*

$$\theta(\overline{D} + \overline{D}') \geq \theta(\overline{D}) + \theta(\overline{D}').$$

(2) *D is ample if and only if $\theta(\overline{D})$ is finite.*

(3) *Let $\overline{D}_1, \overline{D}_2, \dots, \overline{D}_\ell \in \widehat{\text{Div}}(X)_{\mathbb{R}}$. If D is ample, then*

$$\lim_{\max\{|t_1|, \dots, |t_\ell|\} \rightarrow 0} \theta(\overline{D} + t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell) = \theta(\overline{D}).$$

Proof. (1) Clearly we may assume that $\theta(\overline{D}) > -\infty$ and $\theta(\overline{D}') > -\infty$. It suffices to observe that the sum of two w-ample adelic \mathbb{R} -Cartier divisors is w-ample.

(2) If $\theta(\overline{D})$ is finite, then clearly D is ample. Conversely, assume that D is ample. By Lemma 3.14, there exists $t \in \mathbb{R}$ such that $\overline{D}(t)$ is w-ample. Therefore $\theta(\overline{D}) \geq t$ is finite.

(3) If we replace \overline{D} by $\overline{D}(t)$ for some real number t , both sides of the equality differ by $-t$. By Lemma 3.14, we may therefore assume that \overline{D} is w-ample. Let $\varepsilon > 0$ be a real number. For sufficiently small real numbers t_1, \dots, t_ℓ , the adelic \mathbb{R} -Cartier divisors

$$(1 + \varepsilon)\overline{D} - (\overline{D} + t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell) = \varepsilon \overline{D} - (t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell)$$

and

$$\overline{D} + t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell - (1 - \varepsilon)\overline{D} = \varepsilon \overline{D} + (t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell)$$

are w-ample by Lemma 3.11. In particular,

$$\theta(\varepsilon \overline{D} - (t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell)) \geq 0 \quad \text{and} \quad \theta(\varepsilon \overline{D} + (t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell)) \geq 0$$

by definition of θ . By (1), we infer that

$$(1 + \varepsilon)\theta(\overline{D}) \geq \theta(\overline{D} + t_1 \overline{D}_1 + \dots + t_\ell \overline{D}_\ell) \geq (1 - \varepsilon)\theta(\overline{D}),$$

and the result follows. \square

Let us now prove Theorem 6.1. We shall combine Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2] and the continuity property given by Lemma 6.2 (3).

Proof of Theorem 6.1. Since D is ample, we have $\theta(\overline{D}) > -\infty$ by Lemma 6.2 (2). Let $t < \theta(\overline{D})$ be a real number. By definition, $\overline{D}(t)$ is w-ample and Lemma 3.13 gives

$$\zeta_{\text{abs}}(\overline{D}) - t = \zeta_{\text{abs}}(\overline{D}(t)) > 0.$$

By letting t tend to $\theta(\overline{D})$, we conclude that $\zeta_{\text{abs}}(\overline{D}) \geq \theta(\overline{D})$.

For the converse inequality, let us first assume that $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$. By homogeneity of $\theta(\overline{D})$ and $\zeta_{\text{abs}}(\overline{D})$, we may assume that \overline{D} is an adelic Cartier divisor without loss of generality. Let $t < \zeta_{\text{abs}}(\overline{D})$ be a real number. Since $\zeta_{\text{abs}}(\overline{D}(t)) = \zeta_{\text{abs}}(\overline{D}) - t > 0$, we have

$$h_{\overline{D}(t)}(Y) > 0$$

for any subvariety $Y \subseteq X$ by Lemma 5.2. By [Zha95b, Theorem 1.7] (see also [Zha95b, Proof of Theorem 1.8]), for any subvariety $Y \subseteq X$ there exists an integer $n > 0$ such that $\widehat{H}^0(Y, n\overline{D}(t)|_Y) \neq 0$. By [CM18, Theorem 1.2], $\overline{D}(t)$ is w-ample. Therefore $\theta(\overline{D}) \geq t$, and we conclude by letting t tend to $\zeta_{\text{abs}}(\overline{D})$.

Let us now prove the equality $\zeta_{\text{abs}}(\overline{D}) = \theta(\overline{D})$ in full generality. Since D is ample, we can write $D = \sum_{i=1}^{\ell} \lambda_i A_i$ where for each $i \in \{1, \dots, \ell\}$, $\lambda_i \in \mathbb{R}_{>0}$ and A_i is an ample Cartier divisor on X . For each $i \in \{1, \dots, \ell\}$, we equip A_i with a collection

of A_i -Green functions $(g_{i,v})_{v \in \Sigma_K}$ such that $\overline{A}_i = (A_i, (g_{i,v})_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)$ is semi-positive. For any $\varepsilon > 0$, we can find a ℓ -tuple of real numbers $\mathbf{t} = (t_1, \dots, t_\ell) \in [0, \varepsilon]^\ell$ such that

$$\overline{D}_{\mathbf{t}} := \overline{D} + \sum_{i=1}^{\ell} t_i \overline{A}_i = \left(\sum_{i=1}^{\ell} (\lambda_i + t_i) A_i, (g_v + \sum_{i=1}^{\ell} t_i g_{i,v})_{v \in \Sigma_K} \right) \in \widehat{\text{Div}}(X)_{\mathbb{Q}}$$

is an adelic \mathbb{Q} -Cartier divisor. Note that $\overline{D}_{\mathbf{t}}$ is semi-positive. By the above, we have $\zeta_{\text{abs}}(\overline{D}_{\mathbf{t}}) = \theta(\overline{D}_{\mathbf{t}})$. Letting ε tend to zero, we find that $\zeta_{\text{abs}}(\overline{D}) = \theta(\overline{D})$ by continuity of ζ_{abs} and θ (Lemma 4.1 (2) and Lemma 6.2 (3)). \square

Remark 6.3. In the proof of Theorem 6.1, we used a particular case of a theorem of Chen and Moriawaki [CM18], which generalizes Zhang's arithmetic Nakai-Moishezon criterion [Zha95a, Theorem 4.2]. Using Zhang's original result would have required extra work since it involves stronger assumptions on the metrics.

We now deduce a refinement of Theorem 1.1 from Theorems 5.1 and 6.1.

Corollary 6.4. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K})$ be a semi-positive adelic \mathbb{R} -Cartier divisor on X . The following conditions are equivalent:*

- (1) \overline{D} is ample;
- (2) $h_{\overline{D}}(Y) > 0$ for every subvariety $Y \subseteq X$;
- (3) D is ample and $\inf_{Y \subseteq X} \widehat{h}_{\overline{D}}(Y) > 0$, where the infimum is over all subvarieties $Y \subseteq X$;
- (4) D is ample and $\zeta_{\text{abs}}(\overline{D}) > 0$.

Proof. The assertion (2) \Leftrightarrow (3) \Leftrightarrow (4) is given by Lemma 3.9 and Theorem 5.1. The implication (1) \Rightarrow (4) is Lemma 3.13, so it only remains to prove (4) \Rightarrow (1). If (4) holds, then $\theta(\overline{D}) = \zeta_{\text{abs}}(\overline{D}) > 0$ by Theorem 6.1 and therefore \overline{D} is w-ample by definition of $\theta(\overline{D})$. Since \overline{D} is also semi-positive, it is ample. \square

Remark 6.5. In [BGMPS16, Definition 3.18 (2)], the authors defined arithmetic ampleness by using the notion of metrized divisors generated by small \mathbb{R} -sections. It is straightforward to check that if $\overline{D} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ is ample in the sense of Definition 3.10, then it is ample in the sense of [BGMPS16]. On the other hand, if \overline{D} is ample in the sense of [BGMPS16], then clearly $\zeta_{\text{abs}}(\overline{D}) > 0$. Therefore, Corollary 6.4 implies that our definition of arithmetic ampleness coincides with the one of [BGMPS16, Definition 3.18 (2)].

We conclude this article with two direct consequences of our results.

Corollary 6.6. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be semi-positive and let $\overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be w-ample. The following assertions are equivalent:*

- (1) \overline{D} is ample;
- (2) D is ample and there exists a real number $\varepsilon > 0$ such that $\widehat{h}_{\overline{D}}(x) \geq \varepsilon \widehat{h}_{\overline{A}}(x)$ for any $x \in X(\overline{K})$.

Proof. (1) \Rightarrow (2): By Lemma 3.11, there exists a real number $\varepsilon > 0$ such that $\overline{D} - \varepsilon \overline{A}$ is w-ample. By Lemma 3.13, we have

$$\widehat{h}_{\overline{D}}(x) - \varepsilon \widehat{h}_{\overline{A}}(x) = \widehat{h}_{\overline{D} - \varepsilon \overline{A}}(x) > 0$$

for any $x \in X(\overline{K})$.

(2) \Rightarrow (1): Since \overline{A} is w-ample, $\zeta_{\text{abs}}(\overline{A}) > 0$ by Lemma 3.13. Assumption (2) therefore implies that $\zeta_{\text{abs}}(\overline{D} - \varepsilon' \overline{A}) > 0$ for any $\varepsilon' \in (0, \varepsilon)$. By Lemma 4.1 (1), it follows that

$$\zeta_{\text{abs}}(\overline{D}) \geq \zeta_{\text{abs}}(\overline{D} - \varepsilon' \overline{A}) + \zeta_{\text{abs}}(\varepsilon' \overline{A}) > 0,$$

and therefore \overline{D} is ample by Corollary 6.4. \square

Corollary 6.7. *Let $\overline{D} = (D, (g_v)_{v \in \Sigma_K}) \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be semi-positive. The following assertions are equivalent:*

- (1) $\zeta_{\text{abs}}(\overline{D}) \geq 0$;
- (2) $\overline{D} + \overline{A}$ is ample for any ample $\overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$.

Proof. (1) \implies (2): Let $\overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be ample. Then the underlying divisor A of \overline{A} is ample. Since D is nef by semi-positivity of \overline{D} , $D + A$ is ample. Moreover we have

$$\zeta_{\text{abs}}(\overline{D} + \overline{A}) \geq \zeta_{\text{abs}}(\overline{D}) + \zeta_{\text{abs}}(\overline{A}) \geq \zeta_{\text{abs}}(\overline{A}) > 0,$$

where the last inequality is given by Lemma 3.13. By Corollary 6.4, $\overline{D} + \overline{A}$ is ample.

(2) \implies (1): Let $x \in X(\overline{K})$ be a closed point. We want to prove that $\widehat{h}_{\overline{D}}(x) \geq 0$. Let $\overline{A} \in \widehat{\text{Div}}(X)_{\mathbb{R}}$ be ample and semi-positive and let $\varepsilon > 0$ be a real number. Since $\overline{D} + \varepsilon\overline{A}$ is ample, we have

$$\widehat{h}_{\overline{D}}(x) + \varepsilon\widehat{h}_{\overline{A}}(x) = \widehat{h}_{\overline{D} + \varepsilon\overline{A}}(x) > 0,$$

and we conclude by letting ε tend to zero. \square

A semi-positive adelic \mathbb{R} -Cartier divisor satisfying $\zeta_{\text{abs}}(\overline{D}) \geq 0$ is usually called nef in the literature [Mor16, Definition 4.4.1]. Roughly speaking, Corollary 6.7 asserts that an adelic \mathbb{R} -Cartier divisor is nef if and only if it is the limit of a sequence of ample adelic \mathbb{R} -Cartier divisors.

REFERENCES

- [BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.
- [BGMP16] José Ignacio Burgos Gil, Atsushi Moriwaki, Patrice Philippon, and Martín Sombra. Arithmetic positivity on toric varieties. *J. Algebraic Geom.*, 25(2):201–272, 2016.
- [BGPS14] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, (360):vi+222, 2014.
- [CM18] Huayi Chen and Atsushi Moriwaki. Extension property of semipositive invertible sheaves over a non-archimedean field. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(1):241–282, 2018.
- [CP90] Frédéric Campana and Thomas Peternell. Algebraicity of the ample cone of projective varieties. *J. Reine Angew. Math.*, 407:160–166, 1990.
- [Iko16] Hideaki Ikoma. Remarks on arithmetic restricted volumes and arithmetic base loci. *Publ. Res. Inst. Math. Sci.*, 52(4):435–495, 2016.
- [Iko21] Hideaki Ikoma. Adelic Cartier divisors with base conditions and the Bonnesen–Diskant-type inequalities. *Tohoku Mathematical Journal*, 73(3):341 – 401, 2021.
- [Mor15] Atsushi Moriwaki. Semiample invertible sheaves with semipositive continuous hermitian metrics. *Algebra Number Theory*, 9(2):503–509, 2015.
- [Mor16] Atsushi Moriwaki. Adelic divisors on arithmetic varieties. *Mem. Amer. Math. Soc.*, 242(1144):v+122, 2016.
- [Zha95a] Shouwu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8(1):187–221, 1995.
- [Zha95b] Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.

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