SUPPLEMENTARY MATERIAL
to
"Steady state of the KPZ equation in an interval and Liouville quantum mechanics"

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Abstract –We provide some further details about the arguments and results presented in the Letter and review the work of Hariya and Yor [1].

1. Edwards-Wilkinson limit

Consider the measure $\mathcal{D}\widetilde{X}e^{-S[\widetilde{X}]}$ for the rescaled process, up to a normalization, given in Eq.(28) in the Letter, and let us rewrite it in the equivalent form

$$S[\widetilde{X}] = \int_0^1 d\tilde{x} \left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}}\right)^2 + \frac{(\tilde{u}+\tilde{v})}{\sqrt{L}} \log\left(\sqrt{L}\int_0^1 \mathrm{d}\tilde{x}\; e^{-2\sqrt{L}\widetilde{X}(\tilde{x})}\right) + 2\tilde{v}\widetilde{X}(1), \tag{1}$$

with $\widetilde{X}(0) = 0$ and $\widetilde{X}(1)$ free. For $L \to 0$, $S[\widetilde{X}]$ can be expanded as

$$S[\widetilde{X}] = \int_0^1 d\tilde{x} \left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}}\right)^2 + \frac{(\tilde{u}+\tilde{v})}{\sqrt{L}} \log\left(1 - 2\sqrt{L}\int_0^1 \mathrm{d}\tilde{x}\; \widetilde{X}(\tilde{x}) + 2L\int_0^1 \mathrm{d}\tilde{x}\widetilde{X}(\tilde{x})^2 + \dots\right) + 2\tilde{v}\widetilde{X}(1), \tag{2}$$

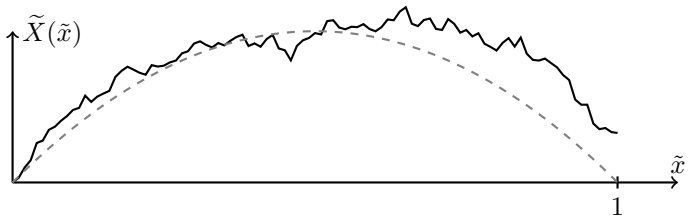

Fig. 1: Steady-state of the Edwards-Wilkinson equation on the interval $[0, 1]$, depicted here for $\tilde{u} = \tilde{v} > 0$. It is given by a parabola plus a standard Brownian motion – see (6).

where we do not keep track of constant terms which are absorbed in the normalization. To the leading order as $L \to 0$, we obtain

$$S[\widetilde{X}] \simeq \int_0^1 d\tilde{x} \left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}} \right)^2 - 2(\tilde{u} + \tilde{v}) \int_0^1 \mathrm{d}\tilde{x}\ \widetilde{X}(\tilde{x}) + 2\tilde{v}\widetilde{X}(1) \tag{3}$$

which we can rewrite, using integration by parts (and up to an irrelevant constant)

$$S[\widetilde{X}] \simeq \int_0^1 d\tilde{x} \left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}} - \tilde{u} + (\tilde{u} + \tilde{v})\tilde{x} \right)^2 \tag{4}$$

This is a Gaussian measure and one can write the convergence in distribution

$$\widetilde{X}(x) \Longrightarrow \tilde{u}\tilde{x} - \frac{1}{2}(\tilde{u} + \tilde{v})\tilde{x}^2 + \frac{1}{\sqrt{2}}\tilde{\mathsf{B}}(\tilde{x}) \tag{5}$$

where $\tilde{\mathsf{B}}(\tilde{x})$ is a standard one sided Brownian motion with $\tilde{\mathsf{B}}(0) = 0$. The mean profile of $\widetilde{X}(\tilde{x})$ is thus quadratic with a curvature proportional to $-(\tilde{u} + \tilde{v})$. Thus the (rescaled) stationary height field $\widetilde{H}$ converges to a parabola plus a Brownian motion,

$$\widetilde{H}(\tilde{x}) \Longrightarrow \tilde{u}\tilde{x} - \frac{1}{2}(\tilde{u} + \tilde{v})\tilde{x}^2 + \mathsf{B}(\tilde{x}) \tag{6}$$

where $\mathsf{B}(\tilde{x})$ is another standard Brownian motion, as stated in the Letter.

Remark: The fact that the RHS of (6) is stationary for the EW equation can be shown by a direct calculation [2] : First, by substracting a parabola, one may reduce to the case $u = v = 0$. Then, we observe that starting from Brownian initial data, the height is a Gaussian process. Finally, using the explicit form of the Green's function of the heat equation with Neumann boundary conditions, the spatial covariance can be computed explicitly and shown to be that of a Brownian motion.

2. Observables of the process X

Let us define, for any process $X(x)$ with $0 \leqslant x \leqslant L$,

$$Z_L[X] = \int_0^L dx e^{-2X(x)}. \tag{7}$$

By definition, the average of any observable $O[X]$ over the process $X(x)$, defined in Eq. (5) of the Letter, can be written in the following path integral forms

$$\mathbb{E}\left[O[X]\right] = \frac{1}{\mathcal{Z}_{u,v,L}} \int_{X(0)=0} \mathcal{D}X O[X] e^{-\int_0^L dx (\frac{dX(x)}{dx})^2} e^{-2vX(L)} Z_L[X]^{-(u+v)}, \tag{8}$$

$$= \frac{e^{v^2 L}}{\mathcal{Z}_{u,v,L}} \int_{X(0)=0} \mathcal{D}X O[X] e^{-\int_0^L dx (\frac{dX(x)}{dx} + v)^2} Z_L[X]^{-(u+v)}. \tag{9}$$

where $\mathcal{Z}_{u,v,L}$ is the normalisation. Equivalently, as in Eq. (7) of the Letter, it can be written as averages over the Brownian motion in two equivalent forms

$$\mathbb{E}\left[O[X]\right] = \frac{1}{\mathcal{Z}_{u,v,L}} \mathbb{E}_B\left[O[B] e^{-2vB(L)} Z_L[B]^{-(u+v)}\right] = \frac{e^{v^2 L}}{\mathcal{Z}_{u,v,L}} \mathbb{E}_{B_{-v}}\left[O[B_{-v}] Z_L[B_{-v}]^{-(u+v)}\right], \tag{10}$$

where $\mathbb{E}_B$ denotes the average over the one-sided Brownian motion of diffusion constant $1/2$ (with $B(0) = 0$), i.e. $B(x) = \frac{1}{\sqrt{2}}\mathsf{B}(x)$ where $\mathsf{B}(x)$ is the standard one-sided Brownian motion, and $\mathbb{E}_{B_{-v}}$ the average over the one-sided Brownian motion of diffusion constant $1/2$ with a drift $-v$, i.e. $B_{-v}(x) = \frac{1}{\sqrt{2}}\mathsf{B}(x) - vx$. The normalization is thus equal to the following expectations.

$$\mathcal{Z}_{u,v,L} = \mathbb{E}_B\left[e^{-2vB(L)} Z_L[B]^{-(u+v)}\right] = e^{v^2 L} \mathbb{E}_{B_{-v}}\left[Z_L[B_{-v}]^{-(u+v)}\right] \tag{11}$$

It admits a simple expression in some cases, we will give examples below.

It is interesting to note that the moments of $Z_L[X]$ for the process X can be expressed from the normalization as (here k can be any real number)

$$\mathbb{E}\left[Z_L[X]^k\right] = \frac{\mathcal{Z}_{u-k,v}}{\mathcal{Z}_{u,v}}. \tag{12}$$

A related property is that if we denote $Q_L^B(Z)$ the PDF of the variable $Z = Z_L[B_{-v}]$ for the Brownian B_{-v} with diffusion coefficient $1/2$ and drift $-v$, then the PDF $Q_L(Z)$ of $Z = Z_L[X]$ for the process $X(x)$ is simply

$$Q_L(Z) = C_{u,v,L} Q_L^B(Z) Z^{-(u+v)}, \tag{13}$$

where $C_{u,v,L}$ is a normalization. One simple application, discussed in the Letter, is for the case $v < 0$. Then we know that $Z = Z_L[B_{-v}]$ has a limit distribution as $L \to +\infty$ given by an inverse Gamma law $1/\Gamma(-2v, 1)$

$$Q_{+\infty}^B(Z) = \frac{1}{\Gamma(-2v)} \frac{1}{Z^{1-2v}} e^{-1/Z}. \tag{14}$$

If $u > v$, we see that $Z = Z_L[X]$ will also have a limit distribution for $L \to +\infty$ given by another inverse Gamma law $1/\Gamma(u - v, 1)$

$$Q_{+\infty}(Z) = \frac{1}{\Gamma(u-v)} \frac{1}{Z^{1+u-v}} e^{-1/Z}. \tag{15}$$

One-point distribution of total height difference. Consider here the height difference between the two boundaries of the interval. From Eq. (4) in the Letter we have

$$H(L) - H(0) = \frac{\sqrt{L}}{\sqrt{2}} G + X(L) \tag{16}$$

where G is an independent unit Gaussian random variable. From (10) choosing $O[X] = \delta(X(L) - Y)$, the PDF $P_L(Y)$ of the random variable $Y = X(L)$ is given by

$$P_L(Y) = \mathbb{E}\left[\delta(X(L) - Y)\right] = \frac{1}{\mathcal{Z}_{u,v,L}} \mathbb{E}_B\left[\delta(B(L) - Y) e^{-2vB(L)} Z_L[B]^{-(u+v)}\right] \tag{17}$$

$$= \frac{1}{\mathcal{Z}_{u,v,L}} e^{-2vY} \frac{e^{-\frac{Y^2}{L}}}{\sqrt{\pi L}} \mathbb{E}_B\left[Z_L[B]^{-(u+v)} \middle| B(L) = Y\right] \tag{18}$$

where the expectation value is over a Brownian motion $B(x)$ with diffusion coefficient $1/2$ and $B(0) = 0$, conditioned to $B(L) = Y$. We give some simple explicit examples below.

An interesting property, which is a consequence of the form of the measure (10), is that the cumulants (respectively the moments) of the random variable $X(L)$ (i.e. the cumulants/moments of $P_L(Y)$ denoted $\langle Y^n \rangle_L^c$ and $\langle Y^n \rangle_L$ respectively) can be obtained from the normalisation. Indeed setting $u + v = w$ fixed and taking successive derivatives w.r.t. v one obtains

$$\langle Y^n \rangle_L^c = \left[\left(-\tfrac{1}{2}\partial_v\right)^n \log \mathcal{Z}_{w-v,v,L}\right]\Big|_{w=u+v}, \tag{19}$$

$$\langle Y^n \rangle_L = \left[\frac{1}{\mathcal{Z}_{w-v,v,L}} \left(-\tfrac{1}{2}\partial_v\right)^n \mathcal{Z}_{w-v,v,L}\right]\Bigg|_{w=u+v}. \tag{20}$$

3. Some explicit formulas

In this section we use our main result, i.e. Eqs. (4) and (5) in the Letter, to obtain some simple explicit formulas in special cases.

3.1 The Brownian case

The simplest case is $u+v=0$, in which case one finds $\mathcal{Z}_{u,v}=e^{v^2L}$ and

$$P_L(Y)=\frac{1}{\sqrt{\pi L}}e^{-\frac{(Y+vL)^2}{L}} \tag{21}$$

which leads to $H(L)-H(0)=\frac{\sqrt{L}}{\sqrt{2}}G+\frac{\sqrt{L}}{\sqrt{2}}G'+uL=\sqrt{L}G''+uL$, as expected (here G and G' are independent unit Gaussian variables, and G'' another unit Gaussian variable). More generally the process $X(x)$ has the same law as $B_{-v}(x)=B_u(x)$.

3.2 Normalisation

There is one case where $\mathcal{Z}_{u,v,L}$ is trivial. For $u=1/2$ and $v=0$ one has from [3, Prop. 5.7]

$$\mathcal{Z}_{u,v,L}=\mathbb{E}_B\left[Z_L[B]^{-1/2}\right]=L^{-1/2}. \tag{22}$$

For $u+v=-n$, where n is a positive integer, it is possible to obtain explicit expressions. One has from [4] (with $\mu=-2v$, $\beta=2$, $\sigma=1/2$, $-\beta F_0=2v$) or from [3, Thm 5.2] (with $\mu=2v$),

$$\mathcal{Z}_{u,v,L}=e^{v^2L}\sum_{k=0}^{n}e^{Lk(k+2v)}(-1)^{n-k}\binom{n}{k}(2k+2v)\frac{\Gamma(k+2v)}{\Gamma(n+1+k+2v)}. \tag{23}$$

For instance, for $n=1$, we have by (11)

$$\mathcal{Z}_{u,v,L}=\mathbb{E}_B\left[\int_0^L dx e^{-2vB(L)-2B(x)}\right]=\int_0^L dx e^{(1+2v)x+v^2L}=\frac{e^{v^2L}}{1+2v}(e^{(1+2v)L}-1) \tag{24}$$

where the average is over a Brownian motion $B(x)$ with $B(0)=0$ and diffusion coefficient $1/2$. It is consistent with (23) for $n=1$.

3.3 PDF of the total height difference

Similarly a direct calculation of $P_L(Y)$ is possible for $u+v=-n$, using (18). To this aim let us define a Brownian bridge $B(x)$ such that $B(L)=Y$, in terms of the standard Brownian bridge $\tilde{\mathsf{B}}(x)$, as

$$B(x)=\frac{Y}{L}x+\frac{1}{\sqrt{2}}\tilde{\mathsf{B}}(x),\quad \tilde{\mathsf{B}}(x)=W(x)-\frac{x}{L}W(L),\quad \mathbb{E}[\tilde{\mathsf{B}}(x_1)\tilde{\mathsf{B}}(x_2)]=\min(x_1,x_2)-\frac{x_1x_2}{L}, \tag{25}$$

where $W(x)$ is a standard Brownian with $W(0)=0$.

Case $u+v=-1$. One has

$$\mathbb{E}_B\left[\int_0^L dx e^{-2B(x)}\right]=\int_0^L dx e^{-2\frac{Y}{L}x+x(1-\frac{x}{L})}=\frac{1}{2}\sqrt{\pi L}e^{\frac{(L-2Y)^2}{4L}}\left(\mathrm{erf}\left(\frac{L-2Y}{2\sqrt{L}}\right)+\mathrm{erf}\left(\frac{L+2Y}{2\sqrt{L}}\right)\right) \tag{26}$$

Inserting in (18) and using (24) we obtain

$$P_L(Y)=\frac{1}{2\mathcal{Z}_{u,v}}e^{-(1+2v)Y+\frac{L}{4}}\left(\mathrm{erf}\left(\frac{L-2Y}{2\sqrt{L}}\right)+\mathrm{erf}\left(\frac{L+2Y}{2\sqrt{L}}\right)\right). \tag{27}$$

One can check that (27) with (24) is normalized to unity $\int_{\mathbb{R}}dY P_L(Y)=1$.

On the transition line at the point $u = v = -1/2$, we obtain a symmetric distribution for $Y = X(L)$

$$P_L(Y) = \frac{\mathrm{erf}\left(\frac{L-2Y}{2\sqrt{L}}\right) + \mathrm{erf}\left(\frac{L+2Y}{2\sqrt{L}}\right)}{2L}. \tag{28}$$

It is easy to see that in the large L limit, $P_L(Y)$ converges to the uniform distribution on the interval $[-L/2, L/2]$ with a boundary layer of size $\sqrt{L}$ near the edges. As mentioned in the Letter, in the context of the directed polymer, this is consistent with the prediction that the polymer is localized at either boundary with probability 1/2. A possible scenario which would explain this limit distribution, is that the process $X(x)$ attains its minimum at a random location x^* uniformly distributed between $x = 0$ and $x = L$, and the process is approximately (as L goes to infinity) a straight line with slope u on $[0, x^*]$ and a straight line with slope $-v$ on $[x^*, 1]$. It would be interesting to confirm this scenario by computing multipoint distributions (see below).

Remark: For TASEP, described as a zero temperature polymer model, a similar scenario was discovered in [5], thus confirming the universality of stationary measures at large scale.

To obtain more detailed information, let us compute the cumulants of $X(L)$ for any v, on the line $u + v = -1$. Using the general formula (19) for $w = u + v = -1$, we obtain

$$\langle Y^n \rangle_L^c = \left(-\tfrac{1}{2}\partial_v\right)^n \left(v^2 L + \log(e^{(1+2v)L} - 1) - \log(1 + 2v)\right) \tag{29}$$

where in the last equation we used (24). One finds for the mean

$$\langle Y \rangle_L = L\left(\frac{1}{1 - e^{(2v+1)L}} - v - 1\right) + \frac{1}{2v+1} \tag{30}$$

The small L behavior is $\langle Y \rangle = -(v + \frac{1}{2})L + O(L^2)$. At large L there are two cases (i) $v > -1/2$ and $\langle Y \rangle \simeq uL$ (using $u + v = -1$) and (ii) $v < -1/2$ and $\langle Y \rangle \simeq -vL$. Around the symmetric point $v = u = -1/2$ the mean vanishes as $\langle Y \rangle = -\frac{1}{6}L(6 + L)(v + \frac{1}{2}) + O((v + \frac{1}{2})^2)$.

The variance is

$$\langle Y^2 \rangle_L^c = \frac{L^2}{2 - 2\cosh((2v+1)L)} + \frac{L}{2} + \frac{1}{(2v+1)^2} \tag{31}$$

The small L behavior is $\langle Y^2 \rangle = \frac{L}{2} + O(L^2)$. For $v \neq -1/2$ the large L behavior is simply $\langle Y^2 \rangle \simeq L/2$. Exactly at $v = -1/2$, the variance goes to a constant of order L^2, more precisely for any L around the symmetric point one has $\langle Y^2 \rangle = \frac{1}{12}L(L + 6) - \frac{1}{60}L^4(v + \frac{1}{2})^2 + O\left((v + \frac{1}{2})^3\right)$. One sees from these results that at large L and near the symmetric point, there is a diverging length scale $L_v = 1/(v + \frac{1}{2})$, and a scaling behavior with L/L_v (it will be studied in more details below). These results are consistent with the observation that from (27), $P_L(Y)$ at large L is approximately $e^{-(1+2v)Y}$ times a uniform distribution on the interval $[-L/2, L/2]$.

For $u + v = -1$ one can also easily obtain the one point PDF of $X(x)$, denoted $P_{x,L}(Y)$ for any $0 < x < L$. By similar manipulations one finds

$$P_{x,L}(Y) = P_x(Y)\frac{\mathcal{Z}_{u,v,x}}{\mathcal{Z}_{u,v,L}} e^{v^2(L-x)} + e^{-2Y}\frac{e^{-\frac{(Y+vx)^2}{x}}}{\sqrt{\pi x}}\frac{\mathcal{Z}_{u,v,L-x}}{\mathcal{Z}_{u,v,L}} e^{v^2 x}, \tag{32}$$

where $P_x(Y)$ is obtained from $P_L(Y)$ in (27) by substituting $L \to x$, and $\mathcal{Z}_{u,v,L}$ is given in (24). It is then easy to compute its mean using the result (30) and one thus finds *the mean profile* as

$$\mathbb{E}[X(x)] = \langle Y \rangle_{x,L} = \langle Y \rangle_x \frac{\mathcal{Z}_{u,v,x}}{\mathcal{Z}_{u,v,L}} e^{v^2(L-x)} - (v+1)x e^{2vx+x}\frac{\mathcal{Z}_{u,v,L-x}}{\mathcal{Z}_{u,v,L}} e^{v^2 x}, \tag{33}$$

$$= \frac{v(2v+1)x + e^{(2v+1)x} - 1 - (v+1)(2v+1)x e^{(2v+1)L}}{(2v+1)\left(e^{(2v+1)L} - 1\right)}. \tag{34}$$

At the symmetric point $v = u = -1/2$ the mean profile is parabolic, i.e. $\langle Y \rangle_{x,L} = -\frac{x(L-x)}{2L}$. This is consistent with the scenario discussed above. Near the symmetric point, and for large L, there is a critical regime with a diverging length scale $L_v = 1/(v + \frac{1}{2})$ and the mean profile takes the scaling form

$$\langle Y \rangle_{x,L} \simeq \frac{L}{4} \left(\frac{\left(e^{2\tilde{v}\tilde{x}} - 1\right)\left(\coth(\tilde{v}) - 1\right)}{\tilde{v}} - 2\tilde{x}\coth(\tilde{v}) \right), \quad \tilde{x} = \frac{x}{L}, \quad \tilde{v} = \frac{L}{L_v} = L(v + \frac{1}{2}). \tag{35}$$

Remark: A similar diverging length scale near $u = v < 0$ was obtained in previous works on TASEP [5, 6], with the same power law dependence with respect to $u - v$.

Case $u + v = -n$. This calculation can be extended to the case $u + v = -n$ for any positive integer n, using that

$$\mathbb{E}_{B(L)=Y}\left[\left(\int_0^L dx e^{-2B(x)} \right)^n \right] = \int_0^L \cdots \int_0^L dx_1 \cdots dx_n e^{-2\frac{Y}{L}\sum_i x_i + \sum_{i,j} \min(x_i, x_j) - \frac{x_i x_j}{L}} \tag{36}$$

although the algebra becomes rapidly tedious. Instead, one can use in [3, Prop. 5.3] and obtain and obtain

$$P_L(Y) = \frac{1}{\mathcal{Z}_{u,v,L}} \frac{2^{n+1} e^{-nY - 2vY}}{n!(\pi L^3)^{1/2}} \int_{|Y|}^{+\infty} dr\, r e^{-r^2/L} (\cosh(r) - \cosh(Y))^n \tag{37}$$

where $\mathcal{Z}_{u,v,L}$ is given in (23). We can give a more explicit form for $u + v = -2$

$$P_L(Y) = \frac{e^{\frac{L}{4} - 2(v+1)Y}}{2\mathcal{Z}_{u,v}} \Bigg(e^{\frac{3L}{4}} \left(\operatorname{erf}\left(\frac{L - Y}{\sqrt{L}} \right) + \operatorname{erf}\left(\frac{L + Y}{\sqrt{L}} \right) \right) - 2\cosh Y \left(\operatorname{erf}\left(\frac{L - 2Y}{2\sqrt{L}} \right) + \operatorname{erf}\left(\frac{L + 2Y}{2\sqrt{L}} \right) \right) \Bigg), \tag{38}$$

where the normalization is

$$\mathcal{Z}_{u,v,L} = e^{v^2 L} \frac{2v + 3 - 4(v+1)e^{2Lv+L} + (2v+1)e^{4L(v+1)}}{2(v+1)(2v+1)(2v+3)}. \tag{39}$$

Returning to arbitrary integer n, we see that at large L the integrand in (37) is dominated by large r, where it behaves as $e^{-r^2/L + nr}$. Hence if $|Y| < Ln/2$ the saddle point at $r = Ln/2$ dominates and the integral is of order $e^{L/4n^2}$, i.e approximately independent of Y. Hence for $u = v = -n/2$ the result is similar to the one obtained above for $n = 1$. The exponential factor $e^{-nY - 2vY} = e^{(u-v)Y}$ will deform the distribution of Y for $u - v \neq 0$ which becomes peaked near $Y = \operatorname{sgn}(u - v)nL/2$.

Case $u + v = 1$. Another case where simplifications occur is $u + v = 1$ where using [3, Prop. 5.9] and (18) one finds

$$P_L(Y) = \frac{1}{\mathcal{Z}_{u,v,L}} \frac{e^{-\frac{Y^2}{L} - 2vY}}{\sqrt{\pi L}} \frac{Y e^Y}{L \sinh Y} \tag{40}$$

and $\mathcal{Z}_{u,v,L}$ can be simply obtained from the normalization condition $\int_{\mathbb{R}} dY P_L(Y) = 1$.

Case $u, v > 0$. In that case, we use the Laplace transform, obtained for $L = 1$ in [7], and which reads for general L and for $-2v < c < 2u$

$$\mathbb{E}[e^{-cX(L)}] = \frac{I(c)}{I(0)}, \quad I(c) = \int_0^{+\infty} dk e^{-\frac{k^2}{4}L} \frac{|\Gamma(\frac{c}{2} + v + \frac{ik}{2})|^2 |\Gamma(-\frac{c}{2} + u + \frac{ik}{2})|^2}{|\Gamma(2ik)|^2}. \tag{41}$$

For $L \to +\infty$ without rescaling $u, v > 0$ one finds, by rescaling $k \to k/\sqrt{L}$

$$\mathbb{E}[e^{-cX(L)}] \xrightarrow[L \to +\infty]{} \frac{\Gamma(\frac{c}{2} + v)^2 \Gamma(-\frac{c}{2} + u)^2}{\Gamma(v)^2 \Gamma(u)^2} \tag{42}$$

Since a Gamma random variable γ_u of parameter u has moments $\mathbb{E}[\gamma_u^a] = \frac{\Gamma(u+a)}{\Gamma(u)}$, we obtain the weak convergence

$$X(L) \Longrightarrow -\frac{1}{2}\log\left(\frac{\gamma_v^{(1)}\gamma_v^{(2)}}{\gamma_u^{(1)}\gamma_u^{(2)}}\right), \tag{43}$$

where the four Gamma random variables are independent. Let us recall that the total height difference is $H(L) - H(0) = \sqrt{\frac{L}{2}}G + X(L)$. Since $X(L)$ is $O(1)$ at large L it is subdominant in the variance of $H(L)$, but it controls the higher cumulants of the height difference. Note that $X(L) = O(1)$ is consistent with the rescaled process $\widetilde{X}(\tilde{x}) = \frac{1}{\sqrt{L}}X(L\tilde{x})$ being a Brownian excursion in the large L limit, as shown in the Letter.

4. Some properties of the steady state of the KPZ fixed point on the interval

4.1 Multipoint probability for the rescaled process, Brownian motion with an absorbing boundary, and quantum mechanics in presence of a hard wall

We have shown in the Letter that, in the limit $L \to +\infty$, upon rescaling the boundary parameters as $u = \tilde{u}/\sqrt{L}$, $v = \tilde{v}/\sqrt{L}$, and rescaling space as $\tilde{x} = x/L$, the rescaled process $\widetilde{X}(\tilde{x}) = \frac{1}{\sqrt{L}}X(\tilde{x})$, $\tilde{x} \in [0,1]$ has the following probability measure, which can be written in the two equivalent forms

$$\frac{e^{-\tilde{v}^2}}{\widetilde{Z}_{\tilde{u},\tilde{v}}}\mathcal{D}\widetilde{X}e^{-\int_0^1 d\tilde{x}\left(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}}+\tilde{v}\right)^2}e^{2(\tilde{u}+\tilde{v})\min_{\tilde{x}}\widetilde{X}(\tilde{x})} = \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}}\mathcal{D}\widetilde{X}e^{-\int_0^1 d\tilde{x}(\frac{d\widetilde{X}(\tilde{x})}{d\tilde{x}})^2}e^{2(\tilde{u}+\tilde{v})\min_{\tilde{x}}\widetilde{X}(\tilde{x})-2v\widetilde{X}(1)} \tag{44}$$

with $\widetilde{X}(0) = 0$ and where $\widetilde{Z}_{\tilde{u},\tilde{v}}$ is a normalisation. Consider an observable $O\left[\widetilde{X}\right]$. Its expectation is given by

$$\mathbb{E}_{\widetilde{X}}\left[O[\widetilde{X}]\right] = \frac{e^{-\tilde{v}^2}}{\widetilde{Z}_{\tilde{u},\tilde{v}}}\int_{-\infty}^0 dbe^{2(\tilde{u}+\tilde{v})b}\mathbb{E}_{B_{-\tilde{v}}}\left[O[B_{-\tilde{v}}]\delta(\min B_{-\tilde{v}} - b)\right], \tag{45}$$

$$= \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}}\int_{-\infty}^0 dbe^{2(\tilde{u}+\tilde{v})b}\mathbb{E}_B\left[O[B]e^{-2\tilde{v}B(1)}\delta(\min B - b)\right], \tag{46}$$

where the averages are over a Brownian motion of diffusion coefficient $1/2$, respectively with drift $-\tilde{v}$ and with zero drift, on the interval $[0,1]$. We denote the minimum of the process B over $[0,1]$ by $\min B$.

Let us consider $\tilde{u} + \tilde{v} > 0$ and perform an integration by part

$$\mathbb{E}_{\widetilde{X}}\left[O[\widetilde{X}]\right] = \frac{e^{-\tilde{v}^2}}{\widetilde{Z}_{\tilde{u},\tilde{v}}}2(\tilde{u}+\tilde{v})\int_{-\infty}^0 dbe^{2(\tilde{u}+\tilde{v})b}\mathbb{E}_{B_{-\tilde{v}}}\left[O[B_{-\tilde{v}}]\theta(\min B_{-\tilde{v}} - b)\right] \tag{47}$$

$$= \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}}2(\tilde{u}+\tilde{v})\int_{-\infty}^0 dbe^{2(\tilde{u}+\tilde{v})b}\mathbb{E}_B\left[O[B]e^{-2\tilde{v}B(1)}\theta(\min B - b)\right] \tag{48}$$

For a Brownian motion it is easy to evaluate the multi-point probability joint with the probability that the minimum is larger than b. Let us introduce the propagator of the Brownian motion in presence of an absorbing wall at position $U = b$, which can be obtained from the image method

$$G_b(U,U',x-x') = G(U-b,U'-b,x-x'), \quad G(U,U',x-x') = \left(\frac{e^{-\frac{(U-U')^2}{x-x'}} - e^{-\frac{(U+U')^2}{x-x'}}}{\sqrt{\pi(x-x')}}\right)\theta(U)\theta(U') \tag{49}$$

One has

$$G_b(U,U',x-x') = \mathbb{E}_B\left[\delta(B(x)-U)\theta\left(\min_{x'<z<x}B(z) - b\right)\middle| B(x') = U'\right]\theta(U'-b). \tag{50}$$

For the propagator with drift, it is similar.

The evaluation of the multi-point probabilities can be performed equivalently using the quantum mechanics of a particle in presence of a hard wall. Indeed one can also write

$$G(U,U',x-x') = \int_0^{+\infty} dk \phi_k(U)\phi_k(U') e^{-(x-x')\frac{k^2}{4}}, \quad \phi_k(U) = \sqrt{\frac{2}{\pi}}\sin(kU)\,\theta(U) \tag{51}$$

This is the Green's function for the Hamiltonian given by Eq. (8) in the Letter, where the Liouville potential $V(U) = e^{2U}$ is replaced by a hard wall potential, i.e. $V(U) = +\infty$ for $U < 0$ and $V(U) = 0$ for $U > 0$, which has eigenfunctions ϕ_k. It is natural that the hard wall potential arises in the large L limit, since the rescaling of the process X leads to $U \to \sqrt{L}U$. The formula for this quantum mechanics can be obtained by considering all LQM formula in the Letter in the regime of small k. One can check that in this limit the LQM eigenfunctions $\psi_k(U)$ given in Eq. (9) in the Letter become equal to the eigenfunction $\phi_k(U)$ in (51) restricted to $U > 0$, upon rescaling $k \to k/\sqrt{L}$ and $U \to \sqrt{L}U$.

Let us return to the computation of the multipoint distribution of the process $\widetilde{X}$. For $\tilde{x}_0 = 0 < \tilde{x}_1 < \cdots < \tilde{x}_m < \tilde{x}_{m+1} = 1$, $b_1, \ldots, b_{m+1} \in \mathbb{R}$, $b \leqslant 0$ and $b_0 = 0$, we may write

$$\begin{aligned} e^{-\tilde{v}^2}\mathbb{E}_{B_{-\tilde{v}}}\left[\delta(B_{-\tilde{v}}(\tilde{x}_1)-b_1)\ldots\delta(B_{-\tilde{v}}(\tilde{x}_m)-b_m)\delta(B_{-\tilde{v}}(1)-b_{m+1})\theta(\min B_{-\tilde{v}}-b)\right] \\ = \mathbb{E}_B\left[\delta(B(\tilde{x}_1)-b_1)\ldots\delta(B(\tilde{x}_m)-b_m)\delta(B(1)-b_{m+1})e^{-2\tilde{v}B(1)}\theta(B-b)\right] \\ = e^{-2\tilde{v}b_{m+1}}\prod_{j=0}^{m} G_b(b_{j+1},b_j,\tilde{x}_{j+1}-\tilde{x}_j). \end{aligned} \tag{52}$$

From this result, one obtains using (48) the multipoint PDF of the rescaled process $\widetilde{X}$ (for $\tilde{u}+\tilde{v} > 0$)

$$\begin{aligned} \mathbb{E}_{\widetilde{X}}\left[\delta(\widetilde{X}(\tilde{x}_1)-\widetilde{X}_1)\ldots\delta(\widetilde{X}(\tilde{x}_m)-\widetilde{X}_m)\delta(\widetilde{X}(1)-\widetilde{X}_{m+1})\right] \\ = \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}} 2(\tilde{u}+\tilde{v})\int_{-\infty}^{0} db e^{2(\tilde{u}+\tilde{v})b} e^{-2\tilde{v}\widetilde{X}_{m+1}} \prod_{j=0}^{m} G_b(\widetilde{X}_{j+1},\widetilde{X}_j,x_{j+1}-x_j) \end{aligned} \tag{53}$$

This formula is suitable to calculate the Laplace transform in the form

$$\mathbb{E}_{\widetilde{X}}\left[e^{-\sum_{j=1}^{m}\tilde{s}_j(\widetilde{X}(\tilde{x}_j)-\widetilde{X}(\tilde{x}_{j+1}))}\right] = \frac{\hat{J}\left(\vec{\tilde{s}}\right)}{\hat{J}(0)} \tag{54}$$

From similar manipulations as in the Letter (from (16) to (20)) but using the eigenfunctions for the hard wall quantum mechanics (51), it is possible to express it as

$$\begin{aligned} \hat{J}\left(\vec{\tilde{s}}\right) = \prod_{j=1}^{m+1}\int_0^{+\infty}\frac{dk_j k_j^2}{4\pi}\prod_{j=1}^{m}\frac{(\tilde{s}_j-\tilde{s}_{j+1})}{((\tilde{s}_j-\tilde{s}_{j+1})^2+(k_j+k_{j+1})^2)((\tilde{s}_j-\tilde{s}_{j+1})^2+(k_j-k_{j+1})^2)} \\ \times \frac{1}{(2\tilde{u}-s_1)^2+k_1^2}\frac{1}{4\tilde{v}^2+k_{m+1}^2}\, e^{-\sum_{j=1}^{m+1}\frac{k_j^2}{4}(x_j-x_{j-1})}, \end{aligned} \tag{55}$$

where $2\tilde{u} > \tilde{s}_1 > \cdots > \tilde{s}_{m+1}$. This formula coincides, up to an irrelevant global multiplicative constant, with the formula (20) in the Letter, where one performs the rescaling $s_j \to \tilde{s}_j/\sqrt{L}$, $k_j \to \tilde{k}_j/\sqrt{L}$, $x_j \to L\tilde{x}_j$, $u \to \tilde{u}/\sqrt{L}$, $v \to \tilde{v}/\sqrt{L}$.

Finally, note that for arbitrary $\tilde{u}+\tilde{v}$ we can obtain the multipoint joint PDF of the rescaled process $\widetilde{X}$ in the form

$$\begin{aligned} \mathbb{E}_{\widetilde{X}}\left[\delta(\widetilde{X}(\tilde{x}_1)-\widetilde{X}_1)\ldots\delta(\widetilde{X}(\tilde{x}_m)-\widetilde{X}_m)\delta(\widetilde{X}(1)-\widetilde{X}_{m+1})\right] \\ = \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}}\int_{-\infty}^{0} db e^{2(\tilde{u}+\tilde{v})b} e^{-2\tilde{v}\widetilde{X}_{m+1}}(-\partial_b)\left[\prod_{j=0}^{m} G_b(\widetilde{X}_{j+1},\widetilde{X}_j,\tilde{x}_{j+1}-\tilde{x}_j)\right] \end{aligned} \tag{56}$$

4.2 Normalisation

It is easy to calculate the normalisation of the measure of the rescaled process (44) using known results about the maximum of the Brownian motion. One finds

$$\widetilde{Z}_{\tilde{u},\tilde{v}} = \int_{-\infty}^{0} dy \int_{y}^{+\infty} dY \frac{4(Y-2y)}{\sqrt{\pi}} e^{-(2y-Y)^2} e^{2(\tilde{u}+\tilde{v})y - 2\tilde{v}Y} = \frac{\tilde{u}e^{\tilde{u}^2}\mathrm{erfc}(\tilde{u}) - \tilde{v}e^{\tilde{v}^2}\mathrm{erfc}(\tilde{v})}{\tilde{u}-\tilde{v}} \tag{57}$$

where we have used that the joint PDF of $\min B = y$ and $B(1) = Y$ is $2p(-\sqrt{2}y, -\sqrt{2}Y)$ where p is given in (1.9) in [8]. It can also be obtained by Laplace inversion from [9, Chap. IV, item 32].

4.3 One point distribution of $\widetilde{X}(1)$

The formula (53) with $m = 0$ gives for $\tilde{u} + \tilde{v} > 0$ the PDF of the random variable $\widetilde{X}(1)$ as

$$P(Y) = \mathbb{E}_{\widetilde{X}}\left[\delta(\widetilde{X}(1) - Y)\right] = \frac{1}{\widetilde{Z}_{\tilde{u},\tilde{v}}} 2(\tilde{u}+\tilde{v}) e^{-2\tilde{v}Y} \int_{-\infty}^{0} db e^{2(\tilde{u}+\tilde{v})b} \frac{1}{\sqrt{\pi}} \left(e^{-Y^2} - e^{-(Y-2b)^2}\right) \theta(Y-b) \tag{58}$$

which leads to

$$P(Y) = \frac{2(\tilde{u}+\tilde{v})}{\widetilde{Z}_{\tilde{u},\tilde{v}}} e^{-2\tilde{v}Y} \left(\frac{e^{-Y^2}}{2\sqrt{\pi}(\tilde{u}+\tilde{v})} (\theta(Y) + \theta(-Y) e^{2Y(\tilde{u}+\tilde{v})}) - \frac{1}{4} e^{\frac{1}{4}(\tilde{u}+\tilde{v})(\tilde{u}+\tilde{v}+4Y)} \mathrm{erfc}\left(\frac{\tilde{u}+\tilde{v}}{2} + |Y|\right) \right) \tag{59}$$

Using the normalisation in (57), one checks that it is correctly normalized to unity $\int_{\mathbb{R}} dY P(Y) = 1$.

Laplace transform. Alternatively, for $\tilde{u} + \tilde{v} > 0$ we can use the Laplace transform, obtained for $L = 1$ in [7], extended here to general L, and which leads, for the rescaled process in the limit $L \to +\infty$ (as discussed in the previous section one needs to rescale $k \to k/\sqrt{L}$) to the formula

$$\mathbb{E}[e^{-\tilde{c}\widetilde{X}(1)}] = \frac{I(\tilde{c})}{I(0)}, \quad I(\tilde{c}) = F(2\tilde{v} + \tilde{c}, 2\tilde{u} - \tilde{c}), \quad F(a,b) = \int_0^{+\infty} dk e^{-\frac{k^2}{4}} \frac{k^2}{(k^2+a^2)(k^2+b^2)} \tag{60}$$

One has

$$F(a,b) = \frac{a^2 g(a) - b^2 g(b)}{a^2 - b^2}, \quad g(a) = \int_0^{+\infty} dk e^{-\frac{k^2}{4}} \frac{1}{k^2+a^2} = \frac{\pi e^{\frac{a^2}{4}} \mathrm{erfc}\left(\frac{a}{2}\right)}{2a} \tag{61}$$

This leads to the formula, for $c \in (-2\tilde{v}, 2\tilde{u})$,

$$\mathbb{E}\left[e^{-c\widetilde{X}(1)}\right] = \frac{e^{\frac{1}{4}c(c-4\tilde{u})}(\tilde{v}-\tilde{u}) \left(e^{\tilde{u}^2}(c-2\tilde{u})\mathrm{erfc}\left(\tilde{u} - \frac{c}{2}\right) + (c+2\tilde{v})\mathrm{erfc}\left(\frac{c}{2} + \tilde{v}\right) e^{c(\tilde{u}+\tilde{v})+\tilde{v}^2}\right)}{(c - \tilde{u} + \tilde{v})\left(2e^{\tilde{v}^2}\tilde{v}\,\mathrm{erfc}(\tilde{v}) - 2e^{\tilde{u}^2}\tilde{u}\,\mathrm{erfc}(\tilde{u})\right)} \tag{62}$$

Expanding in c, this formula leads to expressions for the mean and variance of $\widetilde{X}(1)$. The Laplace inversion of that formula is not a priori obvious. One can check however (through a tedious calculation) that it is indeed the Laplace transform of $P(Y)$ in (59). This shows the consistency with the claims of the previous section, and that the method using the Brownian representation of the measure is quite efficient to obtain the PDF's of the process.

4.4 Properties of the minimum of the rescaled process $\widetilde{X}$

Let $\widetilde{X}_m$ be the minimum of $\widetilde{X}(\tilde{x})$ on the interval $\tilde{x} \in [0,1]$ and let $0 \leqslant \tilde{x}_m \leqslant 1$ be the first point where it is reached. Then the joint PDF of $\widetilde{X}_m$, $\tilde{x}_m$ and of the value $Y = \widetilde{X}(1)$ of the process at the boundary is given by

$$P(\widetilde{X}_m, \tilde{x}_m, Y) = C_{u,v} P^B(\widetilde{X}_m, \tilde{x}_m, Y) e^{2(\tilde{u}+\tilde{v})\widetilde{X}_m - 2\tilde{v}Y}, \tag{63}$$

where $C_{u,v}$ is a normalization, and $P^B(\widetilde{X}_m, x_m, Y)$ is the corresponding joint PDF for the Brownian B with diffusion coefficient $1/2$ and zero drift. Using the main result of [8] we obtain

$$P(y,x,Y) = \frac{1}{\mathcal{Z}_{u,v}} \frac{4|y|(Y-y)}{\pi x^{3/2}(1-x)^{3/2}} e^{-\frac{y^2}{x} - \frac{(y-Y)^2}{1-x} + 2(\tilde{u}+\tilde{v})y - 2\tilde{v}Y} \theta(-y)\theta(Y-y)\theta(0<x<1) \tag{64}$$

Integration over x, the position of the minimum, using the identity below (1.9) in [8] leads to the joint PDF of the minimum and the value at the boundary as

$$P(y,Y) = \frac{1}{\mathcal{Z}_{u,v}} \frac{4(Y-2y)}{\sqrt{\pi}} e^{-(Y-2y)^2+2(\tilde{u}+\tilde{v})y-2\tilde{v}Y} \theta(-y)\theta(Y-y) \tag{65}$$

This PDF is correctly normalized to unity, from (57). Integrating over y we obtain once again (by a different method) the result (59). However, with this method it appears to be valid for any $\tilde{u}$, $\tilde{v}$.

5. Translation of the results of Hariya-Yor

Let us state the results of [1, 3] and translate them into our notations. In the Letter, we consider the measure on continuous real-valued processes $X(x) = \hat{X}(x) - vx$, defined on on $[0, L]$, such that the density of $\hat{X}$ is

$$\mathcal{D}\hat{X} e^{-\int_0^L dx (\frac{d\hat{X}}{dx})^2} \left(\int_0^L dx e^{-2\hat{X}(x)+2vx} \right)^{-(u+v)} . \tag{66}$$

Hariya and Yor [1] studied several probability measures on Brownian paths weighted by exponential functionals of the Brownian motion. In particular, they study the measure on processes $X(s) = \mathsf{B}(s)+\mu s$ defined on $[0, t]$, denoted $\mathcal{P}_t^{(\mu,m)}$ in [1], defined by

$$\mathcal{D}\mathsf{B} e^{-\frac{1}{2}\int_0^t ds (\frac{d\mathsf{B}}{ds})^2} \left(\int_0^t ds e^{2\mathsf{B}(s)+2\mu s} \right)^{-m} . \tag{67}$$

where B denotes the standard Brownian motion, to distinguish it from B which we reserve for the Brownian with diffusion coefficient $1/2$, i.e. $B = \frac{1}{\sqrt{2}}\mathsf{B}$. In order to match the measures, we let $x = 2s$, that is

$$\hat{X}(x) = -B(s = x/2). \tag{68}$$

Then our measure becomes

$$\mathcal{D}\mathsf{B} e^{-\frac{1}{2}\int_0^{L/2} ds (\frac{d\mathsf{B}}{ds})^2} \left(\int_0^{L/2} ds e^{2\mathsf{B}(s)+4vs} \right)^{-(u+v)} . \tag{69}$$

Hence we must identify

$$m = u + v, \quad \mu = 2v, \quad t = L/2. \tag{70}$$

Haryia-Yor [1] define three regions (see Fig. 2), which exactly match the three regions of our phase diagram:

$$R_1 = \{2m > \mu, \mu > 0\} = \{u > 0, v > 0\} \tag{71}$$
$$R_2 = \{m < \mu, 2m < \mu\} = \{u < v, u < 0\} \tag{72}$$
$$R_3 = \{m > \mu, \mu < 0\} = \{u > v, v < 0\} \tag{73}$$

They also defined the borders between these regions

$$L_1 = \{2m = \mu, \mu > 0\} = \{u = 0, v > 0\}, \tag{74}$$
$$L_2 = \{m = \mu, \mu < 0\} = \{u = v, v < 0\}, \tag{75}$$
$$L_3 = \{m > 0, \mu = 0\} = \{u > 0, v = 0\}, \tag{76}$$

so that L_1 is between R_1 and R_2, L_2 is between R_2 and R_3, L_3 is between R_1 and R_3.

Hariya and Yor proved [1, Theorem 1.3] (the statement can also be found in the review [3, Theorem 7.4]) that we have the following weak convergence of probability measures on continuous functions:

$$\mathcal{P}_t^{(\mu,m)} \underset{t\to+\infty}{\Longrightarrow} \begin{cases} \mathcal{W}^{(0)}_{2\gamma_{\nu=m-\mu/2}} = \mathcal{W}^{(0)}_{2\gamma_{\nu=u}} & \text{if } (u,v) \in L_1 \cup R_1 \cup L_3, \\ \mathcal{W}^{(\mu-2m)} = \mathcal{W}^{(-2u)} & \text{if } (u,v) \in L_1 \cup R_2 \cup L_2, \\ \mathcal{W}^{(-\mu)}_{2\gamma_{\nu=m-\mu}} = \mathcal{W}^{(-2v)}_{2\gamma_{u-v}} & \text{if } (u,v) \in L_2 \cup R_3 \cup L_3. \end{cases} \tag{77}$$

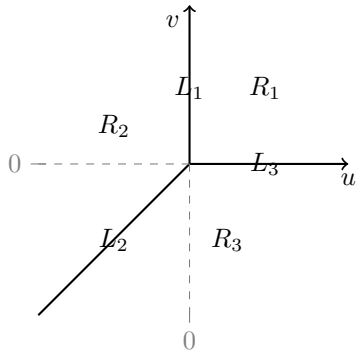

Fig. 2: The regions R_1, R_2, R_3 and their borders L_1, L_2, L_3 defined in [1].

In (77), γ_ν is a Gamma random variable with parameter ν (and by convention, $\gamma_0 = 0$), $\mathcal{W}^{(\mu)}$ denotes the law of a Brownian motion with drift μ , and $\mathcal{W}^{(\mu)}_{2\gamma_\nu}$ denotes the law of the process

$$t \mapsto \mathsf{B}_t + \mu t - \log\left(1 + 2\gamma_\nu \int_0^t e^{2\mathsf{B}_s + 2\mu s} ds\right). \tag{78}$$

Using (68) this implies that the process $X(x)$ converges as L goes to infinity to the following, consistent with Equations (34) and (35) in the Letter.

- When $u, v \geqslant 0$, $X(x)$ weakly converges to

$$B(x) + \log\left(1 + \gamma_u \int_0^x e^{-2B(z)} dz\right). \tag{79}$$

- When $u \leqslant 0$ and $u \leqslant v$, $X(x)$ weakly converges to

$$B(x) + ux. \tag{80}$$

- And, when $v \leqslant 0$, $u \geqslant v$, $X(x)$ weakly converges to

$$B(x) + vx + \log\left(1 + \gamma_{u-v} \int_0^x e^{-2B(z) - 2vz} dz\right). \tag{81}$$

In (79), (80) and (81), $B(x)$ denotes a Brownian motion with diffusion coefficient $1/2$, and γ_u and γ_{u-v} denote independent Gamma distributed random variables with respective shape parameters u and $u-v$, and scale parameter 1.

When $u = -v \geqslant 0$, the process in (81) becomes simply a Brownian motion with drift $u = -v$ and diffusion coefficient $1/2$. This non-trivial identity can be found in [1, Eq. (1.8)], see also (97) below.

6. Review of the main arguments in [1]

Here we describe the derivation of the convergence result (77), that is [1, Theorem 1.3], using the notation of the present paper.

6.1 Overall argument

Consider the process $X(x)$ with $0 \leqslant x \leqslant L$. We are interested in taking $L \to +\infty$ and describing the process $X(z)$ for all $z \leqslant x$, where x is fixed as $L \to +\infty$. For this we can split the process in two parts: $X(z)$ for $0 \leqslant z \leqslant x$ and $X(z)$ for $x < z \leqslant L$. We will average over (i.e. integrate over) the second part of the process, to obtain the measure for the first part. To characterize the measure, we consider

a general bounded functional $F_x[X] = F[\{X(z)\}_{0\leqslant z\leqslant x}]$ which depends only on the first part. One has, from Equation (7) in the Letter

$$\mathbb{E}_X\left[F_x[X]\right] = \frac{1}{\mathcal{Z}_{u,v,L}}\mathbb{E}_{B_{-v}}\left[F_x[B_{-v}]\left(\frac{1}{\int_0^x dz e^{-2B_{-v}(z)} + \int_x^L dz e^{-2B_{-v}(z)}}\right)^{u+v}\right] \tag{82}$$

where in the LHS the average is over the full process $\{X(z)\}_{0\leqslant z\leqslant L}$ and in the RHS it is over a Brownian motion with diffusion coefficient $1/2$ and drift $-v$, denoted $\{B_{-v}(z)\}_{0\leqslant z\leqslant L}$. In the denominator we have splitted the integral into two pieces. Since the increments of the Brownian motion after time x are independent of the value $B_{-v}(x)$, we have the equality in law (for a fixed x)

$$\int_x^L dz e^{-2B_{-v}(z)} = e^{-2B_{-v}(x)}\int_x^L dz e^{-2(B_{-v}(z)-B_{-v}(x))} \overset{(d)}{=} e^{-2B_{-v}(x)}\int_0^{L-x} dz e^{-2\tilde{B}_{-v}(z)} \tag{83}$$

where $\tilde{B}_{-v}(z)$ is an independent Brownian motion with diffusion coefficient $1/2$ and drift $-v$. Hence we can rewrite

$$\mathbb{E}_X[F_x[X]] = \mathbb{E}_{\{B_{-v}(z)\}_{0\leqslant z\leqslant x}}\left[F_x[B_{-v}]Q_x[B_{-v}]\right] \tag{84}$$

in terms of the weight

$$Q_x[B_{-v}] = \frac{\Delta(a,\xi,L-x)}{\Delta(0,1,L)}, \quad a = \int_0^x dz e^{-2B_{-v}(z)}, \quad \xi = e^{-2B_{-v}(x)} \tag{85}$$

where we have defined

$$\Delta(a,\xi,L) = \mathbb{E}_{\tilde{B}_{-v}}\left[(a+\xi Z_L)^{-(u+v)}\right], \quad Z_L = \int_0^L dz e^{-2\tilde{B}_{-v}(z)} = \int_0^L dz e^{-2\tilde{B}(z)+2vz}, \tag{86}$$

which depends only on the random variable Z_L.

The variable $\Delta(a,\xi,L)$ has been much studied and by [1, Theorem 2.2], translated in our variables using (68),(70), we have

$$\lim_{L\to+\infty}\frac{\Delta(a,\xi,L-x)}{\Delta(0,1,L)} = \begin{cases} e^{v^2x}\xi^{-v}\times\frac{1}{\Gamma(u)}\int_0^{+\infty}\frac{r^{u-1}e^{-r}}{(ar+\xi)^u}, & u>0, v\geqslant 0 \\ e^{(v^2-u^2)x}\xi^{-(u+v)}, & u\leqslant 0, u\leqslant v \\ \frac{1}{\Gamma(u-v)}\int_0^{+\infty}\frac{r^{u-v-1}e^{-r}}{(ar+\xi)^{u+v}}, & v\leqslant 0, v<u \end{cases} \tag{87}$$

where the integrals arise as expectation values over independent Gamma variables, e.g. one has

$$\mathbb{E}_{\gamma_b}\left[\frac{1}{(\gamma_b a+\xi)^c}\right] = \frac{1}{\Gamma(b)}\int_0^{+\infty} dr\frac{r^{b-1}e^{-r}}{(ar+\xi)^c} \tag{88}$$

where γ_b denotes the Gamma variable $\Gamma(b,1)$. For instance, for $u>v$, the limit in the third line is easily understood since for $v<0$ one knows that $\lim_{L\to+\infty} Z_L = Z_\infty$, where $1/Z_\infty$ is a Gamma variable $\Gamma(-2v,1)$. For the other cases, see [1, Theorem 2.2].

6.2 Large L limits

Now we are able to take the large L limits of (84). The simplest case to analyze is $u\leqslant 0, u\leqslant v$. Inserting the limit in the second line of (87), with $\xi = e^{-2B_{-v}(x)}$, into (84) one sees that the measure for $X(z)$, $0\leqslant z\leqslant x$ is

$$\mathcal{D}X e^{(v^2-u^2)x} e^{-\int_0^x dz\left(\frac{dX(z)}{dz}+v\right)^2+2(u+v)X(x)} = \mathcal{D}X e^{-\int_0^x dz\left(\frac{dX(z)}{dz}-u\right)^2} \tag{89}$$

with $X(0)=0$, so $X(z)$ is a Brownian with diffusion coefficient $1/2$ and drift u, as stated in the Letter. Note that the multiplication by the weight $e^{2(u+v)X(x)}$ has changed the drift from $-v$ to u. This is a general fact, which is a special case of the Cameron-Martin (CM) theorem.

Consider now the limit in the first line in (87), that is when $u > 0$, $v \geqslant 0$. Inserting it into (84) and using the CM theorem (from the first line to the second) yields

$$\mathbb{E}_X[F_x[X]] = \mathbb{E}_{\{B_{-v}(z)\}_{0\leqslant z\leqslant x}}\left[F_x[B_{-v}]e^{v^2x+2vB_{-v}(x)}\mathbb{E}_{\gamma_u}\left[\frac{1}{(\gamma_u\int_0^x dze^{-2B_{-v}(z)}+e^{-2B_{-v}(x)})u}\right]\right] \tag{90}$$

$$= \mathbb{E}_{\{B(z)\}_{0\leqslant z\leqslant x}}\left[F_x[B]\mathbb{E}_{\gamma_u}\left[\frac{1}{(\gamma_u\int_0^x dze^{-2B(z)}+e^{-2B(x)})u}\right]\right] \tag{91}$$

$$= \mathbb{E}_{\{Y(z)\}_{0\leqslant z\leqslant x}}[F_x[Y]] \tag{92}$$

where in the second line B has no drift. The non-trivial part is the third line, i.e. after expectation over γ_u the average is the same that over the path transformed process,

$$Y(z) = B(z) + \log\left(1+\gamma_u\int_0^z dz'e^{-2B(z')}\right) \tag{93}$$

which gives the result stated in the Letter (and above in (77)). A similar identity allows to obtain the result stated above in the third line in (77), using (87), when $v < u$, $v \leqslant 0$.

6.3 The T-transformation on paths

For completeness, we briefly recall here how the identity (92) is derived in [1]. For simplicity we will return to B_μ, the standard Brownian motion with drift μ used there, and use the correspondence (68),(70) at the end. The argument is based on:

(i) The definition of the path transformation T_z (also used in [10]) defined for any process X as

$$T_z(X)(t) = X_t - \log\left(1+z\int_0^t dse^{2X(s)}\right) \tag{94}$$

which has the following interesting properties

$$\frac{1}{\int_0^t dse^{2T_z(X)(s)}} = \frac{1}{\int_0^t dse^{2X(s)}} + z, \quad T_zT_{z'} = T_{z+z'}, \quad e^{X(t)} = \frac{e^{T_z(X)(t)}}{1-z\int_0^t dse^{2T_z(X)(s)}} \tag{95}$$

The first follows from the total derivative property

$$e^{2T_z(X)(t)} = \frac{e^{2X(t)}}{(1+z\int_0^t dse^{2X(s)})^2} = -\frac{1}{z}\frac{d}{dt}\frac{1}{1+z\int_0^t dse^{2X(s)}}. \tag{96}$$

The second is a consequence of the first. The third is obtained from the first, also consistent with the fact that the inverse path transform of T_z is T_{-z}.

(ii) The non-trivial identity in law, from [11, 12], valid for $\mu < 0$,

$$\left(\{\mathsf{B}_\mu\}_{t\geqslant 0}\,,\,\frac{1}{\int_0^{+\infty} dse^{2\mathsf{B}_\mu(s)}}\right) = \left(\{T_{2\gamma_{-\mu}}(\mathsf{B}_{-\mu})\}_{t\geqslant 0}\,,\,2\gamma_{-\mu}\right) \tag{97}$$

Using these two inputs, the derivation of (92) goes as follows. The simple change of drift

$$e^{-\frac{1}{2}\int_0^t\left(\frac{dX(s)}{ds}-\nu\right)^2} = e^{-\frac{1}{2}(\nu^2-b^2)t}e^{-\frac{1}{2}\int_0^t\left(\frac{dX(s)}{ds}-b\right)^2+(\nu-b)X(t)} \tag{98}$$

gives for any observable

$$\mathbb{E}\left[O(\mathsf{B}_\nu)\right] = e^{-\frac{1}{2}(\nu^2-b^2)t}\mathbb{E}\left[e^{(\nu-b)\mathsf{B}_b(t)}O(\mathsf{B}_b)\right]. \tag{99}$$

Let us apply it with the choice $O(X) := F[T_{2\gamma_b}(X)]$. One obtains

$$\mathbb{E}\left[F[T_{2\gamma_b}(\mathsf{B}_\nu)]\right] = e^{-\frac{1}{2}(\nu^2-b^2)t}\mathbb{E}\left[e^{(\nu-b)\mathsf{B}_b(t)}F[T_{2\gamma_b}(\mathsf{B}_b)]\right] \tag{100}$$

$$= e^{-\frac{1}{2}(\nu^2-b^2)t}\mathbb{E}\left[\frac{e^{(\nu-b)T_{2\gamma_b}(\mathsf{B}_b)(t)}}{(1-2\gamma_b\int_0^t ds e^{2T_{2\gamma_b}(\mathsf{B}_b)(s)})^{\nu-b}}F[T_{2\gamma_b}(\mathsf{B}_b)]\right] \tag{101}$$

$$= e^{-\frac{1}{2}(\nu^2-b^2)t}\mathbb{E}\left[e^{(\nu-b)\mathsf{B}_{-b}(t)}\left(1-\frac{\int_0^t ds e^{2\mathsf{B}_{-b}(s)}}{\int_0^{+\infty} ds e^{2\mathsf{B}_{-b}(s)}}\right)^{b-\nu}F[\mathsf{B}_{-b})]\right] \tag{102}$$

In the second line we have used the last identity in (95). In the third line the identity in law (97) has been used to replace the joint average over $T_{2\gamma_b}(\mathsf{B}_b)$ and γ_b with the same joint average over B_{-b} and $1/\int_0^{+\infty} ds e^{2\mathsf{B}_{-b}}(s)$. Now since we are interested only in the first part of the process (up to time t) we can replace $\int_t^{+\infty} ds e^{2\mathsf{B}_{-b}(s)} \overset{(d)}{=} e^{2\mathsf{B}_{-b}(t)}/(2\gamma_b)$ thus

$$1-\frac{\int_0^t ds e^{2\mathsf{B}_{-b}(s)}}{\int_0^{+\infty} ds e^{2\mathsf{B}_{-b}(s)}} = \frac{\int_t^{+\infty} ds e^{2\mathsf{B}_{-b}(s)}}{\int_0^t ds e^{2\mathsf{B}_{-b}(s)}+\int_t^{+\infty} ds e^{2\mathsf{B}_{-b}(s)}} \overset{(d)}{=} \frac{e^{2\mathsf{B}_{-b}(t)}}{2\gamma_b\int_0^t ds e^{2\mathsf{B}_{-b}(s)}+e^{2\mathsf{B}_{-b}(t)}} \tag{103}$$

Inserting into (102) and using again the CM theorem to relate expectations over B_{-b} to $\mathsf{B}_{-\nu}$ one finally obtains the identity

$$\mathbb{E}[F[T_{2\gamma_b}(\mathsf{B}_\nu), s\leqslant t]] = \mathbb{E}\left[\left(\frac{1}{2\gamma_b\int_0^t ds e^{2\mathsf{B}_{-\nu}(s)}+e^{2\mathsf{B}_{-\nu}(t)}}\right)^{b-\nu}F[\mathsf{B}_{-\nu}, s\leqslant t]\right] \tag{104}$$

which for $\nu = 0$ and $b = u$ and upon the correspondence (68),(70) leads to (92).

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