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Steady state of the KPZ equation on an interval and Liouville quantum mechanics

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Abstract – We obtain a simple formula for the stationary measure of the height field evolving according to the Kardar-Parisi-Zhang equation on the interval $[0, L]$ with general Neumann type boundary conditions and any interval size. This is achieved using the recent results of Corwin and Knizel (arXiv:2103.12253) together with Liouville quantum mechanics. Our formula allows to easily determine the stationary measure in various limits: KPZ fixed point on an interval, half-line KPZ equation, KPZ fixed point on a half-line, as well as the Edwards-Wilkinson equation on an interval.

Introduction. – The Kardar-Parisi-Zhang (KPZ) equation [1] describes the stochastic growth of a continuum interface driven by white noise. In one dimension it is at the center of the so-called KPZ class which contains a number of well-studied models sharing the same universal behavior at large scale. For all these models one can define a height field. For example, in particle transport models such as the asymmetric simple exclusion process (ASEP) on a lattice, the local density is a discrete analog to the height gradient [2, 3]. In the limit of weak asymmetry, ASEP converges [4], upon rescaling space and time to the KPZ equation. In the large scale limit, all models in the KPZ class (in particular ASEP and the KPZ equation) are expected to converge to a universal process called the KPZ fixed point [5, 6]. Note that the KPZ fixed point is universal with respect to the microscopic dynamics but still depends on the geometry of the space considered (full-line, half-line, circle, segment with boundary conditions).

An important question is the nature of the steady state. While the global height grows linearly in time with non trivial $t^{1/3}$ fluctuations, the height gradient, or the height differences between any two points, will reach a stationary distribution. It was noticed long ago [1, 7, 8] that the KPZ equation on the full line admits the Brownian motion as a stationary measure. It was proved rigorously in [4, 9], and in [10] for periodic boundary conditions. For ASEP, stationary measures were studied on the full and half-line [11, 12] and exact formulas were obtained on an interval using the matrix product ansatz [13]. The large scale limit of the stationary measures for ASEP on an interval was studied in [14, 15]. The processes obtained there as a limit

can be described in terms of textbook stochastic processes such as Brownian motions, excursions and meanders, and they should correspond to stationary measures of the KPZ fixed point on an interval.

For the KPZ equation, while the stationary measures are simply Brownian in the full-line and circle case, the situation is more complicated (not translation invariant, not Gaussian) in the cases of the half-line and the interval. One typically imposes Neumann type boundary conditions (that is, we fix the derivative of the height field at the boundary) so that stationary measures depend on boundary parameters and involve more complicated stochastic processes (see below). For the KPZ equation on the half-line with Neumann type boundary condition, it can be shown [16] that a Brownian motion with an appropriate drift is stationary (the drift must be proportional to the boundary parameter). This specific stationary measure was studied in [17] for the equivalent directed polymer problem for which the boundary parameter measures the attractiveness of the wall. However, based on the analysis of stationary measures of ASEP on a half-line [11], it was expected that more complicated stationary measures for the KPZ equation also exist.

The question of the stationary measure for the KPZ equation on the interval $[0, L]$ has also remained open. In a recent breakthrough, Corwin and Knizel obtained [18] an explicit formula for the Laplace transform of the stationary height distribution (for $L = 1$, and for some range of parameters). This Laplace transform formula relates the stationary measure to an auxiliary stochastic process called continuous dual Hahn process. This construction

corresponds to the KPZ equation limit¹ of formulas with a similar structure obtained in [20] of formulas with a similar structure obtained in [20] for the stationary measure of ASEP. However, obtaining a characterization of the process allowing to study properties of the stationary measure remains a challenge. This amounts to invert the complicated Laplace transforms in [18].

In this Letter we obtain a simple formula for the stationary measure of the KPZ equation on an interval of any size L , with general Neumann type boundary conditions. Our result is particularly convenient to study various limits as $L \rightarrow +\infty$. In particular we recover the phase diagram for stationary measures of the KPZ fixed point on an interval, we obtain new crossover regimes near the critical point, and we study the stationary measures of the KPZ equation on a half-line and their large scale limits. We unveil and exploit a surprising connection to Liouville quantum mechanics (LQM), i.e. the Schrodinger equation in an exponential potential. The LQM, which is the 1D limit of the 2D Liouville field theory, notably allows to study the statistics of exponential functionals of the Brownian motion, see [21] for a short review. As such, it appears in several areas of physics such as diffusion in 1D random media [21–27], multifractal eigenfunctions of random Schrodinger and Dirac operators [21, 28–31], diffusion in the hyperbolic plane [32–34], and more recently, and strikingly, in quantum chaos and its relation to gravity [35, 36]. LQM was recently used to obtain multipoint observables for the stationary KPZ equation in a half-space (see Supp. Mat. in [17]).

Model. – The KPZ equation for the height field $h(x, t)$ reads

$$\partial_t h(x, t) = \nu \partial_x^2 h + \frac{\lambda}{2} (\partial_x h)^2 + \sqrt{D} \xi(x, t) \quad (1)$$

where $\xi(x, t)$ is a standard space-time white noise. We use space-time units so that $\nu = 1$ and $\lambda = D = 2^2$. Here we study the problem on the interval so that (1) holds for $0 < x < L$. The solution is defined from the Cole-Hopf mapping $h(x, t) = \log Z(x, t)$, where $Z(x, t)$ equals the partition sum of a continuum directed polymer with endpoint at (x, t) in a random potential $-\sqrt{2}\xi(x, t)$. It satisfies the stochastic heat equation (SHE)

$$\partial_t Z(x, t) = \partial_x^2 Z(x, t) + \sqrt{2} Z(x, t) \xi(x, t), \quad x \in [0, L] \quad (2)$$

in the Ito sense, with Robin boundary conditions

$$\partial_x Z(x, t)|_{x=0} = AZ(0, t), \quad \partial_x Z(x, t)|_{x=L} = -BZ(L, t), \quad (3)$$

and it will be convenient to define boundary parameters $u = A + 1/2$ and $v = B + 1/2$. Although $Z(x, t)$, $t > 0$,

¹The limit from ASEP to KPZ was previously investigated in [19]. Remark 2.11 therein explains how to rescale ASEP boundary parameters to obtain the boundary conditions for the KPZ equation.

²The units chosen in [18] amount to rescaling our time as $t \rightarrow 2t$, immaterial in the steady state.

is not differentiable, the standard way to understand (3) is to impose these conditions on the heat kernel [19], or through a path integral as in [37, 38]. For the DP, $A > 0$ corresponds to a repulsive wall and $A < 0$ an attractive one, and similarly for B at $x = L$.

Main result. – Our main result is the prediction that the KPZ random height profile in the stationary state, denoted $\{H(x)\}_{x \in [0, L]}$, can be written as the sum of two independent random fields

$$H(x) - H(0) = \frac{1}{\sqrt{2}} W(x) + X(x) \quad (4)$$

where $W(x)$ is a one sided Brownian motion (i.e. with $W(0) = 0$ and $W(L)$ free) and the probability distribution of the process $X(x)$ is given by the path integral measure

$$\frac{\mathcal{D}X}{\mathcal{Z}_{u,v}} e^{-\int_0^L dx (\frac{dX(x)}{dx})^2} e^{-2vX(L)} \left(\int_0^L dx e^{-2X(x)} \right)^{-(u+v)} \quad (5)$$

with $X(0) = 0$ and $X(L)$ free, and $\mathcal{Z}_{u,v}$ a normalization such that $\mathcal{Z}_{0,0} = 1$. For the choice $v = -u$, $H(x) + vx$ is thus simply a standard Brownian motion. The Brownian motion with drift u is also stationary for the half-space KPZ equation [16], and on the segment it arises only when the two boundary conditions are compatible.

In mathematical terms, X is a continuous stochastic process on $[0, L]$ whose measure \mathbb{P}_X is absolutely continuous with respect to that of a Brownian motion with diffusion coefficient $1/2$, denoted \mathbb{P}_B , with Radon-Nikodym derivative $\frac{d\mathbb{P}_X}{d\mathbb{P}_B} = \frac{1}{\mathcal{Z}_{u,v}} e^{-\mathcal{E}_{u,v}(X)}$ where

$$\mathcal{E}_{u,v}(X) = u \log \left(\int_0^L dx e^{-2X(x)} \right) + v \log \left(\int_0^L dx e^{2X(L) - 2X(x)} \right). \quad (6)$$

This is a mere reformulation of (5) in a more symmetric form so that it becomes apparent that the process is left invariant after reversing space and exchanging u, v . This reformulation simply means that for any continuous and bounded functional F of the process $X = \{X(x)\}_{x \in [0, L]}$,

$$\mathbb{E}_X [F(X)] = \frac{1}{\mathcal{Z}_{u,v}} \mathbb{E}_B [F(B) e^{-\mathcal{E}_{u,v}(B)}], \quad (7)$$

where in the R.H.S., $B = \{B(x)\}_{x \in [0, L]}$ is a Brownian motion with diffusion coefficient $1/2$. Surprisingly the measure defined here in (5) as the steady state of the KPZ equation has been already introduced and studied in a work of Hariya and Yor in a different context [39]. We will use some of their results below.

Liouville quantum mechanics. – The Liouville Hamiltonian \hat{H} on the real axis $U \in \mathbb{R}$ is defined as

$$\hat{H} = -\frac{1}{4} \frac{d^2}{dU^2} + e^{-2U} \quad (8)$$

Its eigenfunctions $|k\rangle$, in coordinate basis $\psi_k(U) = \langle U|k\rangle$, can be chosen real and indexed by $k \geq 0$ with

$$\hat{H}\psi_k(U) = \frac{k^2}{4}\psi_k(U), \quad \psi_k(U) = N_k K_{ik}(2e^{-U}). \quad (9)$$

where $N_k^2 = \frac{2}{\pi\Gamma(ik)\Gamma(-ik)}$ and K_{ik} is a modified Bessel function. They form a continuum orthonormal basis with $\langle k|k'\rangle = \delta(k-k')$ with $\int_0^{+\infty} dk|k\rangle\langle k| = I$. We will need the matrix elements of the operator

$$e^{-2\alpha\hat{U}} = \int_{\mathbb{R}} dU e^{-2\alpha U} |U\rangle\langle U| \quad (10)$$

which read

$$\begin{aligned} \langle k|e^{-2\alpha\hat{U}}|k'\rangle &= N_k N_{k'} \int_0^{+\infty} \frac{r^{2\alpha} dr}{r} K_{ik}(2r) K_{ik'}(2r) \\ &= \frac{N_k N_{k'}}{8\Gamma(2\alpha)} \Gamma_4\left(\alpha \pm \frac{ik}{2} \pm \frac{ik'}{2}\right) \end{aligned} \quad (11)$$

for any $\alpha > 0$, where $\Gamma_4(\alpha \pm x \pm y) := \prod_{\sigma, \tau = \pm 1} \Gamma(\alpha + \sigma x + \tau y)$ is a product of four Gamma functions.

Multipoint Laplace transform. – We now compute, using the LQM, the multi-point Laplace transform (LT) of the distribution of $H(x)$, as defined in (4), (5). For the moment, we restrict ourselves to $u, v > 0$, and we will discuss the general case later. We consider the increasing sequence of points

$$x_0 = 0 < x_1 < \dots < x_m < x_{m+1} = L \quad (12)$$

Since $W(x)$ is simply a Brownian independent of X the following multipoint expectation, with parameters $\vec{s} = \{2u > s_1 > \dots > s_m > s_{m+1} = 0\}$, takes the form

$$\mathbb{E}[e^{-\sum_{j=1}^m s_j (H(x_j) - H(x_{j-1}))}] = e^{\frac{1}{4} \sum_{j=1}^{m+1} s_j^2 (x_j - x_{j-1})} \frac{J(\vec{s})}{J(0)} \quad (13)$$

where $J(\vec{s})$ is the following expectation over the process $X(x)$, from (5), with $X_0 = 0$

$$\begin{aligned} J(\vec{s}) &= \left[\prod_{j=1}^{m+1} \int_{\mathbb{R}} dX_j \right] e^{-\sum_{j=1}^m s_j (X_j - X_{j-1}) - 2v X_{m+1}} \\ &\times \prod_{j=1}^{m+1} \int_{X(x_{j-1})=X_{j-1}}^{X(x_j)=X_j} \mathcal{D}X e^{-\int_{x_{j-1}}^{x_j} dx \left(\frac{dX(x)}{dx}\right)^2} Z_L[X]^{-(u+v)} \end{aligned} \quad (14)$$

where $Z_L[X] := \int_0^L e^{-2X(x)}$. Now we insert in the integrand the following representation

$$Z_L[X]^{-(u+v)} = \frac{2}{\Gamma(u+v)} \int_{\mathbb{R}} dU_0 e^{-2U_0(u+v) - e^{-2U_0} Z_L[X]} \quad (15)$$

Performing the change of variable $U_j = X_j + U_0$, and defining the process $U(x) = X(x) + U_0$, the RHS of (14), i.e. $J(\vec{s})$, takes the form

$$\begin{aligned} &\frac{2}{\Gamma(u+v)} \prod_{j=0}^{m+1} \int_{\mathbb{R}} dU_j e^{-\sum_{j=1}^m s_j (U_j - U_{j-1}) - 2v U_{m+1} - 2u U_0} \\ &\times \prod_{j=1}^{m+1} \int_{U(x_{j-1})=U_{j-1}}^{U(x_j)=U_j} \mathcal{D}U e^{-\int_{x_{j-1}}^{x_j} dx \left[\left(\frac{dU(x)}{dx}\right)^2 + e^{-2U(x)}\right]} \end{aligned}$$

In the last line we recognize the path integral representation of the imaginary time Green's function of the Liouville Hamiltonian \hat{H} in (8). From the Feynman-Kac formula

$$\begin{aligned} &\int_{U(a)=U}^{U(b)=U'} \mathcal{D}U e^{-\int_a^b dx \left[\left(\frac{dU(x)}{dx}\right)^2 + e^{-2U(x)}\right]} \\ &= \langle U'|e^{-(b-a)\hat{H}}|U\rangle \end{aligned}$$

Hence we can rewrite $J(\vec{s}) = \frac{2}{\Gamma(u+v)} \tilde{J}(\vec{s})$ with

$$\begin{aligned} \tilde{J}(\vec{s}) &= \left[\prod_{j=0}^{m+1} \int dU_j \right] e^{-2v U_{m+1} - (2u - s_1) U_0} \\ &\prod_{j=1}^m e^{-(s_j - s_{j+1}) U_j} \prod_{j=1}^{m+1} \langle U_j | e^{-(x_j - x_{j-1}) \hat{H}} | U_{j-1} \rangle. \end{aligned} \quad (16)$$

Inserting the spectral decomposition

$$e^{-(x_j - x_{j-1}) \hat{H}} = \int_0^{+\infty} dk_j e^{-\frac{1}{4}(x_j - x_{j-1}) k_j^2} |k_j\rangle\langle k_j| \quad (17)$$

and using the definition (10), the RHS of (16) becomes

$$\begin{aligned} &\int_{\mathbb{R}^2} dU_0 dU_{m+1} \prod_{j=1}^{m+1} \int_0^{+\infty} dk_j e^{-2v U_{m+1} - (2u - s_1) U_0} \langle U_{m+1} | k_{m+1} \rangle \\ &\times \langle k_1 | U_0 \rangle \prod_{j=1}^m \langle k_{j+1} | e^{-(s_j - s_{j+1}) \hat{U}} | k_j \rangle e^{-\sum_{j=1}^{m+1} (x_j - x_{j-1}) \frac{k_j^2}{4}} \end{aligned} \quad (18)$$

Now we use that for $w > 0$

$$\begin{aligned} \int_{\mathbb{R}} dU e^{-2wU} \langle U | k \rangle &= N_k \int_{\mathbb{R}} dU e^{-2wU} K_{ik}(2e^{-U}) \\ &= \frac{N_k}{4} \left| \Gamma\left(w + \frac{ik}{2}\right) \right|^2, \end{aligned} \quad (19)$$

and $\langle U | k \rangle = \langle k | U \rangle$ to integrate (18) w.r.t. U_0 and U_{m+1} . We can then substitute the matrix elements $\langle k_{j+1} | e^{-(s_j - s_{j+1}) \hat{U}} | k_j \rangle$ by their explicit expressions from

(11). This leads to

$$\begin{aligned} \tilde{J}(\vec{s}) &= \\ & \frac{1}{2} \prod_{j=1}^{m+1} \int_0^{+\infty} \frac{dk_j}{4\pi|\Gamma(ik_j)|^2} \prod_{j=1}^m \frac{\Gamma_4\left(\frac{s_j-s_{j+1}}{2} \pm \frac{ik_j}{2} \pm \frac{ik_{j+1}}{2}\right)}{\Gamma(s_j-s_{j+1})} \\ & \times \left| \Gamma\left(u - \frac{s_1}{2} + \frac{ik_1}{2}\right) \Gamma\left(v + \frac{ik_{m+1}}{2}\right) \right|^2 e^{\sum_{j=1}^{m+1} \frac{-k_j^2}{4}(x_j-x_{j-1})}. \end{aligned} \quad (20)$$

Comparison with the result of Corwin-Knizel. –

Now, we explain why our formula for the stationary measure in (4) and (5) is equivalent to the Laplace transform formula of Corwin-Knizel [18] for $L = 1$. Again, we assume that $u, v > 0$ with $u < 1$ (though [18] provides formulas for the whole range $u + v > 0$). The Laplace transform of the stationary height $H(x)$ is expressed in [18] in terms of a Markov process $\{\mathbb{T}_s\}_{s \in [0, 2u]}$ with values on \mathbb{R}_+ , called the continuous dual Hahn process (CDHP). The transition probability for $\mathbb{T}_{s_j} = t_j$ given $\mathbb{T}_{s_{j+1}} = t_{j+1}$, with $s_j > s_{j+1}$, is given by

$$\begin{aligned} p_{s_{j+1}, s_j}(t_{j+1}, t_j) &= \frac{1}{8\pi} \\ & \times \frac{|\Gamma(u - \frac{s_j}{2} + \frac{i}{2}\sqrt{t_j})|^2 \Gamma_4(\frac{s_j-s_{j+1}}{2} \pm \frac{i}{2}\sqrt{t_{j+1}} \pm \frac{i}{2}\sqrt{t_j})}{|\Gamma(u - \frac{s_{j+1}}{2} + \frac{i}{2}\sqrt{t_{j+1}})|^2 \Gamma(s_j - s_{j+1}) \sqrt{t_j} |\Gamma(i\sqrt{t_j})|^2} \end{aligned}$$

and the marginal PDF of $\mathbb{T}_s = t$ at time s is given by

$$p_s(t) = \frac{(v+u)(v+u+1)}{8\pi} \frac{|\Gamma(v + \frac{s}{2} + \frac{i\sqrt{t}}{2}) \Gamma(u - \frac{s}{2} + \frac{i\sqrt{t}}{2})|^2}{\sqrt{t} |\Gamma(i\sqrt{t})|^2} \quad (21)$$

Then the Laplace transform is obtained, for $2u > s_1 > s_2 > \dots > s_{m+1} = 0$, as

$$\mathbb{E} \left[e^{-\sum_{j=1}^m s_j (H(x_j) - H(x_{j-1}))} \right] = e^{\frac{1}{4} \sum_{j=1}^{m+1} s_j^2 (x_j - x_{j-1})} \frac{I(\vec{s})}{I_0} \quad (22)$$

with (upon some rewriting of formula (1.12) in [18])

$$\begin{aligned} I(\vec{s}) &= \left[\prod_{j=1}^{m+1} \int_0^{+\infty} dt_j \right] p_0(t_{m+1}) \prod_{j=1}^m p_{s_{j+1}, s_j}(t_{j+1}, t_j) \\ & \times e^{-\frac{1}{4} \sum_{j=1}^{m+1} t_j (x_j - x_{j-1})} \end{aligned} \quad (23)$$

and $I_0 = I(0) = \int dt e^{-\frac{1}{4}t} p_0(t)$. It is now a simple exercise to check, using the change of variables $t_j = k_j^2$, that the above formula implies that

$$I(\vec{s}) = 2(u+v)(u+v+1) \tilde{J}(\vec{s}) = \Gamma(u+v+2) J(\vec{s}) \quad (24)$$

Since the prefactor cancels in the ratio, the multipoint Laplace transform (13) of our formula (4), (5) written

more explicitly in (20), coincides with the result of [18] in the domain $u, v > 0$ and for $L = 1$.

We may now extend the result of Corwin-Knizel in two directions. First, we may assume that L is arbitrary, and reproduce the analysis of [18] with points x_i as in (12) using the same scalings as in [18] starting from the ASEP model on NL sites and arriving at the same formula (22). This shows why (4), (5) are correct for any $L > 0$ and not only $L = 1$ as considered in [18]. Then, we may extend the range of parameters u, v . We expect that the distribution of stationary measures depend analytically on u, v for finite L . However, performing a direct analytic continuation on (22) is intricate and would involve many residues. This is why it is useful to rewrite, through LQM, the results of Corwin-Knizel as in (4), (5) which depends on the parameters in an analytic way for all u, v .

Thus, from now on we will assume that our result for the stationary probability (5), (4) holds for any u, v , and explore the consequences.

Remark: Eq. (22) relates the Laplace transform of the KPZ height field under the stationary measure to the Laplace transform of another Markov process, the CDHP, where the role of time/space parameters and Laplace transform parameters are exchanged. The CDHP can be interpreted as living in the Fourier space dual to the real U space of LQM. A similar duality holds for Brownian excursions [40] and can be obtained as a limit of (22) as $L \rightarrow +\infty$, as we shall see in the sequel. Note also that an analog of (22) has been established for ASEP [20], and it would be interesting to study the connection to discrete variants of LQM [41]. It would be also very interesting to know if such dualities extend to other solvable models in the KPZ class.

Limits and consequences. – To study the various limits, we define the scaled processes $\tilde{H}(\tilde{x})$, $\tilde{W}(\tilde{x})$ and $\tilde{X}(\tilde{x})$, with $\tilde{x} = x/L \in [0, 1]$ as

$$H(x) = \sqrt{L} \tilde{H}(\tilde{x}), \quad W(x) = \sqrt{L} \tilde{W}(\tilde{x}), \quad X(x) = \sqrt{L} \tilde{X}(\tilde{x}) \quad (25)$$

so that one has

$$\tilde{H}(\tilde{x}) = \frac{1}{\sqrt{2}} \tilde{W}(\tilde{x}) + \tilde{X}(\tilde{x}) \quad (26)$$

Clearly $\tilde{W}(\tilde{x})$ is also a standard one-sided Brownian motion. In order to impose Neumann type boundary conditions on $\tilde{H}(\tilde{x})$ (with slopes \tilde{u} , $-\tilde{v}$ respectively at each boundary) we scale boundary parameters as

$$u = \tilde{u}/\sqrt{L}, \quad v = \tilde{v}/\sqrt{L}. \quad (27)$$

The measure for $\tilde{X}(\tilde{x})$ can then be written, up to a normalization, as $\mathcal{D}\tilde{X}e^{-S[\tilde{X}]}$ with the action

$$S[\tilde{X}] = \int_0^1 d\tilde{x} \left(\frac{d\tilde{X}(\tilde{x})}{d\tilde{x}} \right)^2 + \frac{\tilde{u}}{\sqrt{L}} \log \left(\int_0^1 d\tilde{x} e^{-2\sqrt{L}\tilde{X}(\tilde{x})} \right) + \frac{\tilde{v}}{\sqrt{L}} \log \left(\int_0^1 d\tilde{x} e^{2\sqrt{L}(\tilde{X}(1)-\tilde{X}(\tilde{x}))} \right) \quad (28)$$

with $\tilde{X}(0) = 0$ and $\tilde{X}(1)$ free.

Limit $L \rightarrow 0$. In this limit one recovers the Edwards-Wilkinson model. One sees that in (28) we can expand up to linear order in \tilde{X} in the logarithmic terms, and one finds that to leading order in L , i.e. to L^0 , it becomes a Gaussian action with a parabolic mean profile. One finds (see details in [42])

$$\tilde{H}(\tilde{x}) - \tilde{H}(0) \implies \tilde{u}\tilde{x} - \frac{1}{2}(\tilde{u} + \tilde{v})\tilde{x}^2 + \mathbf{B}(\tilde{x}) \quad (29)$$

where $\mathbf{B}(\tilde{x})$ is a standard Brownian motion.

Limit $L \rightarrow +\infty$ (KPZ fixed point). Under the scalings considered above, \tilde{X} converges to a probability measure proportional to

$$\mathcal{D}\tilde{X}e^{-\int_0^1 d\tilde{x} \left(\frac{d\tilde{X}(\tilde{x})}{d\tilde{x}} + \tilde{v} \right)^2 e^{2(\tilde{u}+\tilde{v})\min_{\tilde{x}} \tilde{X}(\tilde{x})}} \quad (30)$$

with $\tilde{X}(0) = 0$. Hence it is a Brownian on $[0, 1]$ with a non trivial Radon Nikodym derivative depending on \tilde{u}, \tilde{v} (equivalently a Brownian with drift $-\tilde{v}$ with derivative depending only on $\tilde{u} + \tilde{v}$). The measure can also be rewritten in a more symmetric form as

$$\mathcal{D}\tilde{X}e^{-\int_0^1 d\tilde{x} \left(\frac{d\tilde{X}(\tilde{x})}{d\tilde{x}} \right)^2 e^{2\tilde{u}\min_{\tilde{x}} \{\tilde{X}(\tilde{x})\} + 2\tilde{v}\min_{\tilde{x}} \{\tilde{X}(\tilde{x}) - \tilde{X}(1)\}}} \quad (31)$$

As detailed in [42] the measure (30) can be studied using a limit of LQM, where the exponential potential is replaced by a hard wall. Accordingly all the above Laplace transform formula (13)-(20), are obtained for the rescaled process and parameters by simply replacing $\Gamma(z) \rightarrow 1/z$. The field $\tilde{H}(\tilde{x})$ should correspond to stationary measures of the KPZ fixed point on the interval $[0, 1]$ with boundary parameters \tilde{u}, \tilde{v} , and it is natural to predict that they arise as scaling limit of stationary measures of all models in the KPZ class on an interval. This is partially confirmed in some special cases that we study next, where we recover results obtained in [14, 15] for the large scale limit of ASEP stationary measures.

Phase diagram. — Now we study the phase diagram in Fig. 1, obtained in the $L \rightarrow \infty$ limit when u, v are fixed (equivalently when \tilde{u}, \tilde{v} go to $\pm\infty$).

When $\tilde{u}, \tilde{v} \rightarrow +\infty$ ($u, v > 0$). In that case the weight in (31) vanishes unless $\min_{\tilde{x}} \tilde{X}(\tilde{x}) = 0$ and $\min_{\tilde{x}} (\tilde{X}(\tilde{x}) - \tilde{X}(1)) = 0$, in which case the weight is 1. The second identity taken at $\tilde{x} = 0$, together with the first taken at $\tilde{x} = 1$

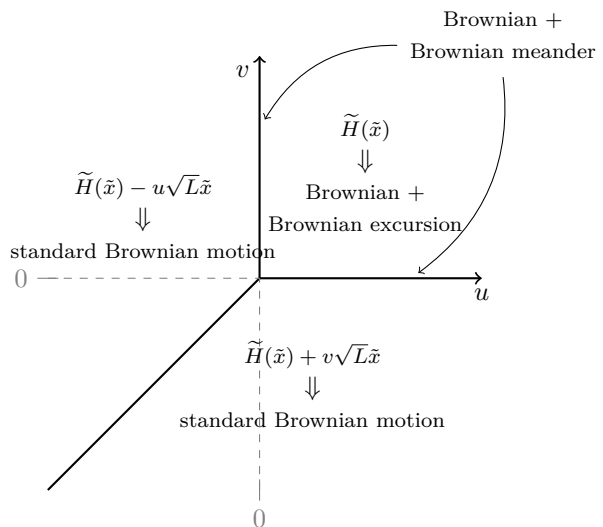


Fig. 1: Phase diagram of the large-scale limit of stationary measures for the KPZ equation on the segment. On the three regions $u, v > 0$ (maximal current phase), $v > u < 0$ (low density phase) and $u > v < 0$ (high density phase), we have indicated the nature of the stationary measure in the large scale limit. For the directed polymer the phases are as follows: For $u, v > 0$ the polymer is delocalized in the bulk, for $v < 0, u > v$ it is bound to $x = L$, and for $u < 0, u < v$ it is bound to $x = 0$. Exactly at the phase boundary $u = v$ the polymer has probability 1/2 to be bound to either side.

implies that $\tilde{X}(1) = 0$. Thus, in the limit $\tilde{u}, \tilde{v} \rightarrow +\infty$, $\tilde{X}(\tilde{x}) \implies \frac{1}{\sqrt{2}}E(\tilde{x})$, where E is a standard Brownian excursion (i.e. a Brownian bridge conditioned to stay positive), so that

$$\tilde{H}(\tilde{x}) \implies \frac{1}{\sqrt{2}}\tilde{W}(\tilde{x}) + \frac{1}{\sqrt{2}}E(\tilde{x}). \quad (32)$$

We recover the same process as in the large scale limit of TASEP [14] or ASEP [15] invariant measures. This shall not be a surprise: TASEP, ASEP and the KPZ equation all converge at large time to the KPZ fixed point [43], so that the process (32) describes the stationary measure of the KPZ fixed point in the so called maximal current phase for ASEP, that is with repulsive boundary conditions in terms of the directed polymer model.

When $\tilde{u} \rightarrow +\infty, \tilde{v} = 0$ or $\tilde{u} = 0, \tilde{v} \rightarrow +\infty$ ($u > 0, v = 0$ or $u = 0, v > 0$). When $\tilde{u} \rightarrow +\infty, \tilde{v} = 0$ (equivalently $u > 0, v = 0$) the weight in (31) vanishes unless $\min_{\tilde{x}} \tilde{X}(\tilde{x}) = 0$, hence \tilde{X} is now a Brownian meander (Brownian motion conditioned to stay positive up to time 1). We recover the same stationary process as in the large scale limit of ASEP stationary measures [15]. Similarly, when $\tilde{u} = 0, \tilde{v} \rightarrow +\infty$ (equivalently $u = 0, v > 0$), $\tilde{X}(1) - \tilde{X}(1-x)$ tends to a Brownian meander, and again, this matches with [15].

When u or v may be negative. By symmetry, we only need to consider the case where v is negative. The measure for $X(x)$ in (5) is a Brownian measure with drift $-v$

weighted by $Z_L[X]^{-(u+v)}$, where $Z_L[X] = \int_0^L dx e^{-2X(x)}$. We may rewrite (7), up to a renormalisation, as

$$\mathbb{E}_{B_{-v}} \left[F(B_{-v}) Z_L[B_{-v}]^{-(u+v)} \right], \quad (33)$$

where B_{-v} denotes a Brownian with drift $-v$ and diffusion coefficient $1/2$. It is well-known that $Z_L[B_{-v}]$ converges as $L \rightarrow \infty$ to a finite random variable $Z_\infty[B_{-v}]$ distributed as an inverse Gamma law with shape parameter $-2v$ [22, 44], which becomes asymptotically independent from the rescaled process \tilde{X} . In order for $Z_\infty[B_{-v}]^{-(u+v)}$ to have a finite expectation, we need to assume that $-(u+v) < -2v$, that is $u > v$. Thus, we are considering the whole high density phase in Fig. 1. At this point, we find that the resulting measure for $\tilde{H}(\tilde{x}) + v\sqrt{L}\tilde{x}$ is simply the standard Brownian motion. In the low density phase where $u < 0$ and $v > u$ (see Figure 1), we deduce by symmetry that the limiting stationary process is a standard Brownian motion with drift $u\sqrt{L}$.

Half-line KPZ equation. – Consider the KPZ equation (1) in \mathbb{R}_+ , with boundary parameter u at $x = 0$. The stationary measures of the height field can be computed as the $L \rightarrow \infty$ limit of the stationary measures defined in (4) and (5) and depending on parameters u, v . It turns out that this limit was studied in [39] (see also the review [45]). We will make considerable use of these results here. The set of stationary measures obtained in the $L \rightarrow \infty$ limit always depend on the boundary parameter u , and sometimes depend on the parameter v . When this is the case v can be interpreted as minus the average drift of the process at infinity (see below).

In the low density phase ($u \leq 0, v \geq u$), the stationary measure (4) simply converge as $L \rightarrow \infty$ to a standard Brownian motion with drift u [39] (so that the limit does not depend on v). In the maximal current phase ($u \geq 0, v \geq 0$), the stationary measures converge [39] to a distribution that we denote $\mathcal{HY}_{\gamma_u}^{(0)}$ – the letters \mathcal{HY} stand for Hariya-Yor. This is the distribution of the process

$$H(x) = B^{(1)}(x) + B^{(2)}(x) + \log \left(1 + \gamma_u \int_0^x e^{-2B^{(2)}(z)} dz \right), \quad (34)$$

where $B^{(1)}(x), B^{(2)}(x)$ are two independent Brownian motions with diffusion coefficient $1/2$ and γ_u is an independent Gamma random variable with shape parameter u . Again, the limit does not depend on v . In the high-density phase ($v \leq 0, u \geq v$), the stationary measures converge [39] to a distribution that we denote $\mathcal{HY}_{\gamma_{u-v}}^{(-v)}$. This is the distribution of the process

$$H(x) = B^{(1)}(x) + B^{(2)}(x) + vx + \log \left(1 + \gamma_{u-v} \int_0^x e^{-2B^{(2)}(z) - 2vz} dz \right), \quad (35)$$

where $B^{(1)}(x), B^{(2)}(x)$ are two independent Brownian motions with diffusion coefficient $1/2$ and γ_{u-v} is an independent Gamma random variable with shape parameter $u-v$.

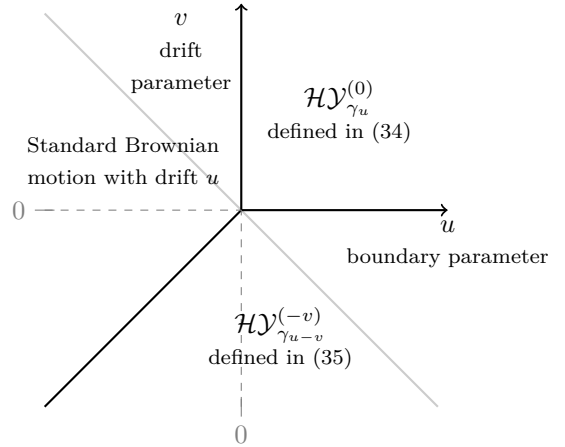


Fig. 2: Phase diagram of stationary measures for the KPZ equation in the half-space \mathbb{R}_+ with boundary parameter u . The diagram means that if the initial condition has drift $-v$ at infinity, the height field should converge at large time under mild assumptions to the stationary measure indicated in one of the three regions of the (u, v) plane. In particular, if $u \leq 0$ and the drift at infinity is 0, which includes the flat initial data, the height field will converge to a Brownian motion with drift u , as predicted in [17]. Along the antidiagonal line $u = -v$, the stationary measure is always a Brownian motion with drift u , see [42].

In this phase, the limit depends on v , and we remark that the drift at infinity of the process (35) is $-v$ (in the sense that $H(x) - H(0) \simeq -vx$ when $x \rightarrow +\infty$).

We expect that for a large class of initial conditions with drift at infinity equal to $-v$, the height field will converge at large time (modulo a global shift) to one of the stationary measures that we have just described, according to the phase diagram in Figure 2. This prediction is based on an analogous convergence result at large time for ASEP on a half-line proved in [11], though the stationary measures were explicitly described much later [13, 46].

Large-scale limit. Now that we have described the stationary measures of the KPZ equation on a half-line, it would be interesting to consider their large scale limit as x goes to infinity. The processes obtained in this limit should be understood as stationary measures of the half-space KPZ fixed point, that is the universal process arising as scaling limit of all half-space models in the KPZ class. In particular, we conjecture that the large scale limit of half-line ASEP stationary measures [46] do converge to the same limit at large scale. Note that this half-space KPZ fixed point has not been defined rigorously (unlike the full-space situation), but its multipoint distributions for some initial conditions are known [47, 48]. The large scale limit of half-line KPZ equation stationary measures are described in the phase diagram of Fig. 3. We will let $x \rightarrow +\infty$ and define $\tilde{H}(y) = \frac{1}{\sqrt{x}} H(xy)$ and scale the boundary parameter as $u = \frac{\tilde{u}}{\sqrt{x}}$. In the low-density phase, clearly, the Brownian motion with drift u becomes a Brownian

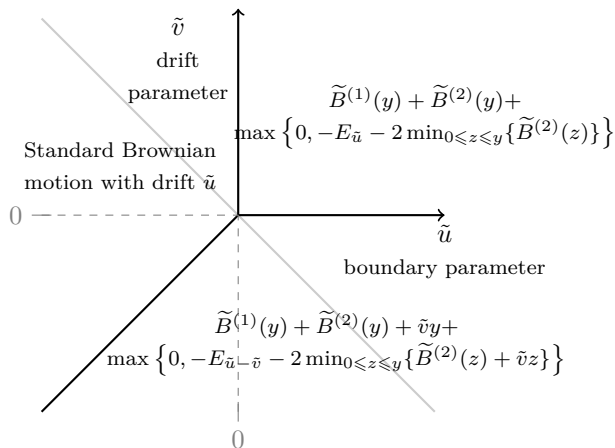


Fig. 3: Large scale limits of stationary measures of the half-space KPZ equation described in Fig. 2. $\tilde{B}^{(1)}(y)$ and $\tilde{B}^{(2)}(y)$ denote independent Brownian motions with diffusion coefficient $1/2$. $E_{\tilde{u}}$ and $E_{\tilde{u}-\tilde{v}}$ denote independent exponential random variables with parameters \tilde{u} and $\tilde{u}-\tilde{v}$. The parameters \tilde{u}, \tilde{v} are rescalings of the parameters u, v in Fig. 2 as explained in the Letter.

motion in the variable y with drift \tilde{u} in the scaling limit. In the maximal current phase ($\tilde{u}, \tilde{v} > 0$), the large scale limit of (34) yields

$$\tilde{H}(y) = \tilde{B}^{(1)}(y) + \tilde{B}^{(2)}(y) + \max\left\{0, -E_{\tilde{u}} - 2 \min_{0 \leq z \leq y} \{\tilde{B}^{(2)}(z)\}\right\}, \quad (36)$$

where $\tilde{B}^{(1)}(y), \tilde{B}^{(2)}(y)$ are two independent Brownian motions with diffusion coefficient $1/2$ and $E_{\tilde{u}}$ is an independent exponential random variable with rate parameter \tilde{u} . Here we have used that $\frac{-1}{\sqrt{x}} \log \gamma_{\tilde{u}/\sqrt{x}}$ converges to an exponential distribution with parameter \tilde{u} . In particular, when $\tilde{u} \rightarrow +\infty$, we obtain the sum of a Brownian motion and a Bessel 3 process [49]³, that is a Brownian motion conditioned to remain positive on $[0, +\infty)$. In the high density phase ($\tilde{v} < 0, \tilde{u} > \tilde{v}$), we scale $v = \frac{\tilde{v}}{\sqrt{x}}$ and the large scale limit of (35) yields

$$\tilde{H}(y) = \tilde{B}^{(1)}(y) + \tilde{B}^{(2)}(y) + \tilde{v}y + \max\left\{0, -E_{\tilde{u}-\tilde{v}} - 2 \min_{0 \leq z \leq y} \{\tilde{B}^{(2)}(z) + \tilde{v}z\}\right\}, \quad (37)$$

where $\tilde{B}^{(1)}(y), \tilde{B}^{(2)}(y)$ are two independent Brownian motions with diffusion coefficient $1/2$ and $E_{\tilde{u}-\tilde{v}}$ is an independent exponential random variable with parameter $\tilde{u}-\tilde{v}$. Again, $-\tilde{v}$ represents the drift at infinity of the process $\tilde{H}(y)$. When $\tilde{u} = -\tilde{v}$, the process (37) becomes a standard Brownian motion with drift \tilde{u} as this was already the case before taking any limit, although this is not immediately obvious from (37).

³We thank A. Comtet for an exchange on this point.

Directed polymer endpoint. – We obtain the endpoint probability for a very long polymer as $\mathcal{Q}(x) = e^{H(x)}/\int_0^L dx e^{H(x)}$, where $H(x)$ is given in (4). The statistics of the ratio $\mathcal{Q}(L)/\mathcal{Q}(0) = e^{\sqrt{\frac{x}{2}}G+Y}$, where G is a standard Gaussian random variable, requires only the PDF $P(Y)$ of $Y = X(L) - X(0)$, which in some special cases takes a simple form [42]. For $u + v = -1$ one obtains

$$P(Y) = \frac{e^{-(1+2v)Y + \frac{Y^2}{4}}}{2\mathcal{Z}_{u,v}} \left[\operatorname{erf}\left(\frac{L-2Y}{2\sqrt{L}}\right) + \operatorname{erf}\left(\frac{L+2Y}{2\sqrt{L}}\right) \right] \quad (38)$$

with $\mathcal{Z}_{u,v} = \frac{e^{v^2 L}}{1+2v}(e^{(1+2v)L} - 1)$. At the transition point, $u = v = -1/2$, $P(Y)$ becomes uniform in $[-L/2, L/2]$ consistent with the polymer being localized near either boundary with probability $1/2$ (see [42] for details, and [50] for an earlier work based on ASEP).

Note. While this work was near completion, the preprint [51] appeared. This paper also performs a Laplace transform inversion of the result of [18], although [51] describes the process X differently, as a Markov process with explicit transition probabilities. Both descriptions are equivalent: indeed, their formula [51, (1.7)] can also be read from (16) above.

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