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# Observer Design for a Class of Delay Systems Using a Fredholm Transform

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**Abstract:** In this paper, we design an observer for a system represented by a general class of Integral Delay Equations (IDE). This class of equations encompasses various systems, from chemical or biological processes to the propagation of electric pulses on excitable media. The available measurement corresponds to a discrete and distributed-delayed value of the state. Under an appropriate spectral observability assumption, an implementable observer is proposed. Inspired by the fact that first-order hyperbolic Partial Derivative Equations (PDE) and time-delay systems are closely related, we use a PDE formulation and the well-known backstepping methodology for the design. However, due to integral terms in the dynamics, the use of a Fredholm integral transform is required. We prove its existence and invertibility by using an operator framework. Both proofs derive from the spectral observability assumption. Some test case simulations end the paper.

*Keywords:* infinite dimensional systems; observer design; backstepping methodology.

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## 1. INTRODUCTION

In this paper, we design an observer for a dynamical system represented by a general class of Integral Delay Equations (IDEs). Such systems are encountered in a wide range of engineering applications, where transport phenomena occur. Delays also naturally appear when considering interconnected and/or networked systems. They are often sources of instabilities in the closed-loop system and can induce poor performance for the resulting feedback control laws Gu and Niculescu (2003). Time-delay systems have then been the object of many research works Niculescu (2001).

With the objective of designing efficient output-feedback (and thus implementable) control laws for such systems, interest for observers has grown significantly in the last decades. For an overview on the subject, the reader is referred to Sename (2001). Even though time-delay systems are inherently infinite-dimensional systems, results on finite-dimensional systems can still be a relevant source of inspiration. Conversions between different representations are of great use to take advantage of analysis tools or algorithms built for one formulation Peet (2021). More specifically, a duality between observation and output feedback for general classes of linear systems with distributed and pointwise delays has been studied Sun (2007). However, it led to restrictive stability assumptions. Though, the control law design framework can still inspire observer designs. In particular, Redaud et al. (2021) used an approach based on a comparison with a hyperbolic PDE system. Indeed, such systems can be seen as neutral differential equations Hale and Verduyn Lunel (1993); Auriol and Di Meglio (2018) that present a structure similar to one of the classes of IDEs under consideration. This representation allows the use of the well-known backstepping

methodology that has been proved to be an efficient tool to design observers Krstic and Smyshlyaev (2008).

More precisely, the contribution of the paper is two-fold. First, unlike classical observer design approaches for time-delay systems Klamka (1982), we use a PDE framework. Then, we introduce an appropriate Fredholm integral transform, similar to the one used in Redaud et al. (2021).

To the best of the authors' knowledge, the operator framework used to prove the existence of the observer gains is a novelty in the literature. It gives a constructive way to design implementable observers.

The proposed approach can be resumed as follows. First, we use a new representation of the delay system based on the state interpretation. This infinite-dimensional linear system class will allow us to use an operator framework. We then define a hyperbolic system and show that its boundaries satisfy the same equations as our initial delay system.

Next, we define an observer state and the resulting error system associated with this hyperbolic system. We use the backstepping methodology to map it to a stable target system. However, the integral terms in the delay system lead to coupling terms in the hyperbolic comparison system, and a Fredholm integral transform must be used. Unlike Volterra transforms traditionally used, the invertibility of such transforms is not guaranteed Yoshida (1960). Utilizing an operator theory framework Coron et al. (2016), we show that the invertibility of the Fredholm transform and the well-posedness of the kernel equations naturally derive from a spectral observability assumption. The existence of observer gains making the estimated state converging towards the actual state is then guaranteed.

The paper's layout is as follows: in Section 2, we present the general class of time-delay equations considered herein

and describe our strategy relying on a spectral observability assumption. Then we show in Section 3 that this equation can be reformulated as a hyperbolic system, for which we design an observer. We use the backstepping methodology with a Fredholm transform to map the error system to a stable target system. Next, we introduce the operator framework and show the well-posedness of the kernel equations. We use the same approach to prove that the Fredholm transform is invertible. Then, we define the observer gains to guarantee the convergence of the estimated state towards the actual state. We illustrate this result by simulations in Section 5. Finally, we end this paper with some concluding remarks in Section 6.

*Notations* : In this paper, we denote  $D_\tau = H^1([-\tau, 0], \mathbb{R})$  the Banach space of  $H^1$  real-valued functions taking values in the compact  $[-\tau, 0]$ , for any fixed time-delay  $\tau > 0$ . For any function  $\phi : [-\tau, \infty) \mapsto \mathbb{R}$ , we define an associated partial trajectory  $\phi_{[t]} \in D_\tau$  by  $\phi_{[t]}(\theta) = \phi(t + \theta)$ ,  $-\tau \leq \theta \leq 0$ . We use  $s$  as the Laplace variable. For all  $a, b, \nu \in [0, 1]$ , the characteristic function is denoted by  $\mathbb{1}_{[a,b]}(\nu)$  (equals 1 if  $\nu \in [a, b]$ , 0 else).

## 2. PROBLEM UNDER CONSIDERATION

### 2.1 Presentation of the system

*Time-delay formulation* Consider the neutral type equation defined for all  $t \geq 0$  by

$$z(t) + az(t - \tau_1) + \int_0^{\tau_1} N_1(\nu)z(t - \nu)d\nu = 0, \quad (1)$$

where  $\tau_1$  is a strictly positive delay, and  $N_1$  a piecewise continuous. The parameter  $a$  is a constant that belongs to  $] -1, 1[$ . The initial data is given by  $z_0 \in D_{\tau_1}$ . A function  $z : [-\tau_1, \infty) \rightarrow \mathbb{R}$  is called a solution of the initial value problem (1) if  $z(0) = z_0$ , and if (1) is satisfied for  $t \geq 0$ . System (1) corresponds to a difference equation with a pointwise-delay term and a distributed-delay term. The class of system (1) can model a wide variety of systems in biology, chemistry, epidemiology, physics, or engineering sciences Niculescu (2001). The condition  $|a| < 1$  guarantees that the open-loop system has a finite number of unstable roots Hale and Verduyn Lunel (1993). If  $|a| > 1$ , any arbitrary small delay would destabilize the potential estimation Logemann et al. (1996). Thus, it would not be possible to implement the resulting observer.

Our objective is to design an observer for the time-delay system (1) using the measurement

$$y(t) = z(t - \tau_2) + \int_0^{\tau_2} N_2(\nu)z(t - \nu)d\nu, \quad (2)$$

with  $\tau_2 > 0$  and  $N_2$  a piecewise continuous function. More precisely, using the delayed output  $y(t)$ , we want to design an observer  $\hat{z}$  such that  $\|z(t) - \hat{z}(t)\|_{L^2([-\max\{\tau_1, \tau_2\}, 0])} \xrightarrow{t \rightarrow \infty} 0$ . The measurement (2) corresponds the knowledge of a diffuse value of the state (distributed delay). An application example can be found with complex systems, whose control is ensured by multiple sensors. Measurements of each sensor are in most cases available after a specific acquisition, processing, and transportation time that varies from one sensor to another. Moreover, all sensors do not acquire data at the same frequency. One challenge is then to synchronize the

different outputs Bak and et al. (1998). Multiple outputs processing could result in a measurement of form (2), where a sum approximates the integral term for high-frequency acquisition.

### 2.2 Spectral observability assumption

To guarantee the possibility to design an observer for (1)-(2), we need an observability assumption Mounier (1998). This condition is obtained considering the spectral properties of the system. Taking the Laplace transform of (1)-(2), we have

$$\begin{aligned} z(s)(1 + ae^{-\tau_1 s} + \int_0^{\tau_1} N_1(\nu)e^{-\nu s}d\nu) &= \mathcal{F}(s)z(s) = 0 \\ y(s) &= (e^{-\tau_2 s} + \int_0^{\tau_2} N_2(\nu)e^{-\nu s}d\nu)z(s) = \mathcal{O}(s)z(s). \end{aligned} \quad (3)$$

Therefore, we obtain the following observability condition.

#### *Hypothesis 1. Spectral observability*

For all  $s \in \mathbb{C}$ ,  $\text{rank}[\mathcal{F}(s), \mathcal{O}(s)] = 1$ .

It guarantees that  $\mathcal{F}$  and  $\mathcal{O}$  are not simultaneously equal to 0. A parallel can be drawn with the spectral controllability condition given in Redaud et al. (2021), since observability and controllability are dual problems.

### 2.3 Observer design strategy

The strategy to design the observer states as follows:

- first, we rewrite the integral delay equation (1) as a hyperbolic PDE system. The distributed measurement (2) corresponds to a boundary measurement for this new hyperbolic system;
- second, we propose an observer for this PDE system as a copy of the original dynamics with some tunable output injection gains;
- then, we use a Fredholm backstepping integral transform that maps the associated error system to a stable target system. This transformation gives a suitable choice for the observer gains;
- next, we show that the existence of this backstepping transform is equivalent to the existence of a solution for some integral equations. We use an operator framework to establish the existence of a solution for this set of equations;
- finally, the invertibility of the backstepping transform is shown using the same operator framework.

## 3. PDE FORMULATION

This section introduces a simple hyperbolic system that will be used as a *comparison system* Niculescu (2001), to simplify the design of the observer. For this PDE system, the measurement (2) rewrites as a pointwise boundary measurement. We show that the estimation of the PDE state implies the estimation of the initial state  $z$ .

### 3.1 Comparison system

Consider the following hyperbolic system, defined by

$$\partial_t u(t, x) + \lambda \partial_x u(t, x) = h_1(x)u(t, 1), \quad (4)$$

$$\partial_t v(t, x) - \mu \partial_x v(t, x) = h_2(x)u(t, 1), \quad (5)$$

where  $(u(t, x), v(t, x))^T$  is the state of the system, the different arguments evolving in  $\{(t, x) \text{ s.t. } t > 0, x \in [0, 1]\}$  with the boundary conditions

$$u(t, 0) = au(t, 1), \quad v(t, 1) = u(t, 1), \quad (6)$$

and where  $\lambda = \frac{1}{\tau_1}, \mu = \frac{1}{\tau_2}$ , and  $\forall \nu \in [0, 1], h_1(\nu) = N_1(\tau_1(1 - \nu)), h_2(\nu) = N_2(\tau_2\nu)$ . The initial conditions  $u_0$  and  $v_0$  belong to  $H^1([0, 1], \mathbb{R})$ . They are defined for all  $x \in [0, 1]$  by  $u_0(x) = u(0, x) = z_0(-\frac{1-x}{\lambda})$  and  $v_0(x) = v(0, x) = 0$ . We consider that we have access to the measurement  $y_1(t) = v(t, 0)$ . We will show below that  $y_1(t)$  corresponds to  $y(t)$ . We now rewrite the system (4)-(6) in the abstract form  $\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$ , where the operator  $A$  is defined by

$$A : D(A) \subset (L^2(0, 1))^2 \rightarrow (L^2(0, 1))^2$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} -\lambda u_x(x) + h_1(x)u(1) \\ \mu v_x(x) + h_2(x)u(1) \end{pmatrix}, \quad (7)$$

with  $D(A) = \{(u, v) \in (H^1(0, 1))^2 | u(0) = au(1), v(1) = u(1)\}$ . The operator  $A$  is well posed and densely defined Bastin and Coron (2016) (the solution remains in  $H^1$  and the trace operator is properly defined). We now show that the system (4)-(6) is a comparison system and that the measurement  $y_1$  corresponds to  $y$ . Indeed, defining  $z(t) = u(t, 1)$  and applying the method of characteristics to (4)-(6), we obtain

$$u(t, 1) = z(t) = az(t - \frac{1}{\lambda}) + \int_0^{\frac{1}{\lambda}} h_1(1 - \lambda s)z(t - s)ds,$$

$$y_1(t) = z(t - \frac{1}{\mu}) + \int_0^{\frac{1}{\mu}} h_2(\mu s)z(t - s)ds.$$

Thus,  $u(\cdot, 1)$  is solution of (1), while  $y_1$  satisfies (2).

### 3.2 Observer and error systems

Let us design an observer for the hyperbolic system (4)-(6). The observer state  $(\hat{u}(t, x), \hat{v}(t, x))$  satisfies the same dynamics as the original hyperbolic system (4)-(6), in which we add output injection gains. We have

$$\partial_t \hat{u}(t, x) + \lambda \partial_x \hat{u}(t, x) = h_1(x)\hat{u}(t, 1) - g_1(x)(\hat{v}(t, 0) - y(t)),$$

$$\partial_t \hat{v}(t, x) - \mu \partial_x \hat{v}(t, x) = h_2(x)\hat{u}(t, 1) - g_2(x)(\hat{v}(t, 0) - y(t)),$$

with the boundary conditions

$$\hat{u}(t, 0) = a\hat{u}(t, 1) \quad \hat{v}(t, 1) = \hat{u}(t, 1), \quad (8)$$

where the observer gains  $g_1$  and  $g_2$  are two real-value functions defined on  $[0, 1]$  that will be determined later. They belong to  $H^1([0, 1])$ . The initial conditions  $\hat{u}(0, \cdot)$  and  $\hat{v}(0, \cdot)$  are arbitrarily  $H^1$  functions. We now define the error between the observer and the state as  $\tilde{u} = u - \hat{u}, \tilde{v} = v - \hat{v}$ . The error state satisfies the set of equations

$$\partial_t \tilde{u}(t, x) + \lambda \partial_x \tilde{u}(t, x) = h_1(x)\tilde{u}(t, 1) - g_1(x)\tilde{v}(t, 0), \quad (9)$$

$$\partial_t \tilde{v}(t, x) - \mu \partial_x \tilde{v}(t, x) = h_2(x)\tilde{u}(t, 1) - g_2(x)\tilde{v}(t, 0), \quad (10)$$

with the boundary conditions

$$\tilde{u}(t, 0) = a\tilde{u}(t, 1) \quad \tilde{v}(t, 1) = \tilde{u}(t, 1). \quad (11)$$

The objective is to design the observer gains  $g_1, g_2$ , so that the error system is exponentially stable in the sense of the  $L^2$ -norm. More precisely, we want to show that  $\exists \nu > 0, C > 0, \|(\tilde{u}, \tilde{v})\|_{L^2(0, 1)} \leq Ce^{-\nu t} \|(\tilde{u}_0, \tilde{v}_0)\|_{L^2(0, 1)}$ . Indeed, denoting  $\hat{z}(t) = \hat{u}(t, 1)$ , we will show that the convergence of  $(\tilde{u}, \tilde{v})$  to zero implies the convergence of

$\hat{z}$  to  $z$ . We can rewrite the error system (9)-(11) using the operator abstract form

$$\frac{d}{dt} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = A \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \mathcal{G}C \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad (12)$$

where  $\mathcal{G} = (g_1, g_2)^T$  corresponds to the product with the observer gains and where the operator  $C$  is the trace operator defined on  $D(A)$  by

$$C : D(A) \subset (L^2(0, 1))^2 \rightarrow \mathbb{R}$$

$$(\tilde{u}, \tilde{v})^T \mapsto \tilde{v}(0). \quad (13)$$

### 3.3 Observability properties

The following lemma is a consequence of Hypothesis 1

*Lemma 2.* Under Hypothesis 1, consider the operators  $A$  and  $C$  respectively defined by (7) and (13). Then, for any  $s \in \mathbb{C}$ , we have

$$\ker(s - A) \cap \ker(C) = \{0\}. \quad (14)$$

The proof is omitted due to space restrictions.

### 3.4 Insights on Fredholm integral operators

The stabilization of the error system (9)-(11) will be done using a backstepping transform of the Fredholm type. Indeed, only a transformation of the Fredholm type gives enough degrees of freedom to map the error system to a stable target system. More precisely, we consider an integral operator  $\mathcal{T} : (L^2([0, 1]))^2 \rightarrow (L^2([0, 1]))^2$  defined by

$$\mathcal{T} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - \int_0^1 N(x, \nu) \begin{pmatrix} u(\nu) \\ v(\nu) \end{pmatrix} d\nu, \quad (15)$$

where  $N$  is a bounded function defined on the unit square  $\mathcal{S} = \{(x, y) \in [0, 1]^2\}$ . Unlike Volterra integral transformations, Fredholm transformations are not always invertible Yoshida (1960). The following lemma (adjusted from (Coron et al., 2016, Lemma 2.2, Proposition 2.6) guarantees (under several conditions) the invertibility of such an integral operator.

*Lemma 3.* Consider a Fredholm integral operator  $\mathcal{T} : (L^2([0, 1]))^2 \rightarrow (L^2([0, 1]))^2$  as defined by (15). Assume that

- (a)  $\ker(\mathcal{T}) \subset D(A)$ ,
- (b)  $\ker(\mathcal{T}) \subset \ker(C)$ ,
- (c)  $\forall z \in \ker(\mathcal{T}), \mathcal{T}Az = 0$ ,
- (d)  $\forall s \in \mathbb{C}, \ker(s - A) \cap \ker(C) = \{0\}$ .

Then, the operator  $\mathcal{T}$  is invertible.

**Proof.** The proof follows the steps of (Coron et al., 2016, Lemma 2.2, Proposition 2.6). Since the integral part of  $\mathcal{T}$  is a compact operator, the Fredholm alternative Brezis (2010) implies that  $\dim \ker(\mathcal{T}) < \infty$ . Suppose that  $\ker(\mathcal{T}) \neq \{0\}$ . Due to assumption (a),  $Az$  is well-defined, and assumption (c) implies that  $\ker(\mathcal{T})$  is stable by  $A$ , that is to say, for all  $z \in \ker(\mathcal{T}), Az \in \ker(\mathcal{T})$ . Since  $\ker(\mathcal{T})$  is finite-dimensional and not reduced to  $\{0\}$ , the restriction  $A|_{\ker(\mathcal{T})}$  of  $A$  to  $\ker(\mathcal{T})$  has at least one eigenvalue  $\nu \in \mathbb{C}$ . Let  $\zeta$  be the corresponding eigenfunction. Thus,  $\zeta \in \ker(\nu - A)$  and  $\zeta \in \ker(C)$  (due to assumption (b)). This is in contradiction with assumption (d). Thus,  $\ker(\mathcal{T}) = \{0\}$  and  $\mathcal{T}$  is injective. Using the Fredholm alternative, we obtain that  $\mathcal{T}$  is invertible. ■

#### 4. OBSERVER DESIGN

In this section, we design the observer gains  $g_1, g_2$  to stabilize the error system (9)-(11). First, we use a Fredholm integral transform to map (9)-(11) to a stable target system. We show that the existence of such a transform is implied by the existence of solutions for some integral equations. Finally, using the operator framework introduced in Section 3, we prove that the proposed Fredholm integral operator is invertible.

##### 4.1 Presentation of the target system

We now follow the backstepping methodology to map (9)-(11) to a stable target system with equivalent stability properties. Consider the candidate target system

$$\partial_t \tilde{\alpha}(t, x) + \lambda \partial_x \tilde{\alpha}(t, x) = 0, \quad (16)$$

$$\partial_t \tilde{\beta}(t, x) - \mu \partial_x \tilde{\beta}(t, x) = 0, \quad (17)$$

with the boundary conditions

$$\tilde{\alpha}(t, 0) = a\tilde{\alpha}(t, 1) \quad \tilde{\beta}(t, 1) = \tilde{\alpha}(t, 1). \quad (18)$$

This target system (16)-(18) is exponentially stable in the sense of the  $L_2$ -norm, since  $|a| < 1$  Bastin and Coron (2016). Let us define

$$\tilde{\alpha}(t, x) = \tilde{u}(t, x) - \int_0^1 L^{11}(x, \nu) \tilde{u}(t, \nu) + L^{12}(x, \nu) \tilde{v}(t, \nu) d\nu,$$

$$\tilde{\beta}(t, x) = \tilde{v}(t, x) - \int_0^1 L^{21}(x, \nu) \tilde{u}(t, \nu) + L^{22}(x, \nu) \tilde{v}(t, \nu) d\nu,$$

with  $L^{ij}$ ,  $i, j \in \{1, 2\}$  four bounded functions defined on  $\mathcal{S}$ . We want to prove that the existence of some kernels such that this transform maps the original system (9)-(11) to the target system (16)-(18). Then, if this transform is invertible, the exponential stability of (16)-(18) will imply the one of (9)-(11).

##### 4.2 Kernel equations

Let us define the kernels  $L^{ij}$  by

$$\partial_x L^{11}(x, \xi) + \partial_\xi L^{11}(x, \xi) = 0, \quad (19)$$

$$\lambda \partial_x L^{12}(x, \xi) - \mu \partial_\xi L^{12}(x, \xi) = 0, \quad (20)$$

$$\mu \partial_x L^{21}(x, \xi) - \lambda \partial_\xi L^{21}(x, \xi) = 0, \quad (21)$$

$$\partial_x L^{22}(x, \xi) + \partial_\xi L^{22}(x, \xi) = 0, \quad (22)$$

where we have

$$\begin{aligned} h_1(x) + \lambda(L^{11}(x, 1) - aL^{11}(x, 0)) - \mu L^{12}(x, 1) \\ = \int_0^1 L^{11}(x, \nu) h_1(\nu) + L^{12}(x, \nu) h_2(\nu) d\nu, \end{aligned} \quad (23)$$

$$\begin{aligned} h_2(x) + \lambda(L^{21}(x, 1) - aL^{21}(x, 0)) - \mu L^{22}(x, 1) \\ = \int_0^1 L^{21}(x, \nu) h_1(\nu) + L^{22}(x, \nu) h_2(\nu) d\nu, \end{aligned} \quad (24)$$

and the boundary conditions

$$L^{11}(0, \xi) = aL^{21}(1, \xi), \quad L^{12}(0, \xi) = aL^{12}(1, \xi), \quad (25)$$

$$L^{11}(1, \xi) = L^{21}(1, \xi), \quad L^{22}(1, \xi) = L^{12}(1, \xi), \quad (26)$$

$$L^{21}(0, \xi) = 0, \quad L^{22}(0, \xi) = 0. \quad (27)$$

These two additional boundary conditions (27) are necessary to guarantee the invertibility of the Fredholm integral

transform. Finally, let us define the observer gains as the solutions of the integral equations

$$\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} - \int_0^1 \begin{pmatrix} L^{11}(x, \nu) & L^{12}(x, \nu) \\ L^{21}(x, \nu) & L^{22}(x, \nu) \end{pmatrix} \begin{pmatrix} g_1(\nu) \\ g_2(\nu) \end{pmatrix} d\nu = \mu \begin{pmatrix} L^{12}(x, 0) \\ L^{22}(x, 0) \end{pmatrix} \quad (28)$$

If (19)-(27) admit a solution, then, following the backstepping approach, we prove that (9)-(11) can be mapped to (16)-(18). Then, if the transformation is invertible, the two systems will have equivalent stability properties. Finally, note that the existence of the functions  $g_1$  and  $g_2$  defined by (28) will be a consequence of the invertibility of the integral operator.

##### 4.3 Well-posedness of the kernel equations

In this section, we prove the existence and uniqueness of a solution to the kernels equations (19)-(27). We have the following theorem

*Theorem 4.* The set of equations (19)-(27) admits a unique bounded on the unit square  $\mathcal{S}$ .

The proof of this theorem will be given in the rest of this section and requires several intermediate results. First, we show that the knowledge of the two functions  $L^{12}(\cdot, 1)$  and  $L^{21}(\cdot, 1)$  is sufficient to properly define the four kernels on  $\mathcal{S}$ . Next, we rewrite the integral equations (23)-(24) using an integral operator acting on these two boundary functions. Then, we show that this integral operator is invertible using Lemma 3. Consequently, it will imply that (19)-(27) admit a unique bounded solution. In what follows, we will only consider the case  $\lambda \leq \mu$ . The case  $\lambda > \mu$  can be done following the same ideas. Due to space restrictions, we will not give computational details.

*Kernels reduction* First, we show that the existence and uniqueness of all the kernels on  $\mathcal{S}$  is implied by the one of  $L^{12}(x, 1)$  and  $L^{21}(x, 1)$  on  $[0, 1]$ .

*Lemma 5.* For all  $(x, \xi) \in \mathcal{S}$ ,  $(i, j) \in \{1, 2\}^2$ ,  $L^{i,j}(x, \xi)$  can be expressed as functions of  $L^{12}(\cdot, 1)$  and  $L^{21}(\cdot, 1)$ .

**Proof.** We apply the method of characteristics to the transport equations (19)-(22) to express the kernels as functions of their boundary values. We obtain

$$L^{11}(x, \xi) = \begin{cases} L^{21}(1 - \frac{\mu}{\lambda}(x - \xi), 1) & \text{if } \xi \leq x \\ aL^{21}(1 - \frac{\mu}{\lambda}(1 + \xi - x), 1) & \text{else} \end{cases} \quad (29)$$

$$L^{12}(x, \xi) = \begin{cases} L^{12}(x - \frac{\lambda}{\mu}(1 - \xi), 1) & \\ \text{if } \max\{0, 1 - \frac{\mu}{\lambda}x\} \leq \xi \leq 1 & \\ aL^{12}(1 + x - \frac{\lambda}{\mu}(1 - \xi), 1) & \text{else} \end{cases} \quad (30)$$

$$L^{21}(x, \xi) = \begin{cases} 0 & \text{if } 0 \leq \xi \leq \min\{1 - \frac{\lambda}{\mu}x, 1\} \\ L^{21}(x - \frac{\mu}{\lambda}(1 - \xi), 1) & \text{else} \end{cases} \quad (31)$$

$$L^{22}(x, \xi) = \begin{cases} L^{12}(1 - \frac{\lambda}{\mu}(x - \xi), 1) & \text{if } 0 \leq \xi \leq x \\ 0 & \text{else} \end{cases} \quad (32)$$

In particular, we get  $L^{21}(1, \xi) = 0 \quad \forall \xi \leq 1 - \frac{\lambda}{\mu}$ , as illustrated on Figure 1. Thus, we can compute  $L^{i,j}(x, \xi)$ ,  $(i, j) \in \{1, 2\}^2$  on  $\mathcal{S}$  from the values of  $L^{12}(x, 1)$  and  $L^{21}(x, 1)$  on  $[0, 1]$ . ■

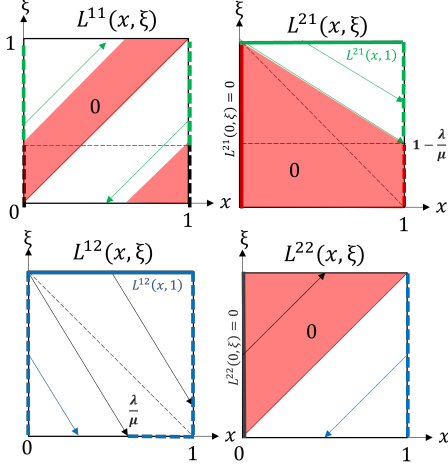


Fig. 1. Representation of kernels  $L^{ij}$  on  $\mathcal{S}$

*Operator formulation* We now rewrite the integral equations (23)-(24) as functions of  $L^{12}(x, 1)$  and  $L^{21}(x, 1)$ . First, we notice that (19) and the boundary condition (25) imply that  $L^{11}(x, 1) - aL^{11}(x, 0) = 0$ . Thus, (23) rewrites

$$\mu L^{12}(x, 1) + \int_0^1 L^{11}(x, \nu) h_1(\nu) + L^{12}(x, \nu) h_2(\nu) d\nu = h_1(x).$$

Using the different kernel equations, we finally obtain

$$\begin{pmatrix} \mu L^{12}(x, 1) \\ -\lambda L^{21}(x, 1) \end{pmatrix} + \int_0^1 I(x, \nu) \begin{pmatrix} L^{12}(\nu, 1) \\ L^{21}(\nu, 1) \end{pmatrix} = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} \quad (33)$$

where the four components  $(I(x, \nu))_{ij}$  are defined by

$$\begin{aligned} I_{11}(x, \nu) &= \mathbb{1}_{[0, \frac{\lambda}{\mu}]}(x) (\mathbb{1}_{[x+1-\frac{\lambda}{\mu}, 1]} a \frac{\mu}{\lambda} h_2(1 + \frac{\mu}{\lambda}(\nu - 1 - x)) \\ &\quad + \mathbb{1}_{[0, x]}(\nu) \frac{\mu}{\lambda} h_2(1 + \frac{\mu}{\lambda}(\nu - x))) \\ &\quad + \mathbb{1}_{[\frac{\lambda}{\mu}, 1]}(x) \mathbb{1}_{[x-\frac{\lambda}{\mu}, x]}(\nu) \frac{\mu}{\lambda} h_2(1 + \frac{\mu}{\lambda}(\nu - x)) \end{aligned} \quad (34)$$

$$\begin{aligned} I_{12}(x, \nu) &= \mathbb{1}_{[0, \frac{\lambda}{\mu}]}(x) (\mathbb{1}_{[0, 1-\frac{\mu}{\lambda}x]} a \frac{\lambda}{\mu} h_1(1 - x - \frac{\lambda}{\mu}(1 - \nu)) \\ &\quad + \mathbb{1}_{[1-\frac{\mu}{\lambda}x, 1]}(\nu) \frac{\lambda}{\mu} h_1(x - \frac{\lambda}{\mu}(1 - \nu))) \\ &\quad + \mathbb{1}_{[\frac{\lambda}{\mu}, 1]}(x) \frac{\lambda}{\mu} h_1(x - \frac{\lambda}{\mu}(1 - \nu)) \end{aligned} \quad (35)$$

$$I_{21}(x, \nu) = \mathbb{1}_{[1-\frac{\lambda}{\mu}, 1]}(\nu) \frac{\mu}{\lambda} h_2(x - \frac{\mu}{\lambda}(1 - \nu)) \quad (36)$$

$$I_{22}(x, \nu) = \mathbb{1}_{[0, x]}(\nu) \frac{\lambda}{\mu} h_1(1 - \frac{\lambda}{\mu}(x - \nu)) \quad (37)$$

To rewrite (33) as an integral operator  $\mathcal{Q}$  of the form (15), we define the new variables  $\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \mu L^{12}(x, 1) \\ -\lambda L^{21}(x, 1) \end{pmatrix}$ .

With kernels

$$N(x, \nu) = \begin{pmatrix} -\frac{1}{\mu} I_{11}(x, \nu) & \frac{1}{\lambda} I_{12}(x, \nu) \\ \frac{1}{\mu} I_{21}(x, \nu) & \frac{1}{\lambda} I_{22}(x, \nu) \end{pmatrix} \quad (38)$$

the Fredholm operator  $\mathcal{Q}$  satisfies  $\mathcal{Q} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ . Thus, by proving that  $\mathcal{Q}$  is invertible, we will show that the integral equations (23)-(24) admit a unique solution.

*Invertibility of the operator  $\mathcal{Q}$*

*Lemma 6.* The Fredholm integral operator  $\mathcal{Q}$  of form (15), with kernels defined by (38) is invertible.

**Proof.** We only need to prove that assumptions (a) – (d) of Lemma 3 are satisfied for the operator  $\mathcal{Q}$ . By evaluating the coupling terms (34)-(37), we immediately get assumptions (a)–(b). Assumption (d) is satisfied under the spectral observability Hypothesis 1. In order to prove that assumption (c) holds, we use assumptions (a) – (b), integration by parts and changes of variables. ■

It is straightforward to show that  $(f, g)$  belongs to  $H^1([0, 1])$  (due to the regularity of the functions  $h$  Coron et al. (2016)). We can now conclude the proof of Theorem 4. The boundary functions  $L^{12}(x, 1)$  and  $L^{21}(x, 1)$  exist and are unique on  $[0, 1]$  by Lemma 6. They define the four kernels  $L^{ij}$ ,  $(i, j) \in \{1, 2\}^2$  on their definition domain  $\mathcal{S}$  by Lemma 5. ■

#### 4.4 Invertibility of the Fredholm transform and existence of the observer gains $g_1$ and $g_2$

Following the same operator framework, we now show

*Theorem 7.* The Fredholm integral transform  $\mathcal{P}$  with kernels defined by (19)-(27) is invertible. Consequently, the observer gains  $g_1$  and  $g_2$  are properly defined in (28).

**Proof.** Since the operator  $\mathcal{P}$  is of form (15) with

$$N(x, \nu) = \begin{pmatrix} L^{11}(x, \nu) & L^{12}(x, \nu) \\ L^{21}(x, \nu) & L^{22}(x, \nu) \end{pmatrix}, \quad (39)$$

we can use Lemma 3 to show its invertibility. As before, assumption (d) derives from the spectral observability hypothesis 1. The boundary conditions (25)-(27) imply that assumptions (a)–(b) hold. Assumption (c) is a direct consequence of (19)-(24) and assumptions (a) – (b). Using equations (28), the gain observers  $g_1, g_2$  satisfy

$$\mathcal{P} \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} \mu L^{12}(x, 0) \\ \mu L^{22}(x, 0) \end{pmatrix}. \quad (40)$$

Since  $\mathcal{P}$  is invertible, the observer gains exist and are uniquely defined on  $[0, 1]$ . Moreover, it is straightforward to show that they belong to  $H^1$ . ■

#### 4.5 Convergence of the observer

We can now end this section with the design of an observer for the initial system (1)-(2).

*Theorem 8.* Any solution  $(\tilde{u}, \tilde{v})$  of (9)-(11) converges towards 0 in the sense of the  $L^2$ -norm.

**Proof.** First, system (16)-(18) is exponentially stable in the sense of the  $L_2$ -norm. Since the backstepping transform is invertible by Theorem 7, system (9)-(11) shares equivalent stability properties. Indeed, the inverse transform is of form (15) with bounded kernels. ■

Thus, the error system (9)-(11) is exponentially stable. The observer state defined with gains  $g_1, g_2$  converges towards the initial state  $(u, v)$ . More precisely, we can show that  $\tilde{u}(t, 1)$  converges towards 0. Thus,  $\tilde{z}(t)$  exponentially converges towards zero. We can thus compute an observer  $\hat{z}(t)$  for (1)-(2) as  $\hat{z}(t) = \hat{u}(t, 1)$  that converges towards  $z$ .

## 5. SIMULATION RESULTS

In this section, we give some simulation results to illustrate the performance of our observer design for the time-delay system (1). Following the PDE approach presented in this paper, we simulate system (1) on Matlab using the hyperbolic comparison system (4)-(6). The evolution of the initial and observer states are computed using a Godunov scheme LeVeque (2002). We chose the parameter values  $\tau_1 = 1 > \tau_2 = 0.5$ ,  $a = 0.8$ , and  $N_1 = 0.4, N_2 = -0.5$  constant functions. They were selected such that the system is slightly unstable, as illustrated on Figure 2. One can verify that Hypothesis 1 is satisfied for these parameter values. We compute the evolution of the time-delay system over 10s, with a time-discretization of 0.01s. The initial conditions are  $z(0) = 1, \hat{z}(0) = 0.5$ . The kernels  $L^{ij}$  and the observer gain functions  $g_1, g_2$  are previously computed using a fixed-point algorithm (successive approximation technique). Their definition domains are discretized with a space mesh equal to 0.02. The integral terms are numerically approximated by the trapezoidal method. Plotting the evolution of the norm of the error state  $\tilde{z}$ , we observe that it towards zero, the observer (red curve) converges towards the real value (blue curve) as expected.

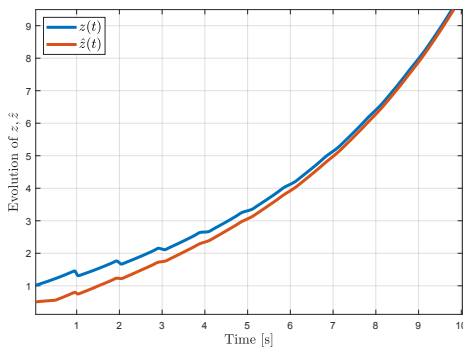


Fig. 2. Evolution of the time-delay system (1)  $z(t), \hat{z}(t)$ .

## 6. CONCLUDING REMARKS

This paper proposed a new approach to design an observer for a general class of integral delay equations under a spectral observability condition. Following the methodology proposed in Redaud et al. (2021), we used a scalar hyperbolic system (sharing the same observability properties) as a comparison system. We designed an observer for this PDE system using the backstepping approach to map the resulting error system to a stable target system. Since the existence of distributed delay terms required the backstepping transform to be of Fredholm type, invertibility needed to be proven. To do so, we used an operator framework inspired by Coron et al. (2016). Interestingly, the existence of solutions to the kernels equations depended on the invertibility of a similar operator. The well-posedness of the kernels defining the Fredholm transform was thus proved using the same results. The invertibility of both integral operators was a consequence of the assumed spectral observability condition. Finally, some simulations illustrated the relevance of this approach to design an observer for a time-delay system with delayed output. In future contributions, we will try to extend our results

to non-scalar systems. It will lead us to a general methodology for designing observers for integral delay dynamics and underactuated hyperbolic systems.

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