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# DYNAMICAL ZETA FUNCTIONS IN THE NONORIENTABLE CASE

YONAH BORNS-WEIL AND SHU SHEN

ABSTRACT. We use a simple argument to extend the microlocal proofs of meromorphy of dynamical zeta functions to the nonorientable case. In the special case of geodesic flow on a connected non-orientable negatively curved closed surface, we compute the order of vanishing of the zeta function at the zero point to be the first Betti number of the surface.

## 1. BACKGROUND

In this note we use a simple geometric argument to extend the results of Dyatlov and Zworski [5, 6] and of Dyatlov and Guillarmou [3, 4] to Axiom A flows with nonorientable stable and unstable bundles. It is classically known that on a closed manifold there are countably many closed orbits of such flows, and therefore one can define the *Ruelle zeta function*

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{i\lambda T_\gamma^\sharp})$$

where the product is taken over all primitive closed geodesics  $\gamma^\sharp$  with corresponding periods  $T_\gamma^\sharp$ . Note that by [3, Lemma 1.17] and [4, Section 3], this product converges for  $\text{Im}(\lambda) \gg 1$  large enough. The meromorphic continuation of  $\zeta_R$  to all of  $\mathbb{C}$  was conjectured by Smale [13], and proved by Fried [8] under analyticity assumptions. The case of smooth Anosov flows was first answered by Giulietti, Liverani and Pollicott [9] and then with microlocal methods by Dyatlov and Zworski [5] for manifolds with orientable stable and unstable bundles, and was extended to Axiom A flows by Dyatlov and Guillarmou [3, 4] under the same orientability assumptions. In [9, Appendix B], the authors also outlined ideas for removing the orientability assumptions.

We remove the orientability assumption and give a full proof for Axiom A flows. Specifically, we shall show

**Theorem 1.** *If  $(\phi_t)_{t \in \mathbb{R}}$  is an Axiom A flow on a closed manifold, the Ruelle zeta function  $\zeta_R$  extends to a meromorphic function on  $\mathbb{C}$ .*

The definition of an Axiom A flow is given as Definition 1.3.

We then restrict to the case of contact Anosov flow on a 3-manifold, and study the order of vanishing of  $\zeta_R$  at  $\lambda = 0$ . An important example is when  $M = S^*\Sigma$ ,

the cosphere bundle of a connected negatively curved closed surface  $\Sigma$ , and  $(\phi_t)_{t \in \mathbb{R}}$  is geodesic flow [1]. This problem was treated in [6] in the case where the stable bundle is orientable, and it was shown that the order of vanishing is  $b_1(M) - 2$ , where  $b_1(M)$  is the first Betti number of  $M$ .

We shall show that for nonorientable stable bundle, the analogous result is the following:

**Theorem 2.** *Let  $(\phi_t)_{t \in \mathbb{R}}$  be the Reeb flow on a connected contact closed 3-manifold. If  $(\phi_t)_{t \in \mathbb{R}}$  is Anosov with nonorientable stable bundle  $E_s$ , the Ruelle zeta function has vanishing order at  $\lambda = 0$  equal to  $b_1(\mathcal{O}(E_s))$ , the dimension of the first de Rham cohomology with coefficients in the orientation line bundle of  $E_s$ .*

The orientation line bundle is reviewed in Definition 1.5.

In the special case of the geodesic flow on  $M = S^*\Sigma$  with  $\Sigma$  nonorientable, the vanishing order at  $\lambda = 0$  is given by  $b_1(\Sigma)$ , as is shown in Proposition 3.10. This is in contrast to the orientable case, in which it is  $b_1(\Sigma) - 2$ .

More precisely, let  $\chi'(\Sigma)$  be the derived Euler characteristic of  $\Sigma$ , i.e.,

$$\chi'(\Sigma) = \sum_{i=0}^2 (-1)^i b_i(\Sigma) = \begin{cases} -b_1(\Sigma) + 2, & \text{if } \Sigma \text{ is orientable,} \\ -b_1(\Sigma), & \text{otherwise.} \end{cases}$$

**Corollary 3.** *If  $(\phi_t)_{t \in \mathbb{R}}$  is the geodesic flow on the cosphere bundle of a connected negatively curved closed surface (orientable or not), the Ruelle zeta function has vanishing order at  $\lambda = 0$  equal to  $-\chi'(\Sigma)$ .*

**1.1. Axiom A Flows.** Let  $M$  be a compact manifold without boundary of dimension  $n$ , and let  $(\phi_t)_{t \in \mathbb{R}}$  be a flow on  $M$  generated by the vector field  $V \in C^\infty(M; TM)$ .

**Definition 1.1.** A  $\phi_t$ -invariant set  $K \subseteq M$  is called *hyperbolic* for the flow  $(\phi_t)_{t \in \mathbb{R}}$  if  $V$  does not vanish on  $K$  and for each  $x \in K$  the tangent space  $T_x M$  can be written as the direct sum

$$T_x M = E_0(x) \oplus E_s(x) \oplus E_u(x)$$

where  $E_0(x) = \text{span}(V(x))$ ,  $E_s, E_u$  are continuous  $\phi_t$ -invariant vector bundles on  $K$ , and for some Riemannian metric  $|\cdot|$ , there are  $C, \theta > 0$  such that for all  $t > 0$ ,

$$\begin{aligned} |d\phi_t(x)v|_{\phi_t(x)} &\leq C e^{-\theta t} |v|_x & v \in E_s(x) \\ |d\phi_{-t}(x)w|_{\phi_{-t}(x)} &\leq C e^{-\theta t} |w|_x & w \in E_u(x). \end{aligned} \tag{1}$$

In the important case where all of  $M$  is hyperbolic, we call  $(\phi_t)_{t \in \mathbb{R}}$  an *Anosov flow*.

There is an analogous notion of hyperbolicity at fixed points.

**Definition 1.2.** A fixed point  $x \in M$ , i.e.,  $V(x) = 0$ , is called *hyperbolic* if the differential  $DV(x)$  has no eigenvalues with vanishing real part.

A generalization of Anosov flows is the following, given first by Smale [13, II.5 Definition 5.1]:

**Definition 1.3.** The flow  $(\phi_t)_{t \in \mathbb{R}}$  is called *Axiom A* if

- (1) all fixed points of  $(\phi_t)_{t \in \mathbb{R}}$  are hyperbolic,
- (2) the closure  $\mathcal{K}$  of the union of all closed orbits of  $(\phi_t)_{t \in \mathbb{R}}$  is hyperbolic,
- (3) the nonwandering set ([4, Definition 2.2]) of  $(\phi_t)_{t \in \mathbb{R}}$  is the disjoint union of the set of fixed points and  $\mathcal{K}$ .

We now recall the definition of a locally maximal set, given in [4, Definition 2.4].

**Definition 1.4.** A compact  $\phi_t$ -invariant set  $K \subseteq M$  is called *locally maximal* for  $(\phi_t)_{t \in \mathbb{R}}$  if there is a neighborhood  $V$  of  $K$  such that

$$K = \bigcap_{t \in \mathbb{R}} \phi_t(V).$$

We may then state the key proposition, which generalises [4, Proposition 3.1] to the case where  $E_s$  or  $E_u$  is not necessarily orientable on  $\mathcal{K}$ .

**Proposition 1.4.** *Let  $K \subseteq M$  be a locally maximal hyperbolic set for  $(\phi_t)_{t \in \mathbb{R}}$ , and let  $\zeta_K$  be defined as the Ruelle zeta function where we only take the product over trajectories in  $K$ . Then  $\zeta_K$  has a continuation to a meromorphic function on all of  $\mathbb{C}$ .*

Theorem 1 follows from Proposition 1.4, as we may remark that by [13, II.5 Theorem 5.2] we can write  $\mathcal{K} = K_1 \sqcup \dots \sqcup K_N$  with  $K_j$  basic hyperbolic<sup>1</sup>. Then the product

$$\zeta_R(\lambda) = \prod_{j=1}^N \zeta_{K_j}(\lambda),$$

which a priori holds for  $\text{Im}(\lambda) \gg 1$ , gives that  $\zeta_R$  also has a meromorphic continuation to all of  $\mathbb{C}$ .

The goal of Section 2 is to prove Proposition 1.4.

**1.2. The Orientation Bundle.** To fix notation we recall the definition of transition functions of a vector bundle. Given a continuous real vector bundle  $E$  of rank  $k$  over a manifold  $M$  with projection map  $\pi$ , let  $U_\alpha, U_\beta \subseteq M$  be two small open sets with nonempty intersection, and let  $\psi_\alpha : \pi^{-1}U_\alpha \rightarrow U_\alpha \times \mathbb{R}^n$ ,  $\psi_\beta : \pi^{-1}U_\beta \rightarrow U_\beta \times \mathbb{R}^n$  be local trivializations. Then the map  $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$  is of the form

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$$

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<sup>1</sup>These are locally maximal hyperbolic by definition (see [4, Definition 2.5]).

where  $\tau_{\alpha\beta} \in C^0(U_\alpha \cap U_\beta, \mathrm{GL}_k(\mathbb{R}))$  is called a *transition function*. If the local trivializations can be chosen such that  $\tau_{\alpha\beta}$  are smooth, then  $E$  is a smooth vector bundle. Similarly, if  $\tau_{\alpha\beta}$  can be chosen to be locally constant functions, then  $E$  is a flat vector bundle.

Furthermore, suppose we are given an open cover  $(U_\alpha)_{\alpha \in A}$  of  $M$  together with a set of continuous (resp. smooth, resp. locally constant)  $\mathrm{GL}_k(\mathbb{R})$ -valued functions  $(\tau_{\alpha\beta})_{\substack{\alpha, \beta \in A \\ U_\alpha \cap U_\beta \neq \emptyset}}$  with  $\tau_{\alpha\alpha} = I$  on  $U_\alpha$ . Then there exists a continuous (resp. smooth, resp. flat) vector bundle  $E$  with transition functions  $\tau_{\alpha\beta}$ , provided the following *triple product property* holds:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p) = I$$

for any  $p \in U_\alpha \cap U_\beta \cap U_\gamma$ .

**Definition 1.5.** If  $E$  is a continuous (but not necessarily smooth) real vector bundle over  $M$  with transition functions  $\tau_{\alpha\beta}$ , the *orientation bundle* of  $E$  is a smooth flat line bundle  $\mathcal{O}(E)$  with transition functions

$$\sigma_{\alpha\beta}(p) = \mathrm{sgn} \det(\tau_{\alpha\beta}(p)) = \begin{cases} 1 & \det(\tau_{\alpha\beta}(p)) > 0 \\ -1 & \det(\tau_{\alpha\beta}(p)) < 0. \end{cases}$$

Recall that if  $f : M \rightarrow M$  is a map, we say  $f$  *lifts* to a bundle map  $F : E \rightarrow E$  if  $\pi \circ F = f \circ \pi$ .

Since  $\mathcal{O}(E)$  is a flat vector bundle, using the associated flat connection, we can lift the flow  $(\phi_t)_{t \in \mathbb{R}}$  to a flow  $(\tilde{\Phi}_t)_{t \in \mathbb{R}}$  on  $\mathcal{O}(E)$ . If the flow  $(\phi_t)_{t \in \mathbb{R}}$  on  $M$  lifts to a flow  $(\tilde{\Phi}_t)_{t \in \mathbb{R}}$  on  $E$ , if  $\psi, \eta$  are distinct trivializations of  $E$  near  $p$ ,  $\phi_t(p)$  respectively, and  $\tilde{\psi}, \tilde{\eta}$  are trivializations of  $\mathcal{O}(E)$  near  $p$ ,  $\phi_t(p)$  respectively, we have for  $p \in M$  and  $l \in \mathcal{O}(E)_p$ :

$$\tilde{\Phi}_t(l) = \tilde{\eta}^{-1} \left( \phi_t(p), \mathrm{sgn} \left( \det \left( \eta \Phi_t \psi^{-1} \right) \Big|_p \right) \mathrm{proj}_2 \tilde{\psi}(l) \right), \quad (2)$$

where  $\mathrm{proj}_2$  is the obvious projection to the second component.

**1.3. Geodesic flows.** Let  $Z$  be a negatively curved closed Riemannian manifold. Let  $M = S^*Z$  be the cosphere bundle on  $Z$ . It is classical that the geodesic flow on  $M$  is Anosov [1].

Let  $\pi : M \rightarrow Z$  be the canonical projection. For  $x \in M$ , we have a morphism of linear spaces

$$\pi_* : T_x M \rightarrow T_{\pi(x)} Z. \quad (3)$$

The following proposition is classical [1, Section 22] and [12, Proposition 6]. We include a proof for the sake of completeness.

**Proposition 1.5.** *The morphism  $\pi_*$  induces an isomorphism of continuous vector bundles on  $M$ ,*

$$E_s \oplus E_0 \simeq \pi^*(TZ). \quad (4)$$

*Proof.* Since both sides of (4) have the same dimension, it is enough to show that  $\pi_*|_{E_s \oplus E_0}$  is injective. We will show this using Jacobi fields. It is convenient to work on the sphere bundle  $M' = SZ$ . We identify  $M'$  with  $M$  via the Riemannian metric on  $Z$ .

We follow [7, Section II.H]. Let  $\mathcal{M}$  be the total space of  $TZ$ . Denote still by  $\pi : \mathcal{M} \rightarrow Z$  the obvious projection. Let  $T^V\mathcal{M} \subset T\mathcal{M}$  be the vertical subbundle of  $T\mathcal{M}$ . The Levi-Civita connection on  $TZ$  induces a horizontal subbundle  $T^H\mathcal{M} \subset T\mathcal{M}$  of  $T\mathcal{M}$ , so that

$$T\mathcal{M} = T^V\mathcal{M} \oplus T^H\mathcal{M}. \quad (5)$$

Since  $T^V\mathcal{M} \simeq \pi^*(TZ)$  and  $T^H\mathcal{M} \simeq \pi^*(TZ)$ , by (5), we can identify the smooth vector bundles,

$$T\mathcal{M} = \pi^*(TZ \oplus TZ).$$

For  $x = (z, v) \in \mathcal{M}$ , let  $\gamma_x$  be the unique geodesic on  $Z$  such that  $(\gamma_x(0), \dot{\gamma}_x(0)) = (z, v)$ . For  $w \in T_x\mathcal{M}$ , let  $J_{x,w} \in C^\infty(\gamma_x, TZ|_{\gamma_x})$  be the unique Jacobi field along  $\gamma_x$  such that  $(J_{x,w}(0), \dot{J}_{x,w}(0)) = w$ , where  $\dot{J}_{x,w}$  is the covariant derivation of  $J_{x,w}$  in the direction  $\dot{\gamma}_x$ . Recall that a Jacobi field  $J$  is called stable, if there is  $C > 0$  such that for all  $t \geq 0$ ,

$$|J(t)| \leq C.$$

By [7, Proposition VI.A], given  $x \in \mathcal{M}$ , for any  $Y_1 \in T_xZ$ , there exists one and only one stable Jacobi field  $J$  along  $\gamma_x$  such that  $J(0) = Y_1$ .

For  $x = (z, v) \in M'$ , we have

$$T_xM' = \{(Y_1, Y_2) \in T_zZ \oplus T_zZ : \langle Y_2, v \rangle = 0\}.$$

The morphism  $\pi_*$  in (3) is just

$$w \in T_xM' \rightarrow J_{x,w}(0) \in T_zZ.$$

By [7, Proposition VI.B],  $w \in E_s(x) \oplus E_0(x)$  if and only if the Jacobi fields  $J_{x,w}$  is stable. By the uniqueness of stable Jacobi fields, we see that  $\pi_*|_{E_s \oplus E_0}$  is injective.  $\square$

Since  $E_0$  is a trivial line bundle, our proposition implies immediately:

**Corollary 1.6.** *We have the isomorphism of smooth flat line bundles*

$$\mathcal{O}(E_s) \simeq \pi^*(\mathcal{O}(TZ)).$$

## 2. PROOF OF PROPOSITION 1.4

We use the notation of [5]. If  $0 \leq k \leq n-1$ , let  $\mathcal{E}_0^k \subset \Lambda^k(T^*M)$  denote the subbundle of  $k$ -forms  $\omega$  such that  $\iota_V \omega = 0$ , where  $\iota$  denotes interior multiplication.

Let  $\widetilde{\mathcal{E}}_0^k = \mathcal{E}_0^k \otimes \mathcal{O}(E_s)$ . We consider the pullback  $\phi_{-t}^*$  on sections of  $\widetilde{\mathcal{E}}_0^k$ . Note that the flow  $(\phi_t)_{t \in \mathbb{R}}$  lifts to a flow  $(\Phi_t)_{t \in \mathbb{R}}$  on  $\mathcal{E}_0^k$ . Indeed, for  $p \in M$ ,  $\omega \in \mathcal{E}_{0,p}^k$ ,  $\Phi_t \omega \in \mathcal{E}_{0,\phi_t(p)}^k$  is defined for  $v_1, \dots, v_k \in T_{\phi_t(p)}M$  by

$$\Phi_t \omega(v_1, \dots, v_k) = \omega((d\phi_t|_p)^{-1}v_1, \dots, (d\phi_t|_p)^{-1}v_k). \quad (6)$$

Note that from the above formula, it is easy to check that  $\iota_V \Phi_t \omega = 0$ . Recall also that the flow  $(\phi_t)_{t \in \mathbb{R}}$  lifts to a flow  $\widetilde{\Phi}_t$  on  $\mathcal{O}(E_s)$  (see (2)). For a section  $s$  in  $\widetilde{\mathcal{E}}_0^k$ , we have

$$\phi_{-t}^* s(p) = \left( \Phi_t \otimes \widetilde{\Phi}_t \right) (s(\phi_{-t}(p))). \quad (7)$$

Let  $\chi \in C^\infty(M)$  be a smooth function whose support is contained in a small neighborhood of  $K$  such that  $\chi(x) = 1$  for all  $x \in K$ . We now invoke the Guillemin trace formula (see [11, pp. 501-502], [5, Appendix B], [3, (4.6)]) which says that the flat trace  $\text{tr}^b \chi \phi_{-t}^* \chi|_{C^\infty(M; \widetilde{\mathcal{E}}_0^k)}$  is a distribution on  $(0, \infty)$  given by

$$\text{tr}^b \chi \phi_{-t}^* \chi|_{C^\infty(M; \widetilde{\mathcal{E}}_0^k)} = \sum_{\gamma \subset K} \frac{T_\gamma^\sharp \text{tr}^{\widetilde{\mathcal{E}}_{0,y}^k} \left( \Phi_{T_\gamma} \otimes \widetilde{\Phi}_{T_\gamma} \right)}{|\det(I - \mathcal{P}_\gamma)|} \delta_{t-T_\gamma}, \quad (8)$$

where the sum is taken over all the periodic trajectories  $\gamma$  in  $K$  with period  $T_\gamma$  and primitive period  $T_\gamma^\sharp$ ,  $y$  is any point on  $\gamma$ , and  $\mathcal{P}_\gamma = d\phi_{-T_\gamma}|_{(E_s \oplus E_u)_y}$  is the linearized Poincaré map at  $y$ . Note that as trace and determinant are invariant under conjugation, the right hand side does not depend on  $y$ .

By (6), the trace of  $\Phi_{T_\gamma}$  on  $\mathcal{E}_{0,y}^k$  is just  $\text{tr} \left( \bigwedge^k \mathcal{P}_\gamma \right)$ . By (2), we may take trivializations  $\psi, \widetilde{\psi}$  of  $E_s, \mathcal{O}(E_s)$  in a neighborhood of  $y$  and have the induced lifting on  $\mathcal{O}(E_s)$  to be  $\text{sgn} \left( \det \left( \psi d\phi_{T_\gamma}|_{E_{s,y}} \psi^{-1} \right) \right)$ . By definition we get this to be equal to

$$\text{sgn} \left( \det d\phi_{T_\gamma}|_{E_{s,y}} \right) = \text{sgn} \left( \det d\phi_{-T_\gamma}|_{E_{s,y}} \right) = \text{sgn} \det \left( \mathcal{P}_\gamma|_{E_s} \right),$$

and as it is a map between one dimensional spaces, the trace is given by that expression as well. By the above consideration, we can rewrite (8) as

$$\text{tr}^b \chi \phi_{-t}^* \chi|_{C^\infty(M; \widetilde{\mathcal{E}}_0^k)} = \sum_{\gamma \subset K} \frac{T_\gamma^\sharp \text{tr} \left( \bigwedge^k \mathcal{P}_\gamma \right) \text{sgn} \left( \det \mathcal{P}_\gamma|_{E_s} \right)}{|\det(I - \mathcal{P}_\gamma)|} \delta_{t-T_\gamma}. \quad (9)$$

Let us follow [4, Section 3]. By [4, Lemma 3.2], we may and we will assume that near  $K$ ,  $(\phi_t)_{t \in \mathbb{R}}$  is an open hyperbolic system in the sense of [3, Assumptions (A1)–(A4)].

By [3, Lemma 1.17], there is  $C > 0$  such that for all  $t \geq 0$ ,

$$|\{\gamma \text{ closed trajectory in } K : T_\gamma \leq t\}| \leq Ce^{Ct}. \quad (10)$$

For  $\text{Im}(\lambda) \gg 1$  big enough, set

$$\zeta_{K,k}(\lambda) = \exp \left( - \sum_{\gamma \subset K} \frac{T_\gamma^\sharp}{T_\gamma} \frac{\text{tr}(\wedge^k \mathcal{P}_\gamma) \text{sgn}(\det \mathcal{P}_\gamma|_{E_s})}{|\det(I - \mathcal{P}_\gamma)|} e^{i\lambda T_\gamma} \right). \quad (11)$$

**Lemma 2.1.** *For  $\text{Im}(\lambda) \gg 1$  big enough, we have*

$$\partial_\lambda \log \zeta_{K,k}(\lambda) = -i \int_0^\infty e^{i\lambda t} \text{tr}^b \chi \phi_{-t}^* \chi|_{C^\infty(M; \widetilde{\mathcal{E}}_0^k)} dt. \quad (12)$$

The function  $\zeta_{K,k}(\lambda)$  has a holomorphic extension to  $\mathbb{C}$ .

*Proof.* Let us first remark that by (9) and (10), the right hand side of (12) is well defined. Taking a logarithm and differentiating (11) and using Guillemin trace formula (9), we get (12). The last part of the lemma follows from the arguments of [3, Section 4].  $\square$

Recall that for  $\text{Im}(\lambda) \gg 1$  big enough, we have

$$\zeta_K(\lambda) = \prod_{\gamma^\sharp \subset K} (1 - e^{i\lambda T_\gamma^\sharp}) = \exp \left( - \sum_{\gamma \subset K} \frac{T_\gamma^\sharp}{T_\gamma} e^{i\lambda T_\gamma} \right). \quad (13)$$

Proposition 1.4 is a consequence of the following lemma. This lemma was stated in [2], but we restate and prove it for convenience.

**Lemma 2.2.** *The following identity of meromorphic functions on  $\mathbb{C}$  holds,*

$$\zeta_K(\lambda) = \prod_{k=0}^{n-1} (\zeta_{K,k}(\lambda))^{(-1)^{k+\dim E_s}}. \quad (14)$$

*Proof.* Following [5, (2.4)-(2.5)], since  $\det(I - \mathcal{P}_\gamma) = \sum_{k=0}^{n-1} (-1)^k \text{tr}(\wedge^k \mathcal{P}_\gamma)$ , by (11) and (13), it is enough to show

$$|\det(I - \mathcal{P}_\gamma)| = (-1)^{\dim E_s} \text{sgn}(\det \mathcal{P}_\gamma|_{E_s}) \det(I - \mathcal{P}_\gamma). \quad (15)$$

Remark that

$$\begin{aligned} \det(I - \mathcal{P}_\gamma) &= \det(I - \mathcal{P}_\gamma|_{E_u}) \det(I - \mathcal{P}_\gamma|_{E_s}) \\ &= (-1)^{\dim E_s} \det(I - \mathcal{P}_\gamma|_{E_u}) \det(I - \mathcal{P}_\gamma^{-1}|_{E_s}) \det(\mathcal{P}_\gamma|_{E_s}). \end{aligned} \quad (16)$$

As time is running in the negative direction, we have by (1) that the eigenvalues  $\lambda$  of  $\mathcal{P}_\gamma|_{E_u}$  have  $|\lambda| < 1$ , and the eigenvalues  $\mu$  of  $\mathcal{P}_\gamma^{-1}|_{E_s}$  have  $|\mu| < 1$ . This gives any



eigenvalues of  $I - \mathcal{P}_\gamma|_{E_u}$  to be either  $1 - \lambda$  for  $\lambda \in (-1, 1)$  or conjugate pairs  $1 - \lambda, 1 - \bar{\lambda}$  when  $\lambda$  is not real. In any case, we get by multiplying that

$$\det(I - \mathcal{P}_\gamma|_{E_u}) > 0$$

and similarly

$$\det(I - \mathcal{P}_\gamma^{-1}|_{E_s}) > 0.$$

Then taking signs in (16), we get (15).  $\square$

*Remark.* The key point of our argument is based on the smoothness of  $\mathcal{O}(E_s)$ . Thanks to this property, most of the analytic arguments in the proof of Proposition 1.4 are reduced to [3]. In [2, Section 2], Baladi and Tsujii used the orientation bundle in a different way for the flow with discrete time.

### 3. VANISHING ORDER AT ZERO ON A CONTACT 3-MANIFOLD

In this section, we assume that  $M$  is a connected closed 3-manifold with a contact form  $\alpha$ , and that  $V$  is the associated Reeb vector field. We suppose also that the flow  $(\phi_t)_{t \in \mathbb{R}}$  of  $V$  is Anosov. One such example would be when  $M = S^*\Sigma$ , the cosphere bundle of a connected closed surface  $\Sigma$  with negative (variable) curvature, and  $(\phi_t)_{t \in \mathbb{R}}$  is geodesic flow.

The following result was proven in [6]:

**Theorem 3.1.** *If  $(\phi_t)_{t \in \mathbb{R}}$  is a contact Anosov flow on a connected closed 3-manifold with orientable  $E_u$  and  $E_s$ , the Ruelle zeta function has vanishing order at  $\lambda = 0$  equal to  $b_1(M) - 2$ , where  $b_1(M)$  denotes the first Betti number of  $M$ .*

The goal of this section is to determine the order of vanishing of  $\zeta_R$  at 0 in the case that  $E_s, E_u$  are not orientable, and hence give a proof of Theorem 2. We remark that since a contact manifold is orientable, orientability of  $E_s$  is equivalent to orientability of  $E_u$ .

**3.1. The twisted cohomology.** Let us recall some background and general facts on the twisted cohomology of a flat vector bundle. Let  $X$  be a closed manifold. Let  $F$  be a flat vector bundle on  $X$  with flat connection  $\nabla$ . It induces a sheaf  $\mathcal{F}$  on  $X$  defined by locally constant sections, i.e., if  $U \subset X$  is an open set, then

$$\mathcal{F}(U) = \{s \in C^\infty(U; F|_U) : \nabla s = 0\}.$$

The twisted cohomology  $H^\bullet(X; F)$  is defined by the cohomology of the sheaf  $\mathcal{F}$  [10, Section II.4.4]. They are the algebraic invariants which describe the rigidity properties of the global flat sections of  $F$ . Let  $b_k(F)$  be the twisted Betti number

$$b_k(F) = \dim H^k(X; F).$$

If  $F$  is the trivial line bundle, we get the classical de Rham cohomology with real coefficients.

To evaluate  $H^\bullet(X; F)$ , one can use the twisted de Rham complex. Indeed, if we denote  $F^k = \Lambda^k(T^*X) \otimes F$ , the flat connection  $\nabla$  extends to an operator  $d_k : C^\infty(X; F^k) \rightarrow C^\infty(X; F^{k+1})$  by Leibniz rule: if  $\alpha \in C^\infty(X; \Lambda^k(T^*X))$  and  $s \in C^\infty(X; F)$ , we have

$$d_k(\alpha \cdot s) = d\alpha \cdot s + (-1)^k \alpha \wedge \nabla s.$$

By the flatness of  $\nabla$ , we have  $d_{k+1}d_k = 0$ , so that  $(C^\infty(X; F^\bullet), d_\bullet)$  is a complex. By the de Rham isomorphism [10, Théorème II.4.7.1], we have

$$H^k(X; F) = \ker d_k / \text{Im} d_{k-1}. \quad (17)$$

As an analogue of [6, Lemma 2.1], using the theory of elliptic operators, we can evaluate  $H^\bullet(X; F)$  using the complex of twisted currents, or more generally twisted currents with wavefront conditions.

More precisely, let  $\Gamma \subset T^*X$  be a closed cone. We denote by  $\mathcal{D}'_\Gamma(X; F^k)$  the space of  $F^k$ -valued distributions whose wavefront set is contained in  $\Gamma$  (see [6, Section 2.1]). By microlocality, we have

$$d_k : \mathcal{D}'_\Gamma(X; F^k) \rightarrow \mathcal{D}'_\Gamma(X; F^{k+1}).$$

For simplicity, we will write  $d$  sometimes.

**Lemma 3.2.** *If  $u \in \mathcal{D}'_\Gamma(X; F^k)$  and  $du \in C^\infty(X; F^{k+1})$ , then there exist  $v \in C^\infty(X; F^k)$  and  $w \in \mathcal{D}'_\Gamma(X; F^{k-1})$  such that*

$$u = v + dw.$$

*In particular, if  $u \in \mathcal{D}'_\Gamma(X; F)$  and  $du \in C^\infty(X; F^1)$ , then  $u \in C^\infty(X; F)$ .*

*Proof.* Take a Riemannian metric on  $X$  and a Hermitian metric on  $F$ . Remark that these two metrics induce a fibrewise scalar product  $\langle \cdot, \cdot \rangle$  on  $F^k$ . For  $u, v \in C^\infty(X; F^k)$ , we can define the  $L^2$ -product by

$$\langle u, v \rangle_{L^2(X; F^k)} = \int_X \langle u, v \rangle d\text{vol}, \quad (18)$$

where  $d\text{vol}$  is a volume form. Let  $\delta_{k+1} : C^\infty(X; F^{k+1}) \rightarrow C^\infty(X; F^k)$  be the formal adjoint of  $d$  with respect to the  $L^2$ -product (18). Define the twisted Hodge Laplacian by

$$\Delta_k = d_{k-1}\delta_k + \delta_{k+1}d_k : C^\infty(X; F^k) \rightarrow C^\infty(X; F^k).$$

Then  $\Delta_k$  is an essentially self-adjoint second order elliptic differential operator. The remainder of the proof carries over identically from that of [6, Lemma 2.1].  $\square$

Remark that if  $F$  is the orientation bundle of certain vector bundle and  $u \in C^\infty(X; F)$ , then for  $x \in X$ ,  $|u(x)|^2$  is independent of the choice of trivializations. It defines a Hermitian metric on  $F$ .

**3.2. Resonant State Spaces.** Let  $M$  be a connected 3-dimensional closed manifold with a contact form  $\alpha \in C^\infty(M, T^*M)$ . Let  $V$  be the associated Reeb vector field. Then,

$$i_V \alpha = 1, \quad \iota_V d\alpha = 0. \quad (19)$$

We assume that the flow  $(\phi_t)_{t \in \mathbb{R}}$  associated to  $V$  is Anosov. Let  $E_u^* \subset T^*M$  be the dual of  $E_s$ . We will apply the results of Section 3.1 to the case where  $(X, F, \Gamma) = (M, \mathcal{O}(E_s), E_u^*)$ .

Since the flow  $(\phi_t)_{t \in \mathbb{R}}$  is Anosov, we have  $K = M$ . For  $0 \leq k \leq 2$ , we write  $\zeta_k = \zeta_{K,k}$ . By (14), we have

$$\zeta_R(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)}. \quad (20)$$

We consider the operator  $P_k = -i\mathcal{L}_V$ , where  $\mathcal{L}_V$  (in a slight abuse of notation) denotes the natural action on sections of  $\widetilde{\mathcal{E}}_0^k$ , given by the Lie derivative on sections of  $\mathcal{E}_0^k$  tensored with the flat connection on  $\mathcal{O}(E_s)$ . For  $\text{Im}\lambda \gg 1$  large enough, the integral  $R_k(\lambda) = i \int_0^\infty e^{i\lambda t} \phi_{-t}^* dt$  converges and defines a bounded operator on the  $L^2$ -space; this is nothing more than the resolvent operator of  $P_k$ . Then by [6, Section 2.3] we have that  $R_k$  extends meromorphically to the entire complex plane,

$$R_k(\lambda) : C^\infty(M; \widetilde{\mathcal{E}}_0^k) \rightarrow \mathcal{D}'(M; \widetilde{\mathcal{E}}_0^k).$$

More precisely, near  $\lambda_0 \in \mathbb{C}$ , we have

$$R_k(\lambda) = R_{k,H}(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(P_k - \lambda)^{j-1} \Pi_k}{(\lambda - \lambda_0)^j}$$

where  $R_{k,H}$  is a holomorphic family defined near  $\lambda_0$ ,  $J(\lambda_0) \in \mathbb{N}$ , and  $\Pi_k$  has rank  $m_k(\lambda_0) < \infty$ . By the arguments at the end of [5], we have that at  $\lambda_0$ , the function  $\zeta_k$  has a zero of order  $m_k(\lambda_0)$ .

We define the space of *resonant states* at  $\lambda_0$  to be

$$\text{Res}_k(\lambda_0) = \left\{ u \in \mathcal{D}'_{E_u^*}(M; \widetilde{\mathcal{E}}_0^k) : (P_k - \lambda_0)u = 0 \right\}.$$

Then a special case of [6, Lemma 2.2] gives the following:

**Lemma 3.3.** *Suppose  $P_k$  satisfies the semisimplicity condition:*

$$u \in \mathcal{D}'_{E_u^*}(M; \widetilde{\mathcal{E}}_0^k), \quad (P_k - \lambda_0)^2 u = 0 \implies (P_k - \lambda_0)u = 0.$$

*Then  $m_k(\lambda_0) = \dim \text{Res}_k(\lambda_0)$ .*

Recall that we are trying to find the order at  $\lambda = 0$  of  $\zeta_R$ , which by (20) is simply

$$m_R(0) = m_1(0) - m_0(0) - m_2(0). \quad (21)$$

We will compute each of these individually, by computing  $\dim \text{Res}_k(0)$  and checking that the semisimplicity condition in Lemma 3.3 holds.

We begin with twisted ‘0-forms’, which are just sections of the orientation bundle  $\mathcal{O}(E_s)$ .

**Proposition 3.4.** *If  $E_s$  is nonorientable, the space  $\text{Res}_0(0)$  is  $\{0\}$ .*

*Proof.* Suppose  $u \in \text{Res}_0(0)$ , i.e.,

$$P_0 u = 0. \quad (22)$$

Since the flow  $(\phi_t)_{t \in \mathbb{R}}$  preserves the contact volume form  $\alpha \wedge d\alpha$ ,  $P_0 : C^\infty(M; \mathcal{O}(E_s)) \rightarrow C^\infty(M; \mathcal{O}(E_s))$  is a symmetric operator with respect to the  $L^2$ -product (18). By [6, Lemma 2.3],  $u \in C^\infty(M; \mathcal{O}(E_s))$ . Using  $\partial_t(\phi_{-t}^* u) = -\phi_{-t}^* \nabla_V u$  (where  $\nabla$  is the flat connection), we see that  $u$  is constant on the flow line: for all  $t \in \mathbb{R}$ ,

$$u = \phi_{-t}^* u. \quad (23)$$

Let  $(x, v) \in TM$ . The pairing  $\langle du(x), v \rangle$  is an element of  $\mathcal{O}(E_s)_x$ . By (7) and (23), we have

$$\langle du(x), v \rangle = \langle \phi_{-t}^*(du)(x), v \rangle = \tilde{\Phi}_t \langle du(\phi_{-t}(x)), d\phi_{-t}(x)v \rangle.$$

If  $v \in E_u(x)$ , then sending  $t \rightarrow \infty$  gives  $\langle du(x), v \rangle = 0$  by (1). Similarly, if  $v \in E_s(x)$ , then sending  $t \rightarrow -\infty$  gives  $\langle du(x), v \rangle = 0$ . This shows that

$$du|_{E_s \oplus E_u} = 0. \quad (24)$$

By Cartan’s formula and by (22), we have  $\iota_V du = 0$ , i.e.,

$$du|_{E_0} = 0. \quad (25)$$

By (24) and (25), we have  $du = 0$ . So  $u \in H^0(M; \mathcal{O}(E_s))$ . Since  $E_s$  is nonorientable, we have  $H^0(M; \mathcal{O}(E_s)) = 0$ , so  $u = 0$  and  $\text{Res}_0(0)$  is trivial.  $\square$

**Corollary 3.5.** *If  $E_s$  is nonorientable, the multiplicity for 0-forms is  $m_0(0) = 0$ .*

*Proof.* If  $P_0^2(u) = 0$ , then  $P_0 u \in \text{Res}_0(0)$ . By Proposition 3.4,  $P_0 u = 0$ , so  $u \in \text{Res}_0(0)$ . This shows semisimplicity, so by Lemma 3.3 we see that  $m_0(0) = \dim \text{Res}_0(0) = 0$ .  $\square$

**Proposition 3.6.** *If  $E_s$  is nonorientable, the space  $\text{Res}_2(0)$  is  $\{0\}$ .*

*Proof.* We claim that

$$\alpha \wedge : \mathcal{E}_0^2 \rightarrow \mathcal{E}^3 \quad (26)$$

is a bundle isomorphism. Indeed, using (19), it is easy to see that the inverse of (26) is given by  $\iota_V$ . Tensoring with  $\mathcal{O}(E_s)$ , we get a bundle isomorphism

$$\alpha \wedge : \widetilde{\mathcal{E}}_0^2 \rightarrow \widetilde{\mathcal{E}}^3. \quad (27)$$

Let  $u \in \text{Res}_2(0)$ . Since  $\mathcal{E}^3$  is generated by  $\alpha \wedge d\alpha$ , by (27), there is  $v \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$  such that  $\alpha \wedge u = v\alpha \wedge d\alpha$ . Applying  $\iota_V$  and using  $\iota_V u = 0$ , we have  $u = v d\alpha$ . Then

$$0 = P_2(u) = (P_0 v) d\alpha.$$

But this gives  $P_0 v = 0$ , so by Proposition 3.4 we have  $v = 0$ . Therefore,  $u = 0$ .  $\square$

The following is then clear for the same reason as Corollary 3.5.

**Corollary 3.7.** *If  $E_s$  is nonorientable, the multiplicity for 2-forms is  $m_2(0) = 0$ .*

We now turn to the case of  $P_1$  acting on the space of twisted 1-form-valued distributions  $\mathcal{D}'_{E_u^*}(M; \widetilde{\mathcal{E}}_0^1)$ . We can now state the analogous proposition for 1-forms:

**Proposition 3.8.** *If  $E_s$  is nonorientable, the space  $\text{Res}_1(0)$  has dimension  $b_1(\mathcal{O}(E_s))$ .*

*Proof.* The proof is analogous to that of [6, Lemma 3.4], but slightly easier due to the holomorphy of the resolvent  $R_0$  near 0. Let  $u \in \text{Res}_1(0)$ . Then  $du \in \text{Res}_2(0)$  by Proposition 3.6, so  $du = 0$ . By Lemma 3.2 there is a  $\phi \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$  such that

$$u - d\phi \in C^\infty(M; \widetilde{\mathcal{E}}^1), \quad d(u - d\phi) = 0.$$

We shall show that the map:

$$\Theta : u \mapsto [u - d\phi] \in H^1(M; \mathcal{O}(E_s))$$

is well-defined, linear and bijective, which is enough to prove the lemma.

**Well-Definedness and linearity:**

Suppose there is another section  $\psi \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$  with  $u - d\psi \in C^\infty(M; \widetilde{\mathcal{E}}^1)$ . Then subtracting gives  $d(\phi - \psi) \in C^\infty(M; \widetilde{\mathcal{E}}^1)$ , so  $\phi - \psi \in C^\infty(M; \mathcal{O}(E_s))$  by Lemma 3.2. This shows that the map  $\Theta$  is well-defined. It is also easy to see that  $\Theta$  is linear.

**Injectivity:**

If  $\Theta(u) = 0$ , then  $u - d\phi$  is exact, so without loss of generality we can assume that  $u = d\phi$ . Combining with  $\iota_V u = 0$ , we get  $\phi \in \text{Res}_0(0)$ , so  $\phi = 0$  by Proposition 3.4. Therefore  $u = 0$ , and this shows  $\Theta$  to be injective.

**Surjectivity:**

Let  $v \in C^\infty(M; \widetilde{\mathcal{E}}^1)$  with  $dv = 0$ . Then as  $m_0(0) = 0$ , the resolvent  $R_0$  is holomorphic near 0. Take  $\phi = iR_0(0)\iota_V v \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$ . Then  $P_0\phi = i\iota_V v$ . This

rearranges to  $\iota_V(v + d\phi) = 0$ , so  $v + d\phi \in \text{Res}_1(0)$ . This gives that  $\Theta$  is surjective, and completes the proof of our proposition.  $\square$

**Proposition 3.9.** *If  $E_s$  is nonorientable, the multiplicity for 1-forms is  $m_1(0) = b_1(\mathcal{O}(E_s))$ .*

*Proof.* By Lemma 3.3, we must only check that the semisimplicity condition is satisfied. Take  $u \in \mathcal{D}'_{E_u^*}(M; \widetilde{\mathcal{E}}_0^1)$  such that  $(P_1)^2 u = 0$ . Then  $v = \iota_V du \in \text{Res}_1(0)$ . It is enough to show that  $v = 0$ .

Recall that in the proof of Proposition 3.8, we have seen that elements in  $\text{Res}_1(0)$  are closed. In particular,

$$dv = 0. \quad (28)$$

Note that  $\alpha \wedge du \in \mathcal{D}'_{E_u^*}(M; \widetilde{\mathcal{E}}^3)$ . We claim that

$$\alpha \wedge du = 0. \quad (29)$$

Indeed, there is some  $a \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$  such that

$$\alpha \wedge du = a \alpha \wedge d\alpha.$$

Since  $\mathcal{L}_V(\alpha) = 0$ , by (28), we have

$$(\mathcal{L}_V a) \alpha \wedge d\alpha = \alpha \wedge \mathcal{L}_V(du) = \alpha \wedge d\iota_V du = \alpha \wedge dv = 0.$$

Then  $\mathcal{L}_V a = 0$ , so  $a = 0$  by Proposition 3.4. This gives (29).

Since  $\alpha(V) = 1$ , we have  $(\alpha \wedge) \circ \iota_V + \iota_V \circ (\alpha \wedge) = \text{id}$ . By (29), we have

$$du = ((\alpha \wedge) \circ \iota_V + \iota_V \circ (\alpha \wedge)) du = \alpha \wedge v. \quad (30)$$

By Lemma 3.2 and by (28), there are  $w \in C^\infty(M; \widetilde{\mathcal{E}}^1)$ ,  $\phi \in \mathcal{D}'_{E_u^*}(M; \mathcal{O}(E_s))$  such that

$$v = w + d\phi, \quad dw = 0. \quad (31)$$

Then

$$\iota_V w = \iota_V(v - d\phi) = -\mathcal{L}_V \phi. \quad (32)$$

In particular,  $\mathcal{L}_V \phi$  is smooth. We compute by Stokes' Theorem and by (30)-(32),

$$\begin{aligned} 0 &= \text{Re} \int_M du \wedge \bar{w} = \text{Re} \int_M \alpha \wedge d\phi \wedge \bar{w} = \text{Re} \int_M \phi \bar{w} \wedge d\alpha \\ &= \text{Re} \int_M \iota_V(\phi \bar{w}) \alpha \wedge d\alpha = -\text{Re} \int_M \phi(\overline{\mathcal{L}_V \phi}) \alpha \wedge d\alpha = -\text{Re} \langle \mathcal{L}_V \phi, \phi \rangle_{L^2(M; \mathcal{O}(E_s))}, \end{aligned}$$

where the fourth equality comes from the fact the

$$(\alpha \wedge) \circ \iota_V(\phi \bar{w} \wedge d\alpha) = ((\alpha \wedge) \circ \iota_V + \iota_V \circ (\alpha \wedge))(\phi \bar{w} \wedge d\alpha) = \phi \bar{w} \wedge d\alpha.$$

In the above formula, we use the fact that a product of two twisted forms is untwisted. By [6, Lemma 2.3] we have  $\phi \in C^\infty(M; \mathcal{O}(E_s))$ , so  $v \in C^\infty(M; \widetilde{\mathcal{E}}_0^1)$ . Then by the same argument as in Proposition 3.4 (see [6, Lemma 3.5]) we have  $v = 0$ .  $\square$

Now Theorem 2 is a consequence of (21), Corollaries 3.5, 3.7, and Proposition 3.9.

Let  $\Sigma$  be a connected negatively curved closed surface. Take  $M = S^*\Sigma$ . By Corollary 1.6, we have

$$H^1(M; \mathcal{O}(E_s)) = H^1(M; \pi^* \mathcal{O}(T\Sigma)).$$

**Proposition 3.10.** *If  $\Sigma$  is a connected negatively curved closed surface (oriented or not), we have*

$$\dim H^1(M; \pi^* \mathcal{O}(T\Sigma)) = \dim H^1(\Sigma). \quad (33)$$

*Proof.* By the Gysin long exact sequence, we have the exact sequence

$$0 \longrightarrow H^1(\Sigma; \mathcal{O}(T\Sigma)) \xrightarrow{\pi^*} H^1(M; \pi^* \mathcal{O}(T\Sigma)) \xrightarrow{\pi_*} H^0(\Sigma) \xrightarrow{e \wedge} H^2(\Sigma; \mathcal{O}(T\Sigma)) \longrightarrow,$$

where  $\pi^*$  is the pullback,  $\pi_*$  is the integration along the fibre of  $M \rightarrow \Sigma$ , and  $e \in H^2(\Sigma; \mathcal{O}(T\Sigma))$  is the Euler class of  $T\Sigma$ .

We claim that the last map

$$e \wedge : H^0(\Sigma) \rightarrow H^2(\Sigma; \mathcal{O}(T\Sigma))$$

in the Gysin exact sequence is an isomorphism. Indeed, since  $\Sigma$  is connected, we have  $\dim H^0(\Sigma) = 1$ , and by Poincaré duality,  $\dim H^2(\Sigma; \mathcal{O}(T\Sigma)) = 1$ . It is enough to show that  $e \in H^2(\Sigma; \mathcal{O}(T\Sigma))$  is non zero, or equivalently  $\int_\Sigma e \neq 0$ . This is a consequence of the fact that  $\Sigma$  has negative curvature, as  $e = K\mu$  where  $\mu$  is the Riemannian density and  $K < 0$  is the Gauss curvature.

Therefore, we get an isomorphism

$$\pi^* : H^1(\Sigma; \mathcal{O}(T\Sigma)) \simeq H^1(M; \pi^* \mathcal{O}(T\Sigma)). \quad (34)$$

By Poincaré duality, we have

$$H^1(\Sigma; \mathcal{O}(T\Sigma)) \simeq (H^1(\Sigma))^*. \quad (35)$$

By (34) and (35), we get (33).  $\square$

Now Corollary 3 is a consequence of Theorem 2 and Proposition 3.10.

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