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**APPENDIX TO “ON AN INTEGRAL MODEL FOR COLEMAN  
FAMILIES USING PSEUDO-REPRESENTATIONS AND IWASAWA  
INVARIANTS” BY JISHNU RAY**

FILIPPO A. E. NUCCIO MORTARINO MAJNO DI CAPRIGLIO AND TADASHI OCHIAI

In his celebrated papers [Col97a] and [Col97b], Coleman constructed  $p$ -adic families of elliptic cuspforms with fixed slope which are often called Coleman families. Coleman’s theory is a non-ordinary generalization of the theory of  $p$ -adic families of ordinary elliptic cuspforms due to Hida called Hida families. However, some of the properties of Coleman families are not completely parallel to those of Hida families. For example, Hida families are usually defined over the whole of the weight space but Coleman families are usually defined only locally in the weight space. In the original work of Coleman, he constructed his families over affinoid algebras corresponding to some affinoid spaces contained in the weight space. A Hida family is defined over an Iwasawa algebra or over an algebra which is finite over an Iwasawa algebra and these algebras are complete and semi-local. On the other hand, a Coleman family as constructed in [Col97b] is defined over an affinoid algebra. One advantage of the fact that the base ring of a Hida family is complete and semi-local is the possibility to apply to it Wiles’ method of pseudo-representations, which is a technique allowing us to attach a family of Galois representation to a Hida family. However, this method does not apply verbatim to attach a family of Galois representations to a Coleman family formulated as in the original work of Coleman since affinoid algebras are not complete nor semi-local.

Hence it is sometimes very important that the theory of Coleman family is formulated over a complete local algebra. Lack of reference for this, we give such an integral formal model for Coleman families of elliptic cuspforms. As an application, we attach a family of Galois representation by the method of pseudo-representation (see Theorem 3.3).

1. SOME BASICS ON AFFINOIDS AND WEIGHT SPACES

We refer the reader to [BGR84] for our conventions and basic results about rigid analytic spaces in the sense of Tate. Let  $K$  be a complete subfield of  $\mathbb{C}_p$ . The field  $K$  can be either a finite extension of  $\mathbb{Q}_p$  or an infinite extension of  $\mathbb{Q}_p$ . Let  $\mathcal{X}$  be an affinoid space defined over  $K$ . We write  $\mathcal{A}_{\mathcal{X}}$  for the ring of analytic functions on  $\mathcal{X}$  and  $\mathcal{A}_{\mathcal{X}}^0$  for the subring of power-bounded elements (see [BGR84, 1.2.5]). They will always be endowed with their Gauß semi-norm (which is a norm and coincides with the sup-norm if  $\mathcal{X}$  is reduced). When  $K$  is a discrete valuation field, the ring of power-bounded elements  $\mathcal{A}_{\mathcal{X}}^0$  is noetherian because it is a quotient of  $\mathcal{O}_K\langle T_1, \dots, T_n \rangle$  and  $\mathcal{O}_K\langle T_1, \dots, T_n \rangle$  is the  $p$ -adic completion of a polynomial algebra over  $\mathcal{O}_K$ . The ring  $\mathcal{A}_{\mathcal{X}} = \mathcal{A}_{\mathcal{X}}^0[\frac{1}{p}]$  is noetherian whether  $K$  is a discrete valuation field or not. For every maximal ideal  $\mathfrak{m} \subseteq \mathcal{A}_{\mathcal{X}}$ ,  $\mathcal{A}_{\mathcal{X}}/\mathfrak{m}$  is a finite extension of  $K$ . Whereas, for every non-zero prime ideal  $\mathfrak{p} \subseteq \mathcal{A}_{\mathcal{X}}^0$ , the quotient  $\mathcal{A}_{\mathcal{X}}^0/\mathfrak{p}$  is finite over  $\mathcal{O}_K$  (for these facts, see [BGR84, Cor. 3, 6.1.2/3 and Theorem 5.2.7/7]).

We consider the following definition:

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**Definition 1.1.** Let  $K$  be a complete subfield of  $\mathbb{C}_p$  and let  $\mathcal{X}$  be an affinoid space over  $K$ . A subset  $\mathcal{Z}$  of the set of  $K$ -valued points  $\mathcal{X}(K)$  is a Zariski-dense subset in  $\mathcal{X}$  if we have  $U(K) \cap \mathcal{Z} \neq \emptyset$  for every non-empty Zariski-open subspace  $U \subseteq \mathcal{X}$ .

Given  $x_0 \in K$  and  $r \in p^{\mathbb{Q}}$ , we denote by  $\mathcal{B}[x_0, r]_K$  and  $\mathcal{B}(x_0, r)_K$ , respectively, the closed and open ball of radius  $r$  and center  $x_0$ , seen as  $K$ -rigid analytic spaces (see [deJ95, Sect. 7] for a description of the second space). We note that we normalize the  $p$ -adic absolute value  $|\cdot|$  so that  $|p| = \frac{1}{p}$ . For example, in case  $r = 1$  and  $x_0 \in K$ ,  $\mathcal{A}_{\mathcal{B}[x_0, 1]_K}^0$  is isomorphic to the following ring of restricted power series with coefficients in  $\mathcal{O}_K$ :

$$\mathcal{O}_K\langle T - x_0 \rangle = \left\{ \sum_{i=0}^{\infty} c_i (T - x_0)^i \in \mathcal{O}_K[[T - x_0]] \mid \lim_{i \rightarrow \infty} |c_i| = 0 \right\}.$$

Finally, given any complete subfield  $L \subseteq \mathbb{C}_p$ , we also need the notation  $B[a, r]_L$  and  $B(a, r)_L$  for the set of all  $x \in L$  such that  $|x - a| \leq r$  (respectively, such that  $|x - a| < r$ ). When  $K = \mathbb{Q}_p$ , we denote  $B[x_0, r]_{\mathbb{Q}_p}$  (resp.  $B(x_0, r)_{\mathbb{Q}_p}$ ) by  $B[x_0, r]$  (resp.  $B(x_0, r)$ ) dropping the subscript.

**Lemma 1.2.** *Let  $K$  be a complete subfield of  $\mathbb{C}_p$  which is a discrete valuation field.*

- (1) *When  $\mathcal{X}$  is a reduced affinoid defined over  $K$  and  $f \in \mathcal{A}_{\mathcal{X}}$  vanishes on every point of a Zariski-dense subset  $Z$ , we have  $f = 0$ .*
- (2) *Let  $x_0 \in K$ . Every infinite set inside  $\mathcal{B}[x_0, 1]_K(K)$  is Zariski-dense.*

*Proof.* For the first assertion, suppose  $f \neq 0$  and consider the Zariski-open subset  $U_f = \{x \in \mathcal{X} \text{ such that } f(x) \neq 0\}$  of  $\mathcal{X}$ . Since  $\mathcal{X}$  is reduced and  $f \neq 0$ , we have  $U_f \neq \emptyset$ . By the assumption that  $Z \subset \mathcal{X}$  is a Zariski-dense subset, we have  $Z \cap U_f(K) \neq \emptyset$ . For any point  $z \in Z \cap U_f(K)$ , we have  $f(z) = 0$ , contradicting the definition of  $U_f$ .

We pass to the second assertion. By Weierstraß preparation Theorem ([BGR84, Theorem 5.2.2/1]), every function  $f \in \mathcal{A}_{\mathcal{B}[0, 1]_K}$  can be factored as  $f = P \cdot U$  where  $P \in K[T - x_0]$  is a polynomial and  $U \in \mathcal{A}_{\mathcal{B}[x_0, 1]_K}^{\times}$  is an invertible power series which does not vanish on  $\mathcal{B}[x_0, 1]_K$ . It follows that every such  $f$  has only finitely many zeroes and that  $\mathcal{A}_{\mathcal{B}[x_0, 1]_K}$  is a PID, showing that non-trivial Zariski-closed sets in  $\mathcal{B}[x_0, 1]_K$  consist of finitely many points.  $\square$

One of the main rigid spaces of interest for us is the weight space  $\mathcal{W}_N$ , which is isomorphic to  $\varphi(Np)$  copies of  $\mathcal{B}(1, 1)_K$  indexed by

$$\mathfrak{D} = \text{Hom}((\mathbb{Z}/Np\mathbb{Z})^{\times}, \mathbb{C}_p^{\times}).$$

For generalities about  $\mathcal{W}_N$ , we refer to [Col97b, Sect. B1], [CM98, Sect. 1.4]. For more detailed accounts, we refer to [Gou88, Chap. I.3, §4 and Appendix] and [Buz07, page 103]. By definition, the weight space satisfies

$$\mathcal{W}_N(\mathbb{C}_p) = \text{Hom}_{\text{cont}}(\varprojlim_n (\mathbb{Z}/Np^n\mathbb{Z})^{\times}, \mathbb{C}_p^{\times}).$$

Following Coleman and Mazur, we give the following definition:

**Definition 1.3.** We denote by  $\omega : \mu_p(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^{\times}$  the Teichmüller character and by  $\langle\langle \cdot \rangle\rangle : \mathbb{Z}_p^{\times} \rightarrow 1 + p\mathbb{Z}_p$  the projection  $x \mapsto x/\omega(x)$ . For every integer  $k$  and  $\chi \in \mathfrak{D}$  of finite order, the point  $\chi \langle\langle \cdot \rangle\rangle^k \in \mathcal{W}_N(\mathbb{Q}_p)$  is called an accessible weight-character with coordinates  $(\chi, k)$ .

As detailed in [CM98, Definition, Section 1.4] the accessible weight-characters are parametrized by the rigid analytic subspace  $\mathcal{W}_N^* = \mathfrak{D} \times \mathcal{B}^* \subseteq \mathcal{W}_N$  where

$$(1.1) \quad \mathcal{B}^* \cong \mathcal{B}(0, p^{\frac{p-2}{p-1}})$$

is the subdisk of  $\mathcal{B}(1, 1)$  which is the image of  $\mathcal{B}(0, p^{\frac{p-2}{p-1}})$  via the map  $s \mapsto (1+p)^s$ . In the notation introduced in Definition 1.3, the character  $\chi \langle \langle \rangle \rangle^k$  is represented by the point  $(\chi, (1+p)^k) \in \mathcal{W}_N(\mathbb{Q}_p)$  which gets mapped to  $(\chi, k)$  by the identification in (1.1); we see that the word “coordinates” comes from seeing  $\mathcal{W}_N^*$  as  $\mathfrak{D}$ -copies of  $\mathcal{B}(0, p^{\frac{p-2}{p-1}})$ . From now on we systematically write points in the weight space through their coordinates. In particular, for every fixed  $\chi \in \mathfrak{D}$ , the weights of characters with Nebentypus  $\chi$  will be points in  $\mathcal{B}(0, p^{\frac{p-2}{p-1}})$  rather than in  $\mathcal{B}(1, 1)$ .

The assumption  $(N, p) = 1$  allows us to look at the group  $\text{Hom}((\mathbb{Z}/p)^\times, \mathbb{C}_p^\times)$  as a subgroup of  $\mathfrak{D}$ . It thus makes sense, for each  $0 \leq j \leq p-2$ , to interpret  $\omega^j$  as an element of  $\mathfrak{D}$ . The characters  $x \mapsto x^k$ , which are accessible with coordinates  $(1, k)$ , are then the elements of  $\mathcal{W}_N(\mathbb{Q}_p)$  which belong to the  $\omega^k$ -th copy of  $\mathcal{B}^*$  and we call them *integral weights* (of trivial Nebentypus).

## 2. $p$ -ADIC FAMILIES OF MODULAR FORMS

We start with a classical eigencuspform  $f \in S_{k_0}(\Gamma_1(Np), \varepsilon)$  of weight  $k_0$ , level  $Np$  and Nebentypus  $\varepsilon$ , which we factor as product  $\varepsilon = \varepsilon_N \omega^{k_0-i}$  for some character  $\varepsilon_N$  of conductor divisible by  $N$  and some  $0 \leq i \leq p-1$ . Our main reference concerning  $p$ -adic modular forms and  $p$ -adic families thereof is [Col97b, Part B] as well as [Gou88], in particular Section II.3 for the definition of the  $U_p$ -operator.

**Definition 2.1.** Let  $f$  be a  $p$ -adic modular form of level  $Np$  which is an eigenvector with respect to the  $U_p$ -operator. We define the slope of  $f$  to be the  $p$ -adic valuation of the  $U_p$ -eigenvalue of  $f$ . It is a non-negative rational number.

We assume that the slope of  $f$  is  $0 \leq \alpha \leq k_0 - 1$ . In [Col97b], [Col97a] and [CM98] Coleman and Coleman–Mazur have built a theory of families of  $p$ -adic modular forms of slope  $\alpha$  interpolating  $f$ , a part of which is stated in Theorem 2.2 below and will be crucial for us in this paper. Let us introduce some notation to state Theorem 2.2. Given an element  $k_0 \in \mathbb{Q}_p$ , an integer  $i$  and  $r < p^{\frac{p-2}{p-1}} \in p^\mathbb{Q}$ , we denote by  $\mathcal{X}_{(k_0, i)}[r]$  the affinoid subspace of the weight space

$$\mathcal{X}_{(k_0, i)}[r_0] := \{\varepsilon_N \omega^i\} \times \mathcal{B}[k_0, r] \subseteq \mathcal{W}_N^*.$$

Let us consider a finite extension  $\mathcal{K}$  of  $\mathbb{Q}_p$ , which plays a role of the field of coefficients of motives associated to cuspforms  $f$  in the given Coleman family. We note that  $\mathcal{K}$  has nothing to do with the field of definition  $K$ , for which we can choose  $K$  to be  $\mathbb{Q}_p$  which corresponds to the field of definition of motives associated to cuspforms  $f$  in the given Coleman family. From now on, we denote by  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}[r]/\mathcal{K}}^0$  (resp.  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}[r]/\mathcal{K}}$ ) the extension of coefficients  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}[r]}^0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{K}}$  (resp.  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}[r]} \otimes_{\mathbb{Q}_p} \mathcal{K}$ ) where  $\mathcal{O}_{\mathcal{K}}$  is the ring of integers of  $\mathcal{K}$ . We have the following result thanks to [Col97b]:

**Theorem 2.2** ([Col97b]). *Suppose that  $f$  is a classical normalized cuspidal eigenform, of weight  $k_0$ , level  $\Gamma_1(Np)$ , slope  $\alpha < k_0 - 1$ , Nebentypus  $\varepsilon = \varepsilon_N \omega^{i-k_0}$  and which is new away from  $p$ . In the case  $i = 0$ , suppose moreover that  $a^2 \neq \varepsilon_N(p)p^{k_0-1}$ , where  $a$  is the  $U_p$ -eigenvalue of  $f$ .*

Then, there is a radius  $r_0 < p^{\frac{p-2}{p-1}}$  lying in  $p^{\mathbb{Q}}$  and rigid analytic functions  $a_n$  on  $\mathcal{A}_{\mathcal{X}_{(k_0,i)}[r_0]/\mathcal{K}}^0$  with some field of coefficient  $\mathcal{K}$  for every natural number  $n$  such that the following statements hold:

- (1) For every integer  $k \in \mathcal{X}_{(k_0,i)}[r_0](\mathbb{Q}_p)$  satisfying  $k > \alpha + 1$  the series  $\sum_{n=1}^{\infty} a_n(k)q^n \in \mathcal{K}[[q]]$  coincides with the  $q$ -expansion of a classical normalized cuspidal eigenform of level  $Np$ , weight  $k$ , slope  $\alpha$  and character  $\varepsilon_N \omega^{i-k}$ .
- (2) The series  $\sum_{n=1}^{\infty} a_n(k_0)q^n \in \mathcal{K}[[q]]$  coincides with the  $q$ -expansion of  $f$  at  $k = k_0$ .
- (3) The space  $\mathcal{X}_{(k_0,i)}[r_0]$  is  $a_p$ -small in the sense of [Kis03, (5.2)].

*Remark 2.3.* At first glance, it might look better to write  $a_n^{(r_0)}$  for the analytic functions appearing in the statement, since the radius  $r_0$  is not uniquely associated to  $f$  and these functions might *a priori* depend on its choice. However, we will prove in Corollary 2.5 below that this is not necessary.

*Proof.* In [Col97b, pp. 465–467] (and especially along the proof of Corollary B5.7.1 *ibidem*), Coleman attaches to the above  $f$  the space  $\mathcal{X}_{(k_0,i)}[r_0]$  with a radius  $r_0 \in p^{\mathbb{Q}}$  small enough. In [Col97b, pp. 465–467], this space is denoted simply by  $B$  and a crucial step is to take a finite étale affinoid algebra  $R_{(k_0,i)}[r_0]$  over  $\mathcal{A}_{\mathcal{X}_{(k_0,i)}[r_0]}$  whose associated affinoid space parametrizes families of  $p'$ -new forms of slope  $\alpha$ . As in the paper [Col97b], we write  $X(R_{(k_0,i)}[r_0]) \rightarrow \mathcal{X}_{(k_0,i)}[r_0]$  for the affinoid space associated to  $R_{(k_0,i)}[r_0]$ .

Then the statement is Corollary B5.7.1 in [Col97b] *verbatim*, except for the condition that all  $a_n$  be power-bounded and that all forms above be normalized. In Lemma B5.3 *ibidem* it is shown that the Hecke eigenvalues of an overconvergent cuspidal eigenform are bounded by 1 if it is normalized. We thus get the result by observing that  $a_1 = 1$ , which follows from  $a_1 = T(1) = 1$  by the construction given in Theorem B5.7 *ibidem*. As discussed in [Kis03, §5.2] it is always possible to shrink a disk around a point in order to get a smaller one which is  $a_p$ -small, and this radius we call  $r_0$ .  $\square$

**Definition 2.4.** Let  $f$  be a form as in Theorem 2.2 and let  $0 < r \leq r_0$  be smaller than or equal to the radius constructed there. We refer to the rigid functions  $\{a_n\}_{n \in \mathbb{N}}$  on  $\mathcal{X}_{(k_0,i)}[r]$  as a Coleman family of slope  $\alpha$  and radius  $r$  passing through the form  $f$ . We refer to the formal power series

$$\sum_{n=1}^{\infty} a_n q^n \in \mathcal{A}_{\mathcal{X}_{(k_0,i)}[r]/\mathcal{K}}^0[[q]]$$

as the Fourier expansion of the Coleman family. For each  $x \in \mathcal{X}_{(k_0,i)}[r]$ , we denote by  $f_x$  the overconvergent modular form whose expansion is  $\sum a_n(x)q^n$ . We also refer to the collection of all these forms as the Coleman family of slope  $\alpha$  through the form  $f$ .

We consider the subset of accessible weight-characters in  $\mathcal{X}_{(k_0,i)}[r]$  as follows:

$$(2.1) \quad \mathcal{Z}_{(k_0,i)}[r] = \left( \{\varepsilon_N \omega^i\} \times \mathbb{Z}_{>\alpha+1} \right) \cap \mathcal{X}_{(k_0,i)}[r] \subseteq \mathcal{W}_N^*(\mathbb{Q}_p).$$

By abuse of notation, we write  $k$  for elements  $(\varepsilon_N \omega^i, k)$  of  $\mathcal{Z}_{(k_0,i)}[r]$  and we denote the  $p$ -adic Deligne representation attached to  $f_k$  in [Del68] by  $\rho_k$ .

Consider a form  $f$  satisfying the assumption of Theorem 2.2 and a radius  $r_0$  as constructed there.

**Corollary 2.5.** *Recall that  $k_0 \in \mathbb{Z}$  is the weight of the classical form  $f_0$  which is the base of our given Coleman family. Let  $r_0$  be a radius as constructed in Theorem 2.2 and let  $r < r_0$  be a smaller radius lying in  $p^{\mathbb{Q}}$ . Then the functions*

$$\operatorname{res}_{\mathcal{X}_{(k_0,i)}[r]}^{\mathcal{X}_{(k_0,i)}[r_0]}(a_n) \in \mathcal{A}_{\mathcal{X}_{(k_0,i)}[r]/\mathcal{K}}^0 \quad \text{for } n \geq 1$$

are the Fourier expansion of a Coleman family of slope  $\alpha$  and radius  $r$  passing through the form  $f$ .

*Proof.* Follows from the definitions, since for every  $k \in \mathcal{Z}_{(k_0,i)}[r](\mathbb{Q}_p)$  the series

$$\sum_{n=1}^{\infty} \operatorname{res}_{\mathcal{X}_{(k_0,i)}[r]}^{\mathcal{X}_{(k_0,i)}[r_0]}(a_n)(k)q^n$$

is the  $q$ -expansion of a form of the required type.  $\square$

Thanks to the above corollary, we can unambiguously speak about its Fourier coefficients  $a_n$  without referring to the radius; observe also that if shrink an  $a_p$ -small disk to another one of smaller radius gives maintains  $a_p$ -smallness. We recall the following lemma:

**Lemma 2.6.** *Recall that  $k_0 \in \mathbb{Z}$  is the weight of the classical form  $f_0$  which is the base of our given Coleman family. Let  $r \in p^{\mathbb{Q}}$  be a radius satisfying  $r \leq r_0$  and  $\mathcal{K}$  a finite extension of  $\mathbb{Q}_p$ . If the function  $G \in \mathcal{A}_{\mathcal{X}_{(k_0,i)}[r]/\mathcal{K}}$  vanishes on  $\mathcal{Z}_{(k_0,i)}[r_0]$ , it is everywhere zero.*

*Proof.* The subset  $\mathcal{Z}_{(k_0,i)}[r] \subseteq \mathcal{X}_{(k_0,i)}[r](\mathbb{Q}_p)$  is Zariski-dense in  $\mathcal{X}_{(k_0,i)}[r]$  thanks to the second assertion of Lemma 1.2. Hence, the assertion is an immediate consequence of the first assertion of Lemma 1.2.  $\square$

### 3. $p$ -ADIC FAMILY OF GALOIS REPRESENTATIONS

The main result of this section is Theorem 3.3 which produces an *integral* Galois representation with values in  $\mathcal{A}_{\mathcal{X}/\mathcal{K}}^0$  – for a given Coleman family and a suitable affinoid  $\mathcal{X}$  – that specializes to the Deligne representations attached to the classical eigenforms which belong to the given Coleman family. The most important ingredient is to construct pseudo-representations associated to a given Coleman family: once we have a pseudo-representation over our affinoid algebras, it is a standard argument to recover Galois representations from this pseudo-representation. Since we do not find a standard reference to construct pseudo-representations over affinoid algebras, we construct them by using a formal structure of Coleman families.

Let us fix a Coleman family as in Theorem 2.2 and let  $r_0 \in p^{\mathbb{Q}}$  be the radius of this Coleman family which appeared there. Let  $K$  be a complete subfield of  $\mathbb{C}_p$  and take an element  $e_0 \in K$  such that  $r_0 = |e_0|$ . Let us define a complete local ring  $\mathbf{A}_K^0$  to be

$$(3.1) \quad \mathbf{A}_K^0 = \mathcal{O}_K \left[ \left[ \frac{T - k_0}{e_0} \right] \right] = \left\{ \sum_{i=0}^{\infty} c_i \left( \frac{T - k_0}{e_0} \right)^i \mid c_i \in \mathcal{O}_K \right\}.$$

For any  $r \in p^{\mathbb{Q}}$  with  $r < r_0$ , power series in  $\mathbf{A}_K^0$  converge on  $B[k_0, r]_K$  which is strictly contained in  $B(k_0, r_0)_K$  and are there bounded by 1. We consider them as functions on  $\mathcal{X}_{(k_0,i)}[r]$ .

By restriction, this induces a ring homomorphism  $\mathbf{A}_K^0 \xrightarrow{\operatorname{res}} \mathcal{A}_{\mathcal{X}_{(k_0,i)}[r]_K}^0$  for any  $r \in p^{\mathbb{Q}}$  with  $r < r_0$ . For each such radius  $r$ , we choose  $e_r \in \mathbb{C}_p$  such that  $|e_r| = r$ . By [BGR84,

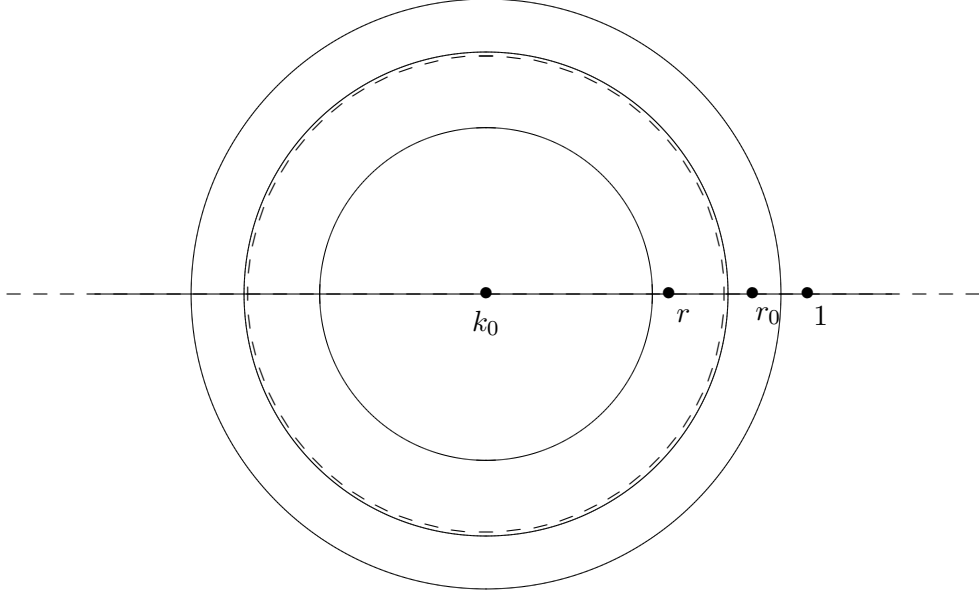


FIGURE 1.

§6.1.5], there is an isomorphism

$$\mathcal{A}_{\mathcal{X}(k_0, i)[r]_{\mathbb{C}_p}}^0 \cong \left\{ \sum_{i=0}^{\infty} c_i \left( \frac{T - k_0}{e_r} \right)^i \in \mathcal{O}_{\mathbb{C}_p} \left[ \left[ \frac{T - k_0}{e_r} \right] \right] \mid \lim_{i \rightarrow \infty} |c_i| = 0 \right\}.$$

Similarly as above, the inclusion  $\{\varepsilon_N \omega^i\} \times \mathcal{B}(k_0, r_0)_K \subset \mathcal{X}(k_0, i)[r_0]_K$  induces a homomorphism  $\mathcal{A}_{\mathcal{X}(k_0, i)[r_0]_K}^0 \rightarrow \mathbf{A}_K^0$  where we have a presentation

$$\mathcal{A}_{\mathcal{X}(k_0, i)[r_0]_K}^0 \cong \left\{ \sum_{i=0}^{\infty} c_i \left( \frac{T - k_0}{e_0} \right)^i \in \mathcal{O}_K \left[ \left[ \frac{T - k_0}{e_0} \right] \right] \mid \lim_{i \rightarrow \infty} |c_i| = 0 \right\}.$$

We have:

$$(3.2) \quad \mathcal{A}_{\mathcal{X}(k_0, i)[r_0]_K}^0 \rightarrow \mathbf{A}_K^0 \xrightarrow{\text{res}} \mathcal{A}_{\mathcal{X}(k_0, i)[r]_{\mathbb{C}_p}}^0$$

which correspond to the inclusions

$$\mathcal{B}[k_0, r]_{\mathbb{C}_p} \subset \mathcal{B}(k_0, r_0)_K \subset \mathcal{B}[k_0, r_0]_K.$$

In Figure 1, there is a sketch of the radii that occurred so far in our construction: When the radius  $r \in p^{\mathbb{Q}}$  with  $r < r_0$  tends to  $r_0$ , we have

$$\mathbf{A}_K^0 \subset \mathbf{A}_{\mathbb{C}_p}^0 = \varprojlim_{r \rightarrow r_0} \mathcal{A}_{\mathcal{X}(k_0, i)[r]_{\mathbb{C}_p}}^0 = \bigcap_{r < r_0} \mathcal{A}_{\mathcal{X}(k_0, i)[r]_{\mathbb{C}_p}}^0.$$

Here  $\mathbf{A}_K^0$  is characterized to be elements in  $\mathbf{A}_{\mathbb{C}_p}^0$  which takes values in  $K$  at a dense subset in  $\mathcal{B}[k_0, r_0]_K$ . From now on, when there is no fear of confusion, we denote  $\mathbf{A}_{\mathbb{Q}_p}^0$  by  $\mathbf{A}^0$  when  $K = \mathbb{Q}_p$  dropping the subscript  $K$ . For a finite extension  $\mathcal{K}$  of  $\mathbb{Q}_p$ , we denote  $\mathbf{A}^0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{K}}$  by  $\mathbf{A}_{\mathcal{K}}^0$ . As stated previously,  $\mathcal{K}$  plays a role of the field of coefficients and  $\mathcal{K}$  has nothing to do with the field of definition  $K$ . By the above observation, we have the following proposition:

**Proposition 3.1.** *Let us take the same assumptions and notations of Theorem 2.2. Then, there exists a finite extension  $\mathcal{K}$  of  $\mathbb{Q}_p$  such that, for every natural number  $n$ , there is a*

unique function  $A_n \in \mathbf{A}^0_{/\mathcal{K}}$  such that the image of  $A_n$  via the natural inclusion  $\mathbf{A}^0_{/\mathcal{K}} \hookrightarrow \mathcal{A}^0_{\mathcal{X}_{(k_0,i)}[r]/\mathcal{K}}$  coincides with  $a_n$  obtained in Theorem 2.2 on a smaller radius  $r$ .

We remark that  $\mathbf{A}^0_{/\mathcal{K}}$  in Proposition 3.1 is isomorphic to the following ring:

$$\mathbf{A}^0_{/\mathcal{K}} \cong \mathcal{O}_{\mathcal{K}} \left[ \left[ \frac{T - k_0}{e_0} \right] \right] = \left\{ \sum_{i=0}^{\infty} c_i \left( \frac{T - k_0}{e_0} \right)^i \mid c_i \in \mathcal{O}_{\mathcal{K}} \right\},$$

where  $e_0 \in \mathbb{Q}_p$  is an element with  $|e_0| = r_0$ .

*Proof.* In Theorem 2.2, we obtained a Coleman family over  $\mathcal{X}_{(k_0,i)}[r_0]$ . By shrinking such a Coleman family to  $\mathcal{X}_{(k_0,i)}[r]$  for any  $r \in p^{\mathbb{Q}}$  with  $r < r_0$  and by taking the limit  $r \rightarrow r_0$ , we showed that there exists  $A_n \in \mathbf{A}^0_{/\mathcal{K}}$  such that the image of  $A_n$  via the natural inclusion  $\mathbf{A}^0_{/\mathcal{K}} \hookrightarrow \mathcal{A}^0_{\mathcal{X}_{(k_0,i)}[r]/\mathcal{K}}$  coincides with  $a_n$  obtained in Theorem 2.2. This completes the proof of the proposition.  $\square$

Based on the existence of the formal structure  $\mathbf{A}^0_{/\mathcal{K}}$ , we introduce the notion of 2-dimensional pseudo-representation based on [Wil88]. We remark that other types of pseudo-representations were later introduced by Taylor in [Tay91] and, much more recently, by Chenevier in [Che11] but for our purposes Wiles’ approach seems to be the best suited. Given a topological group  $G$  and a topological ring  $R$  in which 2 is invertible we say that a triple  $\pi = (A, D, \Xi)$  of continuous functions

$$A, D : G \rightarrow R \quad \Xi : G \times G \rightarrow R$$

satisfying properties (I)–(IV) in [Wil88, Lemma 2.2.3], is a pseudo-representation.

Given a continuous representation  $\rho : G \rightarrow \mathrm{GL}_2(R)$ , by fixing a basis of  $R^2$  we can write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \quad \text{for } \sigma \in G$$

and the triple  $\pi_{\rho} = (A(\sigma) = a(\sigma), D(\sigma) = d(\sigma), \Xi(\sigma, \tau) = b(\sigma)c(\tau))$  is easily checked to be a pseudo-representation; observe that attaching a pseudo-representation to a representation depends on the choice of a basis. Extending the notation introduced in Definition 2.4, we simply denote by  $\pi_k$  the pseudo-representation  $\pi_{\rho_k}$  attached to  $\rho_k$  in some chosen basis.

Let  $Z_{(k_0,i)}[r_0]$  be the set of points of the formal rigid space which is defined to be

$$(3.3) \quad Z_{(k_0,i)}[r_0] = \mathcal{B}(k_0, r_0) \cap \mathcal{Z}_{(k_0,i)}[r_0]$$

where  $\mathcal{Z}_{(k_0,i)}[r_0]$  is the set defined at (2.1). We apply this discussion to the ring  $R = \mathbf{A}^0_{/\mathcal{K}}$  defined in (3.1) to find:

**Proposition 3.2.** *Let  $S$  be the finite set of primes of  $\mathbb{Q}$  consisting of the primes  $\{\ell : \ell \mid N\}$ ,  $p$  and  $\infty$ ,  $G_{\mathbb{Q},S}$  the Galois group of the maximal extension unramified outside  $S$  over  $\mathbb{Q}$ . Then, there exists a continuous pseudo-representation*

$$\pi = (A, D, \Xi) : G_{\mathbb{Q},S} \rightarrow \mathbf{A}^0_{/\mathcal{K}}$$

interpolating the pseudo-representations  $\pi_k$  attached to members of the Coleman family of slope  $\alpha$  through  $f$ . In other words, for each  $k \in Z_{(k_0,i)}[r_0]$ , the evaluation  $\mathrm{ev}_k \circ \pi = (\mathrm{ev}_k \circ A, \mathrm{ev}_k \circ D, \mathrm{ev}_k \circ \Xi)$  coincides with the pseudo representation  $\pi_k$ .



*Proof.* The argument of the pseudo representation of rank two à la Wiles is more or less standard and the proof goes in a quite parallel manner as the proof given in the text book [Hid93, §7.5]. So we only give an outline.

Let us choose a complex conjugation  $c \in G_{\mathbb{Q},S}$ . For each  $k \in Z_{(k_0,i)}[r_0]$ , we fix a a basis for the representation  $\rho_k$  so that  $c$  is represented by the matrix  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . Following [Wil88, Lemma 2.3.3], we define functions

$$A_{(k)}, D_{(k)} : G_{\mathbb{Q},S} \longrightarrow \mathcal{K}$$

by the matrix representation

$$\rho_k(g) = \begin{pmatrix} A_{(k)}(g) & B_{(k)}(g) \\ C_{(k)}(g) & D_{(k)}(g) \end{pmatrix}.$$

We also set  $\Xi_{(k)} : G_{\mathbb{Q},S} \times G_{\mathbb{Q},S} \longrightarrow \mathcal{K}$  by  $\Xi_{(k)}(\gamma_1, \gamma_2) = A_{(k)}(\gamma_1\gamma_2) - A_{(k)}(\gamma_1)A_{(k)}(\gamma_2)$ . The functions  $A_{(k)}, D_{(k)}, \Xi_k$  are continuous for every  $k \in Z_{(k_0,i)}[r_0]$  thanks to continuity of the Deligne representation  $\rho_k$ .

For any  $k \in Z_{(k_0,i)}[r_0]$ , we denote by  $P_k \subset \mathbf{A}_{/\mathcal{K}}^0$  the kernel of the evaluation map  $\mathbf{A}_{/\mathcal{K}}^0 \longrightarrow \mathcal{K}$  at  $k$ . Let us denote the function  $A_{(k)} + D_{(k)}$  by  $\text{Tr}_{P_k}$ . Since  $Z_{(k_0,i)}[r_0]$  is a countable set, we give a numbering

$$Z_{(k_0,i)}[r_0] = \{k_1, k_2, \dots, k_s, \dots\}.$$

For  $k_1, k_2 \in Z_{(k_0,i)}[r_0]$ , we consider the map

$$(3.4) \quad \mathbf{A}_{/\mathcal{K}}^0/P_{k_1} \oplus \mathbf{A}_{/\mathcal{K}}^0/P_{k_2} \longrightarrow \mathbf{A}_{/\mathcal{K}}^0/(P_{k_1} + P_{k_2}), \quad (x, y) \mapsto (x \bmod P_{k_2}) - (y \bmod P_{k_1})$$

whose kernel is isomorphic to  $\mathbf{A}_{/\mathcal{K}}^0/(P_{k_1} \cap P_{k_2})$ . Let  $\ell$  be a prime number outside  $S$ . We have  $\text{Tr}_{P_k}(\text{Frob}_\ell) = a_\ell(k)$  for every  $k \in Z_{(k_0,i)}[r_0]$  and the Fourier coefficients  $a_\ell(k_i)$  glue together when  $k_i$  varies. Hence the values  $\text{Tr}_{P_k}(\text{Frob}_\ell)$  glue together when  $k$  varies, which is true for any prime number  $\ell$  outside  $S$ . By the sequence (3.4), we have a continuous function  $\text{Tr}_{P_{k_1} \cap P_{k_2}} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}_{/\mathcal{K}}^0/(P_{k_1} \cap P_{k_2})$  whose value  $\text{Tr}_{P_{k_1} \cap P_{k_2}}(\text{Frob}_\ell)$  is congruent to  $\text{Tr}_{P_{k_1}}(\text{Frob}_\ell)$  (resp.  $\text{Tr}_{P_{k_2}}(\text{Frob}_\ell)$ ) mod  $P_{k_1}$  (resp. mod  $P_{k_2}$ ) for every prime number  $\ell$  outside  $S$ . Since the set of Frobenius elements is dense in  $G_{\mathbb{Q},S}$ , value  $\text{Tr}_{P_{k_1} \cap P_{k_2}}(\sigma)$  is congruent to  $\text{Tr}_{P_{k_1}}(\sigma)$  (resp.  $\text{Tr}_{P_{k_2}}(\sigma)$ ) mod  $P_{k_1}$  (resp. mod  $P_{k_2}$ ) for every  $\sigma \in G_{\mathbb{Q},S}$ .

By inductive argument, for each natural number  $s$ , we have a continuous function

$$\text{Tr}_{P_{k_1} \cap P_{k_2} \cap \dots \cap P_{k_s}} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}_{/\mathcal{K}}^0/(P_{k_1} \cap P_{k_2} \cap \dots \cap P_{k_s})$$

such that we recover the function  $\text{Tr}_{P_{k_i}}(\sigma) \bmod P_{k_i}$  for  $i = 1, 2, \dots, s$ . Note that we have  $\mathbf{A}_{/\mathcal{K}}^0 = \varprojlim_s \mathbf{A}_{/\mathcal{K}}^0/(P_{k_1} \cap P_{k_2} \cap \dots \cap P_{k_s})$  since  $\mathbf{A}_{/\mathcal{K}}^0$  is complete and local. We define the continuous function  $\text{Tr} : G_{\mathbb{Q},S} \longrightarrow \mathbf{A}_{/\mathcal{K}}^0$  to be  $\varprojlim_s \text{Tr}_{P_{k_1} \cap P_{k_2} \cap \dots \cap P_{k_s}}$ .

By using the function  $\text{Tr}$ , we define the desired function  $A$  and  $D$  as follows:

$$A(\sigma) = \frac{\text{Tr}(\sigma) - \text{Tr}(c \cdot \sigma)}{2}, \quad D(\sigma) = \frac{\text{Tr}(\sigma) + \text{Tr}(c \cdot \sigma)}{2}.$$

Since  $\mathbf{A}_{/\mathcal{K}}^0$  is complete and local, a similar argument is done by first evaluating at pairs  $(g_1, g_2) = (\text{Frob}_{\ell_1}, \text{Frob}_{\ell_2})$  the value  $b(g_1)c(g_2)$  and then observing that the set of these pairs is dense in  $G_{\mathbb{Q},S} \times G_{\mathbb{Q},S}$ .

We thus obtain a continuous function

$$\Xi : G_{\mathbb{Q},S} \times G_{\mathbb{Q},S} \longrightarrow \mathbf{A}_{/\mathcal{K}}^0$$

which recovers the function  $\Xi_{(k)}$  taking the reduction modulo  $P_k$  of the function  $\Xi$  for every  $k \in Z_{(k_0, i)}[r_0]$ .

Setting  $\pi := (A, D, \Xi)$ , we need to check that they verify properties (II)-(IV) of [Wil88] by using the fact that  $Z_{(k_0, i)}[r_0]$  is dense in the reduced  $\mathcal{X}$ . First, let us verify the property (IV), namely we verify that for each  $g_1, g_2, h_1, h_2 \in G_{\mathbb{Q}, S}$  it holds

$$(3.5) \quad \Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2) = 0.$$

We need to check that the function  $\Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2)$  vanishes identically on  $\mathcal{X}$ . Since the above-mentioned property (IV) holds for the pseudorepresentation  $\pi_k = (A_{(k)}, D_{(k)}, \Xi_k)$ , we have

$$\begin{aligned} & (\Xi(g_1, g_2)\Xi(h_1, h_2) - \Xi(g_1, h_2)\Xi(h_1, g_2))(k) \\ &= \Xi_k(g_1, g_2)\Xi_k(h_1, h_2) - \Xi_k(g_1, h_2)\Xi_k(h_1, g_2) \end{aligned}$$

at each point  $k$  in the dense subset  $Z_{(k_0, i)}[r_0]$ . This proves the desired vanishing of (3.5) and the same argument holds for other properties (II) and (III). This completes the proof.  $\square$

Theorem 3.3 below, which is the main result of this section, shows that the pseudo-representation just constructed comes from a true representation.

**Theorem 3.3.** *Let us take the same assumptions and notations as Proposition 3.2. There exists a free  $\mathbf{A}_{/\mathcal{K}}^0$ -module  $\mathbf{T}$  of rank two with a continuous  $G_{\mathbb{Q}, S}$ -action such that the representation*

$$\rho: G_{\mathbb{Q}, S} \longrightarrow \text{Aut}_{\mathbf{A}_{/\mathcal{K}}^0}(\mathbf{T})$$

satisfies  $\pi_\rho = \pi$  (in a suitable basis). In particular,  $\rho$  modulo  $P_k$  is isomorphic to a lattice of  $\rho_k$  for all  $k \in Z_{(k_0, i)}[r_0]$ .

The following proof mainly relies on [Wil88, Lemma 2.2.3], see also [Hid93, Proposition 1, §7.5].

*Proof.* Start with a radius  $r_0$  as in Theorem 2.2 and set  $r = r_0|\varpi|$ . Let  $\pi = (A, D, \Xi)$  be the  $\mathbf{A}_{/\mathcal{K}}^0$ -valued pseudo-representation constructed in Proposition 3.2. There exists a pair of elements  $\sigma, \tau \in G_{\mathbb{Q}, S}$  such that  $\Xi(\sigma, \tau)(k_0) \neq 0$ . If not, the diagonal Galois representation  $g \mapsto A(g)(k_0) \oplus D(g)(k_0)$  would have the same trace and determinant as the representation  $\rho_k$ , which contradicts to Ribet’s Theorem 2.3 of [Rib77] saying that  $\rho_k$  be irreducible (see in [Wil88, Lemma 2.2.3] or [Hid89, Proposition 1.1]). From now on, let us fix a pair  $\sigma, \tau$  such that  $\Xi(\sigma, \tau)(k_0) \in \mathcal{K}$  is of minimal valuation, say  $\mu \in \mathbb{Q}$ . The element  $\Xi(\sigma, \tau)$  can be decomposed as  $\Xi(\sigma, \tau) = p^\mu V^{-1}$  with  $V \in (\mathbf{A}_{/\mathcal{K}}^0)^\times$ . As in [Hid93, Proposition 1], we check that the map

$$\rho: g \mapsto \begin{pmatrix} A(g) & \Xi(g, \tau)Vp^{-\mu} \\ \Xi(\sigma, g) & D(g) \end{pmatrix}$$

is multiplicative, sends  $1 \in G_{\mathbb{Q}, S}$  to  $\mathbf{Id}_2 \in M_2(\mathbf{A}_{/\mathcal{K}}^0)$  and takes values in  $M_2(\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p)$ . Hence we have a continuous group homomorphism  $\rho: G_{\mathbb{Q}, S} \rightarrow \text{GL}_2(\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p)$ . We want to produce a finitely generated  $\mathbf{A}_{/\mathcal{K}}^0$ -submodule of  $\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p$  which is  $\rho$ -stable, we follow the proof of continuity of [Hid89, Proposition 1.1]. Namely, define  $\mathfrak{J} \subseteq \mathbf{A}_{/\mathcal{K}}^0$  to be the ideal generated by all lower-left entries  $\Xi(\sigma, g)$  for  $g \in G_{\mathbb{Q}, S}$  and let  $\mathbf{T}'$  be the  $\mathbf{A}_{/\mathcal{K}}^0$ -submodule of  $(\mathbf{A}_{/\mathcal{K}}^0)^{\oplus 2} \subset (\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p)^{\oplus 2}$  generated by all vectors  $(x, y)$  with  $x \in \mathbf{A}_{/\mathcal{K}}^0, y \in \mathfrak{J}$ . The

$\mathbf{A}_{/\mathcal{K}}^0$ -module  $\mathbf{T}'$  is finitely generated since  $\mathbf{A}_{/\mathcal{K}}^0$  is noetherian and is  $p$ -torsion free since so is  $(\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p)^{\oplus 2}$ . Moreover, given  ${}^t(x, y) \in \mathbf{T}'$  and  $g \in G_{\mathbb{Q}, S}$ , we have

$$\rho(g) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(g)x + \Xi(g, \tau)Vp^{-\mu}y \\ \Xi(\sigma, g)x + D(g)y \end{pmatrix} \in \mathbf{T}'$$

by the definition of  $\mathfrak{J}$  which ensures  $\Xi(\sigma, g)x \in \mathfrak{J}$  and by the definition of  $\mu$  which ensures  $\Xi(g, \tau)Vp^{-\mu} \in \mathbf{A}_{/\mathcal{K}}^0$ . Thus, it follows that  $\mathbf{T}'$  is Galois stable. Finally,  $\mathbf{T}' \otimes \mathbb{Q}_p = (\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p)^{\oplus 2}$  because  $\mathfrak{J}$  contains the element  $\Xi(\sigma, \tau) = p^{-\mu}V$  which becomes a unit after inverting  $p$ . Let us define  $\mathbf{T}$  to be the double dual  $\text{Hom}_{\mathbf{A}^0}(\text{Hom}_{\mathbf{A}^0}(\mathbf{T}', \mathbf{A}_{/\mathcal{K}}^0), \mathbf{A}_{/\mathcal{K}}^0)$ . We have a canonical  $G_{\mathbb{Q}, S}$ -equivariant  $\mathbf{A}_{/\mathcal{K}}^0$ -linear injection  $\mathbf{T}' \hookrightarrow \mathbf{T}$  with finite cokernel. Since  $\mathbf{T}$  is a finitely generated reflexive module over a regular local ring  $\mathbf{A}_{/\mathcal{K}}^0$  of Krull dimension two,  $\mathbf{T}$  is a free  $\mathbf{A}_{/\mathcal{K}}^0$ -module of finite rank. Since  $\mathbf{T} \otimes \mathbb{Q}_p$  is free of rank two over  $\mathbf{A}_{/\mathcal{K}}^0 \otimes \mathbb{Q}_p$ , the rank of  $\mathbf{T}$  over  $\mathbf{A}_{/\mathcal{K}}^0$  is two.

As for the last property, recall that an irreducible representation of  $G_{\mathbb{Q}, S}$  with values in a finite extension of  $\mathbb{Q}_p$  is uniquely determined by its trace and determinant. Thus  $\mathbf{T}/P_k\mathbf{T}$  is isomorphic to a lattice of the Galois drepresentation  $\rho_k$ .  $\square$

Note that  $\iota : \mathbf{A}_{/\mathcal{K}}^0 \hookrightarrow \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$  is continuous. In fact,  $\mathbf{A}_{/\mathcal{K}}^0$  is local with the maximal ideal  $(\frac{T-k_0}{e_0}, \varpi)$  and  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$  is endowed with the  $\varpi$ -adic topology. To show that  $\iota$  is continuous, we need to show that  $\iota^{-1}(\varpi \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0)$  contains the maximal ideal  $(\frac{T-k_0}{e_0}, \varpi)$  of  $\mathbf{A}_{/\mathcal{K}}^0$ . In fact, this implies that  $\iota^{-1}(\varpi^n \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0)$  contains  $(\frac{T-k_0}{e_0}, \varpi)^n$  for every  $n$ . The uniformizer  $\varpi$  is clearly contained in  $\iota^{-1}(\varpi \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0)$  and  $\frac{T-k_0}{e_0}$  is also contained in  $\iota^{-1}(\varpi \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0)$  since we have  $\frac{T-k_0}{e_0} = \varpi \frac{T-k_0}{e_0 \varpi} \in \varpi \mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$ . By extending the coefficient of the result obtained over  $\mathbf{A}_{/\mathcal{K}}^0$  in the above theorem to  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$ , we obtain the following corollary:

**Corollary 3.4.** *Under the same assumptions and notations as Theorem 2.2 and Proposition 3.2, there exists a free  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$ -module  $\mathcal{F}$  of rank two with continuous  $G_{\mathbb{Q}, S}$ -action such that the representation*

$$\rho : G_{\mathbb{Q}, S} \longrightarrow \text{Aut}_{\mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0}(\mathcal{F})$$

satisfies  $\pi_\rho = \pi$  (in a suitable basis). In particular,  $\rho$  modulo  $\mathfrak{m}_k$  is isomorphic to  $\rho_k$  for all  $k \in Z_{(k_0, i)}[r_0]$  where  $\mathfrak{m}_k$  is the unique maximal ideal of  $\mathcal{A}_{\mathcal{X}_{(k_0, i)}^0[r]/\mathcal{K}}^0$  such that  $\mathfrak{m}_k \cap \mathbf{A}_{/\mathcal{K}}^0 = P_k$ .

We stress that every finitely generated module over an affinoid algebra  $\mathcal{A}$  will always implicitly be endowed with the quotient topology induced by any finite presentation (which is independent of the presentation, see [BGR84, Proposition 3.7.3/3]) and that every finitely generated, torsion-free module  $\mathcal{M}$  over a power-bounded affinoid algebra  $\mathcal{A}^0$  will be endowed with the subspace topology induced by its injection into the finitely generated  $\mathcal{A}^0[\frac{1}{p}]$ -module  $\mathcal{M} \otimes \mathbb{Q}_p$ .

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