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AN EXTENSION OF A RESULT OF ERDŐS AND ZAREMBA

MICHEL J. G. WEBER

ABSTRACT. Erdős and Zaremba showed that $\limsup_{n \rightarrow \infty} \frac{\Phi(n)}{(\log \log n)^2} = e^\gamma$, γ being Euler's constant, where $\Phi(n) = \sum_{d|n} \frac{\log d}{d}$.

We extend this result to the function $\Psi(n) = \sum_{d|n} \frac{(\log d)(\log \log d)}{d}$ and some other functions. We show that $\limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2(\log \log \log n)} = e^\gamma$. The proof requires a new approach. As an application, we prove that for any $\eta > 1$, any finite sequence of reals $\{c_k, k \in K\}$, $\sum_{k, \ell \in K} c_k c_\ell \frac{\gcd(k, \ell)^2}{k \ell} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \log \nu)^\eta \Psi(\nu)$, where $C(\eta)$ depends on η only. This improves a recent result obtained by the author.

1. Introduction.

Erdős and Zaremba showed in [4] the following result concerning the arithmetical function $\Phi(n) = \sum_{d|n} \frac{\log d}{d}$,

$$(1.1) \quad \limsup_{n \rightarrow \infty} \frac{\Phi(n)}{(\log \log n)^2} = e^\gamma,$$

where γ is Euler's constant. This function appears in the study of good lattice points in numerical integration, see Zaremba [13]. The proof is based on the identity

$$(1.2) \quad \Phi(n) = \sum_{i=1}^r \sum_{\nu_i=1}^{\alpha_i} \frac{\log p_i^{\nu_i}}{p_i^{\nu_i}} \sum_{\delta|n p_i^{-\alpha_i}} \frac{1}{\delta}, \quad (n = p_1^{\alpha_1} \dots p_r^{\alpha_r}),$$

which follows from

$$(1.3) \quad \sum_{d|n} \frac{\log d}{d} = \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \dots p_r^{\mu_r}} \left(\sum_{i=1}^r \mu_i \log p_i \right).$$

Let $h(n)$ be non-decreasing on integers, $h(n) = o(\log n)$, and consider the slightly larger function

$$(1.4) \quad \Phi_h(n) = \sum_{d|n} \frac{(\log d) h(d)}{d}.$$

In this case a formula similar to (1.3) no longer holds, the "log-linearity" being lost due to the extra factor $h(n)$. The study of this function requires a new approach. We study in this work the case $h(n) = \log \log n$, that is the function

$$(1.5) \quad \Psi(n) = \sum_{d|n} \frac{(\log d)(\log \log d)}{d}.$$

We extend Erdős-Zaremba's result for this function, as well as for the functions

$$\begin{aligned} \Phi_1(n) &= \sum_{p_1^{\mu_1} \dots p_r^{\mu_r} | n} \frac{\sum_{i=1}^r \mu_i (\log p_i) (\log \log p_i)}{p_1^{\mu_1} \dots p_r^{\mu_r}} \\ \Phi_2(n) &= \sum_{d|n} \frac{(\log d) \log \Omega(d)}{d}, \end{aligned}$$

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where $\Omega(d)$ denotes as usual the total number of prime factors of d counting multiplicity. These functions are linked to Ψ .

Throughout, $\log \log x$ (resp. $\log \log \log x$) equals 1 if $0 \leq x \leq e^e$ (resp. $0 \leq x \leq e^{e^e}$), and equals $\log \log x$ (resp. $\log \log \log x$) in the usual sense if $x > e^e$ (resp. $x > e^{e^e}$).

One verifies using standard arguments that

$$(1.6) \quad \limsup_{n \rightarrow \infty} \frac{\Phi_1(n)}{(\log \log n)^2 (\log \log \log n)} \geq e^\gamma, \quad \limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} \geq e^\gamma,$$

and in fact that

$$(1.7) \quad \limsup_{n \rightarrow \infty} \frac{\Phi_1(n)}{(\log \log n)^2 (\log \log \log n)} = e^\gamma.$$

By the observation made after (1.3), the corresponding extension of this result to $\Psi(n)$ is technically more delicate. It follows from (1.1) that

$$(1.8) \quad \limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^3} \leq e^\gamma.$$

The question thus arises whether the exponent of $\log \log n$ in (1.8) can be replaced by $2 + \varepsilon$, with $\varepsilon > 0$ small.

We answer this question affirmatively by establishing the following precise result, which is the main result of this paper.

Theorem 1.1.

$$\limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} = e^\gamma.$$

An application of this result is given in Section 5. The upper bound is obtained, via the inequality

$$(1.9) \quad \Psi(n) \leq \Phi_1(n) + \Phi_2(n),$$

as a combination of an estimate of $\Phi_1(n)$ and the following estimate of $\Phi_2(n)$. Recall that Davenport's function $w(n)$ is defined by $w(n) = \sum_{p|n} \frac{\log p}{p}$. According to Theorem 4 in [2] we have,

$$(1.10) \quad \limsup_{n \rightarrow \infty} \frac{w(n)}{\log \log n} = 1.$$

Let also $\omega(n)$ be the number of prime divisors of n counted without multiplicity.

Theorem 1.2. *For all odd numbers n we have,*

$$\Phi_2(n) \leq C (\log \log \log \omega(n)) (\log \omega(n)) w(n).$$

where C is an absolute constant.

Here and elsewhere C (resp. $C(\eta)$) denotes some positive absolute constant (resp. some positive constant depending only of a parameter η).

The approach used for proving Theorem 1.2 can be adapted with no difficulty to other arithmetical functions of similar type.

Before continuing we mention some other existing extensions, due to Sitaramaiah and Subbarao in [10, 9]. For instance, the case when $\log d$ is replaced by a non-negative additive function g (ie. $S(n) = \sum_{d|n} \frac{g(d)}{d}$) is studied in [9]. In our case we note that $g(d) = (\log d)(\log \log d)$ (see (1.4)), which is obviously not additive. It is proved that if $T(d)$ is one of the three arithmetical functions $\frac{\omega(d)}{d}$, $\frac{\Omega(d)}{d}$, $\frac{\log \tau(d)}{d}$, then

$$(1.11) \quad \limsup_{n \rightarrow \infty} \frac{\sum_{d|n} T(d)}{(\log \log n)(\log \log \log n)} = c_T e^\gamma,$$

where $c_T = 1$ in the two first cases, and $c_T = \log 2$ in the third case. See also Remark 2.3. A basis of their proof lies in the observation that $S(n)/\sigma_{-1}(n)$ is additive. Further it is proved in [10] that for each positive integer k ,

$$(1.12) \quad \limsup_{n \rightarrow \infty} \frac{S_k(n)}{(\log \log n)^{k+1}} = c_k e^\gamma, \quad (S_k(n) = \sum_{d|n} \frac{(\log d)^k}{d})$$

where c_k is a positive explicit constant. The proof is elegant and based on the derivation formula $S_k(n) = (-1)^k f^{(k)}(1)$, where $f(u) = \sigma_{-u}(n)$ and $f^{(k)}$ is the k -th derivative of f , which is specific to these sums.

The paper is organized as follows. Sections 2 and 3 form the main part of the paper, and consist of the proof of Theorem 1.2, which is long and technical and involves the building of a binary tree (subsection 2.2.1). The proof of Theorem 1.1 is given in section 3. Section 4 contains complementary results and the proofs of (1.6), (1.7). Section 5 concerns the afore mentioned application of Theorem 1.1. Additional remarks and results are given in Section 6.

2. Proof of Theorem 1.2.

We use a chaining argument. We make throughout the convention $0 \log 0 = 0$.

Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be an odd number. We will use repeatedly the fact that

$$(2.1) \quad \min_{i=1}^r p_i \geq 3.$$

We note that

$$(2.2) \quad \begin{aligned} \Phi_2(n) &= \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \dots p_r^{\mu_r}} \sum_{i=1}^r \mu_i (\log p_i) \log \left(\sum_{i=1}^r \mu_i \right) \\ &= \sum_{i=1}^r \underbrace{\sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r}}_{\substack{\text{the sum relative} \\ \text{to } \mu_i \text{ is excluded}}} \underbrace{\frac{1}{p_1^{\mu_1} \dots p_r^{\mu_r}}}_{\substack{p_i^{\mu_i} \\ \text{is excluded}}} \left(\sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i (\log p_i)}{p_i^{\mu_i}} \right) \log \left(\sum_{i=1}^r \mu_i \right). \end{aligned}$$

As there is no order relation on the sequence p_1, \dots, p_r , it suffices to study the sum

$$(2.3) \quad \Phi_2(r, n) := \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \dots p_{r-1}^{\mu_{r-1}}} \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left[\sum_{i=1}^{r-1} \mu_i + \mu_r \right].$$

The sub-sums in (2.3) will be estimated by using a recursion argument.

We first explain its principle and examine the structure of the sum $\Phi_2(r, n)$, anticipating somehow the calculations. The last sum $\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left[\sum_{i=1}^{r-1} \mu_i + \mu_r \right]$ is of type

$$\sum_{\mu=0}^{\alpha_r} \alpha \mu (\log(A + \mu)) e^{-\alpha \mu}, \quad \alpha = \log p_r, \quad A = \sum_{i=1}^{r-1} \mu_i.$$

It is easy to observe with (2.6) that the bound obtained in Lemma 2.2, will induce on the sum in μ_{r-1} a logarithmic factor $\log \left[h + \sum_{i=1}^{r-2} \mu_i + \mu_{r-1} \right]$ where h is a positive integer, and so one. More precisely,

$$\begin{aligned} \sum_{\mu=0}^{\infty} \alpha \mu (\log(A + \mu)) e^{-\alpha \mu} &\leq \alpha (\log(A + 1)) e^{-\alpha} + 2\alpha (\log(A + 2)) e^{-2\alpha} \\ &+ \left\{ 3\alpha \log(A + 3) + 3 \log(A + 3) + \frac{1}{\alpha} \log(A + 3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A + 3)} \right\} e^{-3\alpha}. \end{aligned}$$

provided $A \geq 1$, and $\alpha \geq 1$. Whence the bound,

$$\frac{\log p_r}{p_r} (\log(A+1)) + \frac{2 \log p_r}{p_r^2} (\log(A+2)) + \frac{1}{p_r^3} \left\{ 3 \log p_r \log(A+3) + 3 \log(A+3) \right. \\ \left. + \frac{1}{\log p_r} \log(A+3) + \frac{1}{\log p_r} + \frac{1}{(\log p_r)^2 (A+3)} \right\}.$$

By reporting this bound in (2.3), we get sums of type

$$\sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{\log \left[\sum_{i=1}^{r-1} \mu_i + h \right]}{p_1^{\mu_1} \cdots p_{r-1}^{\mu_{r-1}}} \quad h = 1, 2, 3,$$

affected with new coefficients, this is displayed in (2.11). By using (2.7), the last sum is bounded by

$$\log \left[\sum_{i=1}^{r-2} \mu_i + h \right] + \frac{1}{p_{r-1}} \left(\log \left[\sum_{i=1}^{r-2} \mu_i + h + 1 \right] \right. \\ \left. + \frac{\log \left[\sum_{i=1}^{r-2} \mu_i + h + 1 \right]}{\log p_{r-1}} + \frac{1}{(\log p_{r-1})^2 (\sum_{i=1}^{r-2} \mu_i + h + 1)} \right).$$

By recursing once more, this allows one to bound again $\Phi_2(r, n)$. The remaining sums will after be all of same type. The factor $\log \left[\sum_{i=1}^{r-1} \mu_i + h \right]$ induces on the sum of order $(r-2)$ a factor $\log \left[\sum_{i=1}^{r-2} \mu_i + h + 1 \right]$. The whole matter thus consists with understanding how the new coefficients are generated, and in particular to check whether a coefficient of order $1 + \varepsilon$ will not produce by iteration a coefficient of order $(1 + \varepsilon)^r$. A recurrence inequality established in Lemma 2.6 will allow one to control their magnitude efficiently.

2.1. Preparation. Some technical lemmas will be needed.

Lemma 2.1. (i) *Let $\varphi_1(x) = x(\log(A+x))e^{-\alpha x}$, $\varphi_2(x) = (\log(A+x))e^{-\alpha x}$. Then $\varphi_1(x)$ is non-increasing on $[3, \infty)$ if $A \geq 1$ and $\alpha \geq \log 2$. Further, $\varphi_2(x)$ is non-increasing on $[1, \infty)$, if $A \geq 1$ and $\alpha \geq 1$.*

(ii) *Assume that $A \geq 1$ and $\alpha \geq \log 2$. For any integer $m \geq 1$,*

$$(2.4) \quad \alpha \int_m^\infty x(\log(A+x))e^{-\alpha x} dx \leq \frac{1}{\alpha^2(A+m)}e^{-\alpha m} + \frac{1}{\alpha}e^{-\alpha m} + \frac{1}{\alpha}(\log(A+m))e^{-\alpha m} \\ + m(\log A + m)e^{-\alpha m}.$$

(iii) *Assume that $A \geq 1$ and $\alpha \geq 1$. Then,*

$$(2.5) \quad \int_1^\infty (\log(A+x))e^{-\alpha x} dx \leq \frac{\log(A+1)}{\alpha}e^{-\alpha} + \frac{1}{\alpha^2(A+1)}e^{-\alpha}.$$

Proof. (i) We have $\varphi_1'(x) = (\log(A+x))e^{-\alpha x} + \frac{x}{A+x}e^{-\alpha x} - \alpha x(\log(A+x))e^{-\alpha x}$. By assumption and since $\varphi_1'(x) \leq 0 \Leftrightarrow \frac{1}{x} + \frac{1}{(A+x)\log(A+x)} \leq \alpha$, we get

$$\frac{1}{x} + \frac{1}{(A+x)\log(A+x)} \leq \frac{1}{3} + \frac{1}{8 \log 2} \leq \frac{1}{3} + \frac{1}{5} < \log 2 \leq \alpha.$$

Similarly $\varphi_2'(x) = \frac{1}{A+x}e^{-\alpha x} - \alpha(\log(A+x))e^{-\alpha x}$. As $\varphi_2'(x) \leq 0 \Leftrightarrow (A+x)\log(A+x) \geq \frac{1}{\alpha}$, we also get

$$(A+x)\log(A+x) \geq 2 \log 2 > 1 \geq \frac{1}{\alpha}.$$

(ii) We deduce from (i) that

$$(2.6) \quad \alpha x(\log(A+x))e^{-\alpha x} = (\log(A+x))e^{-\alpha x} + \frac{x}{A+x}e^{-\alpha x} - (x(\log A + x)e^{-\alpha x})'.$$

By integrating,

$$\alpha \int_m^\infty x(\log(A+x))e^{-\alpha x} dx = \int_m^\infty x(\log(A+x))e^{-\alpha x} dx + \int_m^\infty \frac{x}{A+x}e^{-\alpha x} dx$$

$$+m(\log A + m)e^{-\alpha m}.$$

Similarly

$$\alpha \int_m^\infty (\log(A+x))e^{-\alpha x} dx = \int_m^\infty \frac{1}{A+x} e^{-\alpha x} dx + (\log(A+m))e^{-\alpha m}.$$

By combining we get,

$$\begin{aligned} \alpha \int_m^\infty x(\log(A+x))e^{-\alpha x} dx &= \frac{1}{\alpha} \int_m^\infty \frac{1}{A+x} e^{-\alpha x} dx + \int_m^\infty \frac{x}{A+x} e^{-\alpha x} dx \\ &+ \frac{1}{\alpha} (\log(A+m))e^{-\alpha m} + m(\log A + m)e^{-\alpha m}. \end{aligned}$$

Therefore,

$$\alpha \int_m^\infty x(\log(A+x))e^{-\alpha x} dx \leq \frac{1}{\alpha^2(A+m)}e^{-\alpha m} + \frac{1}{\alpha}e^{-\alpha m} + \frac{1}{\alpha}(\log(A+m))e^{-\alpha m} + m(\log A + m)e^{-\alpha m}.$$

(iii) We deduce from (i) that

$$\begin{aligned} \int_1^N (\log(A+x))e^{-\alpha x} dx &= \frac{1}{\alpha} \int_1^N \frac{1}{(A+x)} e^{-\alpha x} dx \\ &- \frac{1}{\alpha} \left((\log(A+1))e^{-\alpha} - \log(A+N)e^{-\alpha N} \right). \end{aligned}$$

As $\frac{1}{\alpha} \int_1^N \frac{1}{A+x} e^{-\alpha x} dx \leq \frac{1}{\alpha^2(A+1)}e^{-\alpha}$, letting N tend to infinity gives,

$$\int_1^\infty (\log(A+x))e^{-\alpha x} dx \leq \frac{\log(A+1)}{\alpha}e^{-\alpha} + \frac{1}{\alpha^2(A+1)}e^{-\alpha}.$$

□

Lemma 2.2. *Assume that $A \geq 1$, and $\alpha \geq 1$. Then,*

$$\begin{aligned} \sum_{\mu=0}^\infty \alpha \mu (\log(A+\mu))e^{-\alpha \mu} &\leq \alpha (\log(A+1))e^{-\alpha} + 2\alpha (\log(A+2))e^{-2\alpha} \\ &+ \left\{ 3\alpha \log(A+3) + 3 \log(A+3) + \frac{1}{\alpha} \log(A+3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A+3)} \right\} e^{-3\alpha}. \end{aligned}$$

Proof. As

$$\begin{aligned} \sum_{\mu=0}^\infty \alpha \mu (\log(A+\mu))e^{-\alpha \mu} &= \alpha (\log(A+1))e^{-\alpha} + 2\alpha (\log(A+2))e^{-2\alpha} \\ &+ 3\alpha (\log(A+3))e^{-3\alpha} + \alpha \sum_{\mu=4}^\infty \mu (\log(A+\mu))e^{-\alpha \mu}, \end{aligned}$$

by applying Lemma 2.1-(ii), we get

$$\begin{aligned} \alpha \sum_{\mu=4}^\infty \mu (\log(A+\mu))e^{-\alpha \mu} &\leq \alpha \int_3^\infty x(\log(A+x))e^{-\alpha x} dx \\ &\leq \frac{1}{\alpha^2(A+3)}e^{-3\alpha} + \frac{1}{\alpha}e^{-3\alpha} + \frac{\log(A+3)}{\alpha}e^{-3\alpha} + 3(\log A + 3)e^{-3\alpha}. \end{aligned}$$

Whence,

$$\begin{aligned} \sum_{\mu=0}^\infty \alpha \mu (\log(A+\mu))e^{-\alpha \mu} &\leq \alpha (\log(A+1))e^{-\alpha} + 2\alpha (\log(A+2))e^{-2\alpha} \\ &+ \left\{ 3\alpha \log(A+3) + 3 \log(A+3) + \frac{1}{\alpha} \log(A+3) + \frac{1}{\alpha} + \frac{1}{\alpha^2(A+3)} \right\} e^{-3\alpha}. \end{aligned}$$

□

Lemma 2.3. *Under assumption (2.1) we have*

$$\sum_{\mu_s=0}^{\infty} \frac{\log\left(\sum_{i=1}^s \mu_i + h\right)}{p_s^{\mu_s}} \leq \log\left(\sum_{i=1}^{s-1} \mu_i + h\right) + \frac{1}{p_s} \left(1 + \frac{1}{\log p_s}\right) \log\left(\sum_{i=1}^{s-1} \mu_i + h + 1\right) + \frac{1}{(1 + (\sum_{i=1}^{s-1} \mu_i + 2))(\log p_s)^2 p_s}.$$

In particular,

$$\sum_{\mu_s=0}^{\infty} \frac{\log\left(\sum_{i=1}^{s-1} \mu_i + h\right)}{p_s^{\mu_s}} \leq \left(1 + \frac{1}{p_s} \left(1 + \frac{1}{\log p_s} + \frac{1}{3(\log p_s)^2}\right)\right) \log\left(\sum_{i=1}^{s-1} \mu_i + h + 2\right).$$

Proof. As

$$\begin{aligned} \sum_{\mu=0}^{\infty} (\log(A + \mu)) e^{-\alpha\mu} &= \log A + (\log(A + 1)) e^{-\alpha} + \sum_{\mu=2}^{\infty} (\log(A + \mu)) e^{-\alpha\mu} \\ &\leq \log A + (\log(A + 1)) e^{-\alpha} + \int_1^{\infty} (\log(A + x)) e^{-\alpha x} dx \end{aligned}$$

we deduce from Lemma 2.1-(iii),

$$(2.7) \quad \sum_{\mu=0}^{\infty} (\log(A + \mu)) e^{-\alpha\mu} \leq \log A + e^{-\alpha} \left(\log(A + 1) + \frac{\log(A + 1)}{\alpha} + \frac{1}{\alpha^2(A + 1)} \right).$$

Consequently,

$$\sum_{\mu_s=0}^{\infty} \frac{\log\left(\sum_{i=1}^s \mu_i + h\right)}{p_s^{\mu_s}} \leq \log\left(\sum_{i=1}^{s-1} \mu_i + h\right) + \frac{1}{p_s} \left(1 + \frac{1}{\log p_s}\right) \log\left(\sum_{i=1}^{s-1} \mu_i + h + 1\right) + \frac{1}{(1 + (\sum_{i=1}^{s-1} \mu_i + 2))(\log p_s)^2 p_s}.$$

Finally,

$$\sum_{\mu_s=0}^{\infty} \frac{\log\left(\sum_{i=1}^{s-1} \mu_i + h\right)}{p_s^{\mu_s}} \leq \left(1 + \frac{1}{p_s} \left(1 + \frac{1}{\log p_s}\right) + \frac{1}{3(\log p_s)^2}\right) \log\left(\sum_{i=1}^{s-1} \mu_i + h + 2\right).$$

□

Corollary 2.4. *Assume that condition (2.1) is satisfied.*

(i) *If $\sum_{i=1}^{r-1} \mu_i \geq 1$, then*

$$\begin{aligned} \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log\left(\sum_{i=1}^{r-1} \mu_i + \mu_r\right) &\leq \frac{\log p_r}{p_r} \log\left(\sum_{i=1}^{r-1} \mu_i + 1\right) + \frac{2 \log p_r}{p_r^2} \log\left(\sum_{i=1}^{r-1} \mu_i + 2\right) \\ &+ \frac{1}{p_r^3} \left(3 \log p_r + 3 + \frac{1}{\log p_r}\right) \log\left(\sum_{i=1}^{r-1} \mu_i + 3\right) + \frac{1}{p_r^3 \log p_r} \left(1 + \frac{1}{(\sum_{i=1}^{r-1} \mu_i + 3) \log p_r}\right). \end{aligned}$$

Further,

$$\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log\left(\sum_{i=1}^{r-1} \mu_i + \mu_r\right) \leq 5 \frac{\log p_r}{p_r} \log\left(\sum_{i=1}^{r-1} \mu_i + 3\right).$$

(ii) *If $\sum_{i=1}^{r-1} \mu_i = 0$, then*

$$\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log\left(\sum_{i=1}^{r-1} \mu_i + \mu_r\right) \leq 18 \frac{\log p_r}{p_r}.$$

Proof. (i) The first inequality follows from Lemma 2.2 with the choice $\alpha = \log p_r$ and $A = \sum_{i=1}^{r-1} \mu_i$, noting that by assumption (2.1), $\alpha > 1$. As $p_r \geq 3$, it is also immediate that

$$\begin{aligned} & \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left(\sum_{i=1}^{r-1} \mu_i + \mu_r \right) \\ & \leq \left\{ 3 \frac{\log p_r}{p_r} + \frac{\log p_r}{9p_r} \left(3 + \frac{3}{\log p_r} + \frac{1}{(\log p_r)^2} \right) \right\} \log \left(\sum_{i=1}^{r-1} \mu_i + 3 \right) + \frac{1}{9p_r \log p_r} \left(1 + \frac{1}{4 \log p_r} \right) \\ & \leq 5 \frac{\log p_r}{p_r} \log \left(\sum_{i=1}^{r-1} \mu_i + 3 \right). \end{aligned}$$

(ii) If $\sum_{i=1}^{r-1} \mu_i = 0$, the sums relative to μ_i , $1 \leq i \leq r-1$, do not contribute. Further,

$$\begin{aligned} \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left(\sum_{i=1}^{r-1} \mu_i + \mu_r \right) &= \sum_{\mu_r=2}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \mu_r = \sum_{\mu=1}^{\alpha_r-1} \frac{(\mu+1) \log p_r}{p_r^{\mu+1}} \log(\mu+1) \\ &\leq \frac{1}{p_r} \left\{ \sum_{\mu=1}^{\infty} \frac{\mu \log p_r}{p_r^{\mu}} \log(\mu+1) + \sum_{\mu=1}^{\infty} \frac{\log p_r}{p_r^{\mu}} \log(\mu+1) \right\}. \end{aligned}$$

Lemma 2.2 applied with $A = 1$ and $\alpha = \log p_r$ gives the bound

$$\begin{aligned} \sum_{\mu=1}^{\infty} \frac{\mu \log p_r}{p_r^{\mu}} \log(\mu+1) &\leq \frac{(\log 2) \log p_r}{p_r} + \frac{2(\log 3) \log p_r}{p_r^2} + \frac{1}{p_r^3} \left\{ (6 \log 2)(\log p_r) \right. \\ &\quad \left. + 6 \log 2 + \frac{2 \log 2}{(\log p_r)} + \frac{1}{(\log p_r)} + \frac{1}{4(\log p_r)^2} \right\} \\ &\leq 8 \left(\frac{\log p_r}{p_r} + \frac{1}{p_r^3} \right). \end{aligned}$$

Next estimate (2.7) applied with $A = 1$ and $\alpha = \log p_r$, further gives,

$$\sum_{\mu=1}^{\infty} \frac{\log p_r}{p_r^{\mu}} \log(\mu+1) \leq \frac{1}{p_r} \left(\log 2 + \frac{\log 2}{\log p_r} + \frac{1}{2(\log p_r)^2} \right) \leq \frac{2}{p_r}.$$

Whence, $\sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left(\sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq 18 \frac{\log p_r}{p_r}$. \square

Remark 2.5. As $\log \left(\sum_{i=1}^s \mu_i + h \right) \leq \log (\Omega(n) + 3)$, one can deduce from Corollary 2.4-(ii) that

$$\Phi_2(r, n) \leq 18 \frac{\log p_r}{p_r} \log(\Omega(n) + 3) \prod_{i=1}^r \left(\frac{1}{1 - p_i^{-1}} \right).$$

So that by the observation made at the beginning of section 2,

$$\Phi_2(n) \leq 18 (\log(\Omega(n) + 3)) \left(\sum_{j=1}^r \frac{\log p_j}{p_j} \right) \prod_{i=1}^r \left(\frac{1}{1 - p_i^{-1}} \right).$$

By combining this with the bound for $\Phi_1(n)$ established in Lemma 4.1, next using inequality (1.9), gives

$$(2.8) \quad \Psi(n) \leq \left(\prod_{j=1}^r \frac{1}{1 - p_j^{-1}} \right) \left\{ \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1} + 18 \left(\sum_{i=1}^r \frac{\log p_i}{p_i} \right) \log(\Omega(n) + 3) \right\},$$

recalling that $r = \omega(n)$. Whence by invoking Proposition 4.3, noticing that $\omega(n) \leq \Omega(n) \leq \log_2 n$,

$$\Psi(n) \leq e^\gamma (1 + o(1)) (\log \log n)^2 (\log \log \log n + 18\omega(n)).$$

The finer estimate of $\Psi(n)$ will be derived from a more precise study of the coefficients of $\Psi(r, n)$. This is the object of the next sub-section.

2.2. **Estimates of $\Phi_2(\mathbf{r}, \mathbf{n})$.** We define successively

$$(2.9) \quad \left\{ \begin{array}{l} \mu = (\mu_1, \dots, \mu_r), \quad (\mu_1, \dots, \mu_r) \in \prod_{i=1}^r ([0, \alpha_i] \cap \mathbb{N}), \\ p_\mu(s) = p_1^{-\mu_1} \dots p_s^{-\mu_s}, \quad 1 \leq s \leq r, \\ \Pi_s = \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_s=0}^{\alpha_s} p_\mu(s) = \prod_{\ell=1}^s \left(\frac{1 - p_\ell^{-\alpha_\ell - 1}}{1 - p_\ell^{s-1}} \right). \end{array} \right.$$

Next,

$$\Phi_s(h) = \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_s=0}^{\alpha_s} p_\mu(s) \log \left(\sum_{i=1}^s \mu_i + h \right), \quad 1 \leq s \leq r-1.$$

We also set

$$(2.10) \quad \left\{ \begin{array}{l} c_1 = 1, \quad c_2 = \frac{2}{p_r}, \quad c_3 = \frac{1}{p_r^2} \left(3 + \frac{3}{\log p_r} + \frac{1}{(\log p_r)^2} \right), \\ c_4 = \frac{1}{p_r^3 \log p_r} \left(1 + \frac{1}{3 \log p_r} \right) \\ c_0 = \frac{\log p_r}{p_r}, \quad c = \sum_{i=1}^3 c_i, \\ b_s = \frac{1}{p_s} \left(1 + \frac{1}{\log p_s} \right), \quad \beta_s = \frac{1}{2p_s (\log p_s)^2}. \end{array} \right.$$

2.2.1. *Recurrence inequality.* We deduce from the first part of Lemma 2.3 that

$$\begin{aligned} \Phi_s(h) &= \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{s-1}=0}^{\alpha_{s-1}} p_\mu(s-1) \left\{ \sum_{\mu_s=0}^{\alpha_s} p_s^{-\mu_s} \log \left(\sum_{i=1}^s \mu_i + h \right) \right\} \\ &\leq \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{s-1}=0}^{\alpha_{s-1}} p_\mu(s-1) \left\{ \sum_{\mu_s=0}^{\infty} p_s^{-\mu_s} \log \left(\sum_{i=1}^s \mu_i + h \right) \right\} \\ &\leq \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{s-1}=0}^{\alpha_{s-1}} p_\mu(s-1) \left\{ \log \left(\sum_{i=1}^{s-1} \mu_i + h \right) + \right. \\ &\quad \left. \frac{1}{p_s} \left(1 + \frac{1}{\log p_s} \right) \log \left(\sum_{i=1}^{s-1} \mu_i + h + 1 \right) + \frac{1}{(1 + (\sum_{i=1}^{s-1} \mu_i + 2)) (\log p_s)^2 p_s} \right\} \\ &\leq \Phi_{s-1}(h) + \frac{1}{p_s} \left(1 + \frac{1}{\log p_s} \right) \Phi_{s-1}(h+1) + \frac{1}{2(\log p_s)^2 p_s} \Pi_{s-1}. \end{aligned}$$

Whence with the previous notation,

Lemma 2.6. *Under assumption (2.1), we have for $s = 2, \dots, r-1$,*

$$\Phi_s(h) \leq \Phi_{s-1}(h) + b_s \Phi_{s-1}(h+1) + \beta_s \Pi_{s-1}.$$

Now by using estimate (i) of Corollary 2.4 and the notation introduced, we have, under assumption (2.1), if $\sum_{i=1}^{r-1} \mu_i \geq 1$,

$$\begin{aligned}
& \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left(\sum_{i=1}^{r-1} \mu_i + \mu_r \right) \leq \frac{\log p_r}{p_r} \log \left(\sum_{i=1}^{r-1} \mu_i + 1 \right) + \frac{2 \log p_r}{p_r^2} \log \left(\sum_{i=1}^{r-1} \mu_i + 2 \right) \\
& + \frac{1}{p_r^3} \left(3 \log p_r + 3 + \frac{1}{\log p_r} \right) \log \left(\sum_{i=1}^{r-1} \mu_i + 3 \right) + \frac{1}{p_r^3 \log p_r} \left(1 + \frac{1}{\left(\sum_{i=1}^{r-1} \mu_i + 3 \right) \log p_r} \right) \\
& \leq c_0 c_1 \log \left(\sum_{i=1}^{r-1} \mu_i + 1 \right) + c_0 c_2 \log \left(\sum_{i=1}^{r-1} \mu_i + 2 \right) + c_0 c_3 \log \left(\sum_{i=1}^{r-1} \mu_i + 3 \right) + c_4 \\
(2.11) \quad & = c_0 \sum_{h=1}^3 c_h \log \left(\sum_{i=1}^{r-1} \mu_i + h \right) + c_4,
\end{aligned}$$

since $\frac{1}{p_r^3 \log p_r} \left(1 + \frac{1}{\left(\sum_{i=1}^{r-1} \mu_i + 3 \right) \log p_r} \right) \leq c_4$.

Therefore, under assumption (2.1), if $\sum_{i=1}^{r-1} \mu_i \geq 1$,

$$(2.12) \quad \Phi_2(r, n) \leq c_0 \underbrace{\sum_{h=1}^3 c_h \Phi_{r-1}(h)}_{(1)} + c_4 \Pi_{r-1}.$$

Indeed,

$$\begin{aligned}
\Phi_2(r, n) &= \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \cdots p_{r-1}^{\mu_{r-1}}} \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_r \log p_r}{p_r^{\mu_r}} \log \left[\sum_{i=1}^{r-1} \mu_i + \mu_r \right] \\
&\leq \sum_{\mu_1=0}^{\alpha_1} \cdots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \cdots p_{r-1}^{\mu_{r-1}}} \left\{ c_0 \sum_{h=1}^3 c_h \log \left(\sum_{i=1}^{r-1} \mu_i + h \right) + c_4 \right\} \\
&= c_0 \underbrace{\sum_{h=1}^3 c_h \Phi_{r-1}(h)}_{(1)} + c_4 \Pi_{r-1}.
\end{aligned}$$

By applying the recurrence inequality with $s = r - 1$ to $\Phi_{r-1}(h)$, one gets

$$\Phi_2(r, n) \leq c_0 \underbrace{\sum_{h=1}^3 c_h [\Phi_{r-2}(h) + b_{r-1} \Phi_{r-2}(h+1)]}_{(1)} + c_0 c \beta_{r-1} \Pi_{r-2} + c_4 \Pi_{r-1}.$$

By applying this time the recurrence inequality to $\Phi_{r-2}(h)$, one also gets

$$\begin{aligned}
\Phi_2(r, n) &\leq c_0 \underbrace{\sum_{h=1}^3 c_h \Phi_{r-3}(h)}_{(1)} + c_0 \underbrace{\sum_{h=1}^3 c_h b_{r-2} \Phi_{r-3}(h+1)}_{(3)} + c_0 c b_{r-2} \Pi_{r-3} \\
&+ c_0 \underbrace{\sum_{h=1}^3 c_h b_{r-1} \Phi_{r-3}(h+1)}_{(2)} + c_0 \underbrace{\sum_{h=1}^3 c_h b_{r-1} b_{r-2} \Phi_{r-3}(h+2)}_{(4)} + c_0 c b_{r-1} \beta_{r-2} \Pi_{r-3} \\
&+ c_0 c \beta_{r-1} \Pi_{r-2} + c_4 \Pi_{r-1}.
\end{aligned}$$

One easily verifies (see expressions underlined by (1)) that the coefficient of $\Phi_{r-1}(h)$ is the same as the one of $\Phi_{r-2}(h)$ and $\Phi_{r-3}(h)$. So is also the case for $\Phi_{r-2}(h+1)$, see expressions underlined by (2). New expressions underlined by (3), (4) and linked to $\Phi_{r-3}(h+1)$, $\Phi_{r-3}(h+2)$ appear.

Each new coefficient is kept until the end of the iteration process generated by the recurrence inequality of Lemma 2.6.

We also verify, when applying this inequality, that we pass from a majoration expressed by $\Phi_{r-1}(h)$, Π_{r-1} , *uniquely*, to a majoration expressed by Φ_{r-2} (in h or $h+1$) and Π_{r-2} , Π_{r-1} *uniquely*.

This rule is general, and one verifies that when iterating this recurrence relation, we obtain at each step a bound depending on Φ_{r-d} and the products $\Pi_{r-d}, \Pi_{r-d+1}, \dots, \Pi_{r-1}$ only.

Binary tree: The shift of length h or $h+1$ generates a binary tree whose branches are at each division (steps corresponding to the preceding iterations), either stationary: $\Phi_{r-d}(h) \rightarrow \Phi_{r-d-1}(h)$, or creating new coefficients: $\Phi_{r-d}(h) \rightarrow \Phi_{r-d-1}(h+1)$. One can represent this by the diagram below drawn from Lemma 2.6.

$$\begin{array}{c} \downarrow \text{ shift } +1, \text{ new coefficients } \downarrow \\ \Phi_s(h) \leq \Phi_{s-1}(h) + b_s \Phi_{s-1}(h+1) + \beta_s \Pi_{s-1}. \\ \uparrow \text{ stationarity } \uparrow \end{array}$$

Figure 1.

Before continuing, we recall that by (2.7),

$$\sum_{\mu=0}^{\alpha_s} (\log(A+\mu)) e^{-\alpha\mu} \leq \log A + e^{-\alpha} \left(\log(A+1) + \frac{\log(A+1)}{\alpha} + \frac{1}{\alpha^2(A+1)} \right).$$

Thus

$$\begin{aligned} \Phi_1(v) &\leq \sum_{\mu_1=0}^{\infty} p_{\mu_1}(1) \log \left(\sum_{i=1}^v \mu_i + 1 \right) = \sum_{\mu_1=0}^{\infty} \frac{\log(v+\mu)}{p_1^{\mu}} \\ &\leq \log v + \frac{1}{p_1} \left(\log(v+1) + \frac{\log(v+1)}{\log p_1} + \frac{1}{v(\log p_1)^2} \right) \quad (v \geq 1). \end{aligned}$$

Hence,

$$\Phi_1(h) \leq C \log h.$$

One easily verifies that the d -tuples formed with the b_i have all $\Phi_{r-x}(h+d)$ as factor. The terms having $\Phi_{r-..}(h+\cdot)$ as factor are forming the sum

$$(2.13) \quad c_0 \sum_{d=1}^{r-1} \left(\sum_{1 \leq i_1 < \dots < i_d < r} b_{i_1} \dots b_{i_d} \right) \Phi_1(h+d),$$

once the iteration process achieved, that is after having applied $(r-1)$ times the recurrence inequality of Lemma 2.6.

This sum can thus be bounded from above by (recalling that $h = 1, 2$ or 3)

$$c_0 \sum_{d=1}^{r-1} (\log d) \left(\sum_{1 \leq i_1 < \dots < i_d < r} b_{i_1} \dots b_{i_d} \right).$$

But, for all positive integers a_1, \dots, a_r and $1 \leq d \leq r$, we have,

$$\left(\sum_{i=1}^r a_i \right)^d \geq d! \sum_{1 \leq i_1 < \dots < i_d \leq r} a_{i_1} \dots a_{i_d}.$$

Thus

$$\sum_{d=1}^{r-1} (\log d) \left(\sum_{1 \leq i_1 < \dots < i_d < r} b_{i_1} \dots b_{i_d} \right) \leq \sum_{d=1}^{r-1} \frac{(\log d)}{d!} \left(\sum_{i=1}^r b_i \right)^d.$$

As moreover,

$$b_i = \frac{1}{p_i} \left(1 + \frac{1}{\log p_{i+1}} \right) \leq \frac{1}{p(i)} + \frac{1}{p(i) \log p(i)},$$

one has by means of (4.2),

$$\sum_{i=1}^r b_i \leq \sum_{i=1}^r \left(\frac{1}{i \log i} + \frac{1}{i(\log i)^2} \right) \leq \log \log r + C.$$

Thus

$$\sum_{d=1}^{r-1} (\log d) \left(\sum_{1 \leq i_1 < \dots < i_d < r} b_{i_1} \dots b_{i_d} \right) \leq C \sum_{d=1}^{r-1} \frac{(\log d)}{d!} (\log \log r + C)^d.$$

On the one hand,

$$\begin{aligned} \sum_{\log d \leq 1 + \varepsilon + \log \log \log r} \frac{(\log d)}{d!} (\log \log r + C)^d &\leq (1 + \varepsilon + \log \log \log r) \sum_{d > 1} \frac{(\log \log r + C)^d}{d!} \\ &\leq C(1 + \varepsilon + \log \log \log r) \log r. \end{aligned}$$

On the other, utilizing the classical estimate $d! \geq C\sqrt{d}d^d e^{-d}$, one has

$$\begin{aligned} \sum_{\log d > 1 + \varepsilon + \log \log \log r} \frac{(\log d)}{d!} (\log \log r)^d &\leq \sum_{\log d > 1 + \varepsilon + \log \log \log r} \frac{(\log d)}{\sqrt{d}} e^{-d(\log d - 1 - \log \log \log r)} \\ &\leq \sum_{d > 1} \frac{(\log d)}{\sqrt{d}} e^{-\varepsilon d} < \infty. \end{aligned}$$

One thus deduces, concerning the sum in (2.13) that,

$$(2.14) \quad c_0 \sum_{d=1}^{r-1} \left(\sum_{1 \leq i_1 < \dots < i_d < r} b_{i_1} \dots b_{i_d} \right) \Phi_1(h+d) \leq C \frac{\log p_r}{p_r} (1 + \log \log \log r) \log r;$$

2.3. *Coefficients related to Π_s .* By applying the recurrence inequality (Lemma 2.6), one successively generates

$$\begin{aligned} &c_4 \Pi_{r-1} \\ &c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} \\ &c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} + c_0 c \beta_{r-2} (1 + b_{r-1} b_{r-2}) \Pi_{r-3} \\ &c_4 \Pi_{r-1} + c_0 c \beta_{r-1} \Pi_{r-2} + c_0 c \beta_{r-2} (1 + b_{r-1} b_{r-2}) \Pi_{r-3} \\ &\quad + c_0 c \beta_{r-3} (1 + b_{r-2} + b_{r-1} + b_{r-1} b_{r-2}) \Pi_{r-4}. \end{aligned}$$

Coefficients:

$$\begin{array}{ll} \Pi_{r-1} : c_4 & \Pi_{r-2} : c_0 c \beta_{r-1} \\ \Pi_{r-3} : c_0 c \beta_{r-2} (1 + b_{r-1}) & \Pi_{r-4} : c_0 c \beta_{r-3} (1 + b_{r-2} + b_{r-1} + b_{r-1} b_{r-2}). \end{array}$$

It is easy to check that the coefficients Π_{r-x} are exactly those of $\Phi_{r-x+1}(\cdot)$ affected with the factor $c_0 c \beta_{r-x+1}$. The products form the sum

$$(2.15) \quad c_0 c \sum_{d=0}^{r-2} \beta_{r-d} \left(1 + \sum_{1 \leq i_1 < \dots < i_d < r} b_{r-i_1} \dots b_{r-i_d} \right) \Pi_{r-d-1}.$$

By (4.2), one has

$$(2.16) \quad \beta_j = \frac{1}{2p_j(\log p_j)^2} \leq \frac{1}{2p(j)(\log p(j))^2} \leq \frac{1}{2j(\log j)^3}, \quad \text{if } j \geq 2,$$

Moreover, (4.2) and (4.6) imply that

$$\begin{aligned} \Pi_j &= \prod_{\ell=1}^j \left(\frac{1}{1 - \frac{1}{p^\ell}} \right) \leq \prod_{\ell=1}^j \left(\frac{1}{1 - \frac{1}{p(\ell)}} \right) \leq \prod_{p \leq j(\log j + \log \log j)} \left(\frac{1}{1 - \frac{1}{p}} \right) \\ &\leq C(\log j). \end{aligned}$$

We now note that by definition of Π_j , we also have

$$\Pi_j \leq \max_{\ell \leq 5} \prod_{p \leq p(\ell)} \frac{1}{1 - \frac{1}{p}} = C_0.$$

We deduce that

$$(2.17) \quad \Pi_j \leq C(\log j), \quad \text{if } j \geq 2.$$

Consequently, (2.17) and (2.16) imply that

$$(2.18) \quad \beta_{j+1}\Pi_j \leq \frac{C}{j(\log j)^2}, \quad \text{if } j \geq 2.$$

This implies that the sum in (2.15) can be bounded as follows:

$$\begin{aligned} & c_0 c \sum_{d=0}^{r-2} \beta_{r-d} \left(1 + \sum_{1 \leq i_1 < \dots < i_d < r} b_{r-i_1} \dots b_{r-i_d} \right) \Pi_{r-d-1} \\ \leq & c_0 c \prod_{i=1}^{r-2} (1 + b_{r-i}) \cdot \sum_{d=0}^{r-2} \beta_{r-d} \Pi_{r-d-1} = c_0 c \prod_{j=2}^{r-1} (1 + b_j) \cdot \sum_{d=0}^{r-2} \beta_{r-d} \Pi_{r-d-1} \\ \leq & c_0 c C \prod_{j=2}^{r-1} (1 + b_j) \cdot \sum_{d=0}^{r-2} \frac{1}{(r-d)(\log(r-d))^2} \\ \leq & c_0 c C \prod_{j=2}^{r-1} (1 + b_j) \cdot \sum_{\delta=2}^{\infty} \frac{1}{\delta(\log \delta)^2} \\ (2.19) \quad & \leq c_0 c C \prod_{j=2}^{r-1} (1 + b_j). \end{aligned}$$

We recall that

$$\sum_{p \leq x} \frac{1}{p} \leq \log \log x + C.$$

See for instance [8], inequality (3.20). Thus,

$$(2.20) \quad \prod_{i=1}^r (1 + b_i) \leq C \log r.$$

Now estimate (2.20) implies that

$$\begin{aligned} c_0 c \sum_{d=0}^{r-2} \beta_{r-d} \left(1 + \sum_{1 \leq i_1 < \dots < i_d < r} b_{r-i_1} \dots b_{r-i_d} \right) \Pi_{r-d-1} & \leq c_0 c C \log r \\ (2.21) \quad & \leq C \frac{\log p_r}{p_r} \log r. \end{aligned}$$

We thus deduce from (2.14) and (2.15) that

$$\begin{aligned} \Phi_2(r, n) & \leq C \frac{\log p_r}{p_r} (1 + \log \log \log r) \log r + C \frac{\log p_r}{p_r} \log r \\ (2.22) \quad & \leq C \frac{\log p_r}{p_r} (\log r) (\log \log \log r). \end{aligned}$$

As a result, by taking account of the observation made at the beginning of section 2, we obtain

$$(2.23) \quad \Phi_2(n) \leq C (\log \log \log r) (\log r) \sum_{i=1}^r \frac{\log p_i}{p_i} = C (\log \log \log r) (\log r) w(n).$$

By combining (2.22) with the upper estimate $\Phi_1(n)$ established at Lemma 4.1 and using inequality (1.9), we arrive at

$$(2.24) \quad \Psi(n) \leq \left(\prod_{j=1}^r \frac{1}{1-p_j^{-1}} \right) \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1} + C(\log \log \log r)(\log r)w(n),$$

recalling that $p_j \geq 3$ by assumption (2.1).

3. Proof of Theorem 1.1.

First we prove inequality (1.9). We recall the convention $0 \log 0 = 0$. Inequality (1.9) is an immediate consequence of the following convexity lemma.

Lemma 3.1. *For any integers $\mu_i \geq 0$, $p_j \geq 2$, we have*

$$\begin{aligned} \sum_{i=1}^r (\mu_i \log p_i) \log \left(\sum_{i=1}^r \mu_i \log p_i \right) &\leq \sum_{i=1}^r \mu_i (\log p_i) (\log \log p_i) \\ &\quad + \sum_{i=1}^r \mu_i (\log p_i) \log \left(\sum_{i=1}^r \mu_i \right). \end{aligned}$$

Proof. We may restrict to the case $\sum_{i=1}^r \mu_i \geq 1$, since otherwise the inequality is trivial. Let $M = \sum_{i=1}^r \mu_i$ and write that

$$\begin{aligned} \sum_{i=1}^r \mu_i (\log p_i) \log \left(\sum_{i=1}^r \mu_i \log p_i \right) &= M \left\{ \sum_{i=1}^r \frac{\mu_i}{M} (\log p_i) \log \left\{ \sum_{i=1}^r \frac{\mu_i}{M} \log p_i \right\} \right. \\ &\quad \left. + \sum_{i=1}^r \frac{\mu_i}{M} (\log p_i) (\log M) \right\}. \end{aligned}$$

By using convexity of $\psi(x) = x \log x$ on \mathbb{R}_+ , we get

$$\sum_{i=1}^r \frac{\mu_i}{\sum_{i=1}^r \mu_i} (\log p_i) \log \left\{ \sum_{i=1}^r \frac{\mu_i}{\sum_{i=1}^r \mu_i} \log p_i \right\} \leq \sum_{i=1}^r \frac{\mu_i}{\sum_{i=1}^r \mu_i} (\log p_i) (\log \log p_i).$$

Thus

$$\sum_{i=1}^r (\mu_i \log p_i) \log \left(\sum_{i=1}^r \mu_i \log p_i \right) \leq \sum_{i=1}^r \mu_i (\log p_i) (\log \log p_i) + \sum_{i=1}^r \mu_i (\log p_i) \log \left(\sum_{i=1}^r \mu_i \right).$$

□

The odd case (i.e. condition (2.1) is satisfied) is obtained by combining (2.22) with Corollary 4.2 and utilizing inequality (1.9). Since $r \leq \log n$, by taking account of estimate of $w(n)$ given in (1.10), we get

$$(3.1) \quad \begin{aligned} \Psi(n) &\leq e^\gamma (1 + o(1)) (\log \log n)^2 (\log \log \log n) + C (\log \log \log \log n) (\log \log n)^2 \\ &= e^\gamma (1 + o(1)) (\log \log n)^2 (\log \log \log n). \end{aligned}$$

To pass from the odd case to the general case is not easy. This step will necessitate an extra analysis of some other properties of $\Psi(n)$.

We first exclude the trivial case when n is a pure power of 2, since $\Psi(2^k) \leq C$ uniformly over k , and C is a finite constant.

Now if 2 divides n , writing $n = 2^v m$, $2 \nmid m$, we have

$$\Psi(n) = \sum_{d|n} \frac{(\log d)(\log \log d)}{d} = \sum_{k=0}^v \sum_{\delta|m} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} ..$$

As the function $x \mapsto \frac{(\log x)(\log \log x)}{x}$ decreases on $[x_0, \infty)$ for some positive real x_0 , we can write

$$\sum_{k=0}^v \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{k_0-1} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} + \sum_{k=k_0+1}^v \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} \\
&\leq \sum_{k=0}^{k_0-1} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} + \int_{2^{k_0 \delta}}^{\infty} \frac{(\log u)(\log \log u)}{u^2} du,
\end{aligned}$$

where k_0 is depending on x_0 only. Moreover

$$\left(\frac{(\log u)(\log \log u)}{u} \right)' \geq - \frac{(\log u)(\log \log u)}{u^2}.$$

Thus

$$\begin{aligned}
&\sum_{k=0}^v \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} \\
&\leq \sum_{k=0}^{k_0-1} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta} + \frac{(\log(2^{k_0} \delta))(\log \log(2^{k_0} \delta))}{2^{k_0} \delta},
\end{aligned}$$

whence

$$(3.2) \quad \Psi(n) \leq \sum_{k=0}^{k_0} \sum_{\delta|m} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{2^k \delta}.$$

Let $m = p_1^{b_1} \dots p_\mu^{b_\mu}$. We have by (2.8)

$$\begin{aligned}
\Psi(m) &\leq \left(\prod_{j=1}^{\mu} \frac{1}{1-p_j^{-1}} \right) \sum_{i=1}^{\mu} \frac{(\log p_i)(\log \log p_i)}{p_i - 1} + C (\log \log \log \mu)(\log \mu)w(m) \\
&\leq \left(\prod_{j=2}^{\mu} \frac{1}{1-p(j)^{-1}} \right) \sum_{i=1}^{\mu} \frac{(\log p(i))(\log \log p(i))}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m) \\
&= \frac{1}{2} \left(\prod_{j=1}^{\mu} \frac{1}{1-p(j)^{-1}} \right) \sum_{i=1}^{\mu} \frac{(\log p(i))(\log \log p(i))}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m) \\
&\leq \frac{e^\gamma}{2} (\log \mu + \mathcal{O}(1)) \sum_{i=1}^{\mu} \frac{(\log p(i))(\log \log p(i))}{p(i) - 1} + C (\log \log \log \mu)(\log \mu)w(m),
\end{aligned}$$

by using Mertens' estimate (4.6) and since $p(\mu) \sim \mu \log \mu$. Furthermore by using estimate (4.5), and since $2^\mu \leq m$ we get

$$\begin{aligned}
\Psi(m) &\leq \frac{e^\gamma}{2} (\log \mu + \mathcal{O}(1))(1 + \varepsilon)(\log \mu)(\log \log \mu) + C (\log \log \log \mu)(\log \mu)w(m) \\
&\leq \frac{e^\gamma}{2} \left(\log \frac{\log m}{\log 2} + \mathcal{O}(1) \right) (1 + \varepsilon) \left(\log \frac{\log m}{\log 2} \right) \left(\log \log \frac{\log m}{\log 2} \right) \\
&\quad + C (\log \log \log \frac{\log m}{\log 2}) \left(\log \frac{\log m}{\log 2} \right) (1 + o(1)) \log \log m \\
(3.3) \quad &\leq \frac{e^\gamma}{2} (1 + 2\varepsilon) (\log \log m)^2 (\log \log \log m),
\end{aligned}$$

for m large.

Now let $\psi(2^k m) = \sum_{\delta|m} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{\delta}$, $1 \leq k \leq k_0$. If n is not a pure power of 2, then its odd component m tends to infinity with n . Thus with (3.2),

$$(3.4) \quad \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} \leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{(\log(2^k \delta))(\log \log(2^k \delta))}{\delta (\log \log m)^2 (\log \log \log m)}.$$

But

$$\frac{(\log(2^k \delta))(\log \log(2^k \delta))}{\delta} = \frac{(k(\log 2))(\log \log(2^k \delta)) + (\log \delta)(\log \log(2^k \delta))}{\delta}$$

$$(3.5) \quad \leq k_0(\log 2) \frac{\log(k_0(\log 2) + \log \delta)}{\delta} + \frac{(\log \delta)(\log \log(2^{k_0} \delta))}{\delta}.$$

Now we have the inequality: $\log \log(a+x) \leq \log(b \log x)$ where $b \geq (a+e)$ and $a \geq 1$, which is valid for $x \geq e$. Thus

$$(3.6) \quad \log(k_0(\log 2) + \log \delta) \leq \log(k_0 \log 2 + e) + \log \log \delta.$$

Consequently

$$(3.7) \quad \begin{aligned} & \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{k_0(\log 2) \frac{\log(k_0(\log 2) + \log \delta)}{\delta}}{(\log \log m)^2 (\log \log \log m)} \\ & \leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{k_0(\log 2) \frac{\log(k_0 \log 2 + e)}{\delta}}{(\log \log m)^2 (\log \log \log m)} \\ & \quad + \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{k_0(\log 2) \frac{\log \log \delta}{\delta}}{(\log \log m)^2 (\log \log \log m)} \\ & \leq 2k_0(\log 2) (\log(k_0 \log 2 + e)) \frac{\sigma_{-1}(m)}{(\log \log m)^2 (\log \log \log m)} \\ & \quad + \frac{2k_0(\log 2)}{(\log \log m)^2 (\log \log \log m)} \sum_{\delta|m} \frac{\log \log \delta}{\delta} \\ & \leq C(k_0) \left\{ \frac{1}{\log \log m (\log \log \log m)} + \frac{\sigma_{-1}(m)}{(\log \log m) (\log \log \log m)} \right\} \\ & \leq \frac{C(k_0)}{\log \log \log m} \rightarrow 0 \quad \text{as } m \text{ tends to infinity.} \end{aligned}$$

Further

$$(3.8) \quad \begin{aligned} \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{(\log \delta)(\log \log(2^{k_0} \delta))}{(\log \log m)^2 (\log \log \log m)} & \leq \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{(\log \delta)(\log(k_0 \log 2 + e) + \log \log \delta)}{\delta (\log \log m)^2 (\log \log \log m)} \\ & \leq \frac{\log(k_0 \log 2 + e)}{(\log \log m)^2 (\log \log \log m)} \sum_{k=0}^{k_0} \frac{1}{2^k} \sum_{\delta|m} \frac{(\log \delta)}{\delta} \\ & \quad + 2 \frac{\Psi(m)}{(\log \log m)^2 (\log \log \log m)} \\ & \leq \frac{2 \log(k_0 \log 2 + e) \sigma_{-1}(m)}{(\log \log m) (\log \log \log m)} \\ & \quad + 2 \frac{\Psi(m)}{(\log \log m)^2 (\log \log \log m)} \\ & \leq \frac{C(k_0)}{\log \log \log m} + 2 \frac{e^\gamma}{2} (1 + 2\varepsilon) \frac{(\log \log m)^2 (\log \log \log m)}{(\log \log m)^2 (\log \log \log m)} \\ & \leq \frac{C(k_0)}{\log \log \log m} + e^\gamma (1 + 2\varepsilon), \end{aligned}$$

for m large, where we used estimate (3.3).

Plugging estimates (3.7) and (3.8) into (3.4) finally leads, in view of (3.5), to

$$(3.9) \quad \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} \leq \frac{C}{\log \log \log m} + e^\gamma (1 + 2\varepsilon)$$

for m large, where C depends on k_0 only. As ε can be arbitrary small, we finally obtain

$$(3.10) \quad \limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} \leq e^\gamma.$$

This establishes Theorem 1.1.

4. Complementary results.

In this section we prove complementary estimates Φ_1 , Φ_2 and Ψ , notably estimates (1.6) and (1.7)

4.1. Upper estimates.

Lemma 4.1. *We have the following estimate,*

$$\Phi_1(n) \leq \left(\prod_{j=1}^r \frac{1}{1-p_j^{-1}} \right) \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1}.$$

Proof. We have

$$\Phi_1(n) = \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r} \frac{\mu_1(\log p_1)(\log \log p_1) + \dots + \mu_r(\log p_r)(\log \log p_r)}{p_1^{\mu_1} \dots p_r^{\mu_r}}$$

The i -th term of the numerator yields the sum

$$\underbrace{\sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \dots p_r^{\mu_r}}}_{\substack{\text{the sum relative} \\ \text{to } \mu_i \text{ is excluded}}} \underbrace{\left(\sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i(\log p_i)(\log \log p_i)}{p_i^{\mu_i}} \right)}_{\substack{\text{the sum} \\ \text{relative to } \mu_i \\ \text{is excluded}}}.$$

Consequently,

$$\begin{aligned} \Phi_1(n) &= \sum_{i=1}^r \underbrace{\sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_r=0}^{\alpha_r} \frac{1}{p_1^{\mu_1} \dots p_r^{\mu_r}}}_{\substack{\text{the sum relative} \\ \text{to } \mu_i \text{ is excluded}}} \underbrace{\left(\sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i(\log p_i)(\log \log p_i)}{p_i^{\mu_i}} \right)}_{\substack{\text{the sum} \\ \text{relative to } \mu_i \\ \text{is excluded}}} \\ (4.1) \quad &= \sum_{i=1}^r \prod_{\substack{j=1 \\ j \neq i}}^r \left(\frac{1-p_j^{-\alpha_j-1}}{1-p_j^{-1}} \right) \left[\sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i(\log p_i)(\log \log p_i)}{p_i^{\mu_i}} \right]. \end{aligned}$$

Now as

$$\sum_{\mu=0}^{\alpha_i} \frac{\mu}{p_i^\mu} \leq \sum_{j=0}^{\infty} \frac{j}{p_i^j} = \frac{1}{(p_i-1)(1-p_i^{-1})},$$

we obtain

$$\begin{aligned} \Phi_1(n) &\leq \sum_{i=1}^r \prod_{\substack{j=1 \\ j \neq i}}^r \left(\frac{1-p_j^{-\alpha_j-1}}{1-p_j^{-1}} \right) \cdot \frac{(\log p_i)(\log \log p_i)}{(p_i-1)(1-p_i^{-1})} \\ &\leq \left(\prod_{j=1}^r \frac{1}{1-p_j^{-1}} \right) \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i-1}. \end{aligned}$$

□

Corollary 4.2. *We have the following estimate,*

$$\limsup_{n \rightarrow \infty} \frac{\Phi_1(n)}{(\log \log n)^2 (\log \log \log n)} \leq e^\gamma.$$

Proof. Let $p(j)$ denote the j -th consecutive prime number, and recall that ([8, (3.12-13)],

$$(4.2) \quad \begin{aligned} p(i) &\geq \max(i \log i, 2), & i &\geq 1, \\ p(i) &\leq i(\log i + \log \log i), & i &\geq 6. \end{aligned}$$

Let $\varepsilon > 0$ and an integer $r_0 \geq 4$. If $r \leq r_0$, then

$$(4.3) \quad \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i-1} \leq \delta r_0, \quad \delta = \sup_{p \geq 3} \frac{(\log p)(\log \log p)}{p-1} < \infty.$$

If $r > r_0$, then

$$\begin{aligned} \sum_{i=r_0+1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1} &\leq \left(\max_{i>r_0} \frac{p(i)}{p(i) - 1} \right) \sum_{i=r_0+1}^r \frac{(\log p(i))(\log \log p(i))}{p(i)} \\ &\leq \left(\max_{i>r_0} \frac{p(i)}{p(i) - 1} \right) \sum_{i=r_0+1}^r \frac{(\log(i \log i))(\log \log(i \log i))}{i \log i} \end{aligned}$$

We choose $r_0 = r_0(\varepsilon)$ so that $\log r_0 \geq 1/\varepsilon$ and the preceding expression is bounded from above by

$$(1 + \varepsilon) \sum_{i=r_0+1}^r \frac{\log \log i}{i}$$

We thus have

$$\begin{aligned} \sum_{i=r_0+1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1} &\leq (1 + \varepsilon) \int_{r_0}^r \frac{\log \log t}{t} dt \\ (4.4) \qquad \qquad \qquad &\leq (1 + \varepsilon)(\log r)(\log \log r). \end{aligned}$$

Consequently, for some $r(\varepsilon)$,

$$(4.5) \qquad \sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i - 1} \leq (1 + \varepsilon)(\log r)(\log \log r), \quad r \geq r(\varepsilon).$$

By using Mertens' estimate

$$(4.6) \qquad \prod_{p \leq x} \left(\frac{1}{1 - \frac{1}{p}} \right) = e^\gamma \log x + \mathcal{O}(1) \quad x \geq 2,$$

we further have

$$(4.7) \qquad \prod_{\ell=1}^r \left(\frac{1}{1 - \frac{1}{p^\ell}} \right) \leq \prod_{\ell=1}^r \left(\frac{1}{1 - \frac{1}{p(\ell)}} \right) \leq \prod_{p \leq r(\log r + \log \log r)} \left(\frac{1}{1 - \frac{1}{p}} \right) \leq e^\gamma (\log r) + C,$$

if $r \geq 6$, and so for any $r \geq 1$, modifying C if necessary. As $r = \omega(n)$ and $2^{\omega(n)} \leq n$, we consequently have,

$$\Phi_1(n) \leq e^\gamma (1 + C\varepsilon)^2 (\log \log n)^2 (\log \log \log n),$$

if $r > r_0$. If $r \leq r_0$, we have

$$\Phi_1(n) \leq \delta e^\gamma (1 + \varepsilon) ((\log r_0) + C) := C(\varepsilon).$$

Whence,

$$\Phi_1(n) \leq e^\gamma (1 + \varepsilon)^2 (\log \log n)^2 (\log \log \log n) + C(\varepsilon).$$

As ε can be arbitrary small, the result follows. \square

The following lemma is nothing but the upper bound part of (1.1). We omit the proof.

Lemma 4.3. *We have the following estimate,*

$$\sum_{d|n} \frac{\log d}{d} \leq \prod_{p|n} \left(\frac{1}{1 - p^{-1}} \right) \sum_{p|n} \frac{\log p}{p - 1}.$$

Moreover,

$$\limsup_{n \rightarrow \infty} \frac{1}{(\log \log n)(\log \omega(n))} \sum_{d|n} \frac{\log d}{d} \leq e^\gamma.$$

4.2. Lower estimates. We recall that the smallest prime divisor of an integer n is noted by $P^-(n)$.

Lemma 4.4. *Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $r \geq 1$, $\alpha_i \geq 1$. Then,*

$$\Phi_1(n) \geq \left(1 - \frac{1}{P^-(n)}\right) \prod_{j=1}^r (1 + p_j^{-1}) \left[\sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i} \right]$$

Proof. By (4.1),

$$\begin{aligned} \Phi_1(n) &= \sum_{i=1}^r \prod_{\substack{j=1 \\ j \neq i}}^r \left(\frac{1 - p_j^{-\alpha_j - 1}}{1 - p_j^{-1}} \right) \left[\sum_{\mu_i=0}^{\alpha_i} \frac{\mu_i (\log p_i)(\log \log p_i)}{p_i^{\mu_i}} \right] \\ &\geq \sum_{i=1}^r \prod_{\substack{j=1 \\ j \neq i}}^r \left(\frac{1 - p_j^{-\alpha_j - 1}}{1 - p_j^{-1}} \right) \left[\frac{(\log p_i)(\log \log p_i)}{p_i} \right] \\ &\geq \prod_{j=1}^r (1 + p_j^{-1}) \left[\sum_{i=1}^r \frac{(1 - p_i^{-1})(\log p_i)(\log \log p_i)}{p_i} \right]. \end{aligned}$$

Thus

$$\Phi_1(n) \geq \left(1 - \frac{1}{P^-(n)}\right) \prod_{j=1}^r (1 + p_j^{-1}) \left[\sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i} \right].$$

□

We easily deduce from Lemma 4.1 and Lemma 4.4 the following corollary.

Corollary 4.5. *Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $r \geq 1$, $\alpha_i \geq 1$. Then,*

$$\left(1 - \frac{1}{P^-(n)}\right) \prod_{j=1}^r (1 + p_j^{-1}) \leq \frac{\Phi_1(n)}{\sum_{i=1}^r \frac{(\log p_i)(\log \log p_i)}{p_i}} \leq 2 \prod_{j=1}^r \left(\frac{1}{1 - p_j^{-1}} \right).$$

Proposition 4.6. *We have the following estimates*

- a) $\limsup_{n \rightarrow \infty} \frac{1}{(\log \log n)} \sum_{d|n} \frac{(\log d)}{d} \geq e^\gamma$
- b) $\limsup_{n \rightarrow \infty} \frac{\Phi_1(n)}{(\log \log n)^2 (\log \log \log n)} \geq e^\gamma$,
- c) $\limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2 (\log \log \log n)} \geq e^\gamma$.

Proof. Case a) is Erdős-Zaremba's lower bound of function $\Phi(n)$. Since it is used in the proof of b) and c), we provide a detailed proof for the sake of completeness.

a) Let $n_j = \prod_{p < e^j} p^j$. Recall that $p(i) \geq \max(i \log i, 2)$ if $i \geq 1$. Let $r(j)$ be the integer defined by the condition $p(r(j)) < e^j < p(r(j) + 1)$.

By using (1.2) and following Gronwall's proof [6], we have,

$$\begin{aligned} \sum_{d|n_j} \frac{\log d}{d} &= \sum_{i=1}^{r(j)} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{r(j)} \left(\frac{1 - p(\ell)^{-j-1}}{1 - p(\ell)^{-1}} \right) \left[\sum_{\mu=0}^j \frac{\mu \log p(i)}{p(i)^\mu} \right] \\ &\geq \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left(\frac{1}{1 - p(\ell)^{-1}} \right) \sum_{i=1}^{r(j)} (1 - p(i)^{-1}) \frac{\log p(i)}{p(i)} \left[1 + \frac{1}{p(i)} + \dots + \frac{1}{p(i)^{j-1}} \right] \\ &= \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left(\frac{1}{1 - p(\ell)^{-1}} \right) \sum_{i=1}^{r(j)} \frac{\log p(i)}{p(i)} (1 - p(i)^{-j}). \end{aligned}$$

Recall that $\vartheta(x) = \sum_{p \leq x} \log p$ is Chebycheff's function and that $\vartheta(x) \geq (1 - \varepsilon(x))x$, $x \geq 2$, where $\varepsilon(x) \rightarrow 0$ as x tends to infinity. Thus, $\log n_j = j\vartheta(e^j) = je^j(1 + o(1))$, and thus $\log \log n_j = j(1 + o(1))$.

On the one hand, by (4.6),

$$(4.8) \quad \prod_{\ell=1}^{r(j)} (1 - p(\ell)^{-1}) = \prod_{p < e^j} (1 - p^{-1}) = \frac{e^{-\gamma}}{j} (1 + \mathcal{O}(\frac{1}{j})).$$

And on the other, by Mertens' estimate

$$(4.9) \quad \sum_{p < e^j} \frac{\log p}{p} = j + \mathcal{O}(1) \geq (1 + o(1)) \log \log n_j.$$

Thus

$$(4.10) \quad \sum_{d|n_j} \frac{\log d}{d} \geq (1 + o(1))e^\gamma (\log \log n_j)^2 \quad j \rightarrow \infty,$$

since $\zeta(j+1) \rightarrow 1$ as $j \rightarrow \infty$.

b) Let $\sigma'_{-1}(n) = \sum_{d|n, d \geq 3} 1/d$. Let also X be a discrete random variable equal to $\log d$ if $d|n$ and $d \geq 3$, with probability $1/(d\sigma'_{-1}(n))$. By using convexity of the function $x \log x$ on $[1, \infty)$, we get

$$\begin{aligned} \mathbb{E} X \log X &= \sum_{\substack{d|n \\ d \geq 3}} \frac{(\log d)(\log \log d)}{d\sigma'_{-1}(n)} \geq (\mathbb{E} X) \log (\mathbb{E} X) \\ &= \left(\sum_{\substack{d|n \\ d \geq 3}} \frac{(\log d)}{d\sigma'_{-1}(n)} \right) \log \left(\sum_{\substack{d|n \\ d \geq 3}} \frac{(\log d)}{d\sigma'_{-1}(n)} \right) \\ &\geq \left(\sum_{\substack{d|n \\ d \geq 1}} \frac{(\log d)}{d\sigma'_{-1}(n)} - C \right) \left(\log \left(\sum_{\substack{d|n \\ d \geq 1}} \frac{(\log d)}{d} - C \right) - \log \sigma_{-1}(n) \right). \end{aligned}$$

Whence

$$\sum_{\substack{d|n \\ d \geq 3}} \frac{(\log d)(\log \log d)}{d} \geq \left(\sum_{\substack{d|n \\ d \geq 1}} \frac{(\log d)}{d} - C\sigma_{-1}(n) \right) \left(\log \left(\sum_{\substack{d|n \\ d \geq 1}} \frac{(\log d)}{d} - C \right) - \log \sigma_{-1}(n) \right)$$

Letting $n = n_j$, we deduce from (4.10) that

$$\begin{aligned} \Psi(n) &\geq \sum_{\substack{d|n \\ d \geq 3}} \frac{(\log d)(\log \log d)}{d} \geq \left((1 + o(1))e^\gamma (\log \log n_j)^2 - C \log \log n_j \right) \\ &\quad \times \left(\log \{ (1 + o(1))e^\gamma (\log \log n_j)^2 - C \} - \log C \log \log n_j \right) \\ &\geq (1 + o(1))e^\gamma (\log \log n_j)^2 \log \log \log n_j. \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} \frac{\Psi(n)}{(\log \log n)^2 \log \log \log n} \geq e^\gamma.$$

c) We have

$$\Phi_1(n_j) = \sum_{i=1}^{r(j)} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{r(j)} \left(\frac{1 - p(\ell)^{-j-1}}{1 - p(\ell)^{-1}} \right) \left[\sum_{\mu=0}^j \frac{\mu(\log p(i))(\log \log p(i))}{p(i)^\mu} \right]$$

$$\begin{aligned}
&\geq \frac{1}{\zeta(j+1)} \prod_{\ell=1}^{r(j)} \left(\frac{1}{1-p(\ell)^{-1}} \right) \\
&\quad \times \sum_{i=1}^{r(j)} (1-p(i)^{-1}) \frac{(\log p(i))(\log \log p(i))}{p(i)} \left[1 + \frac{1}{p(i)} + \dots + \frac{1}{p(i)^{j-1}} \right] \\
&\geq \frac{1}{\zeta(j+1)} (e^\gamma j) \left(1 + \mathcal{O}\left(\frac{1}{j}\right) \right) \sum_{i=1}^{r(j)} \frac{(\log p(i))(\log \log p(i))}{p(i)} (1-p(i)^{-j}).
\end{aligned}$$

by (4.8). Let $0 < \varepsilon < 1$. By using (4.9), we also have for all j large enough,

$$\begin{aligned}
\sum_{p < e^j} \frac{(\log p)(\log \log p)}{p} &\geq \sum_{e^{\varepsilon j} \leq p < e^j} \frac{(\log p)(\log \log p)}{p} \\
&\geq (1+o(1))(\log(\varepsilon j)) \sum_{e^{\varepsilon j} \leq p < e^j} \frac{(\log p)}{p} \\
&\geq (1+o(1))(1-\varepsilon)j(\log(\varepsilon j))(1+\mathcal{O}(1/j)) \\
&\geq (1+o(1))(1-\varepsilon)(\log \log n_j)(\log(\varepsilon \log \log n_j)).
\end{aligned}$$

As $\log(\varepsilon \log \log n_j) \sim \log \log \log n_j$, $j \rightarrow \infty$, we have

$$\limsup_{j \rightarrow \infty} \frac{\Phi_1(n_j)}{(\log \log n_j)^2 (\log \log \log n_j)} \geq e^\gamma (1-\varepsilon).$$

As ε can be arbitrarily small, this proves (c). \square

Lemma 4.7. *We have the following estimate*

$$\Phi_2(n) \geq (\log 2) \left(\frac{P^-(n)}{P^-(n)+1} \right) \left(\prod_{i=1}^r \left(1 + \frac{1}{p_i} \right) \right) \sum_{j=1}^r \left(\frac{\log p_j}{p_j} \right).$$

Proof. We observe from (2.3) that

$$\Phi_2(r, n) \geq \sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \dots p_{r-1}^{\mu_{r-1}}} \frac{\log p_r}{p_r} \log \left[\sum_{i=1}^{r-1} \mu_i + 1 \right].$$

It is clear that the above multiple sum can contribute (is not null) only if $\max_{i=1}^{r-1} \mu_i \geq 1$, in which case $\log \left[\sum_{i=1}^{r-1} \mu_i + 1 \right] \geq \log 2$. We thus have

$$\begin{aligned}
\Phi_2(r, n) &\geq (\log 2) \left(\frac{\log p_r}{p_r} \right) \underbrace{\sum_{\mu_1=0}^{\alpha_1} \dots \sum_{\mu_{r-1}=0}^{\alpha_{r-1}} \frac{1}{p_1^{\mu_1} \dots p_{r-1}^{\mu_{r-1}}}}_{\max_{i=1}^{r-1} \mu_i \geq 1} \\
&= (\log 2) \left(\frac{\log p_r}{p_r} \right) \prod_{i=1}^{r-1} \left(1 + \sum_{\mu_i=0}^{\alpha_i} \frac{1}{p_i^{\mu_i}} \right) \\
(4.11) \quad &\geq (\log 2) \left(\frac{\log p_r}{p_r} \right) \prod_{i=1}^{r-1} \left(1 + \frac{1}{p_i} \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Phi_2(n) &\geq (\log 2) \sum_{j=1}^r \left(\frac{\log p_j}{p_j} \right) \prod_{\substack{i=1 \\ i \neq j}}^r \left(1 + \frac{1}{p_i} \right) \\
(4.12) \quad &\geq (\log 2) \left(\frac{P^-(n)}{P^-(n)+1} \right) \left(\prod_{i=1}^r \left(1 + \frac{1}{p_i} \right) \right) \sum_{j=1}^r \left(\frac{\log p_j}{p_j} \right).
\end{aligned}$$

\square

5. An application.

We deduce from Theorem 1.1 the following result.

Theorem 5.1. *Let $\eta > 1$. There exists a constant $C(\eta)$ depending on η only, such that for any finite set K of distinct integers, and any sequence of reals $\{c_k, k \in K\}$, we have*

$$(5.1) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^2}{k\ell} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \log n)^\eta \Psi(\nu).$$

Further,

$$(5.2) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^2}{k\ell} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \nu)^2 (\log \log \log \nu)^{1+\eta}.$$

This much improves Theorem 2.5 in [11] where a specific question related to Gál's inequality was investigated, see [11] for details. The interest of inequality (5.1), is naturally that the bound obtained tightly depends on the arithmetical structure of the support K of the coefficient sequence, while being close to the optimal order of magnitude $(\log \log \nu)^2$.

Theorem 5.1 is obtained as a combination of Theorem 1.1 with a slightly more general and sharper formulation of Theorem 2.5 in [11].

Theorem 5.2. *Let $\eta > 1$. Then, for any real s such that $0 < s \leq 1$, for any sequence of reals $\{c_k, k \in K\}$, we have*

$$(5.3) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \log \nu)^\eta \sum_{\delta | \nu} \frac{(\log \delta)(\log \log \delta)}{\delta^{2s-1}}.$$

The constant $C(\eta)$ depends on η only.

Remark 5.3. From Theorem 2.5-(i) in [11], follows that for every $s > 1/2$,

$$(5.4) \quad \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq \zeta(2s) \inf_{0 < \varepsilon \leq 2s-1} \frac{1+\varepsilon}{\varepsilon} \sum_{\nu \in K} c_\nu^2 \sigma_{1+\varepsilon-2s}(\nu),$$

$\sigma_u(\nu)$ being the sum of u -th powers of divisors of ν , for any real u . As

$$\sum_{\delta | \nu} \frac{(\log \delta)(\log \log \delta)}{\delta^{2s-1}} \ll \sum_{\delta | \nu} \frac{1}{\delta^{2s-1-\varepsilon}} = \sigma_{1+\varepsilon-2s}(\nu),$$

estimate (5.3) is much better than the one given (5.4).

Proof of Theorem 5.2. The proof is similar to that of Theorem 2.5 in [11] and shorter. Let $\varepsilon > 0$ and let J_ε denote the generalized Euler function. We recall that

$$(5.5) \quad J_\varepsilon(n) = \sum_{d|n} d^\varepsilon \mu\left(\frac{n}{d}\right).$$

We extend the sequence $\{c_k, k \in K\}$ to all \mathbb{N} by putting $c_k = 0$ if $k \notin K$. By Möbius' formula, we have $n^\varepsilon = \sum_{d|n} J_\varepsilon(d)$. By using Cauchy-Schwarz's inequality, we successively obtain

$$\begin{aligned} L &:= \sum_{k, \ell=1}^n c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} = \sum_{k, \ell \in K} \frac{c_k c_\ell}{k^s \ell^s} \left\{ \sum_{d \in F(K)} J_{2s}(d) \mathbf{1}_{d|k} \mathbf{1}_{d|\ell} \right\} \\ (k = ud, \ell = vd) &\leq \sum_{u, v \in F(K)} \frac{1}{u^s v^s} \left(\sum_{d \in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud} c_{vd} \right) \\ &\leq \sum_{u, v \in F(K)} \frac{1}{u^s v^s} \left(\sum_{d \in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 \right)^{1/2} \left(\sum_{d \in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{vd}^2 \right)^{1/2} \\ &= \left[\sum_{u \in F(K)} \frac{1}{u^s} \left(\sum_{d \in F(K)} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 \right)^{1/2} \right]^2 \end{aligned}$$

$$(5.6) \quad \leq \left(\sum_{u \in F(K)} \frac{1}{u^s \psi(u)} \right) \left(\sum_{\nu \in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{\substack{u \in F(K) \\ u|\nu}} J_{2s}\left(\frac{\nu}{u}\right) u^s \psi(u) \right),$$

where $\psi(u) > 0$ is a non-decreasing function on \mathbb{R}^+ . We then choose

$$\psi(u) = u^{-s} \psi_1(u) \sum_{t|u} t(\log t)(\log \log t), \quad \psi_1(u) = (\log \log \log u)^\eta.$$

Hence,

$$\begin{aligned} L &\leq \left(\sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{t|u} t(\log t)(\log \log t)} \right) \left(\sum_{\nu \in K} \frac{c_\nu^2}{\nu^{2s}} \sum_{\substack{u \in F(K) \\ u|\nu}} J_{2s}\left(\frac{\nu}{u}\right) \psi_1(u) \sum_{t|u} t(\log t)(\log \log t) \right) \\ &\leq \left(\sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{t|u} t(\log t)(\log \log t)} \right) \left(\sum_{\nu \in K} \frac{c_\nu^2 \psi_1(\nu)}{\nu^{2s}} \sum_{\substack{u \in F(K) \\ u|\nu}} J_{2s}\left(\frac{\nu}{u}\right) \sum_{t|u} t(\log t)(\log \log t) \right). \end{aligned}$$

As $\nu \in K$, we can write

$$\begin{aligned} \sum_{\substack{u \in F(K) \\ u|\nu}} J_{2s}\left(\frac{\nu}{u}\right) \sum_{t|u} t(\log t)(\log \log t) &= \sum_{u|\nu} \sum_{d|\frac{\nu}{u}} d^{2s} \mu\left(\frac{\nu}{ud}\right) \sum_{t|u} t(\log t)(\log \log t) \\ &= \sum_{d|\nu} d^{2s} \sum_{u|\frac{\nu}{d}} \mu\left(\frac{\nu}{ud}\right) \sum_{t|u} t(\log t)(\log \log t) \\ \text{(writing } u = tx) &= \sum_{d|\nu} d^{2s} \sum_{t|\frac{\nu}{d}} t(\log t)(\log \log t) \sum_{x|\frac{\nu}{dt}} \mu\left(\frac{\nu}{dtx}\right) \\ \text{(writing } \frac{\nu}{dt} = x\theta) &= \sum_{d|\nu} d^{2s} \sum_{t|\frac{\nu}{d}} t(\log t)(\log \log t) \sum_{\theta|\frac{\nu}{dt}} \mu(\theta) \\ (5.7) \quad &= \sum_{d|\nu} d^{2s} \left(\frac{\nu}{d}\right) (\log(\frac{\nu}{d})) (\log \log(\frac{\nu}{d})), \end{aligned}$$

where in the last inequality we used the fact that $\sum_{d|n} \mu(d)$ equals 1 or 0 according to $n = 1$ or $n > 1$.

Consequently,

$$\begin{aligned} L &\leq \left(\sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{t|u} t(\log t)(\log \log t)} \right) \left(\sum_{\nu \in K} \frac{c_\nu^2 \psi_1(\nu)}{\nu^{2s}} \sum_{d|\nu} d^{2s} \left(\frac{\nu}{d}\right) (\log(\frac{\nu}{d})) (\log \log(\frac{\nu}{d})) \right) \\ &= \left(\sum_{u \in F(K)} \frac{1}{\psi_1(u) \sum_{t|u} t(\log t)(\log \log t)} \right) \left(\sum_{\nu \in K} c_\nu^2 \psi_1(\nu) \sum_{\delta|\nu} \frac{1}{\delta^{2s}} \delta (\log \delta) (\log \log \delta) \right). \end{aligned}$$

From the trivial estimate $\sum_{t|u} t(\log t)(\log \log t) \geq u(\log u)(\log \log u)$, it follows that

$$\begin{aligned} \sum_{k,\ell=1}^n c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s} &\leq \left(\sum_{u \geq 1} \frac{1}{u(\log u)(\log \log u)(\log \log \log u)^\eta} \right) \\ &\quad \times \left(\sum_{\nu \in K} c_\nu^2 (\log \log \log \nu)^\eta \sum_{\delta|\nu} \frac{(\log \delta)(\log \log \delta)}{\delta^{2s-1}} \right) \\ (5.8) \quad &= C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \log \nu)^\eta \sum_{\delta|\nu} \frac{(\log \delta)(\log \log \delta)}{\delta^{2s-1}}. \end{aligned}$$

□

Proof of Theorem 5.1. Letting $s = 1$ in Theorem 5.2 we get (5.1), next using Theorem 1.1 we obtain,

$$(5.9) \quad \sum_{k, \ell=1}^n c_k c_\ell \frac{(k, \ell)^2}{k\ell} \leq C(\eta) \sum_{\nu \in K} c_\nu^2 (\log \log \nu)^2 (\log \log \log \nu)^{1+\eta},$$

which is (5.2), and thus proves Theorem 5.1. \square

6. Concluding Remarks.

The proof of Theorem 1.2 can be adapted with no difficulty to similar arithmetical functions, for instance with powers of $\log \log d$, but not to the functions $S_k(n)$, $k \geq 1$, which specifically depend on a derivation formula, see after (1.12). We remark that a simple convexity argument shows that

$$(6.1) \quad \limsup_{n \rightarrow \infty} \frac{S_k(n)}{(\log \log n)^{1+k}} \geq e^\gamma.$$

Let indeed X be a discrete random variable equal to $\log d$ if $d|n$, with probability $1/(d\sigma_{-1}(n))$. Then,

$$\mathbb{E} X^k = \sum_{d|n} \frac{(\log d)^k}{d\sigma_{-1}(n)} \geq (\mathbb{E} X)^k = \left(\sum_{d|n} \frac{\log d}{d\sigma_{-1}(n)} \right)^k.$$

Whence,

$$S_k(n) = \sum_{d|n} \frac{(\log d)^k}{d} \geq \sigma_{-1}(n)^{1-k} \left(\sum_{d|n} \frac{\log d}{d} \right)^k.$$

As $\sigma_{-1}(n) \leq (1 + o(1))e^\gamma \log \log n$, by using (4.10) we deduce that

$$\begin{aligned} S_k(n_j) &\geq (1 + o(1))e^{(1-k)\gamma} (\log \log n_j)^{1-k} (e^\gamma (\log \log n_j)^2)^k \\ &= (1 + o(1))e^\gamma (\log \log n_j)^{1+k}. \end{aligned}$$

Moreover for integers n having sufficiently spaced prime divisors, this lower bound is optimal. More precisely, there exists a constant $C(k)$ depending on k only, such that for any integer $n = \prod_{i=1}^r p_i^{\alpha_i}$ satisfying the condition $\sum_{i=1}^r \frac{1}{p_i-1} < 2^{1-k}$, one has

$$(6.2) \quad S_k(n) \leq C(k)(\log \log n)^k \sigma_{-1}(n).$$

As $\sigma_{-1}(n) \leq C \log \log n$, it follows that $S_k(n) \leq C(\eta)(\log \log n)^{1+\eta}$.

We conclude with some remarks concerning Davenport's function $w(n)$. At first, if p_1, \dots, p_r are the r first consecutive prime numbers and $n = p_1 \dots p_r$, then $w(n) \sim \log \omega(n)$. Next, the obvious bound $w(n) \ll \log \log \log n$ holds true when the prime divisors of n are large, for instance when for a given positive number B , these prime divisors, write them p_1, \dots, p_r , satisfy

$$(6.3) \quad \sum_{j=1}^r \frac{\log p_j}{p_j} \leq B \quad \text{and} \quad p_1 \dots p_r \gg e^{e^B}.$$

More generally, one can establish the following result. Let $\{p_i, i \geq 1\}$ be an increasing sequence of prime numbers enjoying the following property

$$(6.4) \quad p_1 \dots p_s \leq p_{s+1} \quad s = 1, 2, \dots$$

Numbers of the form $n = p_1 \dots p_\nu$ with $p_1 \dots p_{i-1} \leq p_i$, $2 \leq i \leq \nu$, $\nu = 1, 2, \dots$ appear as extremal numbers in some divisors questions, see Erdős and Hall [3].

Lemma 6.1. *Let $\{p_i, i \geq 1\}$ be an increasing sequence of prime numbers satisfying condition (6.4). There exists a constant C , such that if $p_1 \geq C$, then for any integer $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ such that $\alpha_i \geq 1$ for each i , we have $w(n) \leq \log \log \log n$.*

Proof. We use the following inequality. Let $0 < \theta < 1$. There exists a number h_θ such that for any $h \geq h_\theta$ and any H such that $e^{\frac{\theta}{(1-\theta)\log 2}} \leq H \leq h$, we have

$$(6.5) \quad h \leq e^h \log \frac{\log(H+h)}{\log H}.$$

Indeed, note that $\log(1+x) \geq \theta x$ if $0 \leq x \leq (1-\theta)/\theta$. Let h_θ be such that if $h \geq h_\theta$, then $h \log h \leq \theta(\log 2)e^h$. Thus

$$\begin{aligned} h &\leq e^h \theta \frac{\log 2}{\log h} \leq e^h \theta \frac{\log 2}{\log H} \leq e^h \log \left(1 + \frac{\log 2}{\log H}\right) = e^h \log \left(\frac{\log 2H}{\log H}\right) \\ &\leq e^h \log \left(\frac{\log H + h}{\log H}\right). \end{aligned}$$

We shall show by a recurrence on r that

$$(6.6) \quad \sum_{i=1}^r \frac{\log p_i}{p_i} \leq \log \log \log(p_1 \dots p_r).$$

This is trivially true if $r = 1$ by the notation made in the Introduction, and since $p \geq 2$. Assume that (6.6) is fulfilled for $s = 1, \dots, r-1$. Then, by the recurrence assumption,

$$\sum_{i=1}^r \frac{\log p_i}{p_i} \leq \log \log \log(p_1 \dots p_{r-1}) + \frac{\log p_r}{p_r}.$$

Put $H = \sum_{i=1}^{r-1} \log p_i$, $h = \log p_r$. It suffices to show that

$$\frac{\log p_r}{p_r} = \frac{h}{e^h} \leq \log \frac{\log \sum_{i=1}^r \log p_i}{\log \sum_{i=1}^{r-1} \log p_i} = \log \frac{\log H + h}{\log H},$$

But $H \leq h$, by assumption (6.4). Choose $C = e^{\frac{\theta}{(1-\theta)\log 2}}$. Then $H \geq \log p_1 \geq e^{\frac{\theta}{(1-\theta)\log 2}}$. The searched inequality thus follows from (6.5).

Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where $\alpha_i \geq 1$ for each i . We have $w(n) \leq \log \log \log(p_1 \dots p_r) \leq \log \log \log n$. \square

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