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No mysterious motor driving time forward

Multiple paths of random motion toward time irreversibility

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Abstract

In this work, I address the century old conflict between the reversible time of classical physics and the irreversible time we observe and experience every day. I formulate a mathematical proof of time irreversibility starting from the reversible time of physics. The key point is a mathematical definition of time, the first one in the history of physics, which implements the intuitive idea that time is an attribute of movement, leading to a relationship between time and path of motion. By considering fundamental principle of dynamics, it is shown that time is necessarily reversible in deterministic motion having unicity of path with given initial condition, and becomes irreversible in indeterministic (random) motion having multiplicity of paths. The proof of this irreversibility is worked out by considering the path probability as well as the path entropy, a measure of probabilistic uncertainty of path of random motion, which turns out to be increasing function of time, and an indicator of the time arrow pointing in the direction of increasing randomness and disorder.

Keywords: Time, space, irreversibility, entropy, random dynamics, path probability, information

1) Introduction

The understanding of time is a millennium old puzzle. Some open questions, still debated to date, waiting for last words, can be traced back to the beginning of our civilization [1][2][3].

The study of time has a remarkable dual character. On the one hand, the concept seems familiar to all. It is everywhere and at every instant in everyday life. Time is one of the most employed word in every language. To see time, it suffices to look at something moving around, or just to close your eyes and feel your heart beating or the tiredness coming slowly into your body. It is the time flowing, you can count it with your clock. You do not need to be mature to feel time. An animal may feel it as well. It is always taken for granted that time is familiar to everybody. This is one of the reasons why many have thought the definition of time is useless. Newton wrote, when he used time as a mathematical variable in the equation of motion, that it is unnecessary to define time because everybody knows it [5]. Pascal wrote approximately the same thing before Newton[1] [6]. As time is everywhere in Nature, in human society, in our everyday life and consciousness, it is a remarkable multidisciplinary concept. Not only physicians can dig into it, philosophers, sociologists, psychologists, writers, presumably all scientists from any domain of sciences are entitled and have the authority to talk about time under different angles and apprehensions, all doing it with lot of ease and confidence.

On the other hand, time is one of the hardest topics in science, especially in physics. Firstly, time does not have unambiguous definition to date. Several widely accepted philosophical definitions, such as time is 'mobile image of eternity' [1], 'number of movement' [2], 'order of succession' [4], or 'measure of movement' [6][7], all acknowledged as the most intuitive and clear-cut concept of time, are not completely immune from confusion. This confusion can easily be seen in the numerous different attributes and interpretations of time (examples shown below), each of them being able to present seemingly convincing arguments to claim to be correct and convincing. Secondly, the intuitive observation that time occurs from motion and seems irreversible, constantly passing from the past to the future, directly goes into conflict with the fundamental laws of physics, all (except one) being time reversible. The only time irreversible law, the second law of thermodynamics, has been interpreted as a statistical

[1] … le temps est de cette sorte. Qui le pourra définir ? Et pourquoi l'entreprendre, puisque tous les hommes conçoivent ce qu'on veut dire en parlant de temps, sans qu'on le désigne davantage ?

property of the time reversible laws, implying it may or may not happen and loses the attribute of a fundamental physics law. This conflict is one of the origins of many disparate even opposite points of view. I can cite, among many others, the following. Time does not exist [8][9]. The passage of time is an illusion [10]. Time irreversibility is a statistical impression, meaning time is reversible, albeit with small probability [11]. Time is reversible in deterministic physics [12]. Time arrow (irreversibility) comes from quantum randomness or uncertainty [13][14]. Time arrow arises from thermodynamics [14]. Time arrow emerges from chaos [15]. Time arrow comes from the initial conditions of any movement [16], or from the initial states of universe at Big Bang [17], so on and so forth. Although each of these theories has its plausible arguments, none of them has yet the last word to date. A remarkable thing common to almost all these theories is that never a theory defines time before talking about it, about its attributes and properties, as if time was something very well known to all and the trouble is only in its property and interpretation. Most debaters start by asking what time is, but go directly to the discussion of its properties (reversible or not, absolute or relative etc.), unwilling or unable to answer the key question (see examples in [21][32]).

It is worth reminding here some of the central questions remaining open to date. What is time? Is time real or illusion? Suppose time is real, is time arrow real (or illusive)? What are the past, the present and the future? Does the past still exists? Is the past still somewhere just as the past of a remote galaxy waiting for our observation? Does the future exist somewhere waiting for us, just as what is happening now around us is the future event for a remote observer in space? Is time irreversible (or is there a time arrow)? If time has been observed irreversible everywhere and all the time, why the time in all physics laws is reversible? Among all these questions and many others, the key enigma of time, to my eyes, is the last question, the conflict between widely observed time irreversibility and the time reversibility of the laws of physics, all being observed facts as well. This conflict is at the origin of many fiendish enigmas and paradoxes, some of them leading directly to non-existence of time [8][10].

This work is limited to the discussion of the time in classical physics. As mentioned above, in its long history, time in physics has never been given a univocal, rigorous and unanimously accepted mathematical definition. The absence of such a definition, to my opinion, is more or less responsible for the impressive divergence and confusion in the understanding of time. It is very often to see that the rivals in a debate are not discussing a single concept of time; each one having different understanding of the term 'time' [21]. In what follows, I propose a definition

of the time of physics, i.e., a time having all the attributes and nature, including the reversibility, of the time used in physics since Galileo and Newton, even before (see below). The simplest way to do this is to mathematize the verbal definition given by Aristotle saying time is the number of motion. This is the most general definition ever of time since motion is in every system in Nature whatever its properties and attributes, making this time usable for any system. Another advantage of this definition is this time can be associated with the path or trajectory of movement, allowing mathematical analysis of its reversibility/irreversibility based on the dynamic property of deterministic/indeterministic motions on their paths from the point of view of fundamental principles of dynamics [18][19][20].

I will avoid here discussing the debate between determinism and indeterminism in physics, which has lasted two millenniums long [37] and is still a hot topics to date [25][26]. In what follows, the analysis of time reversibility and irreversibility is simply based on two well-known phenomenological characteristics of deterministic and indeterministic motion: the uniqueness of path of the former, and the multiplicity of paths of the latter. Detailed discussion of these characteristics is given below.

2) Definition of time

By definition of time, I mean a mathematical expression of time as function of other more fundamental variables. The first purpose of this definition is to give to time a meaning with mathematical restriction and immunity from confusion. The second purpose is more specific to this work: establishing a direct and unambiguous relationship between time and path of motion allowing discussing the time property by considering the reversible and irreversible property of movement on its paths.

I discussed above some reasons for the absence of such a definition of time. Another reason is that time is always considered as a most fundamental and primitive quantity in physics, as primitive as space for example. There is an old viewpoint saying that time cannot be defined by using other more primitive quantities since there are no such quantities[2] [6][21]. This is why, despite the idea relating time to movement [1][2][4][6][7], never is time expressed as a function of space in which the movement takes place. To my eyes, the relationship between time and movement has never been *seriously* considered and taken into account in the study of time. If you look carefully into the assertion that time is an attribute of movement, it becomes obvious, first, that time is at the same epistemic level as movement, and, second, that both time and movement are at a lower epistemic level than spatial position (and space itself). Why? Because movement is defined as variation of position of an object and can only occur after a prior definition of position. You walk because you have a position you can change. The existence of your position does not need your walking, but your walking needs the existence of your position to occur. Space is just an ensemble of fixed positions. By this consideration, time becomes unquestionably less primitive than space, which makes it legitimate to express time as a function of space. The question is how to find out this expression. If you knew already time as a variable, it would be easy to say time can be written as spatial distance divided by velocity because velocity is defined as distance divided by time as we used to do in mechanics. But here we do not know time and velocity yet. We only have space supposedly defined.

It is helpful to think once again about the assertion of Aristotle that time is the number of motion according to before and after [2]. Here the number means measure of time with given unit, day, month, years etc. [7]. Aristotle insisted that the essence of time is movement, as he discussed at length in his Physics [2]. But he did not specify the property and the form of this link to movement. In what follows, I will avoid using the words *before*, *after*, *duration*, *elapse*, *period*, *anterior*, *posterior* etc., because these terms are too often associated to the concept of

[2] … je reviens à l'explication du véritable ordre, qui consiste, comme je disais, à tout définir et à tout prouver. Certainement cette méthode serait belle, mais elle est absolument impossible : car il est évident que les premiers termes qu'on voudrait définir, en supposeraient de précédents pour servir à leur explication, et que de même les premières propositions qu'on voudrait prouver en supposeraient d'autres qui les précédassent; et ainsi il est clair qu'on n'arriverait jamais aux premières. Aussi, en poussant les recherches de plus en plus, on arrive nécessairement à des mots primitifs qu'on ne peut plus définir….C'est ce que la géométrie enseigne parfaitement. Elle ne définit aucune de ces choses, espace, temps, mouvement, nombre … parce que ces termes−là désignent si naturellement les choses qu'ils signifient … que l'éclaircissement qu'on en voudrait faire apporterait plus d'obscurité que d'instruction. - Blaise Pascal, De l'esprit géométrique.

time in most languages. Defining time with them risks giving the definition a circular and self-referential character [21]. I prefer instead use the term *wait*. I mean the *wait* experienced by the observer of a movement, who must wait in order to feel a motion, to perceive the succession of changing positions. It is obvious that the ability of waiting, or the feeling of wait, is an instinctive and innate capability of most animals if not all. Nobody needs learning or knowing time to be able to experience or to feel waiting. The perception of spatial position is, so to speak, instantaneous, but the perception of movement (a succession of changing positions) of a single object needs waiting.

How to find the expression of time as function of space? This is not an easy affaire when you do not know time. In order to proceed, it is helpful to move our mind away from the modern scientific methodology and from the familiar concepts of space, time, velocity, acceleration etc., and to go back to the scientific context at the beginning of modern physics as early as 14th century when there was not yet the calculus using time as variable. Many scholars of that time, Buridan and Oresme among others, tried to investigate the relationship between speed (they called it quality) of motion and time [22]. They took time as a primitive variable without defining it, and expressed the quality, say, Q, as a function of time. To my knowledge, this is the first use of time as mathematical variable because Oresme expressed Q as vertical axis in a figure where time is horizontal axis. A uniform motion is represented by a horizontal straight line (constant velocity) in Oresme's figure [22], in which the spatial position of moving object or the distance L covered by the motion can be defined by the area below the straight line between the initial time t_0 and final time t_f, i.e., $L = Q(t_f - t_0)$.

I will follow this approach but replace time by space in the representation because space is presumably defined by positions, but time not yet. Once again, an object must have a spatial location before changing it to move. Movement is a succession of locations and consequently is at a less primitive level for an object than its spatial position. Spatial position does not need motion to exist, while motion, as well as time, needs a prior definition of spatial position to happen.

First let us consider a space containing immobile objects, each one being at its constant position. No motion at all. To characterize this space, the spatial position x is enough. Suppose a primitive but smart geometer is observing this space, it is enough for him to measure the positions of the objects and to establish a map in order to provide a complete description of the space. He does not need any other parameter than the position x.

Now let the objects move. The space becomes a movie instead of a picture. The map of the geometer is no longer sufficient because a complete description would need an infinity of map, each one corresponding to a configuration of the space, constantly changing. The geometer starts to look for other methods in order to describe the changing space. He looks at a moving body. At a given moment, he measured the body at position x_0. He *waits* a little, the body is located at x_1 away from x_0. He *waits* a little bit more, the body disappeared from x_1 to reach x_2, so on and so forth. The primitive geometer does not have yet the notion of time, he only feels *wait*. As mentioned above, *wait* is an instinct and an innate ability of human being. Our geometer understands he can use this *wait* to describe the changing space.

So he uses t_1 to denote the *wait* from x_0 to x_1, t_2 from x_1 to x_2, t_3 from x_2 to x_3 and so on until the final point of a motion x_N. He is smart enough to start his study by a simplest motion, a motion that seems uniformly traveling the distance. He measure his waits throughout this uniform motion using his heartbeat as Galileo did[3]. He quickly notices the following relationships between the consecutive *waits* and distances:

$$\frac{t_1}{t_2} = \frac{x_1 - x_0}{x_2 - x_1}, \frac{t_2}{t_3} = \frac{x_2 - x_1}{x_3 - x_2}, \ldots \frac{t_i}{t_{i+1}} = \frac{x_i - x_{i-1}}{x_{i+1} - x_i} \ldots$$

which implies an underlying rule linking the *waits* to the distances

$$t_{\mathrm{i}} = q(x_{\mathrm{i}} - x_{\mathrm{i}-1}) \qquad (1)$$

where $i = 1,2 \ldots N$. He calls q, as Oresme did, the quality of the motion determining his *wait* between different positions. He realizes that different uniform motions have different qualities, and that the quality of non-uniform motion varies as function of the position. He then establishes a figure, à la Oresme, where position x is the abscissa and the quality q_1 is the ordinate. A uniform motion has a horizontal straight line in his figure. Different uniform motions are represented by horizontals of different heights. His *wait* is then the area under the lines between the initial and final points (**Figure 1**).

[3] https://en.wikipedia.org/wiki/Galileo_Galilei

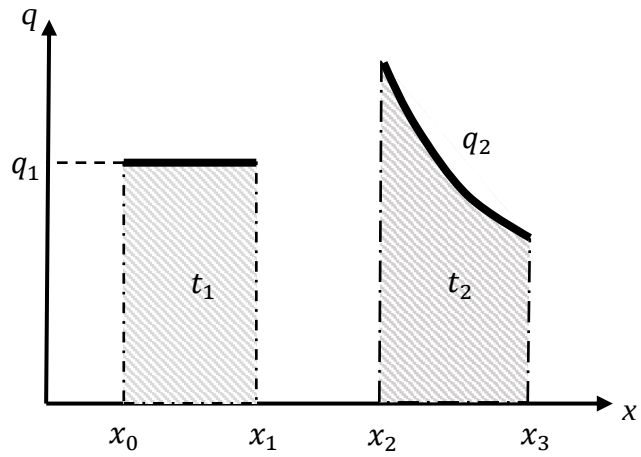

Figure 1: Representation of the quality q of two motions. The left one describes a uniform motion with a horizontal line indicating its quality q_1, The hatched area is the *wait* $t_1 = q_1(x_1 - x_0)$. The right one describes a motion with varying quality $q_2(x)$ as certain function of position *x*. The *wait* of this motion is given by $t_2 = \int_{x_2}^{x_3} q_2(x)\, dx$.

It is worth mentioning that to express the *wait* in Eq.(1) as a function of position and of the quality of motion is not an arbitrary choice by our geometer. He does not have other choice because he has only space or spatial position defined. He can proceed neither like us today to define the quality q (inverse velocity) with space and time in supposing a well-known time, nor like Oresme to define space with q and t because he is looking for t. Once the *wait* t is defined with Eq.(1) and measured as he has done with his heartbeats, the quality q can be determined from Eq.(1). If q is experimentally determined directly as can be done today, the *wait* can be calculated from Eq.(1). Notice that q is an intrinsic quality of movement, directly related to the energy of the moving body. As shown below, Eq.(1) is extremely helpful for giving to the concept of time an immunity against confusion and for studying its irreversibility.

The *wait* defined in Eq.(1) has all the attributes of the concept of time, and can be measured by comparing it to other *waits* associated with more familiar, stable and controllable motions than heart-beating, such as hourglass, the seasons, the apparent motion of the sun, of the moon, and vibratory motions, as happened in the history of metrology of time. Adoption of Eq.(1) implies *time is nothing but an attribute of movement perceived by the observer with his feeling of wait.* The feeling of *wait* is a subjective sensation. This is why time often carries psychological character and may be perceived differently by different observers. However, once related to spatial positions by Eq.(1) in physics, time becomes objective and observer-

independent, and can be measured by different devices using positions or other changing quantities in different clocks.

Eq.(1) also implies time is a relative quantity between two events. Absolute time, or the time of only one event, does not make sense in physics, being just a superfluous invention from both epistemological and practical point of view. In order to give a time moment to a single event, one must fixe the time of another one. This is a trivial convention in science as well as in everyday life.

3) What behind the name of time?

In the context of the long and difficult quest for the understanding of time, the choice of Eq.(1) has *dual advantages*. First, it offer a crystal-clear and unambiguous definition of time, objective and observer-independent. No confusion is possible about its nature. Time is nothing but a duration delimited by two positions of moving object perceived through the *wait* of the observer from one position to another, or more generally, from one value to another of any measurable changing quantity, such as temperature, pressure, light intensity, frequency etc.

No doubt is possible about its objectivity, as long as space (or the changing observable used in the definition) is objective. You can use another name, or just the name of the quantity used in the definition, the position or the configuration of the system for instance. But you cannot deny its existence and objectivity. Killing time is killing space and killing motion.

No place for the confusion about the past, the present and the future. For a given observer, the past is just the configuration of the world that have already happened, no matter whether they are observed and recorded or not. The future is the configuration to come. The observer must wait to be able to observe the future. The present is the configuration of the world he is observing or feeling. Let us look at a moving object for example, the past is all the positions of the object that have been already observed by the observer on a path $(x_0, x_1, x_2, \dots x_{i-1})$ before the present position x_i he is watching, and the future is all the positions $(x_{i+1} \dots x_N)$ the object will occupy after present position x_i and the observer has to wait to see. This discussion is quite trivial but must be limited to only one observer using one clock in a given circumstance, as

different observers may see different time moments due to relativistic effect[4]. Nevertheless, the definition of time I proposed is valid for every observer whatever his situation is.

However, complications come up when questions arise about time passage, about its reversibility and irreversibility. Intuitively, the present we are observing is flying away into the past. The past has disappeared. The future is to come. Can an observer reverse time and transform the past into future, the future into past, and recover all the present that is already in the past? Is this time reversibility possible? Can we travel back into the past? Can we remember the future just as the past? The answer from the fundamental laws of physics is a great YES! Where is the problem between the time irreversibility we observed all the time and the reversibility suggested by the laws of physics, themselves being all observed facts? These questions are the major puzzles of the concept of time. The main objective of my effort in this work is to find out an answer.

4) Time and path of movement

The main method of this work is to associate time with the trajectory of motion, which is the *second advantage* of Eq.(1) leading directly to $t = \sum_{i=1}^{N} t_i = q(x_N - x_0)$ for uniform motion and

$$t = \sum_{i=1}^{N} q_i(x_i - x_{i-1}) \tag{2}$$

for non-uniform one, where t is the time duration of the motion over the trajectory $(x_0, x_1, x_2, \dots x_i \dots x_N)$. If the path is a continuous line between x_0 and x_N, we have

$$t = \int_{x_0}^{x_N} q(x)dx \tag{3}$$

Eqs.(2) and (3) associate the time of a movement with its path. This is the starting point of my interpretation of time irreversibility.

At this stage, the sign of the time t in Eqs.(2) and (3) can be arbitrary. As time of a motion is the duration of *wait*, it is quite natural to consider it positive along a given, say, forward, direction on a trajectory. Let $q(x)$ be the quality of a forward motion along its trajectory in the

[4] The discussion of different relativistic time measured by different observers is out of scope.

forward direction with positive t, then $-q(x)$ implies the inversion of the sign of forward quality at every point on the path, yielding an opposite motion along the same trajectory, just as a rewind of the video of the motion. If a motion can be rewound in this way, it can go back to its starting point x_0, in other words, its time t is reversible. Let t' be the time of the rewound motion, we have $t' = \int_{x_0}^{x_N} -q(x)dx = -\int_{x_0}^{x_N} q(x)dx = -t$, implying a backward motion for the object to backtrack all the past positions one by one, that is, recover its past.

5) From uniqueness of path to time reversibility

The question is when and how a motion is time reversible. It is easy to prove, mathematically, that all motions obeying Newtonian equation of motion $F = m\, d^2x/dt^2$ are time reversible since this equation does not change when you replace t by $-t$ in the second derivative. As a consequence, both $x(t)$ and $x(-t)$ are solutions of Newtonian equation, i.e., movements that can actually happen. Put it differently, there is no time arrow in classical mechanics, including relativistic motion. This time symmetry is due to its deterministic character of the theories and common to all deterministic motions whatever their nature, mechanical, electromagnetic, or optic. One of the important characteristics of deterministic motion is the uniqueness of path between two spatial points for given time duration (or for given initial states or conditions). This path, in the case of a Hamiltonian system conserving its energy, is the path of least (or stationary) action[5] [23], a rule announced for the first time in the writings of the French scientist Maupertuis[6]. The action, denoted by A, can be defined by the time integral of Lagrangian functional $\mathcal{L}$ along a path between two spatial points[7], say, a and b :

$$A = \int_a^b \mathcal{L} dt \tag{4}$$

where the Lagrangian $\mathcal{L}$ has different forms for motions of different nature, and turns out to be time symmetric with $\mathcal{L}(t) = \mathcal{L}(-t)$ [23]. Hence, along the same path, the forward action from a to b, $A_{ab} = \int_a^b \mathcal{L}(t)dt$, turns out to be equal to the action of the backward motion: $A_{ba} =$

[5] https://en.wikipedia.org/wiki/Stationary-action_principle

[6] https://en.wikipedia.org/wiki/Pierre_Louis_Maupertuis

[7] https://en.wikipedia.org/wiki/Action_(physics)

$\int_b^a \mathcal{L}(-t)d(-t) = -\int_b^a \mathcal{L}dt = \int_a^b \mathcal{L}dt = A_{ab}$. As this action is the least one, the two opposite motion must follow the same unique path whatever the direction of the motion is, forward or backward. Hence, the basic principle and the concomitant laws of motion do not have preference of direction of time on the unique path. In other words, along the least action path, the past can be the future, and the future can be the past. From the known states of a past point, the movement is predictable for any moment in the future. In the same manner, from a known state of any moment, you can find out the state of any prior moment in the past. Not only the past states of a movement can be found by calculation, the moving object is also allowed to revisit its past states since the backward motion is a solution of the equation of motion on the unique path. This is a typical characteristic of the time reversibility of deterministic motion prescribed by the fundamental principle.

More generally, this time symmetry can be shown in the Euler-Lagrange equation[8], which is not limited to mechanical motion because it directly stems from the calculus of variation of least action leading to the uniqueness of the path of minimal (stationary in general) action [23].

6) Reversibility vs irreversibility, deterministic vs indeterministic

The above reasoning shows that the link between uniqueness of path in deterministic motion and time reversibility is an unquestionable mathematical certainty. Uniqueness of path, or deterministic motion, or perfect predictability, all entails time reversibility. This has been widely taken for granted both in physics and in mathematics. In what follows, I will show an approach to time irreversibility in considering the unpredictable character of indeterministic or random motion [11][15]. The term 'indeterministic' is used here to distinguish the movements losing the property of uniqueness of path from the movements having uniqueness of path. A movement is random or indeterministic whenever it has many possible paths between two points, or, in other words, whenever it can evolve into many different states from a single state, and go to a single state from many different states. The future states of such a movement can no longer be predictable with certainty. Its past states can no more be found and revisited with certainty neither.

[8] https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation

Indeed, time irreversibility is a ubiquitous property of nature. It is often interpreted as the consequence of the second law of thermodynamics stipulating that entropy always increases in closed and isolated systems evolving in the presence of random or indeterministic motion. Entropy is used in this case as a measure of the dynamic randomness. Hence, time irreversibility is here directly related to indeterministic motion. This is a result of the pioneer work by Boltzmann [11] and Maxwell [24] and remains one of today's mainstream interpretations of time irreversibility.

However, this approach initiated by Boltzmann and Maxwell is only, so to speak, quasi-indeterministic, meaning the indeterministic property of the motion is emergent from the underlying deterministic motion of the particles component of the system [11][15][24]. As well known, deterministic motion is incompatible with probabilistic description [25][26]. There is no place for probability in deterministic motion since everything there is in principle traceable back into the past and predictable towards the future with arbitrary precision. If one introduces, from practical point view, indeterministic character, or randomness, and probability, into deterministic motion of particles due to his incomplete knowledge about the motion, this randomness and the concomitant probability inevitably carry a subjective character and may differ for observers of different ability of observation. This is certainly incompatible with the objectivity of entropy and time (or motion), both being independent of the observer or of the way they are observed or measured. If the probability is subjective, the entropy, as a measure of probabilistic uncertainty or randomness, becomes subjective as well. In this case, the time stemming directly from entropy increase inevitably becomes subjective. This is one of the reasons for many to kill time or to consider time illusory [8][9][10].

On the other hand, the underlying deterministic dynamics in the thermodynamic system necessarily leads to a probabilistic interpretation of entropy increase and time irreversibility [11][12][15][23], meaning that the entropy increase and time arrow have an overwhelmingly large probability compared to entropy decrease and time inversion which, nevertheless, have non zero probability to occur. In other words, there is any way a tiny probability for the second law of thermodynamics to be violated. Tiny probability does not mean the thing does not happen (see the numerous winners of lottery each year!). Similarly, an overwhelmingly probable event may not happen. Hence, the second law can be one day violated somewhere. The violation of the second law of thermodynamics would not be that harmful if it was an isolated phenomenon during a very short time period in a small nook somewhere in the universe, and if it did not

have impact on other laws of physics. Nobody would know and care about it and everything would come back to order in a fraction of second. Nothing dramatic at all. We have even heard many talking about other universes where entropy constantly decreases and everything, including time and causality, reverses [21]. Concretely, if you please, you could imagine that a person was formed from the dust somewhere underground in an inverse process of disintegration of biological matters, came out of his tomb, got younger and younger, in spitting foods from his mouth and absorbing something elsewhere. In this way, he would finish his life (if this was a life) back into the uterus of his mother, and become a sperm escaping from an egg and entering back into the organ of his father who, in turn, would get younger and younger and experience the same backward (and awkward) life as his children. You can ask what happens to the usual laws and causality in this inverse physical, chemical and biological as well as social and sentimental process. Where are the laws of physics when a force is applied to a body to decelerate it in the opposite direction of the force? Where is the causality when you spit things from your stomach in order to feel hunger?

Notice that this scenario was imagined as if the only thing changed, compared to our normal world, was the decreasing entropy and inverse time. All the rest, the other physical and biological laws, the atoms, molecules, planet, gravitation, heat etc. would remain the same and follow the same laws as in our world of increasing entropy. This is a pure imagination. Entropy is not an isolated quantity from the rest of the world. Many often forget the second law of thermodynamics is definitively related to other laws of physics, especially the law of energy conservation. This is a unbreakable relationship marked up in the history of entropy from Carnot to Clausius, in the infallible mathematical tricks leading, from the first hypothesis of nonexistence of perpetual motion, to Carnot's reversible engine and then to the discovery of the quantity named entropy by Clausius [26]. If entropy started to decrease in an isolated world, the energy would not be conserved anymore. Why? Because, if entropy could decrease, perpetual motion would occur, giving rise to the creation of energy from nothing. Energy would lose its conserved character, implying you might fall off you planet, or the planet would collapse into its sun, or you might be burnt by a piece of ice, or simply you quickly die because you could not cook as the fire absorbs heat. Worse, your planet even could not exists since no atom would be stable as its electrons would either fly apart from or collapse onto the nuclei.

We quickly understand to what degree the second law of thermodynamics is an inviolable iron rule. The world might collapse into nil even if the rule was violated during only a fraction

of second. This perspective, together with the incompatibility between the subjectivity of the probability of deterministic systems and the objectivity of entropy and time, is, so to speak, the dead end of the probabilistic interpretation of increasing entropy and time irreversibility. This difficult situation, as well as the paradoxes conjuring up demons after demons [16], is obviously a direct consequence of the conflict between the deterministic character of the fundamental laws of physics, applied to thermodynamic systems [11][24], and the indeterministic nature of the motion in these systems.

In what follows, I propose a possible way out from this impasse, a pathway passing by the multiplicity of paths of indeterministic motion, for which the mathematics of the unique path of deterministic motion has to be extended to a probabilistic description of the dynamics over multiple paths. I will avoid the debate of determinism vs indeterminism dating back to ancient Greek[9] and showing no last words to date [15][25][26], and present below a phenomenological approach to random motion.

7) Multiple paths of random motion

As mentioned above, one of the characteristic of deterministic motion is the uniqueness of path between two spatial points with fixed time duration. This is equivalent to saying that a moving body having a given initial state (position and velocity) has only a single path. This unique path is dictated by the principle of least action in the case of Hamiltonian system.

In order to tackle random motion in similar manner, we should notice that one of its important characteristics is the multiplicity of paths between two spatial points. Whenever the destination of a moving body with given initial state is no longer predictable with certainty, it is in the realm of indeterministic motion because it may move on along different paths. In other words, leaving a given point with given velocity, the moving body can reach a given destination through different paths, as shown in **Figure 2**.

[9] https://en.wikipedia.org/wiki/Clinamen

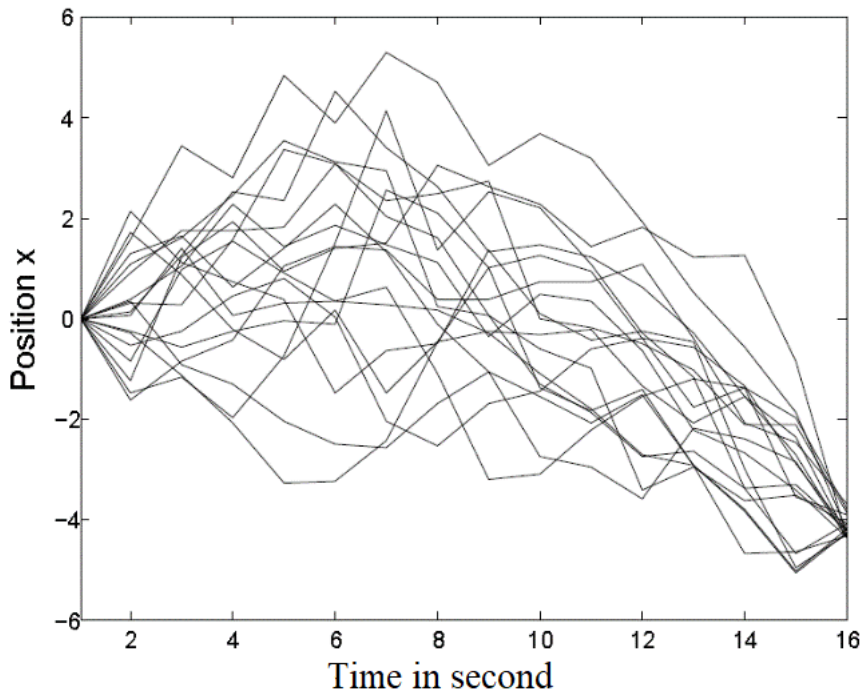

Figure 2: A simulation of the paths between two points on x-axis, $x = 0\ m$ and $x = -4\ m$, travelled during 16 seconds, by the random motion of a body subject to a weight and to a random perturbation by Gaussian noise.

Numerical experiments have been carried out to investigate the probabilities for a moving body to follow different paths [20][28]. Let $p_j = p(Q_j)$ be the probability for a body to follow the path j represented by $(x_a, x_1, x_2, \dots x_i \dots x_b)_j$ in discrete step i=0, 1, 2, …N (N can be very large) between two spatial points a and b, where Q_j is a quantity determining the likelihood of the path. The quantity Q_j, as well as the functional form of p_j, is presumably different for motions of different nature (energy conservative or not, for example) and different randomness (Gaussian noise, Lévy noise etc). Numerical experiments have shown that [20][28], if the motion of a particle is perturbed by Gaussian noise and statistically conserves its energy (Hamiltonian system), Q_j is the action A_j along the path j, and p_j is an exponential function of the action, i.e., $p_j \propto \exp(-\gamma A_j)$ [20], where γ is a positive constant characterizing the randomness. The action A_j is given by Eq.(4) where the Lagrangian is $\mathcal{L} = E_k - E_p$, E_k the kinetic energy and E_p the potential energy. If the noise perturbing the particle is not Gaussian, the function of p_j can take other forms [28].

The discussion below about time irreversibility is independent of the functional form of p_k. The result is then necessarily general and system independent.

8) How does time irreversibility arise?

The position *x* is in general a continuous variable, implying there are an infinite number of paths $(x_a, x_1, x_2, \dots x_i \dots x_b)$ between fixed *a* and *b* when the positions x_i (*i*=1, 2, …*N-1*) of the *N* steps changes continuously. This case necessitates a path probability density $\rho_j = \rho(Q_j)$. The probability of the path *j* reads $dp_j = \rho(Q_j)Dx$, where the differential Dx represents the "size" of the path (see **Figure 3**) given by the product of all the tolerances dx_i at each one of the *N* positions x_i: $Dx = dx_1 dx_2 \dots dx_N$. Note that $x_a = x_0$ is the fixed initial point so $dx_a = dx_0$ may or may not be included. The normalization of this path probability is calculated with the path integral technique[10] [33]:

$$1 = \sum_j dp_j = \int \rho(Q_j)Dx = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_{N-1}\, \rho[Q(x_a, x_1, x_2, \dots x_i \dots x_b)]$$

where the sum over the index *j* is replaced by the integral over all dx_i (*i*=1, 2, …*N-1*, $x_N = x_b$ is fixed).

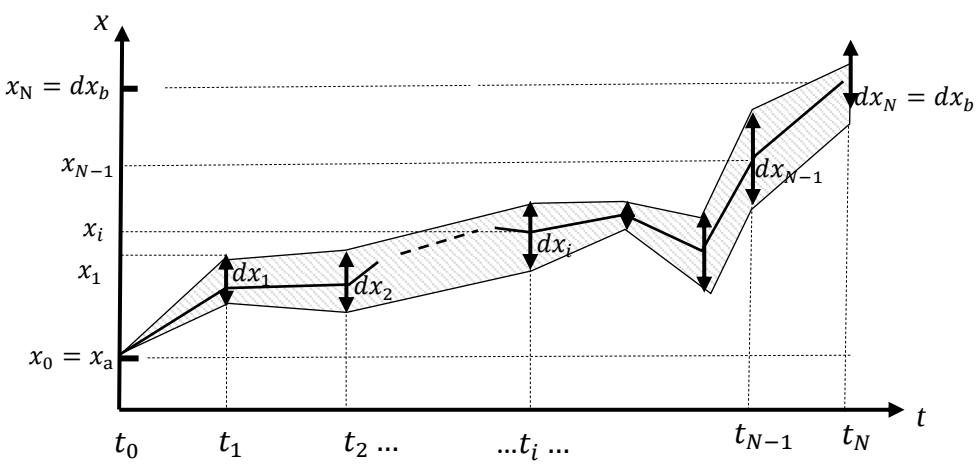

Figure 3: A 1-dimensional path of *N* steps along *x* axis from *a* and *b*, illustrated in space-time *x-t*. The hatched band represents the size $Dx = dx_1 dx_2 \dots dx_N$. The width of the path at the step on x_i is given by the tolerance dx_i. It is obvious that if one of the *N* tolerances dx_i is zero, the path is at once deadlocked since its size $Dx = 0$.

[10] http://www.scholarpedia.org/article/Path_integral

Now consider Eq.(2) defining the time $t = \sum_{i=1}^{N} q_{ij}\,(x_i - x_{i-1})_j$ over a path j between a and b. Suppose t is positive during the forward motion from a to b, and becomes negative for the inverse motion from b to a (when the sign of all the q_i 's is reversed). Let $\rho[Q(x_a, x_1, x_2 \dots x_b)_j]$ be the probability density of the forward motion along the path $(x_a, x_1, x_2 \dots x_{N-1}, x_b)_j$, and $\rho[Q(x_b, x_{N-1} \dots\ x_1, x_a)_j]$ the probability density for the inverse motion on the same path. The probability for the forward motion along the path of size $Dx_{ab} = dx_1 dx_2 \dots dx_N$ is given by $dp_j(a,b) = \rho[Q(x_a \dots x_b)_j]\, Dx_{ab}$. The probability of the backward motion on the same path j with the size $Dx_{ba} = dx_{N-1} \dots dx_1 dx_a$ is given by $dp_j(b,a) = \rho[Q(x_b \dots x_a)_j] Dx_{ba}$.

Notice that the probability of a given path, in the case of continuous variation of position and of infinite number of paths, is proportional to the size Dx of the path. Smaller is Dx, thinner is the path, and smaller is its probability, implying that the probability is zero along a path of size $Dx = 0$. More precisely, the path probability $dp_j = \rho(Q_j) Dx = 0$ if anyone of the N sizes $dx_1, dx_2, \dots dx_N$ is zero, i.e., the path is deadlocked (see figure 3). This mathematical property of the path probability does not affect the motion (forward or backward) of a particle that is free to move on and to explore all possible space with liberty, without constraint on the size of any step.

However, thing is different if constraints are imposed on the size of the paths. Some constraints come from time inversion or returning to the past positions. I will discuss separately two constraints coming from two different kinds of returnings to the past: *the strong reversibility* and the *weak reversibility of time*.

By *strong time reversibility*, I mean the motion must be reversed in such a way that the moving object recovers, step by step, all its past positions. In this way, the moving object experiences again all its past, as if nothing had happened and the world is the same as before the forward motion. Let $(x_a \rightarrow\ x_1 \rightarrow\ x_2 \dots \rightarrow x_{N-1} \rightarrow\ x_b)$ be the forward path, and $(x_b \rightarrow\ x_{N-1} \rightarrow\ \dots x_1 \rightarrow x_a)$ the backward path, where each reversed position x_i must be exactly the same as the forward x_i. No deviation of backward x_i from the forward x_i is tolerated. Any deviation, even infinitesimal one, would mean the moving body is not recovering its past position. Hence, the size of the inverse path must be strictly zero at every inverse step, leading to $Dx_{ba} = dx_{N-1} \dots dx_1 dx_0 = 0$ and to the zero probability for the backward motion along a certain path j: $dp_j(b,a) = \rho[Q(x_b \dots x_a)_j] Dx_{ba} = 0$. Notice the inverse motion along a path

is only possible with nonzero size $Dx_{ba} \neq 0$, meaning that the deviation of backward x_i from the forward x_i for all *i* is nonzero, the backward path is not exactly the same as the forward one, and the moving object is not experiencing its past, hence time is not reversed. However, the same backward path becomes a dead-end whenever you want to produce time inversion by imposing $Dx_{ba} = 0$. Therefore, *the probability of inverse motion producing time reversal must be zero along any path*.

The *weak time reversibility*, by definition, means the moving body continues its forward motion without reversing the sign of q_i's in Eq.(2), and turns out to recover or revisit at least one of its past positions, making a circular motion. Similar return to the past occurs in science fiction when someone travels in time and meets himself of twenty years ago, or his parents when they were younger. This circular motion also occurs with the eternal return of Poincaré in his recurrence theorem [29]. For simplicity, let us suppose that the moving body starts its motion at the point $x_a = x_0$, and, without reversing its time course, comes back to x_a at the step N ($x_N = x_a$) on the path ($x_a \rightarrow x_1 \rightarrow x_2 \ldots \rightarrow x_{N-1} \rightarrow x_a$). The probability of this path, say, *j*, is given by $dp_j(a,a) = \rho\left[Q(x_a, x_1, x_2 \ldots x_{N-1}, x_a)_j\right]Dx_{aa}$ with $Dx_{aa} = dx_1 dx_2 \ldots dx_{N-1} dx_a$. If we do not impose the time reversal of the whole path, none of the *N* sizes in the product $dx_1 dx_2 \ldots dx_{N-1} dx_a$ is zero. However, if you want to produce the weak time reversal for the moving body to revisit the initial position x_a, you cannot but impose zero tolerance $dx_a = 0$ at $x_N = x_a$, leading to $Dx_{aa} = 0$, and to zero path probability $dp_j(a,a) = 0$ of the return, making it impossible for the body to revisit even a single past point.

This result does not contradict the Poincaré's recurrence theorem stipulating a Hamiltonian system of finite energy must pass by any point arbitrarily close to its initial point (in finite phase space) [29]. The theorem only works for deterministic motion characterized by the uniqueness of path, while the above result is obtained for indeterministic motion characterized by the multiplicity of path. In this realm, the Liouville's theorem [30], which is the base of Poincaré's theorem, is modified, leading to the invalidation of the proof of Poincaré's theorem, as well as a constant increase of entropy during the motion [31]. The eternal return of Poincaré can be recovered whenever the randomness of the indeterministic motion cancels out, leading to the collapse of all the different paths onto a single path, the unique path of least action of deterministic motion. This point is addressed in detail below.

I would like to stress the generic character of the above proof of time irreversibility based on a universal and system independent property of indeterministic motion: multiplicity of paths.

Whenever a motion shows this property, it definitively loses the possibility to reverse its time. In what follows, this impossibility of returning to the past is regarded under a different angle related to the dynamic uncertainty of random motion.

9) Path entropy, a measure of irreversibility and of aging

When the randomness is present in a movement, there is a dynamic uncertainty associated with the path probability $p_j = p(Q_j)$, which can be called path uncertainty because it is just the uncertainty in the choice of path by the moving object. Larger is the uncertainty, more there is paths to choose by the motion between two given positions. We can use Shannon information $S_p = -\sum_i p_j \ln p_j$ as a measure of the path uncertainty, referred to as *path entropy* [31]. When the randomness of the movement gradually cancels out, the paths will all collapse onto the single path of a deterministic motion (the least action path for a Hamiltonian system). In this case, the path probability, whatever is its functional form, tends to unity $p_j \to 1$, and the path entropy tends to zero $S_p \to 0$ since $\ln p_j \to \ln 1 = 0$.

A trouble occurs with Shannon information measure when the path probability is a continuous distribution $\rho[Q]$ whose uncertainty or information measure reads $S_p = -\int \rho(Q) \ln\rho(Q) Dx$ called differential entropy[11]. The trouble is this information measure is negative when ρ >1, making it a nonphysical quantity, since a uncertainty measure or information must be non-negative. In what follows, I propose another information measure called varentropy defined by $\delta S_p = \int Q\, \delta\rho(Q) Dx$ [34]. Varentropy was motivated by the quest of an optimal (or maximizable) measure of probabilistic uncertainty [34] and has been proved to be positive in many cases where Shannon formula yields negative measure [35].

Note that the path entropy defined by $\delta S_p(a,b) = \int_{a,b} Q \delta\rho(Q) Dx$ [31] is a relative quantity between the starting point a and the endpoint b of a motion. Obviously, $\delta S_p(a,a) = 0$ because there is no motion, and $S_p(a,b) = 0$ as well if the randomness cancels out, since the

[11] The measure of the uncertainty in continuous probability distribution is still a topic of discussion. It is usually measured with Shannon formula, albeit its questionable negative value. See differential entropy.

movement is deterministic with a single path j of unitary probability $p_j = 1$. Whenever $S_p(a,b) > 0$, the movement is indeterministic and random. Larger is $S_p(a,b)$, more the movement is uncertain. Put it differently, larger is $S_p(a,b)$, more the paths are dispersed away from the unique path of deterministic counterpart of the motion. Notice that, as the path integral in the definition of $S_p(a,b)$ is done over all possible paths from a to b, $S_p(a,b)$ is only a function a and b. An example of calculation of path entropy is given in **Appendix** for Hamiltonian systems.

Suppose now a movement from a to b, but passing by a given point c. The path entropy of the motion on this path, say, j, should be $S_p(a,b)_j = S_p(a,c) + S_p(c,b)$. In this way, the path entropy of a movement from a to b passing by a series of N fixed points ($x_a \rightarrow x_1 \ldots x_{N-1} \rightarrow x_b$) should be $S_p(x_a,x_b)_j = S_p(x_a,x_1) + S_p(x_1,x_2) \ldots + S_p(x_{N-1},x_b)$. If N is sufficiently large with very small step from one fixed point to the next one, the path entropy of the movement along this path j reads

$$S_p(x_a,x_b)_j = \int_a^b dS_p(j) \tag{5}$$

where $dS_p(j)$ is an increment of path entropy at each step along the path j. As $dS_p(j) > 0$ according to its definition [31][34][35], $S_p(x_a,x_b)_j$ must be positive and increases in time as the movement goes on or as the path j gets longer and longer with point b going forwards in time. The path entropy as increasing function of time can also be proven by reductio ad absurdum as follows. Suppose $S_p(x_a,x_b)_j$ starts to decrease from a point b over M steps ($x_b \rightarrow x_{b+M}$). As S_p is a relative quantity between two points, we can define a path entropy $S_p(x_b,x_{b+M})_j$ over the segment from x_b to x_{b+M}. Since the path entropy decreases over these steps and $S_p(x_b,x_b)_j = 0$, $S_p(x_b,x_{b+M})_j$ must be negative, which contradicts the positivity of path entropy.

Hence, as time goes on and the path length from x_a to x_b grows, the movement experiences more and more dynamic uncertainty cumulated in the path entropy. In this regards, path entropy can be viewed as the age of the movement. The increase of path entropy is the aging of the dynamics, indicating the unique evolution arrow pointing in the direction of increasing

uncertainty and disorder, as Eddington asserted when he introduced the term 'time's arrow'[12] [38].

10) Concluding remarks

In this work, I have addressed the conflict between the time of physics, which is reversible according to the fundamental laws of physics, and the time we experience and observe every day, which is, obviously, irreversible. I could prove, mathematically, the time of classical physics, necessarily reversible in deterministic movement, inevitably becomes irreversible whenever the movement becomes indeterministic characterized by multiplicity of paths between two points for given duration of motion or with given initial condition. For this purpose, I proposed a mathematical definition of time to implement the intuitive idea that time is an attribute of movement, perceived by an observer through his instinctive action of waiting. This approach allows time to be expressed as a function of the path of motion, opening the way to mathematical analysis of the property of time through the dynamical property of the motion on the paths. With the help of the path probability, it is shown that time is inevitably irreversible in indeterministic motion having multiplicity of paths. Mathematically, the multiplicity of paths forbids the inversion of time by reducing the probability of revisiting exactly the past positions to zero.

In other words, one cannot rewind time when it goes on in disordered dynamics. You can never undo the randomness of an indeterministic movement, which turns out to be the motor driving time in the direction towards increasing disorder and uncertainty. This impossibility of time reversal can be explained in a popular style as follows, to the detriment of mathematical rigor. You leave your car walking away in an open field for a while, blindfolded. Then you are asked to return to your car by walking backwards along whatever path as you please, you will end up finding your car, perhaps after a long walk. Now you are imposed a constraint to walk backwards on the same footprints as your walking away. With the presence of all the uncertainty

[12] Without any mystic appeal to consciousness it is possible to find a direction of time … Let us draw an arrow arbitrarily. If as we follow the arrow we find more and more of the random element in the state of the world, then the arrow is pointed towards the future; if the random element decreases, the arrow points towards the past. That is the only distinction known to physics. This follows at once if our fundamental contention is admitted that the introduction of randomness is the only thing which cannot be undone (Eddington, The Nature of the Physical World).

in the measure of step size, in the way you move your legs, and in the irregular form of the ground which you cannot see, without mentioning all the other chances you can meet on a wild field, the probability for you to recover your car will be very small, eventually almost zero. Now let the size of your footprints and of your feet become infinitesimally small (required by the strong reversibility), I am sure that you will lose your car. If, moreover, the size of your car tends to zero (required by the weak reversibility), the probability for you to meet again your car is zero whichever path you take.

I would like to emphasize that this framework agrees with the causal arrow of time: the cause naturally precedes its effect in time in a succession of irreversible order. There is no uniqueness of effect from a given cause. A single cause can have multiple effects. Inversely, there is no uniqueness of cause to an effect. A single effect can have several causes. This is the multiplicity of paths from a cause to an effect. The courses from a cause to its effects cannot be reversed, in the same way as the time is irreversible in case of multiple paths. Hence, an effect can never be the cause of its cause, and a cause can never be the effect of its effect, contrary to the symmetric two-way road of cause-effect, and past-future in deterministic processes. This is equivalent to saying that the birth of a thing is a cause (not the only one of course) of its death, but its death can never be a cause of its birth

The result of this work is also in accordance with the thermodynamic arrow of time and entropy increase, with no need of probabilistic interpretation of the second law. In the framework of the probabilistic Hamiltonien mechanics based on the path probability, Poincaré's recurrence theorem is absent due to the modification of the Liouville's theorem (conservation of phase density distribution along the path of the motion), leading to Boltzmann's *H* theorem (no need of molecular chaos [11]) and the concomitant increase of entropy [31]. The probabilistic Hamiltonian mechanics invalidates all the arguments of deterministic mechanics against the pioneer work of Boltzmann [29][36], including the Maxwell's demon (and other ones [16]) haunting the second law of thermodynamics. Indeed, in the context of indeterministic motion with multiplicity of paths and uncontrollable randomness of the gas molecules, the demon can no more measure the velocity of a molecule with certainty and foresee its direction towards his trapdoor. It is useless for him to open the trapdoor to let a fast-moving molecule pass through because the molecule may change its direction and its velocity just in front of the door instead of going through. In this random world, does the demon himself also show more or less disorder in his action? Yes to my eyes because he is living in a ubiquitous randomness.

It is also worth emphasizing that the indeterministic nature of movement considered in this work must be objective in order for the path probability to be objective, leading to observer-independent time arrow. Subjective and observer dependent randomness due to incomplete knowledge of deterministic motion, including chaos [11][15], necessarily leads to subjective and observer dependent probability and then to the illusory character of entropy increase and time arrow.

This view is closely related to a philosophy-laden question about the origin of the objective randomness in the context of all the deterministic laws of physics ruling the world. The millennium old puzzles opposing determinism against indeterminism, and necessity against chance, are well known to all [15][25][26][37]. The debate seems deadlocked nowadays because the absence of irrefutable evidences persists for both deterministic and indeterministic view of nature [15][25][26]. One may question the validity of this work because the view of objective randomness of motion may turn out to be only a belief without justification.

In this context, I suggest an alternative approach to this question on the following ground. In view of the infallible mathematics linking time arrow to multiplicity of path, it can be said that, if a motion has multiplicity of paths, time loses its reversibility. Inversely, if a motion is time irreversible, it is certainly happening over multiple paths, since unicity of path necessarily yields reversible time. We have here two attributes of motion, one entailing another. On this ground, it is reasonable, and more logical from epistemological view, to justify the objectivity, as well as the ubiquity, of indeterministic nature of the world by the objectivity and the ubiquity of its time irreversibility, which is a unquestionable observed fact having never encountered evidence of counter-examples to date. Put it shortly, *time irreversibility is an evidence of the indeterministic nature of the world*.

I remember the answer of a student of mine when I ask the class why a motion can be easily reversed but not the time in the motion.

Sir, he said, the order of time is imposed by God.

It turns out that God plays dice, even in classical world, although he still hesitates in quantum world!

Appendix

In the case of the random motion of Hamiltonian systems following *least action* path, according to [20], the path probability density, around a path of action A between two given spatial points, reads

$$\rho(A) = \frac{\exp(-\gamma A)}{Z}$$

where the normalization constant $Z = \int \exp(-\gamma A)\, Dx$ which can be easily calculated when the action on the paths between a and b is known [33]. $\gamma \geq 0$ is a parameter characterizing the randomness of the motion and assuring that the least action path is the most probable path among all possible ones. If γ tends to zero, it is the case of maximum randomness and uniform probability distribution $\rho(A) = \frac{1}{Z}$. When γ is very large, the randomness cancels out. The motion comes back to deterministic and predictable motion with only one possible path of least action A_l with the probability $dp(A_l) = \frac{\exp(-\gamma A_l)}{Z} Dx$ where $Z = \int \exp(-\gamma A)\, Dx \to \exp(-\gamma A_l)\, Dx$ for $\gamma \to \infty$, leading to $dp(A_l) = 1$.

According to the definition of varentropy $\delta S_p = \int A\, \delta\rho(A) Dx$ [34], we can calculate the path entropy S_p for the exponential path probability density in the path probability distribution $\rho(A)$ given above

$$\delta S_p = -\frac{1}{\gamma}\int (\ln Z + \ln\rho)\, \delta\rho Dx = -\frac{1}{\gamma}\delta\int \rho(\ln Z + \ln\rho - 1)\, Dx$$
$$= \delta\left[-\frac{1}{\gamma}\int \rho \ln(Z\rho)\, Dx + C\right]$$

This implies

$$S_p = -\frac{1}{\gamma}\int \rho \ln(Z\rho)\, Dx + C = -\frac{1}{\gamma}\int \rho(-\gamma A)\, Dx + C = \bar{A} + C$$

where the average action reads

$$\bar{A} = \int A\, \rho(A)\, Dx = \frac{1}{Z}\int A \exp(-\gamma A)\, Dx = \frac{1}{Z}\int -\frac{d}{d\gamma}\exp(-\gamma A)\, Dx$$
$$= -\frac{1}{Z}\frac{d}{d\gamma}\int \exp(-\gamma A)\, Dx = -\frac{d(\ln Z)}{d\gamma}$$

leading to

$$S_p = -\frac{d(\ln Z)}{d\gamma} + C$$

When $\gamma \to \infty$ for deterministic motion having only one single path of least action A_l,

$$Z = \int \exp(-\gamma A)\, Dx \to \exp(-\gamma A_l)\, Dx$$

and

$$S_p = A_l + C$$

Considering the absence of uncertainty in deterministic motion, i.e., $S_p = 0$, we get $C = -A_l$, and

$$S_p = \bar{A} - A_l$$

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