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No mysterious motor driving time forward
Multiple paths of randomness toward time irreversibility

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Abstract

This work focus on a mathematical proof of time irreversibility based on indeterministic motion. The starting point is a mathematical definition of time associating time with movement and establishing a univocal link between time and trajectory of motion. Time irreversibility emerges inevitably in indeterministic and unpredictable process. I propose a measure of irreversibility characterizing time arrow in the direction of increasing randomness and disorder of motion.

Keywords: Time, space, irreversibility, entropy, random dynamics, path probability, information

1) Introduction

The understanding of time is a millennium old puzzle. Some open questions, still debated to date, waiting for last words, can be traced back to the beginning of our civilization [1][2][3].

The study of time has a remarkable dual character. On the one hand, the concept seems familiar to all. It is everywhere and at every instant in everyday life. Time is one of the most employed word in every language. To see time, it suffices to look at something moving around, or just to close your eyes and feel your heart beating or the tiredness coming slowly into your body. It is the time flowing, you can count it with your clock. You do not need to be mature to feel time. An animal may feel it as well. It is always taken for granted that time is familiar to everybody. This is one of the reasons why many have thought the definition of time is useless. Newton wrote, when he used time as a mathematical variable in the equation of motion, that it is unnecessary to define time because everybody knows it [5]. Pascal wrote approximately the same thing before Newton¹ [6]. As time is everywhere in Nature, in human society, in our everyday life and consciousness, it is a remarkable multidisciplinary concept. Not only physicians can dig into it, philosophers, sociologists, psychologists, writers, presumably, all scientists from any domain of sciences are entitled and have the authority to talk about time under different angles, all doing it with lot of ease and confidence.

On the other hand, time is one of the hardest topics in science, especially in physics. Firstly, time does not have unambiguous definition to date. Several widely accepted philosophical definitions, such as time is 'mobile image of eternity' [1], 'number of movement' [2], 'order of succession' [4], or 'measure of movement' [6][7], all acknowledged as the most intuitive and clear-cut concept of time, are not completely immune from confusion, as showed in the numerous different attributes and interpretations of time (examples shown below), each of them being able to present seemingly convincing arguments to claim to be correct and convincing. Secondly, the intuitive observation that time occurs from motion and seems irreversible, constantly passing from the past to the future, directly goes into conflict with the fundamental laws of physics, all (except one) being time reversible. The only time irreversible law, the second law of thermodynamics, has been interpreted as a statistical property of the time reversible laws. This conflict is one of the origins of many disparate even opposite points of

¹ ... le temps est de cette sorte. Qui le pourra définir ? Et pourquoi l'entreprendre, puisque tous les hommes conçoivent ce qu'on veut dire en parlant de temps, sans qu'on le désigne davantage ?

view. I can cite, among many others, the following major points. Time does not exist [8][9]. The passage of time is an illusion [10]. Time irreversibility is a statistical impression, meaning time is reversible, albeit with small probability [11]. Time is reversible in deterministic physics [12]. Time arrow (irreversibility) comes from quantum randomness or uncertainty [13][14]. Time arrow arises from thermodynamics [14]. Time arrow emerges from chaos [15]; Time arrow comes from the initial condition of universe at Big Bang [16], so on and so forth. Although each of these theories has its plausible arguments, none of them has the last word to date. A remarkable thing common to almost all these theories is that never a theory defines time before talking about it and about its attributes and properties, as if time was something very well known to all and the trouble is only in its property. Most debaters start by asking what time is, but go directly to the discussion of its properties (reversible or not, absolute or relative etc.) without answering the question (see examples in [20][31]).

It is worth reminding here some of the central questions remaining open to date. What is time? Is time real or illusion? Is time irreversible (or is there time arrow)? Suppose time is real, Is time arrow real or illusive given a real time? What are the past, the present and the future? Where is the past? Does the future exist somewhere waiting for us? Among all these puzzles, and many others, the key enigma, to my opinion, is about the time arrow, or time irreversibility. If this question finds its answer, the others will be easier to tackle.

This work is limited to the discussion of the time in classical physics. As mentioned above, in its long history, time has never been given a univocal, rigorous and unanimously accepted mathematical definition. The absence of such a definition, to my opinion, is more or less responsible for the impressive divergence and confusion in the understanding of time. It is not that rare to see that the rivals in a debate are not discussing a single concept of time; each one has his own understanding of time [20], making it impossible to avoid confusion and to reach a consensus.

In what follows, I will start by proposing such a definition of the time in physics. The time defined must have all the attributes and nature of the time of physics. It is nothing but a mathematization of the verbal definition given by Aristotle saying time is the number of motion. In this definition, time turns out to be associated with the path or trajectory of movement, allowing discussion of its property, with rigorous mathematics, through the property of the motion on the path (or path probability) of deterministic (or indeterministic) motion [17][18][19]. The notions of determinism and indeterminism in physics have a long history [35]

and are still hot topics of debate to date [24][25]. In what follows, without entering into the debate, I investigate two important characteristics of deterministic and indeterministic motions: the uniqueness of path for the former, and the multiplicity of paths for the latter, and their impacts on the property of time. More detailed discussion is given below.

2) Definition of time

By definition of time, I mean a mathematical expression of time based on other fundamental variables. The first purpose of this definition is to give to time a meaning with mathematical restriction and to avoid confusion in discussion. The second purpose is more specific to this work: establishing a direct and unambiguous relationship between time and path of motion, and discussing the properties of time by considering the reversible and irreversible properties of the paths.

I discussed above some reasons for the absence of such a definition of time. There is another reason: time is always considered as a most primitive quantity in physics, as primitive as space for example. There is an old viewpoint saying that time cannot be defined by using other more primitive quantities since there are no quantities² [6][20]. This is why, despite the idea relating time to movement [1][2][4][6][7], never is time expressed as a function of space. In what follows, I will argue that, if time is an attribute of movement, and if movement is composed of succession of positions (change of position of a single object), then time and movement become a little bit less primitive than space, allowing a mathematical implementation of the link time-movement, as Aristotle put it: time is the number of motion according to before and after [2]. This number obviously means measure of time with given unit, day, month, years etc [7]. Aristotle's statement clarifies the essence of time is movement, as he discussed at length in his Physics [2]. Nevertheless, he did not specify the property or the form of the link.

² ... je reviens à l'explication du véritable ordre, qui consiste, comme je disais, à tout définir et à tout prouver. Certainement cette méthode serait belle, mais elle est absolument impossible : car il est évident que les premiers termes qu'on voudrait définir, en supposeraient de précédents pour servir à leur explication, et que de même les premières propositions qu'on voudrait prouver en supposeraient d'autres qui les précédassent; et ainsi il est clair qu'on n'arriverait jamais aux premières. Aussi, en poussant les recherches de plus en plus, on arrive nécessairement à des mots primitifs qu'on ne peut plus définir....C'est ce que la géométrie enseigne parfaitement. Elle ne définit aucune de ces choses, espace, temps, mouvement, nombre ... parce que ces termes-là désignent si naturellement les choses qu'ils signifient ... que l'éclaircissement qu'on en voudrait faire apporterait plus d'obscurité que d'instruction. - Blaise Pascal, De l'esprit géométrique.

In what follows, I will avoid using the words *before*, *after*, *duration*, *elapse*, *period*, *anterior*, *posterior* etc., because these terms are too often associated to the concept of time in most languages. Defining time with them risks giving the definition a circular and self-referential character [20]. I prefer instead use the term *wait*. I mean the *wait* experienced by the observer of movement, who must wait in order to feel a motion, a change or a succession of positions. It is obvious that the ability of waiting, or the feeling of wait, is an instinctive and innate capability of most animals if not all. Nobody needs learning or knowing time to be able to experience or to feel waiting. The perception of spatial position is, so to speak, instantaneous, but the perception of movement (a succession of positions) of a single object needs this innate ability of waiting.

To follow the following reasoning, it is helpful to move our mind away from the modern scientific methodology and the familiar concepts of space, time, velocity, acceleration etc., and to go back to the scientific context at the beginning of modern physics as early as 14th century when there was not yet the calculus using time as variable. Many scholars of that time, Buridan and Oresme among others, tried to investigate the relationship between speed (they called it quality) of motion and time [21]. They took time as a primitive variable without defining it, and expressed the quality, say, Q , as a function of time. To my knowledge, this is the first use of time as mathematical variable because Oresme expressed Q as vertical axis in a figure where time is horizontal axis. A uniform motion is represented by a horizontal straight line (constant velocity) in Oresme's figure [21], in which the spatial position of moving object or the distance L covered by the motion can be defined by the area below the straight line between the initial time t_0 and final time t_f , i.e., $L = Q(t_f - t_0)$.

I will follow this approach but replace time by space in the representation. To me, space, or spatial position, as an attribute of physical object, is a little bit more basic than motion. It is obvious that an object must have a location before changing it to move. Movement is a variation of location, or a succession of locations of a single object. Given a location, an object may or may not move. In this sense, movement is a secondary attribute of an object compared to its spatial position. Spatial position does not need motion to exist, while motion, as well as time, needs the concept of spatial position to happen.

Let us suppose a universe containing immobile objects, each one being at its constant position. No motion at all. To characterize this universe, the spatial position x is enough. Suppose a primitive but smart geometer is observing the universe, it is enough for him to

measure the positions of the objects and to establish a map of the universe, in order to provide a complete description of the universe. He does not need any other parameter than the position x .

Now let the objects in the universe move. The universe now becomes a movie instead of a picture. The map of the geometer is no longer sufficient because a complete description would need an infinity of map, each one corresponding to a configuration of the universe, constantly changing. The geometer starts to look for other methods in order to describe the changing universe. He looks at a moving body. At a given moment, he measured the body at position x_0 . He *waits* a little, the body is located at x_1 away from x_0 . He *waits* a little bit more, the body disappeared from x_1 to reach x_2 , so on and so forth. The primitive geometer does not have yet the notion of time, he only feels *wait*. As mentioned above, *wait* is an instinct and an innate ability of human being. Our geometer does not need to learn to feel *wait*.

Now our geometer uses t_1 to denote the *wait* from x_0 to x_1 , t_2 from x_1 to x_2 , t_3 from x_2 to x_3 and so on until the final point of the motion x_N . He is smart enough to start his study by a simplest motion, the uniform one, and to measure his waits using his heartbeat as Galileo did³. He quickly notices the following relationships between the consecutive *waits* and distances:

$$\frac{t_1}{t_2} = \frac{x_1 - x_0}{x_2 - x_1}, \frac{t_2}{t_3} = \frac{x_2 - x_1}{x_3 - x_2}, \dots, \frac{t_i}{t_{i+1}} = \frac{x_i - x_{i-1}}{x_{i+1} - x_i} \dots$$

which implies an underlying rule linking the *waits* to the distances

$$t_i = q(x_i - x_{i-1}) \tag{1}$$

where $i = 1, 2 \dots N$. He calls q , as Oresme did, the quality of the motion determining his *wait* between different positions. He realizes that different uniform motions have different qualities, and that the quality of non-uniform motion varies as function of the position. He then establishes a figure, à la Oresme, where position x is the abscissa and the quality q_1 is the ordinate. A uniform motion has a horizontal straight line in his figure. Different uniform motions are represented by horizontals of different heights. His *wait* is then the area under the lines between the initial and final points (**Figure 1**).

³ https://en.wikipedia.org/wiki/Galileo_Galilei

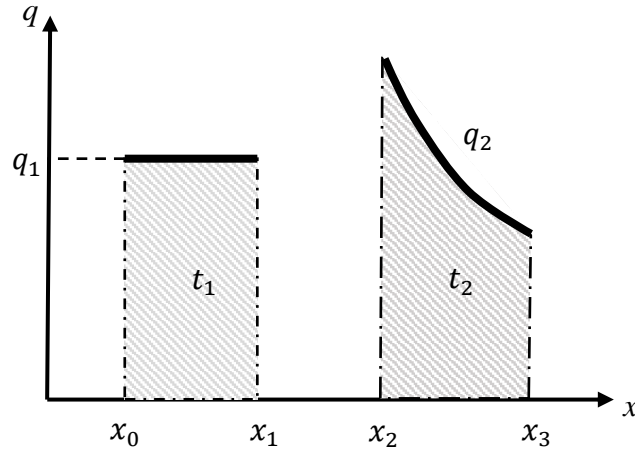


Figure 1: Representation of the quality q of two motions. The left one describes a uniform motion with a horizontal line indicating its quality q_1 , The hatched area is the *wait* $t_1 = q_1(x_1 - x_0)$. The right one describes a motion with varying quality $q_2(x)$ as certain function of position x . The *wait* of this motion is given by $t_2 = \int_{x_2}^{x_3} q_2(x) dx$.

It is worth mentioning that to express the *wait* in Eq.(1) as a function of position and of the quality of motion is a choice. One can perfectly choose any two of the three quantities as basic variables to define the third one. Oresme chose t and q to define distance or position as mentioned above. In modern physics t and x have been chosen to define q . Of course, definition of the *wait* t with Eq.(1) needs the knowledge of the quality q (inverse velocity) which can be experimentally determined either directly or by the measurement of the positions and the *wait*. Calculating q with measured *wait* t does not mean *wait* is more basic or primitive than q . q should be regarded as an intrinsic quality of movement, directly related to its energy of the moving body, and is as primitive as *wait*. The choice of Eq.(1) to define *wait* is necessary in the context of the study of time, more discussion of this point will be given in what follows.

The *wait* defined in Eq.(1) has all the attributes of the concept of time, and can be measured by comparing it to other *waits* associated to more familiar, controllable and periodic motions, such as hourglass, heartbeat, the seasons, the apparent motion of the sun, of the moon, and vibratory motions for example, as happened in the history of metrology of time. Adoption of Eq.(1) implies *time is nothing but an attribute of movement perceived by the observer with his (or its) ability of wait*. As *wait* is usually a subjective sensation, it can carry a psychological character and may be perceived differently by different observers. However, once related to spatial positions by Eq.(1), this *wait* becomes objective, measurable and independent from the

observer. Eq.(1) also implies time is a relative quantity between two events. Absolute time, or the time of only one event, does not make sense in physics, being just a superfluous invention from both epistemological and practical point of view. In order to give a time moment to a single event, one must fix the time of another one. This is a trivial convention in science as well as in everyday life.

3) What behind the name of time?

In the context of the long and difficult quest for the understanding of time, the choice of Eq.(1) has *dual advantages*. First, it offer a crystal-clear and unambiguous definition of time. No confusion is possible about its nature. Time is nothing but a duration delimited by two positions of moving object corresponding to the wait of the observer from one position to another. More generally, it is the wait between any two events or two values of a changing quantity, such as temperature, pressure, light intensity, frequency etc.

No doubt is possible about its objectivity, as long as space (or the changing observable used in the definition) is objective. You can use another name, or just the quantity in the definition, the position or the configuration of the system of interest for instance. But you cannot deny its existence and objectivity. Killing time is killing space and killing motion.

No place for the confusion about the past, the present and the future. For a given observer, the past is just the configuration of the world that have already happened, no matter whether they are observed and recorded or not. The future is the configuration to come. The observer must wait to be able to observe the future. The present is the configuration of the world he is observing or feeling. Let us look at a moving object for example, the past is all the positions of the object that have been already observed by the observer on a path $(x_0, x_1, x_2, \dots x_{i-1})$ before the present position x_i he is watching, and the future is all the positions $(x_{i+1} \dots x_N)$ the object will occupy after present position x_i and the observer has to wait to see. This discussion is quite trivial but must be limited to only one observer using one clock in a given circumstance, as different observers may see different time moments due to relativistic effect⁴. Nevertheless, the definition of time I proposed is valid for every observer whatever his situation is.

⁴ The discussion of different relativistic time measured by different observers is out of scope.

However, complications concerning the passage of time come up when questions arise about time irreversibility. Intuitively, the present is flying away into the past, the past has disappeared, the future is to come, unless the observer can reverse time and transform the past into future, the future into past, and recover all the present that is already in the past, as predicted by all the fundamental laws of physics. Is this time reversibility possible? Can we travel back into the past? Can we remember the future just as the past? These questions are the major puzzles of the concept of time, and the main objective of my effort in this work.

The main method of this work is to associate time with the trajectory of motion, which is the *second advantage* of Eq.(1) leading directly to $t = \sum_{i=1}^N t_i = q(x_N - x_0)$ for uniform motion and

$$t = \sum_{i=1}^N q_i(x_i - x_{i-1}) \quad (2)$$

for non-uniform one, where t is the time duration of the motion over the trajectory $(x_0, x_1, x_2, \dots, x_i \dots x_N)$. If the path is a continuous line between x_0 and x_N , we have

$$t = \int_{x_0}^{x_N} q(x) dx \quad (3)$$

Eqs.(2) and (3) associate the time of a motion to its path. This is the starting point of my interpretation of time irreversibility.

At this stage, the sign of the time t in Eqs.(2) and (3) can be arbitrary. As time of a motion is the duration of *wait*, it is quite natural to consider it positive along a given, say, forward, direction on a trajectory. Let $q(x)$ be the quality of a motion along its trajectory in the forward direction with positive t , then $-q(x)$ must be the quality of the opposite motion along the same trajectory corresponding to $-t$. Nevertheless, $-q(x)$ does not necessarily mean the time is inverted. It is trivial to invert a forward motion but the time continues in the same positive direction. However, $-q(x)$ offers the possibility to invert the time t of a forward motion to get $-t$ of a backward motion for the object to recover all the past positions one by one, that is, recover its past. When a motion has this property, it is called time reversible.

4) From uniqueness of path to time reversibility

The question is when and how a motion can be time reversible. It is easy to prove, mathematically, that all motions obeying Newtonian equation of motion $F = m d^2x/dt^2$ are

time reversible since this equation does not change when you replace t by $-t$ in the second derivative. This time symmetry of Newtonian equation is due to its deterministic character and common to all deterministic motion whatever its nature, mechanical, electromagnetic, or relativistic. One of the important characteristics of deterministic motion is the uniqueness of path between two spatial points (or for given initial condition). This path, in the case of a Hamiltonian system conserving its energy, is the path of the least (or stationary) action⁵ [22], a rule announced for the first time in the writings of the French scientist Maupertuis⁶. The action, denoted by A , can be defined by the time integral of Lagrangian functional \mathcal{L} along a path between two spatial points⁷, say, a and b :

$$A = \int_a^b \mathcal{L} dt \quad (4)$$

where the Lagrangian \mathcal{L} has different forms for motions of different nature, and turns out to be time symmetric in general[22]. Hence, along the same path, the forward action from a to b , $A_{forward} = \int_a^b \mathcal{L} dt$, turns out to be equal to the action of the backward motion: $A_{backward} = \int_b^a \mathcal{L} d(-t) = - \int_b^a \mathcal{L} dt = \int_a^b \mathcal{L} dt = A_{forward}$. As this action is the least one, the two opposite motion must follow the same unique path whatever the direction of the motion is, forward or backward. Hence, the basic principle and the concomitant laws of motion do not have preference of direction of motion on the unique path. In other words, along the least action path, the past can be the future, and the future can be the past, which a typical characteristic of the time reversibility of deterministic motion prescribed by the fundamental principle. This time symmetry can also be seen in the Euler-Lagrange equation⁸ which is a direct consequence of calculus of variation of least action leading to the uniqueness of path [22].

5) Reversibility vs irreversibility, deterministic vs indeterministic

The above reasoning shows that the link between deterministic motion and time reversibility is an unquestionable mathematical certainty, and widely taken for granted both in physics and

⁵ https://en.wikipedia.org/wiki/Stationary-action_principle

⁶ https://en.wikipedia.org/wiki/Pierre_Louis_Maupertuis

⁷ [https://en.wikipedia.org/wiki/Action_\(physics\)](https://en.wikipedia.org/wiki/Action_(physics))

⁸ https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation

in mathematics. In what follows, I will take a different route to time irreversibility in considering the indeterministic nature of random motion [11][15]. I use the term 'indeterministic' in the sense that the uniqueness of path is broken down. In other words, a motion is indeterministic when there are multiple paths between two given points or for given initial condition, i.e., when the motion becomes in principle unpredictable, or random.

Indeed, time irreversibility is a ubiquitous property of nature, interpreted as the consequence of the second law of thermodynamics for systems in which indeterministic motion rules. In this sense, the pioneer work by Boltzmann [11] and Maxwell [23] is a typical example and remains one of today's mainstream interpretations of time irreversibility and of the second law of thermodynamics stipulating that entropy always increase or remains constant in isolated systems.

However, this approach initiated by Boltzmann and Maxwell is only quasi-indeterministic, meaning the indeterministic property is emergent from the underlying deterministic motion of the particles component of the macroscopic system [11][15][23]. As well known, the term deterministic is incompatible with the term probabilistic [24][25]. There is no place for probability in deterministic motion since everything there is in principle traceable back into the past and predictable towards the future with precision. If one introduces, from practical point view, indeterministic character, or randomness, and probability into deterministic motion of particles due to his incomplete knowledge about the motion, this randomness and the concomitant probability inevitably carry a subjective character and may differ for observers of different ability of observation. This is certainly incompatible with the objectivity of entropy and time (or motion), changing independently from the observer or from the way they are observed or measured. This incompatibility is one of the reasons for many to kill time or to give illusory character to time [8][9][10].

On the other hand, the probabilistic interpretation of time irreversibility and entropy increase inevitably leave place to time inversion and entropy decrease, albeit with small probability, meaning a possible violation of the second law of thermodynamics. Small probability does not mean the thing does not happen (see the numerous winners of lottery each year!). Similarly, an overwhelmingly probable event may not happen. The violation of the second law of thermodynamics would not be harmful if it was an isolated phenomenon during a very short time period in a small nook somewhere in the universe, and did not have impact on other laws of physics. Nobody would know and care about it and everything would come

back to order in a fraction of second. Nothing dramatic at all. We have even heard many talking about other universes where entropy constantly decreases and everything, including time and causality, reverses [20]. Concretely, if you please, you could imagine that a person was formed from the dust somewhere underground in an inverse process of disintegration of biological matters, came out of his tomb, got younger and younger, in spitting foods from his mouth and absorbing something elsewhere. In this way, he would finish his life (if this was a life) back into the uterus of his mother, and become a sperm escaping from an egg and entering back into the organ of his father who, in turn, would get younger and younger and experience the same backward (and awkward) life as his children. You can ask what happens to the usual laws and causality in this inverse physical, chemical and biological as well as social and sentimental process. Where are the laws of physics when you apply a force to a body to decelerate it in the opposite direction of the force? Where is the causality when you spit things from your stomach in order to feel hunger?

Notice that this scenario was imagined as if the only thing changed, compared to our normal world, was the decreasing entropy and inverse time. All the rest, the other physical and biological laws, the atoms, molecules, planet, gravitation, heat etc. would remain the same and follow the same laws as in our world. This is a pure imagination. Entropy is not an isolated quantity from the rest of the world. Many often forget the second law of thermodynamics is definitively related to other laws of physics, especially the law of energy conservation. This is a unbreakable relationship marked up in the history of entropy from Carnot to Clausius, in the infallible mathematical tricks leading, from the first hypothesis of nonexistence of perpetual motion, to Carnot's reversible engine and then to the discovery of entropy [26]. If entropy started to decrease in an isolated world, the energy would not be conserved anymore. Why? Because, if entropy could decrease, perpetual motion would occur, giving rise to the creation of energy from nothing. Energy would lose its conserved character, implying you might fall off you planet, or the planet would collapse into its sun, or you might be burnt by a piece of ice, or simply you quickly die because you could not cook as the fire absorbs heat. Worse, your planet even could not exist since no atom would be stable as its electrons would either fly apart from or collapse onto the nuclei.

We quickly understand to what degree the second law of thermodynamics is an inviolable iron rule. The world might collapse into nil even if the rule was violated during only a fraction of second. This perspective, together with the incompatibility between the subjectivity of the

probability of deterministic systems and the objectivity of entropy and time, is, so to speak, the dead end of the probabilistic interpretation of increasing entropy and time irreversibility. This difficult situation is obviously a direct consequence of the conflict between the deterministic character of the fundamental laws of physics, applied to thermodynamic systems [11][23], and the indeterministic nature of the motion in these systems.

In what follows, I propose a possible way out from this impasse, a pathway passing by the multiplicity of paths of indeterministic motion, for which we have to extend the mathematics of the unique path of deterministic motion. Without entering into the old debate, dating back to ancient Greek⁹ and showing no last words to date [15][24][25], about determinism vs indeterminism, necessity vs chance, objectivity vs subjectivity of probability etc. I present below a phenomenological approach to random motion as an extension of the formalism of deterministic motion mentioned in the above section in relation with the basic principle of least action.

6) Multiple paths of random motion

As mentioned above, one of the characteristic of a deterministic motion is the uniqueness of its path between two spatial points, the unique path being dictated by the least action.

In order to tackle random motion in similar manner, we should notice that one of its important characteristics is the multiplicity of its paths between two spatial points. Whenever the destination of a moving body with given initial condition (position and velocity) is no longer predictable, it is in the realm of indeterministic motion because it may move on along different paths. In other words, leaving from a given point with given velocity, the moving body can reach a given destination through different paths, as shown in **Figure 2**.

⁹ <https://en.wikipedia.org/wiki/Clinamen>

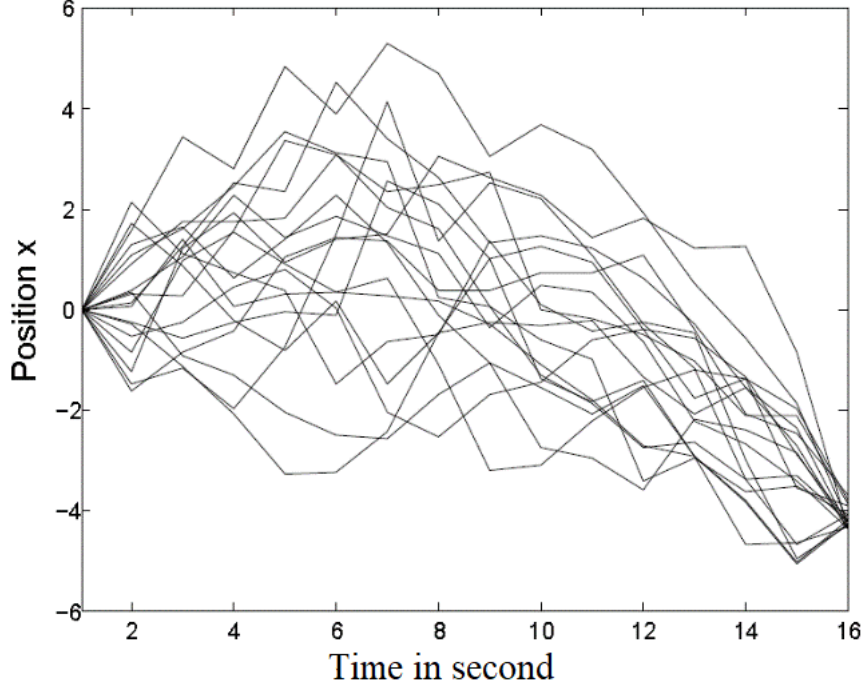


Figure 2: A simulation of the paths between two points on x -axis, $x = 0 \text{ m}$ and $x = -4 \text{ m}$, travelled during 16 seconds, by the random motion of a body subject to a weight and to a random perturbation by Gaussian noise.

It is possible to investigate experimentally the probabilities for the moving body to follow different paths [19][27]. Let $p_j = p(Q_j)$ be the probability for a body to follow the path j represented, for the sake of convenience, by $(x_a, x_1, x_2, \dots, x_i \dots, x_b)_j$ in discrete step $i=0, 1, 2, \dots, N$ (N may be very large) between two spatial points a and b , where Q_j is a quantity as random variable determining the likelihood of the path. The quantity Q_j , as well as the functional form of p_j , is presumably different for motions of different nature (energy conservative or not, for example) and different randomness (Gaussian noise, Lévy noise etc). Numerical experiments have shown that, if the motion of a particle is perturbed by Gaussian noise and statistically conserves its energy (Hamiltonian system), Q_j is the action A_j along the path j , and p_j is an exponential function of the action, i.e., $p_j \propto \exp(-\gamma A_j)$ [19], where γ is a positive constant characterizing the randomness. The action A_j is given by Eq.(4) where the Lagrangian is $\mathcal{L} = E_k - E_p$, E_k the kinetic energy and E_p the potential energy. If the noise perturbing the particle is not Gaussian, the function of p_j can take other forms [27].

The discussion below about time irreversibility is independent of the functional form of p_k , and necessarily general and system independent.

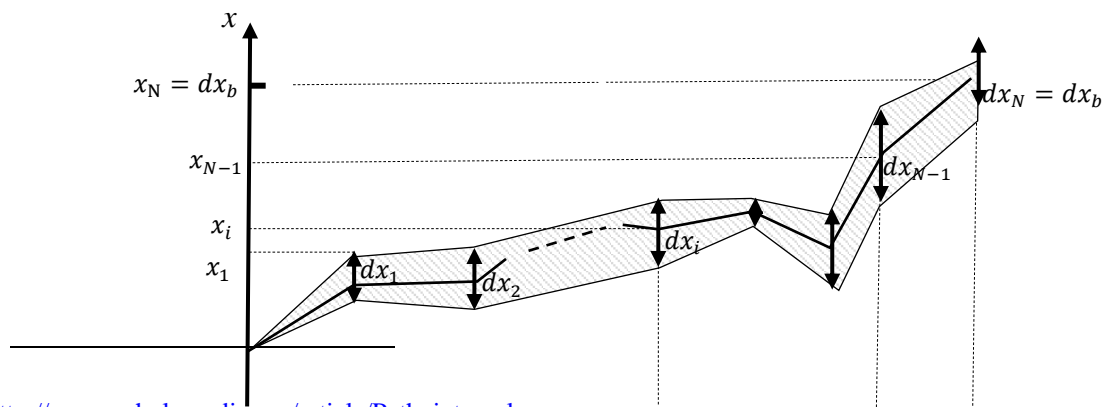
7) How does time irreversibility arise?

The position x is in general a continuous variable, implying there are an infinite number of paths $(x_a, x_1, x_2, \dots, x_i \dots x_b)$ between fixed a and b when the positions x_i ($i=1, 2, \dots, N-1$) of the N steps changes continuously. This case necessitates a path probability density $\rho_j = \rho(Q_j)$, and the probability of the path j reads $dp_j = \rho(Q_j)Dx$, where the differential Dx represents the "size" of the path (see Figure 3) given by the product of all the tolerances dx_i at each one of the N positions x_i : $Dx = dx_1 dx_2 \dots dx_N$. Note that $x_a = x_0$ is the fixed initial point so $dx_a = dx_0$ may or may not be included. The normalization of this path probability is calculated with the path integral technique¹⁰ [32]:

$$1 = \sum_j dp_j = \int \rho(Q_j)Dx = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_{N-1} \rho[Q(x_a, x_1, x_2, \dots, x_i \dots x_b)]$$

where the sum over the index j is replaced by the integral over all dx_i ($i=1, 2, \dots, N-1$, $x_N = x_b$ is fixed).

Now consider Eq.(2) defining the time $t = \sum_{i=1}^N q_{ij} (x_i - x_{i-1})_j$ over a path j between a and b . Suppose t is positive during the forward motion from a to b , and *may be* negative for the inverse motion from b to a (when the sign of all the q_i 's is reversed). Let $\rho[Q(x_a, x_1, x_2 \dots x_b)_j]$ be the probability density of the forward motion along the path $(x_a, x_1, x_2 \dots x_{N-1}, x_b)_j$, and $\rho[Q(x_b, x_{N-1} \dots x_1, x_a)_j]$ the probability density for the inverse motion on the same path. The probability for the forward motion along the path of size $Dx_{ab} = dx_1 dx_2 \dots dx_N$ is given by $dp_j(a, b) = \rho[Q(x_a \dots x_b)_j] Dx_{ab}$. The probability of the backward motion on the same path j with the size $Dx_{ba} = dx_{N-1} \dots dx_1 dx_a$ is given by $dp_j(b, a) = \rho[Q(x_b \dots x_a)_j] Dx_{ba}$.



¹⁰ http://www.scholarpedia.org/article/Path_integral

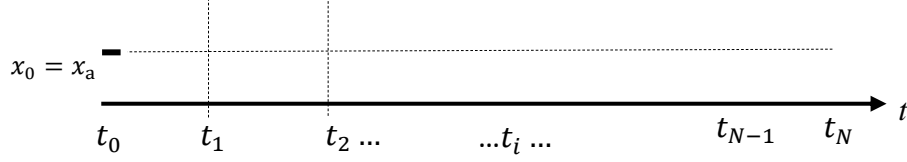


Figure 3: A 1-dimensional path of N steps along x axis from a and b , illustrated in space-time $x-t$. The hatched band represents the size $Dx = dx_1 dx_2 \dots dx_N$. The width of the path at the step on x_i is given by the tolerance dx_i . It is obvious that if one of the N tolerances dx_i is zero, the path is at once deadlocked since its size $Dx = 0$.

Notice that the probability of a given path, in the case of continuous variation of position and of infinite number of paths, is proportional to the size Dx of the path. Smaller is Dx , thinner is the path, and smaller is its probability, implying that the probability is zero along a path of size $Dx = 0$. More precisely, the path probability $dp_j = \rho(Q_j)Dx = 0$ if anyone of the N sizes dx_1, dx_2, \dots, dx_N is zero, i.e., the path is deadlocked (see figure 3). This mathematical property of the path probability does not affect the motion (forward or backward) of a particle that is free to move on and to explore all possible space with liberty, without constraint on the size of any step.

However, thing is different if some constraints are imposed on the size of the paths. A possible constraint comes from time inversion or return to the past of the motion. I will discuss separately two constraints coming from two different kinds of returns to the past: the strong reversibility and the weak reversibility of time.

By *strong time reversibility*, I mean the motion must be reversed in such a way that the moving object recovers, step by step, all its past positions. In this way, the moving object experiences again all its past, as if nothing had happened and the world is the same as before the forward motion. In term of path, if the forward path is denoted by $(x_a \rightarrow x_1 \rightarrow x_2 \dots \rightarrow x_{N-1} \rightarrow x_b)$, the backward path must be $(x_b \rightarrow x_{N-1} \rightarrow \dots x_1 \rightarrow x_a)$, where each reversed position x_i must be exactly the same as the forward x_i . No deviation of backward x_i from the forward x_i is tolerated. Any deviation, even infinitesimal one, would mean the moving body is not recovering its past position. Hence, the size of the inverse path must be strictly zero at every inverse step, leading to $Dx_{ba} = dx_{N-1} \dots dx_1 dx_0 = 0$ and to the zero probability of the backward motion along a path j : $dp_j(b, a) = \rho[Q(x_b \dots x_a)_j]Dx_{ba} = 0$. Notice the inverse

motion along a path is only possible with nonzero size $Dx_{ba} \neq 0$, meaning that the deviation of backward x_i from the forward x_i for all i is nonzero, the backward path is not exactly the same as the forward one, and the moving object is not experiencing its past, hence time is not reversed despite the reversed motion. However, the same backward path becomes a dead-end whenever you want to produce time inversion by imposing $Dx_{ba} = 0$. *The probability of inverse motion producing time reversal must be zero along any path.*

What about the *weak time reversibility*? By definition, weak reversibility means the moving body is free to continue its forward motion without reversing the sign of q_i 's in Eq.(2), and turns out to recover or revisit at least one of its past positions, making a circular motion. Similar return to the past occurs in science fiction when someone travels in time and meets himself of twenty years ago, or his parents when they were younger. This circular motion also occurs with the eternal return of Poincaré in his recurrence theorem [28]. For simplicity, let us suppose that the moving body starts its motion at the point $x_a = x_0$, and, without reversing its time course, comes back to x_a at the step N ($x_N = x_a$) on the path ($x_a \rightarrow x_1 \rightarrow x_2 \dots \rightarrow x_{N-1} \rightarrow x_a$). The probability of this path, say, j , is given by $dp_j(a, a) = \rho[Q(x_a, x_1, x_2 \dots x_{N-1}, x_a)_j]Dx_{aa}$ with $Dx_{aa} = dx_1 dx_2 \dots dx_{N-1} dx_a$. If we do not impose the time reversal of the whole path, none of the $N-1$ sizes in the product $dx_1 dx_2 \dots dx_{N-1} dx_a$ is constrained to be zero. But if you want to produce the weak time reversal for the moving body to revisit the initial position x_a , you cannot but impose zero tolerance $dx_a = 0$ at $x_N = x_a$, leading to $Dx_{aa} = 0$, and to zero path probability $dp_j(a, a) = 0$ of the return, making it impossible for the body to revisit even a single past point.

This result does not contradict the Poincaré's recurrence theorem stipulating a Hamiltonian system of finite energy can come to any point arbitrarily close to its initial point (in finite phase space) [28]. The theorem only works for deterministic motion characterized by the uniqueness of path between a and b , while the above result is obtained for indeterministic motion characterized by the multiplicity of path. In this realm, the Liouville's theorem [29], which is the base of Poincaré's theorem, is modified so as to invalidate the proof of Poincaré's theorem and to yield a constant increase of entropy during the motion [30]. The eternal return of Poincaré is recovered whenever the randomness of the indeterministic motion cancels out, leading to the collapse of all the different paths onto a single path, the unique path of least action of deterministic motion. This point is addressed in detail below.

I would like to stress the generic character of the above proof of time irreversibility based on a universal and system independent property of indeterministic motion: multiplicity of paths. Whenever a motion shows this property, it loses the possibility to reverse its time.

8) A measure of time irreversibility

Whatever is the form of the probability $p_j = p(Q_j)$, or of its density $\rho[Q_j]$, when the randomness gradually cancels out, p_j must approach to unity since there is only one path for deterministic motion. In the case of the mechanical motion of Hamiltonian systems following *least action* path, the single surviving path after the collapse of the multiple paths must be the path of least action. This is a case where we can discuss quantitatively the measure of uncertainty of indeterministic motion.

According to [19], the path probability density, around a path of action A between two given spatial points, reads

$$\rho(A) = \frac{\exp(-\gamma A)}{Z} \quad (5)$$

where the normalization constant $Z = \int \exp(-\gamma A) D\mathbf{x}$ which can be easily calculated when the action on the paths between a and b is known [32]. $\gamma \geq 0$ is a parameter characterizing the randomness of the motion and assuring that the least action path is the most probable path among all possible ones. If γ tends to zero, it is the case of maximum randomness and uniform probability distribution $\rho(A) = \frac{1}{Z}$. When γ is very large, the randomness cancels out. The motion comes back to deterministic and predictable motion with only one possible path of least action A_l with the probability $dp(A_l) = \frac{\exp(-\gamma A_l)}{Z} D\mathbf{x}$ where $Z = \int \exp(-\gamma A) D\mathbf{x} \rightarrow \exp(-\gamma A_l) D\mathbf{x}$ for $\gamma \rightarrow \infty$, leading to $dp(A_l) = 1$.

The probabilistic uncertainty of path, say, U , in this distribution can be calculated by using the variational formula $\delta U = \int A \delta \rho(A) D\mathbf{x}$, referred to as varentropy [33], as an alternative of Shannon formula which risks yielding, in the case of continuous distribution, negative measure

of uncertainty¹¹ without physical meaning. For the distribution given by Eq.(5), this path uncertainty reads

$$U = -\frac{d(\ln Z)}{d\gamma} - A_l \quad (6)$$

or $U = \bar{A} - A_l$ where $\bar{A} = \int A \rho(A) Dx = -\frac{d(\ln Z)}{d\gamma}$ is the average action (See **appendix**). U is always positive since $\bar{A} \geq A_l$ and tends to zero when $\gamma \rightarrow \infty$ and $\bar{A} \rightarrow A_l$.

This formulation of the dynamical uncertainty implies that, whenever $U > 0$, the motion must be indeterministic with multiple paths, and lose time reversibility. In other words, time is reversible iff $U = 0$ when the uniqueness of path is recovered with $\gamma \rightarrow \infty$.

The positivity of U over any spatial interval between two points guarantees it is an increasing function of time. To show this, suppose that the path uncertainty U of a motion over a certain long trajectory starts to decrease over a segment from a point c to a point d . As U is a relative quantity between any two points, we can choose the point c as its zero point in a new reference, leading to negative U between c and d , which contradicts the positivity of U . Hence, as time goes on, the path uncertainty U always increases, and the motion experiences more and more randomness. In other words, time arrow is pointing in the direction of increasing uncertainty of motion, as Eddington indicated when he introduced the term 'time's arrow'¹² [36].

9) Concluding remarks

I have proposed a mathematical definition of time in order to address time irreversibility in relation with the trajectory of motion. This definition is a mathematical implementation of the millennium old idea that time is an attribute of movement and an order of succession (of changing events) [2][4]. Through this definition, time is definitively laid down on the path of motion, with a crystal-clear attribute linked to the *wait* of the observer, and a distinct behavior

¹¹ The measure of the uncertainty in continuous probability distribution is still a topic of discussion. It is usually measured with Shannon formula, albeit it questionable negative value. See [differential entropy](#).

¹² Without any mystic appeal to consciousness it is possible to find a direction of time ... Let us draw an arrow arbitrarily. If as we follow the arrow we find more and more of the random element in the state of the world, then the arrow is pointed towards the future; if the random element decreases, the arrow points towards the past. That is the only distinction known to physics. This follows at once if our fundamental contention is admitted that the introduction of randomness is the only thing which cannot be undone.

described with mathematical rigor. This distinct behavior from any other physical quantity is its reversibility in deterministic motion on a single path between two positions, and its irreversibility whenever the uniqueness of path is broken in indeterministic motion having multiplicity (infinity) of paths forbidding its inversion.

In other words, time can never come back when it goes on in disorder. You can never undo the randomness of a movement which turns out to be the motor driving time in the direction towards increasing disorder and uncertainty. This time can be explained in popular terms as follows, to the detriment of mathematical rigor. You leave your car by walking in an open field, blindfolded. You are now asked to return to your car by walking backwards along whatever path as you please, you will end up finding your car, perhaps after a long walk. Now you are imposed a constraint to walk backwards on the same footprints as your walking away, within all the uncertainty in your measure of step length, in the way you move your legs, and in the form of the ground which you cannot see, without mentioning all the other chances you can meet on a wild field, the probability for you to recover your car will be very small, eventually almost zero. Now if the size of your footprints and of your feet become infinitesimally small (required by the strong irreversibility), I am sure that you will lose your car. If, moreover, the size of your car tends to zero (required by the weak irreversibility), the probability for you to meet again your car is zero whichever path you take.

The car is your past you can no longer revisit whether it is happy or not, and you are walking in your life, in a direction you can never invert, towards an unavoidable destination.

I would like to emphasize that this framework agrees with the causal arrow of time: the cause naturally precede its effect in a succession of irreversible order. There is no uniqueness of effect from a given cause. A single cause can have multiple effects. The courses from a cause to its effects can never be reversed, in the same way as the paths. An effect can never be the cause of its cause, and a cause can never be the effect of its effect, contrary to the symmetric two-way road of cause-effect, and past-future in deterministic processes.

The result of this work is also in accordance with the thermodynamic arrow of time and entropy increase, with no need of probabilistic interpretation of the second law. In the framework of the probabilistic Hamiltonien mechanics based on the path probability, Poincaré's recurrence theorem is absent due to the modification of the Liouville's theorem (conservation of phase density distribution along the path of the motion), leading to Boltzmann's H theorem (no need of molecular chaos [11]) and the concomitant increase of entropy [30]. Probabilistic

Hamiltonian mechanics invalidates the arguments of deterministic mechanics against the pioneer work of Boltzmann [28][34].

It is also worth emphasizing that the indeterministic nature of movement considered in this work must be objective in order for the path probability to be objective, leading to observer independent time arrow. Subjective and observer dependent randomness due to incomplete knowledge about the deterministic motion, including chaos [11], necessarily leads to subjective or observer dependent probability and then to the illusory character of entropy increase and of time arrow. This discussion is closely related to a philosophy-laden question about the origin of the objective randomness in the context of all the deterministic laws of physics ruling the world. The millennium old puzzles opposing determinism and indeterminism [35], as well as necessity and chance, are well known to all. The debate seems deadlocked nowadays because the absence of irrefutable evidences persists for either deterministic or indeterministic view of nature [15][24][25], and no progress is in view. So one may question the validity of this work because the view of objective randomness of motion may turn out to be only a belief without justification.

I suggest an alternative approach to this question on the following ground. In view of the infallible mathematics linking time arrow to multiplicity of path I shown above, it can be said with certainty that, if a motion has an infinite number of paths between any two points, time loses its reversibility. Inversely, if a motion is time irreversible, it certainly has multiplicity of paths (among other properties of random motion), since uniqueness of path necessarily yields reversible time. We have two attributes of motion, one entailing another. On this ground, it is reasonable, and more logical from epistemological view, to justify the objectivity, as well as the ubiquity, of indeterministic nature of motion by the objectivity and the ubiquity of time irreversibility, especially because the latter is universally acknowledged as an everywhere observed fact in nature and has never encountered evidence of counter-examples to date.

I never forget an answer of a student of mine when I ask the class why a motion can be easily reversed but not the time of the same motion.

Sir, he said, order of time is imposed by God.

It turns out that God must play dice, even in classical world, although He is still hesitating in quantum world!

Appendix

According to the definition of varentropy $\delta U = \int A \delta \rho(A) Dx$ [33], we can calculate U for the exponential path probability density given by Eq.(5)

$$\begin{aligned} \delta U &= -\frac{1}{\gamma} \int (\ln Z + \ln \rho) \delta \rho Dx = -\frac{1}{\gamma} \delta \int \rho (\ln Z + \ln \rho - 1) Dx \\ &= \delta \left[-\frac{1}{\gamma} \int \rho \ln(Z\rho) Dx + C \right] \end{aligned}$$

This implies

$$U = -\frac{1}{\gamma} \int \rho \ln(Z\rho) Dx + C = -\frac{1}{\gamma} \int \rho(-\gamma A) Dx + C = \bar{A} + C$$

where the average action reads

$$\begin{aligned} \bar{A} &= \int A \rho(A) Dx = \frac{1}{Z} \int A \exp(-\gamma A) Dx = \frac{1}{Z} \int -\frac{d}{d\gamma} \exp(-\gamma A) Dx \\ &= -\frac{1}{Z} \frac{d}{d\gamma} \int \exp(-\gamma A) Dx = -\frac{d(\ln Z)}{d\gamma} \end{aligned}$$

leading to

$$U = -\frac{d(\ln Z)}{d\gamma} + C$$

When $\gamma \rightarrow \infty$ for deterministic motion having only one single path of least action A_l ,

$$Z = \int \exp(-\gamma A) Dx \rightarrow \exp(-\gamma A_l) Dx$$

and

$$U = A_l + C$$

Considering the absence of uncertainty in deterministic motion, i.e., $U = 0$, we get $C = -A_l$,

and

$$U = \bar{A} - A_l$$

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