No mysterious motor driving time forward Multiple paths of randomness toward time irreversibility

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Abstract

In this work, we address the concept of time within classical physics with the aim of solving some old questions concerning the definition of time and the conflict between the time reversibility of classical physics and the time irreversibility observed in every physics system. We start from a mathematical definition of time based on the intuitive idea that time is an attribute of movement or change, an attribute perceived by the observer through the action of *wait*. In this definition, the *wait*, a mind-dependent concept, is mathematically related to successive positions or path of movement, giving birth to an objective time. It is shown that time defined in this way possesses all the properties of the time of physics, and is necessarily reversible for deterministic motion having uniqueness of path prescribed by the principle of least action. However, time irreversibility emerges for indeterministic (random) motion occurring over multiple paths. A mathematical proof of time irreversibility is formulated by using path probability. The possibility to characterize time irreversibility using path entropy is also discussed.

Keywords: Time; space; irreversibility; entropy; random motion; path probability; information

1) Introduction

As a millennium-old puzzle, time continues to elude us; many open questions can be traced back to the beginning of our civilization [1][2][3][4]. Today we know very well how to measure time with moving objects (moon, earth, sun, heart-beating or clocks). However, we do not know what time is. We do not know how to define it. Many even refuse or do not want to define it. We do not understand many of its essential properties and behaviors. Despite all the progress in science, we can hardly say that the apprehension of time has progressed since Aristotle [2] and Saint Augustine [3]. This paper presents the result of an effort to make some progress in this seemingly dead-end situation in the understanding of time.

We have noticed that time has a remarkable dual character. On the one hand, the concept seems familiar to all. It is everywhere and at every instant in everyday life. Time is one of the most employed word in every language. To feel time, we just need to look at a clock or something moving around, or to close our eyes and to wait for the break of dawn in bed during a sleepless night. An animal can feel time as well when, for example, a hungry dog waiting for food with impatience. It is taken for granted that time is familiar to everybody. This is one of the reasons as to why many have thought that the definition of time is useless. Newton wrote, when he used time as a mathematical variable in the equation of motion, that it is unnecessary to define time because everybody knows it [5]. Pascal wrote approximately the same thing before Newton[1] [6]. As time is everywhere in Nature, in human society, in our everyday life and consciousness, it is a remarkable multidisciplinary concept. Not only physicians can dig into it, philosophers, sociologists, neuroscientists, psychologists, and presumably all scientists from any domain of sciences are entitled and have the authority to talk about time from different angles and apprehensions.

On the other hand, time is one of the hardest topics in science, especially in physics. It’s true nature is still a mystery today. Strangely enough, as a so familiar concept and a indispensable mathematical variable in science, time does not have a rigorous and unanimous definition. There are some conceptual definitions, such as time is 'mobile image of eternity' [1], 'number of movement' [2], 'order of succession' [4], or 'measure of movement' [6][7]. None of

[1] … le temps est de cette sorte. Qui le pourra définir ? Et pourquoi l'entreprendre, puisque tous les hommes conçoivent ce qu'on veut dire en parlant de temps, sans qu'on le désigne davantage ? [6]

these definitions is clear-cut enough to be immune from confusion. Today the confusion is such that we can mention a crisis of time: it would be very difficult to make an exhaustive list of the different interpretations and understandings of time [2] [9]-[21].

One of the fundamental mysteries of time is the conflict between the time *irreversibility* observed everywhere in nature where time eternally heads from the past to the future, and the time *reversibility* in the fundamental laws of nature in which the past and the future are symmetric. The only time irreversible law, the second law of thermodynamics, has been interpreted as a statistical property emergent from the time reversible laws [17], meaning that the second law is no longer a fundamental law of physics; it may or may not happen, and can be violated with small likelihood[3] [22]. This conflict is one of the causes of the impressive divergence of understanding of time.

In this work, we propose a solution of this conflict by introducing mathematical tools into the study of time within classical physics[4]. We think that a rigorous definition of time is now necessary to make progress in the understanding of time. Thus, our starting point is a mathematical definition of time relating time to the path of motion. Such a definition, the first one in physics to our knowledge, is helpful for eliminating confusion about the nature of time and for proving rigorously time irreversibility with mathematical tools.

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[2] We just give some examples here. Time does not exist [9][10]. The passage of time is an illusion [11]. Time irreversibility is a statistical impression, meaning time is reversible, albeit with small probability [12]. Time is reversible in deterministic physics [13]. Time arrow (not time itself) comes from quantum randomness or uncertainty [14][15][16]. Time arrow emerges from thermodynamics as a statistical property [18]. Time arrow emerges from chaos [18], and from the initial conditions [19][20]. Time itself (not only its arrow) emerges from quantum world [19][21].

[3] In [22] it is proved that the second law of thermodynamics has an unbreakable connection to the laws of energy and mass conservation, so that its violation would lead to the failure of these laws and of the most if not all of the laws of physics and chemistry. Therefore the second law is as fundamental as the laws of conservation of energy and mass.

[4] As mentioned in the above footnote, there has been efforts to understand time from quantum considerations [14][15][16][19][21]. In our opinion, quantum interpretation of time is premature in the context of today's incomplete understanding of quantum world itself. Besides, time flowing from past to future is omnipresent in the macroscopic world where quantum efforts are absent or negligible. It is hence necessary to solve the time puzzle in classical world in first place before looking into the time of quantum world.

2) What is time?

a) Can we define time?

By definition of time, we mean a mathematical expression of time as a function of other more fundamental quantities than time. Such a definition has been considered unnecessary or impossible [5][6][25] simply because time is always considered one of the most fundamental and primitive quantities in physics[5], as primitive as space for example. This explains why, despite the intuitive and reasonable idea associating time with movement [1][2][4][6][7], time has never been considered a function of space, or more precisely, a function of movement in space, and has always been regarded as a quality independent of and parallel to space [7]. We would remark that this old belief that time cannot be defined is, as a matter of fact, not well founded because the proponents themselves ignore what time is, making their assertion about time little credible.

Our proposition about the definition of time is the following. The universally accepted point of view that time is associated with movement comes directly from the intuition that time is only quantitatively perceptible when something moves or changes [7]. If nothing changed, there would be no measurable time. Thus, time is always measured with moving objects, examples include the sun, earth, moon (the very first clocks of human beings), the pendulum, the arms of clocks, and today's most developed atomic clocks. If we look carefully into this connection between time and movement, it becomes obvious that time is placed at the same epistemic level as movement. Movement is changing position. We must admit that movement can only occur or be well defined when there is well defined positions. Position is defined in space, or positions form space [7]. By this consideration, time becomes unquestionably less primitive than space,

[5] … je reviens à l'explication du véritable ordre, qui consiste, comme je disais, à tout définir et à tout prouver. Certainement cette méthode serait belle, mais elle est absolument impossible : car il est évident que les premiers termes qu'on voudrait définir, en supposeraient de précédents pour servir à leur explication, et que de même les premières propositions qu'on voudrait prouver en supposeraient d'autres qui les précédassent; et ainsi il est clair qu'on n'arriverait jamais aux premières. Aussi, en poussant les recherches de plus en plus, on arrive nécessairement à des mots primitifs qu'on ne peut plus définir….C'est ce que la géométrie enseigne parfaitement. Elle ne définit aucune de ces choses, espace, temps, mouvement, nombre … parce que ces termes−là désignent si naturellement les choses qu'ils signifient … que l'éclaircissement qu'on en voudrait faire apporterait plus d'obscurité que d'instruction. - Blaise Pascal, De l'esprit géométrique.

not the inverse, as argued in [7][6]. This point of view means that defining time as a function of space is not impossible.

b) How to define time?

It is helpful to think once again about the assertion of Aristotle that time is the number of motion according to before and after [2]. Here the number means measure of time with given unit, day, month, years etc. [7]. Aristotle insisted that the essence of time is movement, as he discussed at length in his Physics [2]. But he did not specify the property and the form of this link to movement. Bergson went farther in the analysis of the concept of time by introducing the pure duration of consciousness, kind of sensation of wait with patience or impatience, defined as a succession of certain states of consciousness [7]. He illustrated the pure duration with the feeling of wait for a sugar to melt in water, but he differentiated it from the time of physics[7] [8]. Certainly, state of consciousness is not state of physics of a movement; but a state of consciousness of the observer can be connected to a state of physics (position) by the action of observation of the position. Similarly, a succession of states of consciousness, represented by the feeling of wait, can be connected to a succession of positions by a series of simultaneous observations. To put it differently, the pure duration can be spread out, by the action of wait

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[6] For if time, as the reflective consciousness represents it, is a medium in which our conscious states form a discrete series so as to admit of being counted, and if on the other hand our conception of number ends in spreading out in space everything which can be directly counted, it is to be presumed that time, understood in the sense of a medium in which we make distinctions and count, is nothing but space ([7] p.91).

If, then, one of these two supposed forms of the homogeneous, namely time and space, is derived from the other, we can surmise *a priori* that the idea of space is the fundamental datum ([7] p.99).

[7] Si je veux me préparer un verre d'eau sucrée, j'ai beau faire, je dois attendre que le sucre fonde. Ce petit fait est gros d'enseignements. Car le temps que j'ai à attendre n'est plus ce temps mathématique qui s'appliquerait aussi bien le long de l'histoire entière du monde matériel, lors même qu'elle serait étalée tout d'un coup dans l'espace. Il coïncide avec mon impatience, c'est-à-dire avec une certaine portion de ma durée à moi, qui n'est pas allongeable ni rétrécissable à volonté. Ce n'est plus du pensé, c'est du vécu (If I prepare a cup of sweet water, I must wait for the sugar to melt …the time of my wait is no more the mathematical time which would apply along the whole history of the material world when it is spread out over space. It coincides with my impatience, i.e. with a portion of my own duration which is neither extensible nor shrinkable. It is no longer thinking, it is lived [8] … Pure duration is the form which the succession of our conscious states assumes when our ego lets itself *live*… ([7] p.100).

and observation, over a succession of positions of a motion. This consideration opens the way to a definition of time as presented below.

We will avoid the words such as before, after, duration, elapse, period, anterior, posterior etc. in defining time, because these terms are too often associated to time in common languages. Using them may give an impression of self-referential definition of time [25]. We propose instead to use the term *wait*. As mentioned above, *wait* is the sensation experienced by the observer of movement, i.e., the observer must wait in order to perceive the succession of positions one after another. The ability of waiting, or the action of wait for something in the future, is an instinctive and innate capability of human being and of most if not all animals; it is a basic capability guaranteeing their survival, just as the capability of perceiving position correctly to find things necessary for their life or to protect themselves from danger. The word *wait* is familiar to everybody and easier to understand than the abstruse concept of pure duration of consciousness[8] whose origin is still under active investigation today [26].

c) Definition of time

To start the introduction to our approach to a definition of time, it is helpful to move our mind away from the familiar concepts of space-time, velocity, acceleration etc., and to go back to the cognitive context before modern physics as early as 14th century when there was no calculus using time. Many scholars of that time, Buridan and Oresme among others, tried to investigate the relationship between the mobility (they called it quality) of motion and time [28]. They took time as a primitive variable without defining it, and expressed the quality (Q) as a function of time. To our knowledge, this is the first use of time as mathematical variable because Oresme expressed Q as vertical axis in a figure where time is horizontal axis. A uniform motion is represented by a constant quality (horizontal straight line) in Oresme's figure [28], in which the spatial position of moving object or the distance L covered by the motion can be defined by the area below the straight line between the initial time t_0 and final time t_f, i.e., $L =$

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[8] The concept of *wait* does coincide with the pure duration of Bergson. The difference is that *wait* is a concrete action of our innate ability of perceiving successive positions (or states of physics) with the help of the memory recording the past positions until the present one, while the pure duration of Bergson is an abstract and obscure concept belonging to consciousness and independent of the external physics world. The innate ability of perceiving successive positions is a potential of action of wait. There is no duration in it. The feeling of duration can take place when the action of *wait* occurs with the perception of successive positions or events.

$Q(t_f - t_0)$. We will follow this approach in replacing time by space as horizontal axis because we do not know yet time, but space is already well defined.

First, let us consider a whole space containing immobile objects, each one being at its fixed position. There is no motion at all. To characterize this space, a fixed spatial position *x* for each object is enough. Let us suppose a primitive but smart geometer who is observing this space, it is enough for him to measure the positions of the objects and to establish a map as a complete description of the space (an ensemble of *x* values). He does not need any other parameter than the positions *x*. There is no change, no duration, no elapsed time.

Now the objects start to move. The space becomes a movie instead of a picture. The map of the geometer is no longer sufficient. In order to describe the changing space, the geometer starts to look for some rules of the motions by studying the moving bodies. At a given moment, he measured a body at position x_0. He *waits* a little, the body has changed its position from x_0 to x_1. He *waits* a little bit more, the body reaches x_2, so on and so forth. The primitive geometer does not have yet the notion of time, he only feels *wait* between successive positions. He is smart enough to realize that this *wait* can be used to describe the changing positions of the objects in the space. He begins to study of the relationship between his *wait* and the changing positions by studying a simplest motion: a moving body that seems uniformly traveling the distance with a constant quality. He measures his *wait* throughout this uniform motion using his heartbeat as Galileo did [29], using t_1 to denote the *wait* from x_0 to x_1, t_2 from x_1 to x_2, t_3 from x_2 to x_3 and so on until the final point of a motion x_N. He quickly notices the following relationships between the consecutive *waits* and positions:

$$\frac{t_1}{t_2} = \frac{x_1 - x_0}{x_2 - x_1}, \frac{t_2}{t_3} = \frac{x_2 - x_1}{x_3 - x_2}, \ldots \frac{t_i}{t_{i+1}} = \frac{x_i - x_{i-1}}{x_{i+1} - x_i} \ldots$$

which implies an underlying rule linking the *waits* to the distances

$$t_{\mathrm{i}} = q(x_{\mathrm{i}} - x_{\mathrm{i}-1}) \qquad (1)$$

where $i = 1,2 \ldots N$, and q is the quality (mobility) of the motion relating his *wait* to different distances.

He realizes that different uniform motions have different qualities, and that the quality of non-uniform motion varies as a function of the position. He then establishes a figure, à la Oresme, where position *x* is the abscissa and the quality q_1 is the ordinate. A uniform motion has a horizontal straight line in his figure. Different uniform motions are represented by horizontals of different heights. His *wait* is then the area under the lines between the initial and final points (Figure 1).

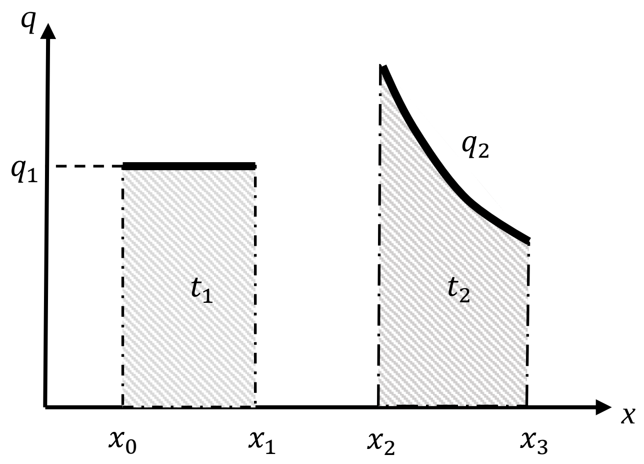

Figure 1: Representation of the quality q of two motions. The left one describes a uniform motion with a horizontal line indicating its quality q_1, The hatched area is the *wait* $t_1 = q_1(x_1 - x_0\,)$. The right one describes a motion with varying quality $q_2(x)$ as certain function of position *x*. The *wait* of this motion is given by $t_2 = \int_{x_2}^{x_3} q_2(x)\,dx$.

It is worth mentioning that to express the *wait* in Eq.(1) as a function of positions and of the quality of motion is not an arbitrary choice by our geometer. He does not have other choice because only spatial positions are well defined. Once the *wait* t is defined with Eq.(1) and measured as he did with his heartbeats, the quality q can be determined from Eq.(1). Notice that q (inverse speed) is an intrinsic quality directly related to the energy of the moving body.

Now the geometer wants to communicate and discuss his study of moving bodies with colleagues, who say the *wait* he has counted is not credible because his heart rhythm is unstable. These colleagues cannot repeat his measurements because different persons in different mental states would have different heart rhythm and read differently the *wait* of a given movement. The only solution is a compromise to take a stable, controllable and familiar motion as standards of *wait* and to compare other waits of different movements to the standard. Examples of the standard includes the apparent motion of the sun, of the moon, hourglass, pendulum, arms of clock, atomic clock etc. as happened in the history of metrology of time. In this way, the *wait* defined by Eq.(1) gives birth to the time of physics, losing its mental dependence, and obtaining its objective and mind-independent character from the path of motion.

To summarize, *time is nothing but an attribute of movement corresponding to the wait of the observer for perceiving successive positions. More generally, time is the wait necessary for perceiving successive states (values) of any changing quantity.*

d) Time and path of movement

One of the objectives of this work is to associate time with the trajectory of motion. From Eq.(1), we can naturally write $t = \sum_{i=1}^{N} t_i = q(x_N - x_0)$ for uniform motion and

$$t = \sum_{i=1}^{N} q_i(x_i - x_{i-1}) \tag{2}$$

for non-uniform one, where t is the time of the motion over the path $(x_0, x_1, x_2, \dots x_i \dots x_N)$. Eq.**(2)** relates the time t of a movement to its path. If the path is a continuous line between x_0 and x_N, we have

$$t = \int_{x_0}^{x_N} q(x)dx \tag{3}$$

In case where $q(x)$ is position independent (uniform motion for instance), Eq.(3) reads $t = ql$ where $l = \int_{x_0}^{x_N} dx$ is the length of the path of the motion. A very special uniform motion whose quality is not only independent of position but also independent of the reference frame is the light motion for which $t = l/c$ $(q = 1/c)$ is valid for any reference, where c is the speed of light in vacuum. This allows using light as a universal standard of time, as discussed below.

3) What behind the name of time?

a) Objectivity of time

Eqs.(1) to (3) offer an unambiguous definition of an objective time for physics, since this time is completely determined by the mind-independent path of motion. No confusion is possible about its nature. As stated above, time is just an interval, perceived by the observer via the feeling of *wait*, between two consecutive positions of a same moving object, or two consecutive values of any measurable changing quantity, such as temperature, pressure, light intensity, frequency etc. No doubt is possible about its objectivity, as long as space (or the changing observables used in the definition) is objective. In this way, the objectivity of time is guaranteed by the objectivity of the observed changing quantities. Within the definition Eq.(1), one cannot say time does not exist or is illusionary [9][10][11], unless he considers position illusionary, which is irrelevant in physics.

b) Subjective time dilation

Nevertheless, as the time of Eq.(1) is a *wait*, and the feeling of wait is experienced by our mind always affected by emotion or by the nature of the observed movement [27], a given time duration may be perceived differently, depending on the mental state of the observer or on the nature of the movement [27]. It is a common knowledge that our emotions warp our perception of time (one minute seems longer for someone suffering than for someone enjoying himself). However, this mental effect is by no means a proof of the subjectivity of the time of physics. An advantage of Eq.(1) is that it removes the mind-dependent character of *wait* by connecting it to the objective positions.

c) Relativity of time

The relativity of time is a very important property of the time of physics. It is related to the famous time dilation due to relative motion and gravitation [23][24]. This property of time is sometimes puzzling for many because, if time is an objective quantity in nature, why is it observed differently by different observers? Does this hint that time is not objective at all and just a mind-dependent variable and spurious concept[9] [9][10][25]? Here we will show that the objective time of Eq.(1) inevitably possesses this relativistic property.

One of the important characteristics of Eq.(1) is that time is a relative quantity delimited by at least two consecutive events. Therefore, on the one hand, absolute time of a single position or event does not make sense. What we call time is nothing but an interval (duration) between two successive events. *It is useless to imagine a time other than duration between events*. It is a common knowledge that a time moment associated with a single event in our common language is always calculated with respect to another time moment of an event (reference).

On the other hand, as positions and paths movements are reference-dependent, *reference-independent absolute time does not exist*. Time defined by Eq.(1) must inherit the relativistic character of position and path. This naturally leads to the dilation of time due to relative motion.

[9] We may therefore surmise that time, conceived under the form of a homogeneous medium, is some spurious concept, due to the trespassing of the idea of space upon the field of pure consciousness ([7] p.98).

Considering that light motion is reference-independent, we can use it as a universal standard to measure time in all references with $t = ql$ ($q = 1/c$) as has been done in special relativity [23] using light clock with two mirrors reflecting a photon in the vertical direction. Now use $t = ql$ for two inertial references , we obtain $t = \frac{t_0}{\sqrt{1-(V/c)^2}}$ [10], where t_0 is the proper time of a clock during the up-down motion (measured in the reference co-moving with the clock), and t is the time of the clock during the same motion but observed in the other reference in which the clock is flying at a speed V in the horizontal direction. Similarly, using $t = ql$, we can show that the perception of the simultaneity of two events also depends on references [24]. The gravitational time dilation is natural to understand as well with light clocks (the speed of light is independent of gravity). Gravity warps space and the paths of movement, time defined by Eq.(1) is therefore naturally affected by gravity [24].

We mention here these well-known results of relativity theories in order to show that they are a natural consequence of the definition of time by Eq.(1) relating time to the path of motion. They are by no means the reasons for killing time as asserted by many [9][10][25].

a) Zeno's paradoxes of motion

Now we have connected time to the path of motion, it is of interest to say some words about Zeno's paradoxes of motion which are directly related to time and space [2]. The solution of these paradoxes has motivated many philosopher from Aristotle [2] until Bergson [7] during more than two thousand years, and is still open questions today [30]. It should be noticed that in Zeno's paradoxes and all the relative discussions, time and space are always separated and considered independent from each other. Here we will show that, if time is connected to the path of motion, many Zeno's paradoxes simply disappear.

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[10] For an observer who is immobile with respect to the light clock, the length of the path of the photon in an up-down movement is $2h$ where h is the distance between the two mirrors. The time of this motion is given by Eq.(3), i.e. $t_0 = \frac{2h}{c}$, also called the proper time of the clock. Now suppose that the clock starts to move in the horizontal direction with respect to the observer at a speed V. As the speed of light is the same for the observer and in the reference co-moving with the moving clock, the proper time does not change. But the length of the path of the photon in an up-down motion recorded by the observer is now $\sqrt{(ct_0)^2 + (Vt)^2}$ where t is the time of the moving clock during an up-down motion read by the observer, we must have $t = \frac{\sqrt{(ct_0)^2+(Vt)^2}}{c}$ and $t = \frac{t_0}{\sqrt{1-(V/c)^2}}$.

A typical Zeno's paradox is Dichotomy proposed to show that all motions are illusory. A version of it states that a motion can *never* start because the moving body, in order to reach its goal at a distance L, must travel the first half the distance $\frac{L}{2}$. And before doing this, the body must travel the first quarter $\frac{L}{2^2}$, and the first eighth $\frac{L}{2^3}$, first sixteenth $\frac{L}{2^4}$, so on and so forth ad infinitum. Finally, the distance d covered by the moving body becomes $d = \frac{L}{2^\infty} = 0$, implying the body cannot move, so the motion can *never* start. *Never* meaning *forever*. There is a forever or eternity because there are indeed infinite operations to halve the distance to travel. Each operation takes a finite duration of time in general, leading to infinite amount of time or eternity.

However, Zeno's arguments can no longer hold once time is related to the path of motion as in Eq.(3). For simplicity, suppose a uniform motion with a constant velocity V (quality), the time τ of the motion should be $\tau = \frac{L}{V}$. If now we halve the distance L to travel, the time t of the motion is also halved $t = \frac{\tau}{2}$. If we halve the distance ad infinitum, we also halve the time of the motion ad infinitum, leading to $t = \frac{\tau}{2^\infty} = 0$. The time for the body to make the first move becomes zero, implying that the motion does not have time to start, which is totally different from that the motion can *never* start. There is no *Never* (infinite amount of time) at all because the time necessary for the motion to occur is fuirtively reduced to zero by taking advantage of the disconnection between time and space. Zeno's sophistic trick is clear. This solution also applies to the paradox of Achilles and Tortoise and the Arrow paradox.

4) From uniqueness of path to time reversibility

Defining time by Eqs.(1) to (3) is as natural and intuitive as reading time in the changing positions of the arms of a clock. It is the simplest way to see what is called time. Nevertheless, people have tried to find it elsewhere outside movement for many reasons. One of them may be the paradox between the *reversibility of movement* (and time as well) prescribed by the laws of physics and the *irreversibility of time* in nature where these laws seem to apply perfectly. Finding a solution of this paradox is the main objective of this work. Let us first discuss the sign of time, which is necessary since there seems no unanimous understanding of the negative sign ($-t$) we are used to write in front of the (forward) time t [34].

a) What is time reversal

As defined in Eqs.(1) to (3), time of a motion is perceived by the observer through his sensation of wait. Wait gives the feeling of increasing duration. So naturally, we tend to give a positive sign to the time of the observed motion along a trajectory. We accept the convention that a forward motion has positive and increasing time t. Given this choice, what does mean if we write $-t$?

Now let $q(x)$ in Eq.(3) be the quality of a motion along its trajectory in the direction with positive t, then $-q(x)$ implies the inversion of the sign of forward quality at every point on the path, yielding an opposite motion along the same trajectory, just as the rewind of a video of motion. If a real motion can be rewound in this way, it is able to go back and retrieve all its past positions. We say that the motion is reversible. If a system completely recovers its past, we can say that its time is reversed. Mathematically, this reversal can be represented by a negative time. Let t' be the time of the inverse motion, we have $t' = \int_{x_N}^{x_0} -q(x)d(-x) = -\int_{x_0}^{x_N} q(x)dx = -t$. This means that the reversibility of time in a motion is a consequence of the reversibility of that motion. More generally, the time reversibility of a process means that the process itself is reversible, i.e. capable of retrieving its past states[11].

b) Reversibility, uniqueness of path, principle of least action

It is trivial to show that all motions obeying Newtonian equation of motion $F = m\, d^2x/dt^2$ are time reversible since this equation does not change when you replace t by $-t$ in the second derivative. As a consequence, both $x(t)$ and $x(-t)$ are solutions of Newtonian equation, i.e., movements that can actually happen. There is no time arrow in classical mechanics. The origin of this time symmetry is in a fundamental principle of classical mechanics. To explain this, let us notice that all classical mechanical motions are characterized by the uniqueness of path between two spatial points for given time duration (or for given initial states). This path, in the

[11] In case of biological processes, the growth of a plant for example, if time of the plant was reversible, it would be able to go back to its past, i.e., retrieving all its past states of a younger plant. We suppose that the states of plant are completely determined by the ensemble of the states of all the molecules and atoms in the plant. Therefore, the time reversal of a plant to a given past moment is a process having $-t$, retrieving all the molecules and atoms it has lost since (the dead leaves for example) in shrinking (abandoning all the molecules and atoms it has gained since that time), but also get them back at the same states as that past moment.

case of a Hamiltonian system conserving energy, is the path of least (or stationary) action, a rule announced for the first time by French scientist Maupertuis then developed further by Lagrange and Hamilton [31]. The action, denoted by A, can be defined by the time integral of Lagrangian function $\mathcal{L}$ along a path between two spatial points [31] a and b :

$$A = \int_a^b \mathcal{L} dt \tag{4}$$

where the Lagrangian $\mathcal{L}$ has different forms for motions of different nature, and turns out to be time symmetric with $\mathcal{L}(t) = \mathcal{L}(-t)$ [31]. Hence, along the same path, the forward action from a to b, $A_{ab} = \int_a^b \mathcal{L}(t) dt$, turns out to be equal to the action of the backward motion: $A_{ba} = \int_b^a \mathcal{L}(-t) d(-t) = -\int_b^a \mathcal{L} dt = \int_a^b \mathcal{L} dt = A_{ab}$. As this action is the least one, the forward motion and the backward one must follow the same unique path. Hence, the basic principle and the concomitant laws of motion do not have preference of direction of time on the unique path. In other words, along the least action path, t and $-t$ are both possible, the past can be the future, and the future can be the past. From a known present state, the state of a movement is predictable with precision for any moment in the future, and can as well be traced back to any moment in the past. We call this kind of motions deterministic. All deterministic motions prescribed by the principle of least action are time reversible and characterized by uniqueness of path.

5) When and how can time be irreversible?

According to the above analysis, all motions obeying the principle of least action must be time reversible. This explains why the laws of physics which can be derived from this basic principle are all time reversible. The question is how and why, in all systems supposedly governed by those time reversible laws, time is always irreversible? If each of the components of an ensemble is white, but the ensemble is not white, there must be something else in play. This point is now tackled in what follows.

a) Multiplicity of path of random motion

Strictly speaking, deterministic motion as described above does not exist in nature. The uniqueness of path has never been observed in real motions, all motions in nature being more or less random with unpredictable future states. One may object to this assertion by saying that all motions in nature obeying Newtonian laws have uniqueness of path and predictability. The movement of the earth around the sun is an example. This is true only for sufficiently short time period. If the period of observation is sufficiently long, all the uncertainties in the unpredictable variation of the masses of the sun and of the planets of solar system will make earth's motion uncertain, finishing by canceling the uniqueness of its path (orbit) prescribed by the principle of least action. All random motions have a common character, i.e., leaving a given state, a system can go to different destination along multiple paths, or go to a given final state along different paths as illustrated in Figure 2. We also call this kind of motion indeterministic.

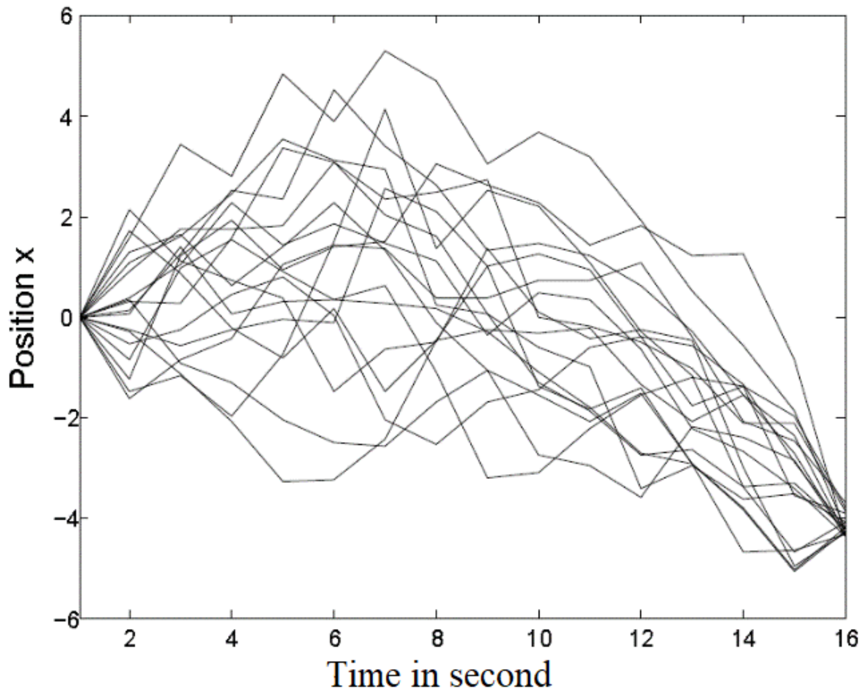

Figure 2: A simulation of the paths of a random motion between two points on x-axis, $x = 0\ m$ and $x = -4\ m$, for a given duration of 16 seconds. The moving body is subject to a weight and to a random perturbation by Gaussian noise.

Indeterministic motion along the different paths is no longer predictable. As a probabilistic motion, it must be described with the help of path probability, i.e. each time the moving body leaves the starting point ($x = 0\ m$ in Figure 2), it takes a given path with a given probability towards the end point ($x = -4\ m$ in Figure 2). Let $p_j = p(Q_j)$ be the probability for the body to follow the path j represented by $(x_a, x_1, x_2, \dots x_i \dots x_b)_j$ in discrete step i=0, 1, 2, …N (N can be very large) between two spatial points a and b, where Q_j is a path-dependent quantity along the path j and determines the probability for the path j to be taken.

However, the existence of the path probability $p_j = p(Q_j)$ with a well defined Q_j is not self-evident for all random motions. For the time being, existence of this probability has only been proved for Hamiltonian systems in random motion with statistically conserved energy [35]-[38]. In the case of randomness created by Gaussian noises, Q_j is just the action A_j along the path j, and p_j is an exponential function of action: $p_j \propto \exp(-\gamma A_j)$, where γ is a positive constant characterizing the randomness [35]-[38].

Our following discussions of time in indeterministic motion are limited to the systems having this path probability as a function of action. If a system is not Hamiltonian or exchanges energy with its environment, such as a Brownian particle, the existence of $p_j = p(A_j)$ is not guaranteed. In this case, we should include the environment of the system in such a way to form a larger Hamiltonian system having path probability for a sufficiently long period. Logically, if a large system is time reversible, each of its subsystems must be time reversible. If a large system is not time reversible, its subsystems, being interdependent in general, cannot be time reversible either. For example, the motion of a Brownian particle contained in water, considered isolated, cannot be time reversible if the ensemble of water+Brownian particle is not, even if the Brownian particle seems to retrieve several of its past states. This is because there are always something missing in the retrieved past states due to the irreversibility of the ensemble (a molecule of the water has not come back to its past position in interaction with the Brownian particle for instance).

b) Irreversibility from multiple paths

When position and paths are continuous, the probability of a path j is given by $dp_j = \rho(Q_j)Dx$, where $\rho(Q_j)$ is the path probability density distribution and the differential Dx represents the "width" of the path (see **Figure 3**) given by the product of all the tolerances dx_i at each one of the N positions x_i: $Dx = dx_1 dx_2 \dots dx_N$. Note that $x_a = x_0$ is the fixed initial point so $dx_a = dx_0$ may or may not be included. The normalization of this path probability is calculated with the path integral technique [41]:

$$\sum_j dp_j = \int \rho(Q_j)Dx = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_{N-1}\, \rho[Q(x_a, x_1, x_2, \dots x_i \dots x_b)] = 1$$

where the sum over the index j is replaced by the integral over all dx_i (i=1, 2, …*N-1*, $x_N = x_b$ is fixed).

Now consider Eq.(2) defining time $t = \sum_{i=1}^{N} q_{ij} \, (x_i - x_{i-1})_j$ over a path *j* between *a* and *b*. Suppose t is positive during the forward motion from a to b, and becomes negative for the inverse motion from *b* to *a* (when the sign of all the q_i's is reversed). Let $\rho[Q(x_a, x_1, x_2 \dots x_b)_j]$ be the probability density of the forward motion along the path $(x_a, x_1, x_2 \dots x_{N-1}, x_b)_j$, and $\rho[Q(x_b, x_{N-1} \dots \; x_1, x_a)_j]$ the probability density for the inverse motion on the same path. The probability for the forward motion along the path of width $Dx_{ab} = dx_1 dx_2 \dots dx_N$ is given by $dp_j(a, b) = \; \rho[Q(x_a \dots x_b)_j] \; Dx_{ab}$. The probability of the backward motion on the same path *j* with the width $Dx_{ba} = dx_{N-1} \dots dx_1 dx_a$ is given by $dp_j(b, a) = \rho[Q(x_b \dots x_a)_j] Dx_{ba}$.

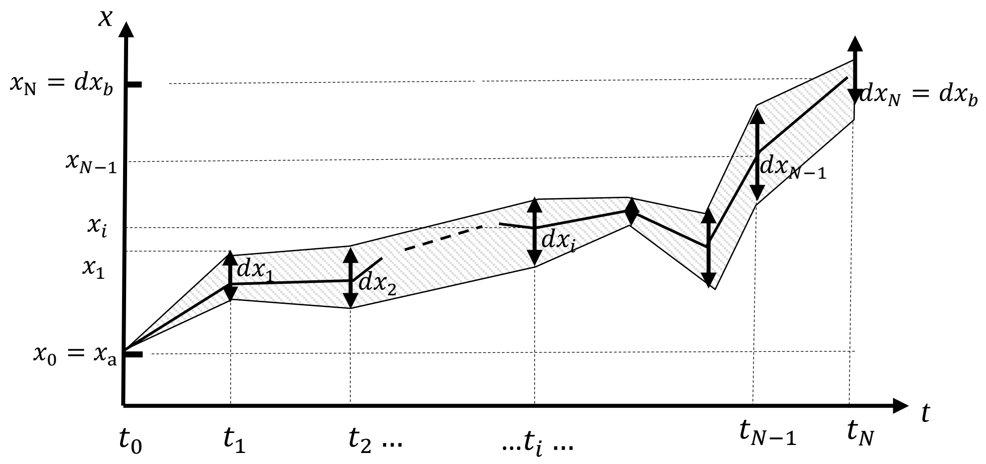

Figure 3: A 1-dimensional path of *N* steps along *x* axis from *a* and *b*, illustrated in space-time *x*-*t.* The hatched band represents the width $Dx = dx_1 dx_2 \dots dx_N$. The width of the path at the step at x_i is given by the tolerance dx_i. It is obvious that if one of the *N* tolerances dx_i is zero, the path is at once deadlocked since its width $Dx = 0$.

Notice that the probability of a given path, in the case of continuous variation of position and of infinite number of paths, is proportional to the width Dx of the path. The smaller Dx, the thinner the path, the smaller its probability. This implies that the probability is zero along a path of width $Dx = 0$. More precisely, the path probability $dp_j = \; \rho\left(Q_j\right) Dx = 0$ if anyone of the *N* widths $dx_1, dx_2, \dots dx_N$ is zero, i.e., the path is deadlocked (see figure 3). This property of the path probability does not affect the motion (forward or backward) of a particle that is free to move on and to explore all possible space with liberty, with no constraint on the width of any step. However, constraints on the width of path may occur for the motions that are required to go somewhere with certain precision. To reverse a motion for example, we require the motion to reach its past states. Below we discuss two constraints coming from two different kinds of reversed motion which we referred to as *strong reversibility* and *weak reversibility*.

c) Strong time reversibility,

Strong time reversibility means the motion is reversed in such a way that the moving object retrieves, step after step, all its past positions. In this way, the moving object experiences again all its past, as if nothing had happened and the world is the same as before. Let $(x_a \rightarrow x_1 \rightarrow x_2 \ldots \rightarrow x_{N-1} \rightarrow x_b)$ be the forward path, and $(x_b \rightarrow x_{N-1} \rightarrow \ldots x_1 \rightarrow x_a)$ the backward path, where each reversed position x_i must be exactly the same as the forward x_i. No deviation of backward x_i from the forward x_i is tolerated. Any deviation, even infinitesimal one, would mean the moving body is not returning to its past so that its time is not reversed. Hence, the width of the inverse path must be strictly zero at every inverse step, leading to $Dx_{ba} = dx_{N-1} \ldots dx_1 dx_0 = 0$ and to the zero probability for the backward motion along a certain path *j*: $dp_j(b,a) = \rho[Q(x_b \ldots x_a)_j]Dx_{ba} = 0$. The inverse motion along a path is possible only with nonzero width $Dx_{ba} \neq 0$, meaning that the deviation of backward x_i from the forward x_i for all *i* is nonzero. This implies that the backward path is not exactly the same as the forward one, so the moving object is not experiencing a return to its past, and time is not reversed. The backward path becomes dead-ended whenever we want to produce time inversion by imposing $Dx_{ba} = 0$. To summarize, *the probability of inverse motion producing time reversal is always zero along any already realized path*.

d) Weak time reversibility

Weak time reversibility means the moving body continues its forward motion without reversing the sign of q_i's in Eq.(3), and turns out to revisit at least one of its past positions, making a circular motion. Similar return to the past occurs in science fiction when someone travels back in time and meets himself of twenty years ago, or his parents when they were younger. This circular motion also occurs with the eternal return of Poincaré in his recurrence theorem [39]. For simplicity, let us suppose that the moving body starts its motion at the point $x_a = x_0$, and, without reversing its time course, comes back to x_a at the step N ($x_N = x_a$) on the path $(x_a \rightarrow x_1 \rightarrow x_2 \ldots \rightarrow x_{N-1} \rightarrow x_a)$. The probability of this path, numbered by *j*, is given by $dp_j(a,a) = \rho\big[Q(x_a, x_1, x_2 \ldots x_{N-1}, x_a)_j\big]Dx_{aa}$ with $Dx_{aa} = dx_1 dx_2 \ldots dx_{N-1} dx_a$. If we do not impose time reversal to the whole path, none of the *N* widths in the product

$dx_1 dx_2 \dots dx_{N-1} dx_a$ is zero. However, if you want to produce the weak time reversal for the moving body to revisit the initial position x_a, you cannot but impose zero tolerance $dx_a = 0$ at $x_N = x_a$, leading to $Dx_{aa} = 0$, and to zero path probability $dp_j(a,a) = 0$ for the return. To summarize, it is impossible for the body to revisit even a single past point.

This result does not contradict the Poincaré's recurrence theorem [39] because it is valid only for deterministic motion characterized by the uniqueness of path, while the above result is obtained for indeterministic motion characterized by the multiplicity of path. A detailed discussion of the modification of Poincaré's theorem has been given in [40].

I would like to stress the generic character of the above proof of time irreversibility based on a system-independent property of indeterministic motion: multiplicity of paths with well-defined path probability. *Whenever a motion shows this property, it definitively loses the possibility to revisit its past*. This is the main result of this work.

In what follows, we discuss this irreversibility from the angle of informational entropy as a measure of the dynamic uncertainty of random motion.

6) Path entropy, a possible measure of irreversibility

The dynamic uncertainty or the randomness of indeterministic motion can be reflected by the informational entropy associated with the path probability $dp_j = \rho(Q_j)Dx$. This entropy has been referred to as *caliber* or *path entropy* S_p expressed as a function of path probability with Shannon formula $S_p(a,b) = -\int_{a,b} \rho(Q) \ln\rho(Q) Dx$ [12] [42][43]. $S_p(a,b)$ is of course a relative quantity between the starting point *a* and the endpoint *b* of the motion. Obviously, $S_p(a,b) = 0$ if the randomness cancels out between a and b, meaning that the motion is deterministic with a single path *j* of unitary probability $p_j = 1$. $S_p(a,b) > 0$ whenever the motion is random. The larger $S_p(a,b)$ is, the more the motion is uncertain, and the more the paths are dispersed around the deterministic path of the system.

Suppose now a movement from a to b, but passing by a given point c. The path entropy $S_p(a,b)_c$ of the motion on this path (a,c,b) should be $S_p(a,b)_c = S_p(a,c) + S_p(c,b)$. As

[12] This formula risks giving negative information measure. A solution to this trouble is proposed in [44] to guarantee the positivity of the informational entropy of continuous probability distributions.

$S_p(a,c)$ and $S_p(c,b)$ are positive by definition [42], $S_p(a,b)_c$ as a total entropy must be larger than each of them. This implies that, as a motion goes on, its total path entropy is an increasing function of time. This time increasing property of path entropy can also be proven by reductio ad absurdum as follows. Suppose $S_p(x_a,x_b)_c$ starts to decrease from the point c to b. This means $S_p(c,b) < 0$, which contradicts the positivity of path entropy. Hence, as time goes on, a system in random motion experiences more and more dynamic uncertainty. This growing uncertainty is cumulated in path entropy, making it increasing with time. It seems possible to use path entropy to mark the time arrow and to measure the aging of the system in random motion. This arrow is always pointing in the direction of increasing uncertainty and disorder, as Eddington asserted when he introduced the term 'time's arrow'[13] [45]. However, for the time being, we don't know yet how to relate path entropy, a mathematical tool for calculating path probability uncertainty, to some measurable physical quantity, thermodynamic entropy or heat for example if it concerns thermodynamic systems. Further work is necessary in this direction.

7) Concluding remarks

In this work, we have addressed the conflict between the time of physics, which is reversible according to the fundamental laws of classical physics, and the irreversible time of nature we experience every day. Our starting point is a mathematical definition of time to implement the intuitive idea that time is an attribute of movement perceived through the instinctive action of wait of the observer to see successive positions. The concept of wait is thus related to the path of motion to give birth to time of physics. This definition opens the way to mathematical investigation of the properties of time from the properties of the paths of motion. We have shown that the time defined in this way has all the attributes of the time of classical physics, and is necessarily reversible in deterministic movement.

The second step of this work is to consider the role of random motion characterized by multiplicity of path. We have shown with mathematical proof that, in the case of multiple paths

[13] Without any mystic appeal to consciousness it is possible to find a direction of time … Let us draw an arrow arbitrarily. If as we follow the arrow we find more and more of the random element in the state of the world, then the arrow is pointed towards the future; if the random element decreases, the arrow points towards the past. That is the only distinction known to physics. This follows at once if our fundamental contention is admitted that the introduction of randomness is the only thing which cannot be undone (Eddington, The Nature of the Physical World).

having path probability, time inevitably becomes irreversible. To put it differently, one cannot rewind time when it goes on in disordered dynamics. You can never undo the randomness of an indeterministic movement. This randomness turns out to be the motor driving time in the direction towards increasing disorder and uncertainty[14].

The result of this work is in accordance with the thermodynamic arrow of time and entropy increase, with no need of probabilistic interpretation of the second law. In the framework of the probabilistic Hamiltonien mechanics based on the path probability, Poincaré's recurrence theorem is absent due to the modification of the Liouville's theorem (conservation of phase density distribution along the path of the motion), leading to Boltzmann's *H* theorem (no need of molecular chaos [12]) and the concomitant increase of entropy [40]. The probabilistic Hamiltonian mechanics invalidates all the arguments of deterministic mechanics against the pioneer work of Boltzmann [39][46], including the Maxwell's demon (and other ones [19]) haunting the second law of thermodynamics. Indeed, in the context of indeterministic motion with multiplicity of paths and uncontrollable randomness of the gas molecules, the demon can no more measure the velocity of a molecule with certainty and foresee its direction towards his trapdoor. It is useless for him to open the trapdoor to let a fast-moving molecule pass through because the molecule may change its direction and its velocity just in front of the door instead of going through. In this random world, does the demon himself also show more or less disorder in his action? Yes to our eyes because he is living in a ubiquitous randomness.

It is also worth emphasizing that the indeterministic nature of movement considered in this work must be objective in order for the path probability to be objective, leading to observer-independent time arrow. Subjective and observer-dependent randomness due to incomplete

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[14] This impossibility of time reversal can be popularized as follows. You leave your car walking away in an open field for a while, blindfolded. Then you are asked to return to your car by walking backwards along whatever path as you please, you will end up finding your car, perhaps after a long walk. Now you are imposed a constraint to walk backwards on the same footprints as your walking away. With the presence of all the uncertainty in the measure of step size, in the way you move your legs, and in the irregular form of the ground which you cannot see, without mentioning all the other chances you can meet on a wild field, the probability for you to recover your car will be very small, eventually almost zero. Now let the size of your footprints and of your feet become infinitesimally small (required by the strong reversibility), I am sure that you will lose your car. If, moreover, the size of your car tends to zero (required by the weak reversibility), the probability for you to meet again your car is zero whichever path you take.

knowledge of deterministic motion, including chaos [12][18], necessarily leads to subjective and observer-dependent probability and then to the illusory character of entropy increase and time arrow.

This view is closely related to a philosophy-laden question about the origin of the objective randomness in the context of the deterministic laws of classical physics ruling the world. The millennium old puzzles opposing determinism against indeterminism, and necessity against chance, are well known to all [18][33][34]. The debate seems deadlocked nowadays because the absence of irrefutable evidences persists for both deterministic and indeterministic view of nature [18][33][34]. In this context, we suggest the following alternative approach to the question. In view of the infallible mathematics linking time arrow to multiplicity of path, it is a certainty that time loses its reversibility in random motion. Inversely, if a motion is time irreversible, it certainly takes place over multiple paths of random motionsd, since the uniqueness of path of deterministic motions necessarily yields reversible time. It's mathematics. Therefore, a plausible method to justify the objectivity and the ubiquity of indeterministic nature of the world by its time irreversibility, observed everywhere with no counter-example to date. Put it shortly, *time irreversibility is an evidence of the indeterministic nature of the world*.

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