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Global dynamics of a spore producing pathogens epidemic system with nonlocal diffusion process

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Abstract

In this paper, we analyze a nonlocal spatial epidemic model presenting the diffusion process of a spore producing plant pathogens responsible of one of the most destructive cocoa pods disease. The global existence, compactness and dissipativity of the semiflow generated by the system are established. By defining a threshold number (the basic reproduction number), we first express conditions for the existence of non-trivial stationary states. Next, we show that the qualitative dynamics of the model range from the extinction (*i.e.* the global stability of the disease-free stationary state), to the persistence (*i.e.* the global stability of the endemic stationary state) of the epidemics. Finally, we go through some simulations of our general analysis.

Keywords: nonlocal diffusion; spore producing pathogens; population dynamics

1 Introduction

The objective of this work is to analyze asymptotic properties of an integro-differential epidemiological model representing the diffusion process of spore producing plant pathogens, typically the major causal agents of cocoa pods damage. The World cacao –known as *Theobroma cacao L.*– production has doubled in the last thirty years. Almost all of this production comes from the four leading world producers countries located in the west and central Africa: Côte d’Ivoire, Ghana, Cameroon and Nigeria in order [1]. According to the latest statistics from the ICCO (International Cocoa Organization), Cameroon’s production has increased by more than 50 thousand tonnes the last twenty years [1]. However, it is noted that the production is not optimal because of several factors among which the various diseases that damage the cocoa trees or fruits and reduce the production. Black pod rot and mirids are the two main diseases plaguing the production in Cameroon and causing over 80% loss on some farms [2]. *Phytophthora megakarya* is the major pathogen agent of cocoa black pod rot present in the field, and has successfully replaced *Phytophthora palmivora* which was initially the main pathogen causing that disease [3].

Several models, based on ordinary differential equations, have been proposed to model the cocoa black pods rot epidemic dynamics, *e.g.* [3–7]. In this work, we use a system of integro-differential

equations to model the spatial diffusion process of spore producing pathogens in a homogeneous cocoa population. The plant population is subdivided into two compartments, Susceptible plant tissue (S) and Infected tissue (I). Healthy cocoa tissue is transformed into infected tissue with the successful germination of a single fungal spore from the spore pool compartment (Q). The host plant population does not represent individual cocoa plants, but rather cocoa pod surface area densities (e.g. cocoa pod surface area per m^2). The cocoa pod surface is viewed as a set of individual patches corresponding to a restricted host surface area that can be colonized by a single pathogen individual. The model is a set of integro-differential equations with nonlocal term representing nonlocal diffusion. The model describes the epidemic dynamic of spore-producing pathogens with nonlocal states structure. Table 1 lists the state variables and parameters of the model. We consider a spatial region –the field– Ω where are found host plant –cocoa pods. The free spore, $Q(t, x)$, is responsible for the black pod rot epidemic at any time t and at location $x \in \Omega$. The latent time between the infection of pods by *P. megakarya* and the manifestation of the first macroscopic symptoms being really short ($< 3-4$ days) [8], there is often no way to distinguish an infected pod from an infectious one. This is why the delay has not been considered in the infection dynamic. The pod surface is then viewed as a set of individual patches corresponding to a restricted host surface area. The cocoa pod areas are divided into two epidemiological states : susceptible $S(t, x)$, and infectious $I(t, x)$. The model explores the damage caused by *P. megakarya* in the considered region through the following epi-evolutionary model with non local diffusion

$$\left\{ \begin{array}{l} \frac{\partial S(t, x)}{\partial t} = \Lambda(x) - \beta(x) \frac{Q(t, x)}{K + Q(t, x)} S(t, x) - \mu(x) S(t, x), \quad t > 0, \quad x \in \Omega \\ \frac{\partial I(t, x)}{\partial t} = \beta(x) \frac{Q(t, x)}{K + Q(t, x)} S(t, x) - (\gamma(x) + \mu(x)) I(t, x), \quad t > 0, \quad x \in \Omega \\ \frac{\partial Q(t, x)}{\partial t} = d \int_{\Omega} J(x - y) [Q(t, y) - Q(t, x)] dy + B[I(t, \cdot)](x) - \delta(x) Q(t, x), \quad t > 0, \quad x \in \Omega. \end{array} \right. \quad (1.1)$$

The above system is coupled with nonnegative initial conditions such that

$$S(0, \cdot) = S_0(\cdot) \in C_+(\Omega), \quad I(0, \cdot) = I_0(\cdot) \in C_+(\Omega), \quad Q(0, \cdot) = Q_0(\cdot) \in C_+(\Omega). \quad (1.2)$$

The region Ω is a closed bounded domain of \mathbb{R}^n ($n \geq 1$) with smooth boundary $\partial\Omega$. We only consider an inflow in the pods compartment of new susceptible area with the rate $\Lambda(x)$ at the location x . Model (1.1) assumes that there is no disease-induced mortality of infected pod areas, *i.e.* the pathogen considered has a biotrophic phase in his life cycle [9]. Thus, death rate of both susceptible and infectious pod areas, at location x , is the same and denoted $\mu(x)$. Parameters $\gamma(x)$ and $\delta(x)$ respectively represents the rate of phytosanitary pod removal and the natural decay of free spores. The rate at which susceptible pod areas become infected is given by $\beta(x) \frac{Q(t, x)}{K + Q(t, x)} S(t, x)$, where $\beta(x)$ is the *p. megakarya* infection efficiency, and K is the *Mickaelis-Mentens* constant. Such a form of infectious rate is supported by previous studies [4], and allow taking into account that the total number of free spores produced by infected pod areas in the field is extremely large, such that saturation occurs in the transmission rate. The integral operator, $d \int_{\Omega} J(x - y) [Q(t, y) - Q(t, x)] dy$, describes diffusion process see [10, 11] where, the positive parameter d is a constant representing the dispersal rate of spores in the field, and $J(x - y)$ is thought of as the probability distribution of jumping from location y to location x . Moreover, $\int_{\Omega} J(x - y) Q(t, y) dy$ is the rate at which free

spores are arriving at position x from all other places and, $-\int_{\Omega} J(x-y)Q(t,x)dy$, is the rate at which they are leaving location x to travel to all other sites. Since integrals are taken over the domain Ω , we assume that diffusion takes places only in Ω . Infected pod areas, $I(t, \cdot)$, produce spores in the free environment at time t through a process modeled by the term $B[I(t, \cdot)]$, where $B : C(\Omega) \rightarrow C(\Omega)$ is a linear positive, bounded operator, and

Assumption 1.1 *One of the below properties is satisfied,*

(i) *B is a multiplication operator, i.e. $B[I](x) = b(x)I(x)$, where b is a positive and continuous function in Ω ,*

(ii) *B is a positive compact operator on $C(\Omega)$ and injective on $C_+(\Omega)$.*

Model (1.1) appears, in a slightly different form, in [12] where the authors estimated parameter for the spatial spread of cocoa black pod disease. Indeed, while the diffusion is local and explicitly modeled by the Laplace operator describing a random walk movement in [12], here the diffusion is modeled by an integro-differential operator with nonlocal terms. Importantly, we here address a mathematical analysis of Model (1.1), and results obtain in this current work can be applied to the model proposed in [12]. The description of spatial propagation systems has a long history. As far as epidemic problems are concerned, we refer for instance to [13] for a survey. For *e.g.*, we refer to [14–17] for systems with nonlocal diffusion. We also mention the work in [18], where authors investigate the asymptotic speed of spread for a spatially distributed system modelling the dynamics of spore producing plant pathogen.

This article is organized as follows. In Section 2, we state and discuss the main results that are obtained in this work. These include the existence and uniqueness of bounded solutions, the pathogen invasion process, bifurcation and global stability of an endemic stationary state. Numerical simulations are provided in Section 3 to illustrate the above main results and a discussion is also proposed. Next sections are dedicated to the proofs of main results. More precisely, Section 5 concerns the existence and uniqueness of solutions for (1.1)-(1.2). The global disease extinction result is proof in Section 6. Finally, details for the existence of endemic endemic stationary state and a bifurcation analysis are handled in Section 7.

2 Main results

In this section, we state the main results that will be proved in this work. Such results include the existence and uniqueness of bounded solutions, the disease invasion process and the threshold global asymptotic dynamics of System (1.1)-(1.2). Overall, Model (1.1) will be considered under the following general assumption.

Assumption 2.1 • *The dispersal kernel J is lipschitz continuous in Ω and satisfies : $J(\cdot) \in C(\Omega)$, $J(0) > 0$, $J(-x) = J(x)$, $x \in \Omega$ and $\int_{\Omega} J(x)dx = 1$*

- Λ , μ , β , and δ are positive continuous functions in Ω .
- γ is a nonnegative continuous function in Ω .

In a pathogen-free environment, the cocoa pod susceptible area dynamics is such that

$$\frac{\partial S(t, x)}{\partial t} = \Lambda(x) - \mu(x)S(t, x), \quad t > 0, \quad x \in \Omega. \quad (2.1)$$

Then, $(\bar{S}, 0, 0, 0) = \left(\frac{\Lambda(\cdot)}{\mu(\cdot)}, 0, 0, 0 \right)$ is the unique stationary solution of System (2.1).

Next, let us introduce the bounded linear operator $A : C(\Omega) \rightarrow C(\Omega)$, defined for each $\phi \in C(\Omega)$ by

$$A[\phi](x) = d \int_{\Omega} J(x-y)[\phi(y) - \phi(x)]dy - \delta(x)\phi(x), \quad \forall x \in \Omega. \quad (2.2)$$

Thanks to assumption 2.1, the linear operator A is resolvent-positive with $s(A) < 0$ i.e. $(-A)^{-1}$ exists with $(-A)^{-1}C_+(\Omega) \subset C_+(\Omega)$. We define

$$I^\diamond(\nu) = \frac{\beta}{\gamma + \mu}(\bar{S} + \nu), \quad \forall \nu \geq 0, \quad (2.3)$$

and consider the family of sets parametrized by $\nu \geq 0$ as follow

$$\Theta_\nu = \left\{ (S_0, I_0, Q_0) \in C_+(\bar{\Omega})^3 : \begin{array}{l} S_0 \leq \bar{S} + \nu \\ I_0 \leq I^\diamond(\nu) \\ Q_0 \leq (-A)^{-1} \circ B[I^\diamond(\nu)] \end{array} \right\}. \quad (2.4)$$

2.1 Existence, dissipativity, and compactness of the semiflow

Since System (1.1)-(1.2) is designed to model a biological process, its solutions should remain positive. The positivity and boundedness of solutions of System (1.1)-(1.2) is given by the following result :

Theorem 2.2 *Let Assumption 2.1 be satisfied. Then,*

(i) *For each initial conditions $u_0 = (S_0, I_0, Q_0)^T \in C_+(\Omega)^3$, there exists a globally defined solution*

Table 1: States variables and parameters of the model

Variable	Description	
t	Time	
x	Space location	
$S(t, x)$	Susceptible pods area at location x at time t	
$I(t, x)$	Infectious pods area at location x at time t	
$Q(t, x)$	Free spores density at location x at time t	
Parameter	Description (unit)	Value/Range [Ref.]
$\Lambda(\cdot)$	Pods recruitment rate (PAD · Tu ⁻¹)	2.22/ 0.5 – 3 [12]
$\mu(\cdot)$	Pods death rate (Tu ⁻¹)	0.03/ 0.0005 – 0.3 [12]
$\gamma(\cdot)$	Rate of phytosanitary pods removal (Tu ⁻¹)	0.11/ 0.0005 – 0.2 [12]
$\beta(\cdot)$	Infection efficiency (Tu ⁻¹)	10 [Assumed]
$\delta(\cdot)$	Natural decay of free spores (Tu ⁻¹)	0.24/ 0.0005 – 0.5 [12]
d	Diffusion rate (Tu ⁻¹)	0.05/ 0.01 – 0.3 [12]
$J(x-y)$	Jumping prob. from y to x (dimensionless)	Eq. (3.1) [Assumed]
$B[\cdot]$	Spores production function (SAD · Tu ⁻¹)	Eq. (3.2) [Assumed]
K	Mickaelis-Mentens constant (SAD)	10 ⁴ [Assumed]

Tu=Time unit. PAD= Pod area density. SAD= Spore area density.

to system (1.1)-(1.2) defined by

$$\mathcal{U}(t)[u_0] = (S(t, \cdot), I(t, \cdot), Q(t, \cdot))^T \in C_+(\Omega)^3, \quad \forall t \geq 0.$$

(ii) *Dissipativity*: Θ_ν , $\nu \geq 0$ is positively invariant with respect to the semiflow generated by (1.1)-(1.2). Moreover, it is bounded dissipative in the sense that, if $\widehat{\Theta}$ is a bounded subset of $C_+(\Omega)^3$, and for each $\nu > 0$, there exists $t_0 > 0$ (depending only on $\widehat{\Theta}$) such that

$$\mathcal{U}(t)(\widehat{\Theta}) \subset \Theta_\nu, \quad \forall t \geq t_0.$$

(iii) *Asymptotic compactness*: In addition, assume that Assumption 1.1(ii) holds. Then, the semiflow \mathcal{U} is asymptotically smooth, i.e., for any nonempty, closed, bounded set $B \subset C_+(\Omega)^3$, there exists a nonempty compact set $D = D(W)$ such that D attracts $\{u_0 \in B : \mathcal{U}(t)[u_0] \in W, \forall t \geq 0\}$.

We recall that a nonempty set $D \subset C_+(\Omega)^3$ is said to *attract* a nonempty set $W \subset C_+(\Omega)^3$ if $\delta_{C_+(\Omega)^3}(\mathcal{U}(t)[W], D) \rightarrow 0$ as $t \rightarrow +\infty$, where $\delta_{C_+(\Omega)^3}(W, D) = \sup_{u \in W} \inf_{v \in D} \|u - v\|_{C_+(\Omega)^3}$ is a semi-distance on $C_+(\Omega)^3$. Moreover, notice that the notion of *asymptotically smooth* is used here within the terminology of Hale, Lasalle and Slemrod [19]. However, this is strictly equivalent to the notion of *asymptotically compact* within the terminology of Ladyzhenskaya [20].

The proof of Theorem 2.2 (see Section 5) is based on the semigroup formulation of System (1.1)-(1.2). Moreover, notice the asymptotic compactness of the semiflow is guaranteed only when Assumption 1.1(ii) holds, i.e. the linear operator B is compact on $C(\Omega)$.

2.2 Pathogen invasion process in uninfected environment and threshold dynamics

When the host population is initially uninfected, the spread of a single pathogen strain can be determined by calculating the basic reproduction number of this strain. The basic reproduction number, usually denoted as \mathcal{R}_0 , is defined as the total number of infections arising from one newly infected individual introduced into a healthy (disease-free) host population [21, 22]. For all $\epsilon \in (-1, 1)$, let $(\overline{S}_\epsilon, \overline{I}_\epsilon, \overline{Q}_\epsilon) = ((1 + \epsilon)\overline{S}, 0, 0)$, a small perturbation of the disease-free equilibrium. Then, linearizing System (1.1)-(1.2) at $(\overline{S}_\epsilon, \overline{I}_\epsilon, \overline{Q}_\epsilon)$, we have for all $x \in \Omega$,

$$\begin{cases} \frac{\partial \widehat{S}(t, x)}{\partial t} = -\frac{\beta(x)\overline{S}_\epsilon(x)}{K}\widehat{Q}(t, x) - \mu\widehat{S}(t, x) \\ \frac{\partial \widehat{I}(t, x)}{\partial t} = \frac{\beta(x)\overline{S}_\epsilon(x)}{K}\widehat{Q}(t, x) - (\gamma(x) + \mu)\widehat{I}(t, x) \\ \frac{\partial \widehat{Q}(t, x)}{\partial t} = A[\widehat{Q}(t, x)] + B[\widehat{I}(t, x)], \quad t > 0, \\ \widehat{S}(0, \cdot) = \widehat{S}_0 \in C(\Omega), \quad \widehat{I}(0, \cdot) = \widehat{I}_0 \in C(\Omega), \quad \widehat{Q}(0, \cdot) = \widehat{Q}_0 \in C(\Omega). \end{cases} \quad (2.5)$$

Note that when $\epsilon = 0$, the above system corresponds to the linearized equation around the disease-free equilibrium. Furthermore, the infective component $(\widehat{I}, \widehat{Q})$ of (2.5) writes

$$\begin{cases} \frac{d\widehat{\psi}(t)}{dt} = \widehat{\mathcal{A}}[\widehat{\psi}(t)] + \mathcal{B}_\epsilon[\widehat{\psi}(t)], \quad t > 0 \\ \widehat{\psi}(0) = \widehat{\psi}_0 \in Y, \end{cases} \quad (2.6)$$

where $\widehat{\mathcal{A}}, \mathcal{B}_\epsilon : C(\Omega)^2 \rightarrow C(\Omega)^2$ are linear operators defined by

$$\widehat{\mathcal{A}}[\widehat{\psi}] = \begin{pmatrix} -(\gamma + \mu)\widehat{I} \\ A[\widehat{Q}] \end{pmatrix}, \quad \mathcal{B}_\epsilon[\widehat{\psi}] = \begin{pmatrix} \frac{\beta \overline{S}_\epsilon}{K} \widehat{Q} \\ B[\widehat{I}] \end{pmatrix}, \quad \forall \widehat{\psi} = (\widehat{I}, \widehat{Q})^T \in C(\Omega)^2. \quad (2.7)$$

Since the linear operator $\widehat{\mathcal{A}}$ is resolvent positive with $s(\widehat{\mathcal{A}}) < 0$, and \mathcal{B}_ϵ is a positive operator, one can use the theory developed in [23] to define the threshold

$$\mathcal{R}_\epsilon = r(\mathcal{L}_\epsilon), \quad \forall \epsilon \in (-1, 1) \quad (2.8)$$

as the spectral radius of the linear operator \mathcal{L}_ϵ given by

$$\mathcal{L}_\epsilon = \mathcal{B}_\epsilon(-\mathcal{A})^{-1} = \begin{pmatrix} 0 & \frac{\beta \overline{S}_\epsilon}{K} (-A)^{-1} \\ B \left[\frac{\cdot}{\mu + \gamma} \right] & 0 \end{pmatrix}, \quad \forall \epsilon \in (-1, 1). \quad (2.9)$$

Lemma 2.3 *Let Assumption 2.1 be satisfied. Then, for all $\epsilon \in (-1, 1)$,*

$$r(\mathcal{L}_\epsilon)^2 = r(\mathcal{H}_\epsilon(-A)^{-1}) = r((-A)^{-1}\mathcal{H}_\epsilon),$$

where $\mathcal{H}_\epsilon : C(\Omega) \rightarrow C(\Omega)$ is the operator defined by

$$\mathcal{H}_\epsilon[\psi] = B \left[\frac{\beta \overline{S}_\epsilon \psi}{K(\mu + \gamma)} \right], \quad \forall \psi \in C(\Omega). \quad (2.10)$$

Proof. Since \mathcal{L}_ϵ is a bounded linear operator, we have $\sigma(\mathcal{L}_\epsilon^2) = \sigma(\mathcal{L}_\epsilon)^2$. From where, $r(\mathcal{L}_\epsilon)^2 = r(\mathcal{L}_\epsilon^2)$. Observing that

$$\mathcal{L}_\epsilon^2 = \begin{pmatrix} \mathcal{H}_\epsilon(-A)^{-1} & 0 \\ 0 & (-A)^{-1}\mathcal{H}_\epsilon \end{pmatrix}$$

it comes $r(\mathcal{L}_\epsilon)^2 = \max(r(\mathcal{H}_\epsilon(-A)^{-1}), r((-A)^{-1}\mathcal{H}_\epsilon))$. Since $r(\mathcal{H}_\epsilon(-A)^{-1}) = r((-A)^{-1}\mathcal{H}_\epsilon)$, the proof is completed. \blacksquare

Next, the limiting case (*i.e.* $\epsilon \rightarrow 0$) corresponds to the linearized equation around the disease-free equilibrium, and thus, the basic reproductive number \mathcal{R}_0 of (1.1) is defined as follows

$$\mathcal{R}_0 = r(\mathcal{L}_0). \quad (2.11)$$

From the above notations, our next result expresses that the invasion and spread of the pathogen is not possible when $\mathcal{R}_0 < 1$. More precisely, we have

Theorem 2.4 *Let Assumptions 2.1 and 1.1 be satisfied. Also assume that $\mathcal{R}_0 < 1$. Then,*

- i) The disease-free equilibrium is the unique stationary solution of System (1.1).*
- ii) The disease-free equilibrium of (1.1) is globally asymptotically stable in $C_+(\Omega)^3$, *i.e.* the semi-flow $\mathcal{U}(t)[u_0] = (S(t, \cdot), I(t, \cdot), Q(t, \cdot))$ of System (1.1)-(1.2) converges, in $C_+(\Omega)^3$, to $(\overline{S}, 0, 0)$ as $t \rightarrow \infty$.*

We have proven that the trivial stationary state $(\bar{S}, 0, 0)$ is always a solution of System (1.1) –see Section 6 for the proof of Theorem 2.4. However, we will show that under some conditions, the model can also have nontrivial stationary solutions. By constructing a suitable Lyapunov function, we will show that such an endemic stationary solution, when it exists, is unique and globally asymptotically stable. The precise result reads as follows

Theorem 2.5 *Let Assumptions 1.1(ii) and 2.1 be satisfied. Also assume that $\mathcal{R}_0 > 1$. Then,*
 (i) *There exists at least one endemic stationary state (S^*, I^*, Q^*) of Model (1.1).*
 (ii) *We have $\lim_{t \rightarrow \infty} (S(t, \cdot), I(t, \cdot), Q(t, \cdot)) = (S^*(\cdot), I^*(\cdot), Q^*(\cdot))$ in the topology of $C(\Omega)^3$.*

Again, notice that Assumption 1.1(ii), *i.e.* the compactness of B is required for Theorem 2.5. Indeed, without such assumption, the strong uniform persistence of the semiflow is technically difficult to establish. We refer to Section 7.

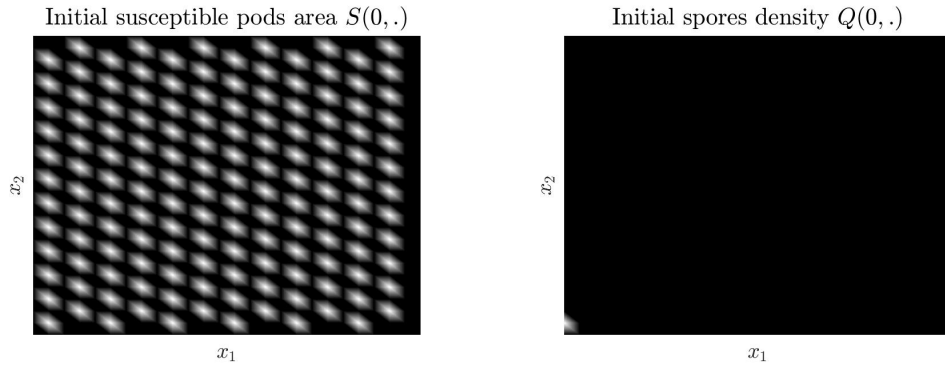


Figure 1: Initial (at $t = 0$) susceptible pods area and free spores density on the domain $\Omega = \{(x_1, x_2)\}$. Grey colour corresponds to areas with susceptible pods (resp. spores), and black colour is free of susceptible pods (resp. spores).

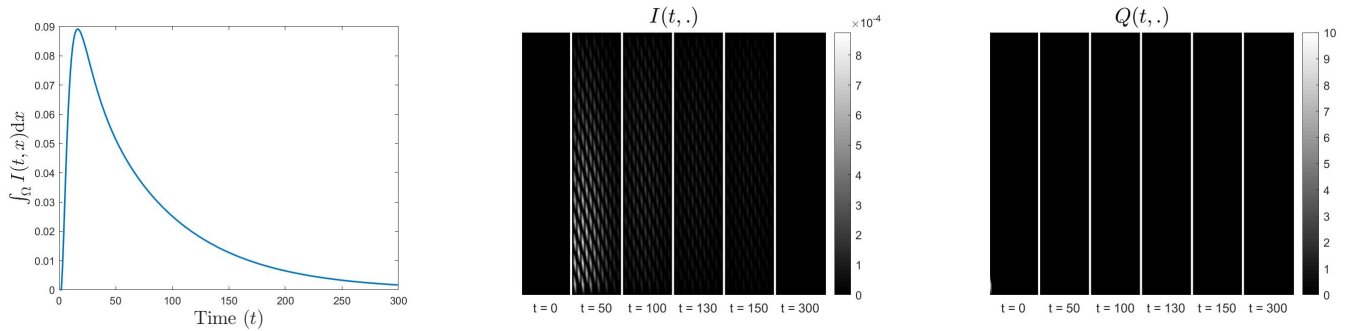


Figure 2: Asymptomatic extinction of the disease and convergence to the disease-free equilibrium when $\mathcal{R}_0 < 1$. For I and Q dynamics, the whole domain Ω is illustrated at different times $t = 0, 50, 100, 130, 150, 300$.

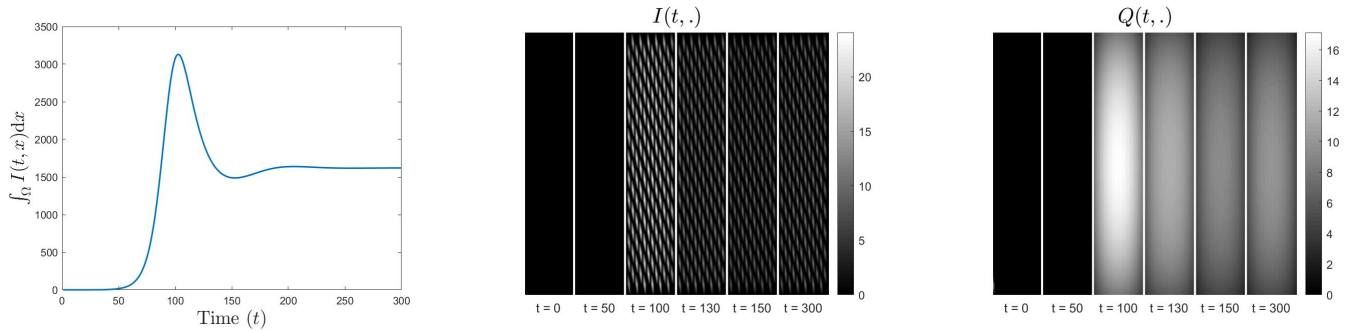


Figure 3: Asymptotic persistence of the disease and convergence to the endemic equilibrium when $\mathcal{R}_0 > 1$. For I and Q dynamics, the whole domain Ω is illustrated at different times $t = 0, 50, 100, 130, 150, 300$.

3 Numerical simulations and discussion

3.1 Description of the model numerical simulation

In this section we provide some numerical simulations, by using finite volume numerical schemes to illustrate bifurcation results of the endemic equilibrium of System (1.1)-(1.2) under Assumptions 1.1-2.1. The domain Ω is a square $[0, 0.05] \times [0, 0.05]$. We fix the jumping probability distribution as

$$J(z) = \mathcal{N}[(0, 0), \text{diag}(0.01, 0.01)](z), \quad (3.1)$$

where $\mathcal{N}[\mu, \Sigma]$ stands for the multivariate normal distribution with the mean vector μ and covariance matrix Σ . Furthermore, the linear operator for the spores producing process is assumed to take the form

$$B[I](x) = \int_{\Omega} b_0 \kappa(x - y) I(y) dy, \quad (3.2)$$

where $\kappa(z_1, z_2) = \frac{1}{4 \times 0.01^2} \times e^{-|z_1|/0.01} \times e^{-|z_2|/0.01}$, and b_0 is a positive parameter. All other parameters are given by Table 1.

For the model initial conditions, we assume that susceptible pods are localized at some points of the domain Ω (Figure 1). Initially, all the domain is free of infected pods area, *i.e.* $I(t = 0, \cdot) \equiv 0$, and the epidemics is initiated by a localized pool of free spores (Figure 1).

A forward bifurcation occurs at $\mathcal{R}_0 = 1$, which means that whenever $\mathcal{R}_0 < 1$, then the disease-free equilibrium is globally asymptotically stable by Theorem 2.4 and no endemic equilibrium exists. Asymptotically, the disease will go extinct (Figure 2, where $b_0 = 0.9$ and $\mathcal{R}_0 \approx 0.92$). Next, when $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and a globally stable endemic equilibrium exists by Theorem 2.5. The disease will asymptotically persist and the semiflow of (1.1)-(1.2) will converge to such endemic equilibrium (Figure 3, where $b_0 = 4$ and $\mathcal{R}_0 \approx 4.12$).

3.2 Discussion

In this paper, we built an integro-differential system modeling the epidemiological spread of spore producing pathogens with nonlocal diffusion process in a closed and bounded subset of \mathbb{R}^n ($n \geq 1$). We proved that the model was mathematically well-posed by using the classical integrated

semigroups theory. We characterized the basic reproduction number \mathcal{R}_0 of the model as a spectral radius of a linear operator. An explicit formula of the \mathcal{R}_0 is difficult to obtain in general within this context, but can be basically computed numerically. We proved that a forward bifurcation occurs around $\mathcal{R}_0 = 1$ where an epidemics can only arise if $\mathcal{R}_0 > 1$. However, the uniform persistence and the global dynamics of the model when $\mathcal{R}_0 > 1$ is technically guaranteed only when the linear operator B for the spores production process is compact.

Information on the many sources of infections either environmental (primary infection from the environmental spores reservoir) or direct transmission (secondary infection from free spores initiated by infected host tissue) [6, 12, 24, 25]. However, in the model presented here, only the latter source of infection is considered. Considering both forms of infections is quite challenging, particularly for the uniform persistence of the system when $\mathcal{R}_0 > 1$. This remains an interesting open problem within this context.

Seasonality is also an important aspect of the environmental spores reservoir hosting pathogens year-round. Indeed, the evolution of the environmental spores reservoir, viewed as a reflection of the epidemics during the past cropping seasons is also known to impact the evolutionary epidemiology in agricultural landscape [26]. This is an important issue that merits further investigation within the context of the model presented here.

Another potential limitation is the lack of host heterogeneity and pathogen diversity in the model formulation. Indeed, in order to control cocoa pods epidemics, one sustainable option is the use of partially resistant or tolerant cocoa cultivars and appropriate cultural practices [5]. Such host heterogeneity can then allow investigating strategies for the deployment of different cultivars in agricultural landscapes in order to control epidemics damage during successive cropping seasons. Within this context, see *e.g.* [26]. Finally, pathogen diversity (*i.e.* genetic variants or strains) can have fundamental impact on the evolutionary dynamics of the model presented here, see *e.g.* [27]. Indeed, each strain is characterized, on a given host class, by pathogenicity traits that describe the basic steps of the disease infection cycle: (i) the infection efficiency, *i.e.*, the probability that a spore deposited on a receptive host surface produces a lesion and (ii) the total number of spores produced. Those strain-specific pathogenicity traits are basic components of the \mathcal{R}_0 which mainly determine the outcome of the disease in an agricultural landscape.

4 General remarks

It is useful to write System (1.1)-(1.2) into a more compact form. To that end, let us consider the following function spaces $X = C(\Omega)$, $Y = C(\Omega) \times C(\Omega)$, and the corresponding positive cones $X_+ = C_+(\Omega)$ and $Y_+ = C_+(\Omega) \times C_+(\Omega)$. The norm is the usual supremum norm and the norm in $C_+(\Omega)^3$ is the usual product norm that is :

$$\left\| \begin{pmatrix} S \\ \psi \end{pmatrix} \right\| = \|S\|_\infty + \|\psi\|_Y,$$

with $\|\psi\|_Y = \|Q\|_\infty + \|I\|_\infty$, $\forall \psi = (I, Q)^T \in Y$. Here z^T is set for the transpose of a vector or a matrix. Next, recall the operator A introduced by (2.2), and consider the nonlinear maps $F_1 : X_+ \times X_+ \rightarrow X_+$ and $F_2 : X_+ \times Y_+ \rightarrow Y_+$ such that

$$F_1 \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \Lambda - N(\psi)\phi, \quad \forall \phi \in X_+, \quad \forall \psi \in X_+$$

and

$$F_2 \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} N(\varphi)\phi \\ b\varpi \end{pmatrix}, \quad \forall \phi \in X_+, \quad \forall \psi = (\varpi, \varphi)^T \in Y_+,$$

where $N : X_+ \rightarrow X_+$, is defined by

$$N(\phi) = \frac{\beta\phi}{K + \phi}, \quad \phi \in X_+. \quad (4.1)$$

Next, we identify the vector $(I(t, \cdot), Q(t, \cdot))^T$ together with $\psi(t)$. Therefore, with $\psi_0 = (I_0, Q_0)^T$, System (1.1)-(1.2) rewrites

$$\begin{cases} \frac{\partial S(t, \cdot)}{\partial t} = -\mu S(t, \cdot) + F_1 \begin{pmatrix} S(t, \cdot) \\ \psi(t) \end{pmatrix}, \quad t > 0 \\ \frac{d\psi(t)}{dt} = \begin{pmatrix} -(\gamma(\cdot) + \mu) & 0 \\ 0 & A \end{pmatrix} \psi(t) + F_2 \begin{pmatrix} S(t, \cdot) \\ \psi(t) \end{pmatrix}, \quad t > 0 \\ S(0, \cdot) = S_0 \in C_+(\Omega), \quad \psi(0) = \psi_0 \in C_+(\Omega)^2. \end{cases} \quad (4.2)$$

Moreover, by setting $v(t) = \begin{pmatrix} S(t, \cdot) \\ \psi(t) \end{pmatrix}$, and $v_0 = \begin{pmatrix} S_0 \\ \psi_0 \end{pmatrix}$, System (1.1)-(1.2) also rewrite as the following abstract Cauchy problem

$$\begin{cases} \frac{dv(t)}{dt} = \mathcal{A}v(t) + F(v(t)), \quad t > 0 \\ v(0) = v_0 \in C_+(\Omega)^3, \end{cases} \quad (4.3)$$

with $\mathcal{A} = \begin{pmatrix} -\mu(\cdot) & 0 & 0 \\ 0 & -(\gamma(\cdot) + \mu(\cdot)) & 0 \\ 0 & 0 & A \end{pmatrix}$ and $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.

5 Proof of Theorem 2.2

Existence of the semiflow. Using the semigroup formulation (4.3), we give the proof of Theorem 2.2(i) into several steps.

Step 1: Recall that the spectral bound $s(A)$ of a given operator A is defined by

$$s(A) = \{\Re(\lambda) : \lambda \in \sigma(A)\}.$$

The following lemma holds

Lemma 5.1 *Let Assumption 2.1 be satisfied. Then,*

- i) $s(\mathcal{A}) < 0$, and \mathcal{A} generates an uniformly continuous semigroup $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$ such that, $T_{\mathcal{A}}(t)(X_+ \times Y_+) \subset X_+ \times Y_+$ for all $t \geq 0$.
- ii) for each $\eta \in (0, -s(\mathcal{A}))$ there exists $M := M(\eta) \geq 1$ such that

$$\|T_{\mathcal{A}}(t)\|_{\mathcal{L}(X \times Y)} \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

iii) For each $v \in X_+ \times Y_+$

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \text{dist} \left(v + h \begin{pmatrix} F_1(v) \\ F_2(v) \end{pmatrix}, X_+ \times Y_+ \right) = 0. \quad (5.1)$$

where $v \mapsto \text{dist}(v, X_+ \times Y_+)$ is the Hausdorff semi-distance defined by $\text{dist}(v, X_+ \times Y_+) = \inf_{w \in X_+ \times Y_+} \|v - w\|$.

Proof. For item i), notice that the bounded linear operator A defined in (2.2) is resolvent positive with spectral bound $s(A) < 0$ [11][Proposition 2.9]. Thus, the diagonal matrix of bounded linear operators \mathcal{A} is resolvent positive and generates a uniformly continuous semigroup that maps $X_+ \times Y_+$ into itself. For item ii), since $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$ is norm continuous, we have $s(\mathcal{A}) = \omega(\mathcal{A})$ [?, Proposition 9.2] with $\omega(\mathcal{A})$ the growth bound of $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$, i.e.

$$\omega(\mathcal{A}) = \lim_{t \rightarrow +\infty} \frac{\ln(\|T_{\mathcal{A}}(t)\|_{\mathcal{L}(X \times Y)})}{t}.$$

Item ii) follows.

Finally, for item iii), let $v = \begin{pmatrix} S \\ \psi \end{pmatrix} \in X_+ \times Y_+$ with $\psi = (I, Q)^T$ be given. Let us first note that for all $h > 0$

$$\begin{aligned} \text{dist} \left(v + h \begin{pmatrix} F_1(v) \\ F_2(v) \end{pmatrix}, X_+ \times Y_+ \right) &= \text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ \psi + hF_2(v) \end{pmatrix}, X_+ \times Y_+ \right) \\ &\leq \text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) + \text{dist} \left(\begin{pmatrix} 0 \\ \psi + hF_2(v) \end{pmatrix}, X_+ \times Y_+ \right) \end{aligned}$$

and since

$$\begin{pmatrix} 0 \\ \psi + hF_2(v) \end{pmatrix} \in X_+ \times Y_+$$

it follows that

$$\text{dist} \left(v + h \begin{pmatrix} F_1(v) \\ F_2(v) \end{pmatrix}, X_+ \times Y_+ \right) \leq \text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right), \quad \forall h > 0.$$

Moreover, using the fact that the distance function is 1-Lipschitz, we also have for each $\epsilon > 0$ and $h > 0$

$$\text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) \leq \epsilon + \text{dist} \left(\begin{pmatrix} S + \epsilon + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right). \quad (5.2)$$

Next, we prove that

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \text{dist} \left(\begin{pmatrix} S + \epsilon + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) = 0. \quad (5.3)$$

To do this, recall that

$$S + \epsilon + hF_1(v) = S + \epsilon + h(\Lambda - \beta N(Q)S)$$

and since $(\Lambda - \beta N(Q)S)$ is bounded in Ω , we have for $h > 0$ small enough

$$S + \epsilon + h(\Lambda - \beta N(Q)S) \in X_+.$$

Hence, for $h > 0$ small enough

$$\begin{pmatrix} S + \epsilon + hF_1(v) \\ 0 \end{pmatrix} \in X_+ \times Y_+ \Rightarrow \text{dist} \left(\begin{pmatrix} S + \epsilon + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) = 0 \quad (5.4)$$

and (5.3) follows. Thus, the equality (5.4) together with (5.2) imply that

$$\overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) \leq \epsilon, \forall \epsilon > 0 \Rightarrow \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \text{dist} \left(\begin{pmatrix} S + hF_1(v) \\ 0 \end{pmatrix}, X_+ \times Y_+ \right) = 0.$$

The proof of the lemma is completed. \blacksquare

Step 2: Let $S_0 \in X_+$ and $(I_0, Q_0)^T \in Y_+$ be given. Combining item iii) of Lemma 5.1 together with the fact that the nonlinear maps F_k , $k = 1, 2$, are Lipschitz continuous on bounded subsets of $X_+ \times Y_+$, we infer from [23, 28] that there exists a maximally defined solution to (4.3). Let $\tau \in (0, +\infty]$ such that the solution exists in $[0, \tau)$. To obtain that the solution is globally defined in time, we add up the first two components of system (1.1) to obtain for each $x \in \Omega$

$$\frac{\partial}{\partial t} (S(t, x) + I(t, x)) \leq \Lambda(x) - \mu(x) (S(t, x) + I(t, x)), \quad t \in [0, \tau) \quad (5.5)$$

Hence

$$0 \leq S(t, x) + I(t, x) \leq \max \left(S_0(x) + I_0(x), \frac{\Lambda(x)}{\mu} \right), \quad \forall t \in [0, \tau), \quad \forall x \in \Omega. \quad (5.6)$$

Moreover we have for $x \in \Omega$

$$\frac{\partial Q(t, x)}{\partial t} = A[Q(t, \cdot)](x) + B[I(t, \cdot)], \quad t \in [0, \tau)$$

which implies that

$$Q(t, \cdot) = T_A(t)[Q_0] + \int_0^t T_A(t-s)B[I(s, \cdot)]ds, \quad t \in [0, \tau).$$

Using item ii) of Lemma 5.1 we have

$$\|Q(t, \cdot)\|_\infty \leq M\|Q_0\|_\infty + M\|B\| \int_0^t \|I(s, \cdot)\|_\infty ds, \quad t \in [0, \tau). \quad (5.7)$$

Hence (5.6) and (5.7) shows that the solution cannot blow up in finite time. This ends the proof of the existence of a globally defined solution of System (4.3) in $[0, +\infty)$.

Dissipativity. By summing up the S - and I -equation of (1.1), it comes

$$S(t, \cdot) + I(t, \cdot) \leq e^{-\mu t} (S_0(\cdot) + I_0(\cdot)) + \bar{S}(\cdot) (1 - e^{-\mu t}), \quad \forall t \geq 0. \quad (5.8)$$

Let $\nu > 0$ be given. Then, from (5.8), there exists $t_0 = t_0(\nu, S_0, I_0) > 0$ such that

$$S(t, x) + I(t, x) \leq \bar{S}(x) + \nu, \quad \forall t \geq t_0, \quad \forall x \in \Omega. \quad (5.9)$$

The I -equation and the positivity of the semilow generated by (1.1), combining with estimate (5.9), give

$$\frac{\partial I(t, x)}{\partial t} \leq \beta(x) (\bar{S}(x) + \nu) - (\gamma(x) + \mu(x))I(t, x), \quad t > t_0, \quad x \in \Omega.$$

From where, we also find,

$$\begin{aligned} I(t, x) &\leq e^{-(\gamma(x)+\mu(x))t} I(0, x) + \frac{\beta(x)}{\gamma(x) + \mu(x)} (\bar{S}(x) + \nu) (1 - e^{-(\gamma(x)+\mu(x))t}), \quad \forall t \geq t_0, \quad \forall x \in \Omega \\ &\leq \frac{\beta(x)}{\gamma(x) + \mu(x)} (\bar{S}(x) + \nu), \quad \forall t \geq t_0, \quad \forall x \in \Omega. \end{aligned} \tag{5.10}$$

Since

$$\frac{\partial Q(t, x)}{\partial t} = A[Q(t, \cdot)](x) + B[I(t, \cdot)](x), \quad t \geq 0,$$

we have

$$Q(t, \cdot) = T_A(t - t_0)[Q(t_0, \cdot)] + \int_{t_0}^t T_A(t - s) \circ B[I(s, \cdot)] ds, \quad \forall t \geq t_0.$$

Setting $I^*(x) = \frac{\beta(x)}{\gamma(x) + \mu(x)} (\bar{S}(x) + \nu)$, inequality (5.10) leads to

$$Q(t, \cdot) \leq T_A(t - t_0)[Q(t_0, \cdot)] + \int_{t_0}^t T_A(t - s) \circ B[I^*(\cdot)] ds, \quad \forall t \geq t_0. \tag{5.11}$$

Next, the properties of the generator of a semigroup ensure that for each $t \geq t_0$

$$T_A(t - t_0)B[I^*] = B[I^*] + A \int_0^{t-t_0} T_A(s) \circ B[I^*] ds = B[I^*] + A \int_{t_0}^t T_A(s)B[I^*] ds \tag{5.12}$$

and since $s(A) < 0$, i.e. $0 \in \rho(A)$, it comes

$$\int_{t_0}^t T_A(s) \circ B[I^*] ds = (-A)^{-1} \circ B[I^*] - (-A)^{-1} \circ T_A(t - t_0) \circ B[I^*], \quad \forall t \geq t_0.$$

Plugging the last equality into (5.11) we get

$$Q(t, \cdot) \leq T_A(t - t_0)[Q(t_0, \cdot)] + (-A)^{-1} \circ B[I^*] - (-A)^{-1} \circ T_A(t - t_0) \circ B[I^*], \quad \forall t \geq t_0. \tag{5.13}$$

Since the resolvent $(-A)^{-1}$ commutes with the semigroup $\{T_A(t)\}_{t \geq 0}$, we have

$$Q(t, \cdot) \leq T_A(t - t_0)[Q(t_0, \cdot)] + (-A)^{-1} \circ B[I^*] - T_A(t - t_0) \circ (-A)^{-1} \circ B[I^*], \quad \forall t \geq t_0. \tag{5.14}$$

Asymptotic compactness. Let $\mathcal{F} = \{u_j = (S_{0j}, I_{0j}, Q_{0j}) \in C_+(\Omega)^3\}_{j \geq 1}$ be equibounded. Let t_j be sequence such that $t_j \rightarrow +\infty$ as $j \rightarrow +\infty$. In the following, we will prove that there exists a convergent subsequence of $\{\mathcal{U}(t_j)(u_j)\}_{j \geq 1}$. We first note that, the set

$$\mathcal{F}_x := \{\mathcal{U}(t_j)(u_j)(x) : j \in \mathbb{N}\} \tag{5.15}$$

is bounded for all $x \in \Omega$. Then by the Arzela-Ascoli theorem, we only need to prove that \mathcal{F} is equicontinuous, that is for each $\epsilon > 0$, there exists $\zeta > 0$ such that

$$|x - z| < \zeta \Rightarrow \|\mathcal{U}(t_j)(u_j)(x) - \mathcal{U}(t_j)(u_j)(z)\| < \epsilon, \quad \forall j \in \mathbb{N}. \tag{5.16}$$

Let us denote for each $j \in \mathbb{N}$, by $t \rightarrow (\tilde{S}_j(t, \cdot), \tilde{I}_j(t, \cdot), \tilde{Q}_j(t, \cdot))$ the solution of (1.1) with initial condition $(S_{0j}, I_{0j}, Q_{0j}) \in C_+(\Omega)^3$, that is $(\tilde{S}_j(t, x), \tilde{I}_j(t, x), \tilde{Q}_j(t, x)) = \mathcal{U}(t)(u_j)(x)$, $\forall t \geq 0, \forall x \in \Omega$. Next, we set for each $j \in \mathbb{N}$, $(S_j(t, x), I_j(t, x), Q_j(t, x)) = \mathcal{U}(t + t_j)(u_j)(x)$, $\forall t \geq -t_j, \forall x \in \Omega$, that is

$$(S_j(t, x), I_j(t, x), Q_j(t, x)) = (\tilde{S}_j(t + t_j, x), \tilde{I}_j(t + t_j, x), \tilde{Q}_j(t + t_j, x)), \quad \forall t \geq -t_j, \quad \forall x \in \Omega. \quad (5.17)$$

Note that for all $x \in \Omega$ and $t \geq -t_j$ we have

$$\begin{cases} \frac{\partial S_j(t, x)}{\partial t} = \Lambda(x) - \beta(x) \frac{Q_j(t, x)}{K + Q_j(t, x)} S_j(t, x) - \mu(x) S_j(t, x) \\ \frac{\partial I_j(t, x)}{\partial t} = \beta(x) \frac{Q_j(t, x)}{K + Q_j(t, x)} S_j(t, x) - (\gamma(x) + \mu(x)) I_j(t, x) \\ \frac{\partial Q_j(t, x)}{\partial t} = A[Q_j(t, \cdot)](x) + B[I_j(t, \cdot)](x). \end{cases} \quad (5.18)$$

We now proceed in several steps.

Step 1: Let $t \in \mathbb{R}$ be given and fixed. Let $j_0 \in \mathbb{N}$ be large enough such that

$$t \geq -t_j, \quad \forall j \geq j_0. \quad (5.19)$$

In the following, we will prove that the sequence $\{Q_j(t, \cdot)\}_{j \geq j_0}$ is equicontinuous. Note that in the particular case $t = 0$, we have $j_0 = 0$ and $\{Q_j(0, \cdot)\}_{j \geq 0} = \{\tilde{Q}(t_j, \cdot)\}_{j \geq 0}$ will be equicontinuous. To that aim, let us note that Theorem 2.2(ii) together with the boundedness of \mathcal{F} implies that the set

$$\{(S_j(s, \cdot), I_j(s, \cdot), Q_j(s, \cdot)) : j \geq 0, s \geq -t_j\} \subset C_+(\Omega) \quad (5.20)$$

is bounded. Therefore, there exists $m_0 > 0$ (depending only on \mathcal{F}) such that

$$\|S_j(s, \cdot)\|_\infty + \|I_j(s, \cdot)\|_\infty + \|Q_j(s, \cdot)\|_\infty \leq m_0, \quad \forall s \geq -t_j, \quad \forall j \in \mathbb{N}. \quad (5.21)$$

Then the compactness of B ensures that

$$\{B[I_j(s, \cdot)] : j \geq j_0, s \geq -t_j\} \subset C_+(\Omega) \quad (5.22)$$

is relatively compact. Hence, by the Arzela-Ascoli theorem, one knows that the set defined in (5.22) is equicontinuous. This means that for each $\eta > 0$, there exists $\delta > 0$ such that

$$x, z \in \Omega, \|x - z\| < \zeta \Rightarrow \|B[I_j(s, \cdot)](x) - B[I_j(s, \cdot)](z)\| < \eta, \quad \forall s \geq -t_j, \quad j \geq j_0. \quad (5.23)$$

Next, we note that for each $j \geq j_0$ we have

$$Q_j(t, \cdot) = T_A(t + t_j)[Q_j(-t_j, \cdot)] + \int_{-t_j}^t T_A(t - s) \circ B[I_j(s, \cdot)] ds. \quad (5.24)$$

Let $\eta > 0$ be given. We deduce from (5.21) and (5.23) that there exists $\zeta > 0$ such that if $x, z \in \Omega$ with $\|x - z\| < \zeta$ then,

$$\|Q_j(t, x) - Q_j(t, z)\| \leq 2m_0 \|T_A(t + t_j)\| + \eta \int_{-t_j}^t \|T_A(t - s)\| ds.$$

Recalling that $s(A) = \omega(A) < 0$, it follows that

$$\begin{aligned} \|Q_j(t, x) - Q_j(t, z)\| &\leq 2m_0\|T_A(t + t_j)\| + \eta \int_{-\infty}^t \|T_A(t - s)\| ds \\ &\leq 2m_0\|T_A(t + t_j)\| + \eta \int_{-\infty}^0 \|T_A(s)\| ds \end{aligned}$$

Since we have $\lim_{j \rightarrow +\infty} 2m_0\|T_A(t + t_j)\| = 0$ it follows that $\{Q_j(t, \cdot)\}_{j \geq j_0}$ is equicontinuous.

Step 2: Let $x, z \in \Omega$ be given. Then we have for all $t \geq -t_j$ with $j \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2} \frac{\partial(S_j(t, x) - S_j(t, z))^2}{\partial t} &= (\Lambda(x) - \Lambda(z))[S_j(t, x) - S_j(t, z)] - \mu(x)[S_j(t, x) - S_j(t, z)]^2 \\ &\quad - S_j(t, z)[S_j(t, x) - S_j(t, z)](\mu(x) - \mu(z)) \\ &\quad - N(Q_j(t, x))S_j(t, x)[S_j(t, x) - S_j(t, z)][\beta(x) - \beta(z)] \\ &\quad - \beta(z)N(Q_j(t, x))[S_j(t, x) - S_j(t, z)]^2 \\ &\quad - \beta(z)[S_j(t, x) - S_j(t, z)]S_j(t, z)[N(Q_j(t, x)) - N(Q_j(t, z))] \end{aligned}$$

Next, using the inequality

$$ab = (a\sqrt{\epsilon}) \left(\frac{b}{\sqrt{\epsilon}} \right) \leq \frac{1}{2} \left(a^2\epsilon + \frac{b^2}{\epsilon} \right), \quad \forall \epsilon > 0$$

and (5.21) we obtain for each $\epsilon > 0$, $t \geq -t_j$ with $j \in \mathbb{N}$

$$\begin{aligned} \frac{\partial(S_j(t, x) - S_j(t, z))^2}{\partial t} &\leq \frac{|\Lambda(x) - \Lambda(z)|^2}{\epsilon} + (\epsilon + 2\epsilon\|\beta\|^2m_0^2 + \epsilon m_0^2 - \inf_{x \in \Omega} \mu(x)) |S_j(t, x) - S_j(t, z)|^2 \\ &\quad + \frac{|\beta(x) - \beta(z)|^2}{\epsilon} + \frac{|N(Q_j(t, x)) - N(Q_j(t, z))|^2}{\epsilon} + \frac{|\mu(x) - \mu(z)|^2}{\epsilon}. \end{aligned}$$

Choosing $\epsilon > 0$ small enough such that $\mu_\epsilon = -(\epsilon + 2\epsilon\|\beta\|^2m_0^2 + \epsilon m_0^2 - \inf_{x \in \Omega} \mu(x)) > 0$ and recalling that

$$|N(Q_j(t, x)) - N(Q_j(t, z))| \leq \frac{|\beta(x)|}{K} |Q_j(t, x) - Q_j(t, z)| + |\beta(x) - \beta(z)|$$

it comes $\forall t \geq -t_j$

$$\begin{aligned} (S_j(t, x) - S_j(t, z))^2 &\leq e^{-\mu_\epsilon(t+t_j)} (S_j(-t_j, x) - S_j(-t_j, z))^2 + \int_{-t_j}^t e^{-\mu_\epsilon(t-s)} \frac{|\beta(x) - \beta(z)|^2}{\epsilon} ds \\ &\quad + \int_{-t_j}^t e^{-\mu_\epsilon(t-s)} \frac{|\mu(x) - \mu(z)|^2}{\epsilon} ds + \int_{-t_j}^t e^{-\mu_\epsilon(t-s)} \frac{|\Lambda(x) - \Lambda(z)|^2}{\epsilon} ds \\ &\quad + \frac{1}{K^2\epsilon} \int_{-t_j}^t e^{-\mu_\epsilon(t-s)} (|Q_j(s, x) - Q_j(s, z)| |\beta(x)| + |\beta(x) - \beta(z)|)^2 ds \end{aligned} \tag{5.25}$$

(5.26)

Setting $t = 0$ into (5.26), we obtain

$$(\tilde{S}(t_j, x) - \tilde{S}(t_j, z))^2 \leq 2m_0e^{-\mu_\epsilon t_j} + \int_{-\infty}^0 e^{\mu_\epsilon s} ds \frac{|\Lambda(x) - \Lambda(z)|^2}{\epsilon} + \int_{-\infty}^0 e^{\mu_\epsilon s} ds \frac{|\mu(x) - \mu(z)|^2}{\epsilon}$$

$$\begin{aligned}
& + \int_{-\infty}^0 e^{\mu_\epsilon s} ds \frac{|\beta(x) - \beta(z)|^2}{\epsilon} \\
& + \frac{1}{K^2 \epsilon} \int_{-t_j}^0 e^{\mu_\epsilon s} (|Q_j(s, x) - Q_j(s, z)| |\beta(x)| + |\beta(x) - \beta(z)|)^2 ds, \quad \forall j \geq 0.
\end{aligned}$$

Assume that $\{\tilde{S}(t_j, \cdot)\}_{j \in \mathbb{N}}$ is not equicontinuous. Then, there exist z_j, x_j with $x_j - z_j \rightarrow 0$, $j \rightarrow +\infty$ such that

$$\limsup_{j \rightarrow +\infty} |\tilde{S}(t_j, x_j) - \tilde{S}(t_j, z_j)| > 0 \quad (5.27)$$

after choosing a subsequence of $\{S(t_j, \cdot)\}_{j \in \mathbb{N}}$. Thus, using (5.26) together with (5.27) we get

$$0 < \limsup_{j \rightarrow +\infty} |\tilde{S}(t_j, x_j) - \tilde{S}(t_j, z_j)| \leq \limsup_{j \rightarrow +\infty} \int_{-t_j}^0 e^{\mu_\epsilon s} \frac{1}{K^2 \epsilon} |Q_j(s, x_j) - Q_j(s, z_j)|^2 |\beta(x_j)|^2 ds.$$

Let $\tau > 0$ be given. Then there exists $j_0 \in \mathbb{N}$ such that $t_j \geq \tau$ for all $j \geq j_0$. Therefore, for all $j \geq j_0$ we have

$$\int_{-t_j}^0 e^{\mu_\epsilon s} \frac{|\beta(x_j)|}{K^2 \epsilon} |Q_j(s, x_j) - Q_j(s, z_j)|^2 ds \leq \int_{-t_j}^{-\tau} e^{\mu_\epsilon s} \frac{\|\beta\|}{K^2 \epsilon} (2m_0)^2 ds + \int_{-\tau}^0 e^{\mu_\epsilon s} \frac{\|\beta\|}{K^2 \epsilon} |Q_j(s, x_j) - Q_j(s, z_j)|^2 ds. \quad (5.28)$$

Finally, combining the above estimate with Fatou's lemma and the equicontinuity of $\{Q_j(t, \cdot)\}_{j \in \mathbb{N}}$ for any fixed $t \in \mathbb{R}$ and j large enough we have

$$0 < \limsup_{j \rightarrow +\infty} |\tilde{S}(t_j, x_j) - \tilde{S}(t_j, z_j)| \leq \frac{e^{-\mu_\epsilon \tau} \|\beta\|}{\mu_\epsilon K^2 \epsilon} (2m_0)^2, \quad \forall \tau > 0. \quad (5.29)$$

This gives a contradiction and we conclude that $\{\tilde{S}(t_j, \cdot)\}_{j \in \mathbb{N}}$ is equicontinuous. A similar argument applies to the equicontinuity of $\{I_j(t, \cdot)\}_{j \in \mathbb{N}}$.

6 Proof of Theorem 2.4

Before going through the proof of Theorem 2.4, let us introduce some useful notations and results. For all $\epsilon \in (-1, 1)$, recall $(\bar{S}_\epsilon, \bar{I}_\epsilon, \bar{Q}_\epsilon) = ((1 + \epsilon)\bar{S}, 0, 0)$, the small perturbation of the disease-free equilibrium. We now give some properties of the following eigenvalue problem:

$$\lambda_\epsilon \phi_\epsilon = (\mathcal{H}_\epsilon + A)[\phi_\epsilon], \quad \forall \epsilon \in (-1, 1), \quad (6.1)$$

where A and \mathcal{H}_ϵ are respectively defined by (2.2) and (2.10).

Lemma 6.1 *Let Assumption 2.1 be satisfied. Under Assumption 1.1(ii), i.e. B is a compact operator, we have,*

- (i) $\text{sign}(s(A + \mathcal{H}_\epsilon)) = \text{sign}(\mathcal{R}_\epsilon - 1)$.
- (ii) $\mathcal{R}_\epsilon \rightarrow \mathcal{R}_0$ as $\epsilon \rightarrow 0$.

(iii) For each $\epsilon \in (-1, 1)$, $\text{sign}(s(A + \mathcal{H}_\epsilon)) = \text{sign}(\omega_0(\mathcal{A} + \mathcal{B}_\epsilon))$, where $\omega_0(\mathcal{A} + \mathcal{B}_\epsilon) = \lim_{t \rightarrow +\infty} \frac{\ln \|T_{\mathcal{A} + \mathcal{B}_\epsilon}(t)\|}{t}$,

and $\{T_{\mathcal{A} + \mathcal{B}_\epsilon}(t)\}_{t \geq 0}$ is the positive semigroup generated by $\mathcal{A} + \mathcal{B}_\epsilon$.

(iv) If $\mathcal{R}_\epsilon > 1$, then $s(A + \mathcal{H}_\epsilon)$ is an eigenvalue of $A + \mathcal{H}_\epsilon$ associated with an eigenfunction $\phi_\epsilon \in C_+(\Omega)$ with $\phi_\epsilon(x) > 0$ for all $x \in \Omega$.

Proof. We first prove item (i). Recalling that $\mathcal{R}_\epsilon = r(\mathcal{H}_\epsilon(-A)^{-1})^2$, it is clear that $\mathcal{R}_\epsilon - 1$ and $r(\mathcal{H}_\epsilon(-A)^{-1}) - 1$ have the same sign. Moreover, A is resolvent positive with $s(A) < 0$ and \mathcal{H}_ϵ is a positive perturbation of A . Therefore, we infer from Theorem 3.5 in [23] that $r(\mathcal{H}_\epsilon(-A)^{-1}) - 1$ and $s(A + \mathcal{H}_\epsilon)$ have the same sign. This completes the proof of item (i). Next, we prove item (ii). The compactness of B implies the compactness of \mathcal{H}_ϵ providing that $\mathcal{H}_\epsilon(-A)^{-1}$ is a compact positive operator on $C(\Omega)$. Since $\mathcal{H}_\epsilon(-A)^{-1}$ converges to $\mathcal{H}_0(-A)^{-1}$ in the operator norm, item (ii) follows from the continuity of spectral radius of compact operators. To obtain item (iii), let us recall that \mathcal{R}_ϵ is also given by $\mathcal{R}_\epsilon = r(\mathcal{B}_\epsilon(-\mathcal{A})^{-1})$. Since $s(\mathcal{A}) < 0$ and \mathcal{B}_ϵ is a positive operator, one knows that $\mathcal{R}_\epsilon - 1$ and $s(\mathcal{A} + \mathcal{B}_\epsilon)$ have the same sign. Moreover, $\mathcal{A} + \mathcal{B}_\epsilon$ being a generator of a uniformly continuous semigroup, we have $s(\mathcal{A} + \mathcal{B}_\epsilon) = \omega_0(\mathcal{A} + \mathcal{B}_\epsilon)$ and the result follows from item (i). For the proof of item (iv), we recall that $\lambda \rightarrow r(\mathcal{H}_\epsilon(\lambda - A)^{-1})$ is convex and strictly decreasing in $(s(A), +\infty)$. Since $\mathcal{R}_\epsilon = r(\mathcal{H}_\epsilon(-A)^{-1}) > 1$, by the intermediate value theorem, there exists $\lambda_\epsilon > 0$ such that $r(\mathcal{H}_\epsilon(\lambda_\epsilon - A)^{-1}) = 1$. Therefore we have $s(\mathcal{H}_\epsilon + A - \lambda_\epsilon) = 0$ so that $s(\mathcal{H}_\epsilon + A) = \lambda_\epsilon$. Next, using the compactness of B , it follows that $(\lambda_\epsilon - A)^{-1}\mathcal{H}_\epsilon$ is a compact positive operator on $C(\Omega)$ and $r((\lambda_\epsilon - A)^{-1}\mathcal{H}_\epsilon) = r(\mathcal{H}_\epsilon(\lambda_\epsilon - A)^{-1}) = 1$. From the Krein-Rutman's theorem, there exists an eigenfunction $\phi_\epsilon \in C_+(\Omega)$ such

$$(\lambda_\epsilon - A)^{-1}\mathcal{H}_\epsilon[\phi_\epsilon] = \phi_\epsilon \Leftrightarrow A[\phi_\epsilon] + \mathcal{H}_\epsilon[\phi_\epsilon] = \lambda_\epsilon\phi_\epsilon.$$

The above equality explicitly writes

$$d \int_{\Omega} J(x-y)\phi_\epsilon(y)dy + B[\phi_\epsilon](x) - (d + \delta(x))\phi_\epsilon(x) = \lambda_\epsilon\phi_\epsilon(x)$$

or equivalently

$$\frac{d}{\lambda_\epsilon + d + \delta(x)} \int_{\Omega} J(x-y)\phi_\epsilon(y)dy + \frac{1}{\lambda_\epsilon + d + \delta(x)} B[\phi_\epsilon](x) = \phi_\epsilon(x), \quad \forall x \in \Omega.$$

Since $\int_{\Omega} J(x-y)\phi_\epsilon(y)dy > 0$ for all $x \in \Omega$, it comes $\phi_\epsilon(x) > 0$ for all $x \in \Omega$. ■

Lemma 6.2 *Let Assumption 2.1 be satisfied. Under Assumption 1.1(i), i.e. B is a multiplication operator, we have,*

(i) $\text{sign}(s(A + \mathcal{H}_\epsilon)) = \text{sign}(\mathcal{R}_\epsilon - 1)$.

(ii) $s(A + \mathcal{H}_\epsilon) \rightarrow s(A + \mathcal{H}_0)$ as $\epsilon \rightarrow 0$.

(iii) For each $\epsilon \in (-1, 1)$, $\text{sign}(s(A + \mathcal{H}_\epsilon)) = \text{sign}(\omega_0(\mathcal{A} + \mathcal{B}_\epsilon))$, where $\omega_0(\mathcal{A} + \mathcal{B}_\epsilon) = \lim_{t \rightarrow +\infty} \frac{\ln \|T_{\mathcal{A} + \mathcal{B}_\epsilon}(t)\|}{t}$,

and $\{T_{\mathcal{A} + \mathcal{B}_\epsilon}(t)\}_{t \geq 0}$ is the positive semigroup generated by $\mathcal{A} + \mathcal{B}_\epsilon$.

(iv) If $\mathcal{R}_\epsilon > 1$, then $s(A + \mathcal{H}_\epsilon)$ is an eigenvalue of $A + \mathcal{H}_\epsilon$ associated with an eigenfunction $\phi_\epsilon \in C_+(\Omega)$ with $\phi_\epsilon(x) > 0$ for all $x \in \Omega$.

Proof. The property (i) is similar to the proof of item (i) in Lemma 6.1. To obtain item (ii), we observe that $\mathcal{H}_\epsilon = (1 + \epsilon)\mathcal{H}_0$. Noting that \mathcal{H}_0 is a multiplication operator represented by a continuous real-valued function, we deduce from Cohen's Theorem (see Theorem 6.1 in [29]) that the map $\epsilon \mapsto s(\mathcal{H}_\epsilon + A)$ is convex. This implies the continuity of $\epsilon \mapsto s(\mathcal{H}_\epsilon + A)$ in any closed sub interval of $(-1, 1)$, and thus ends the proof of item (ii). item (iii) can be proved as for item (iii) of Lemma 6.1. Finally if $\mathcal{R}_\epsilon > 1$ then $s(A + \mathcal{H}_\epsilon) > 0$ so that $s(A + \mathcal{H}_\epsilon) > s(A)$ and the result follows from Theorem 3.2 in [15] ■

We now turn to the proof of Theorem 2.4.

Proof of Theorem 2.4. Proof of item i). Let (S^*, I^*, Q^*) a stationary solution of (1.1). Then we have

$$\begin{cases} 0 = \Lambda(x) - \beta(x) \frac{Q^*(x)}{K + Q^*(x)} S^*(x) - \mu(x) S^*(x) \\ 0 = \beta(x) \frac{Q^*(x)}{K + Q^*(x)} S^*(x) - (\gamma(x) + \mu(x)) I^* \\ 0 = A[Q^*](x) + B[I^*](x) \end{cases}$$

so that, for all $x \in \Omega$,

$$S^*(x) \leq \bar{S}(x) \quad \text{and} \quad I^*(x) = \frac{\beta(x)}{(\gamma(x) + \mu(x))} \frac{Q^*(x)}{K + Q^*(x)} S^*(x) \leq \frac{\beta(x)}{K(\gamma(x) + \mu(x))} Q^*(x) \bar{S}(x).$$

Therefore,

$$0 \leq A[Q^*](x) + \mathcal{H}_0[Q^*](x), \quad \forall x \in \Omega$$

that is

$$Q^* \leq (-A)^{-1} \mathcal{H}_0[Q^*], \quad (6.2)$$

where A and \mathcal{H}_0 are defined by (2.2) and (2.10) respectively. Since $(-A)^{-1} \mathcal{H}_0$ is a positive operator, (6.2) gives

$$Q^* \leq ((-A)^{-1} \mathcal{H}_0)^n [Q^*], \quad \forall n \geq 1. \quad (6.3)$$

Assuming that $\|Q^*\|_\infty > 0$, from (6.3) it comes

$$\|Q^*\|_\infty \leq \|((-A)^{-1} \mathcal{H}_0)^n [Q^*]\|_\infty, \quad \forall n \geq 1 \implies 1 \leq \|((-A)^{-1} \mathcal{H}_0)^n\|, \quad \forall n \geq 1.$$

Thus,

$$1 \leq \lim_{n \rightarrow +\infty} \|((-A)^{-1} \mathcal{H}_0)^n\|^{\frac{1}{n}} = r((-A)^{-1} \mathcal{H}_0).$$

Since $r((-A)^{-1} \mathcal{H}_0) = r(\mathcal{H}_0(-A)^{-1}) < 1$, a contradiction holds. Consequently, $Q^* \equiv 0$. From where $I^* \equiv 0$ and $S^* \equiv \bar{S}$.

Proof of item ii). Let $\epsilon \in (-1, 1)$ be given. Thanks to Lemmas 6.2 and 6.1, $\text{sign}(\mathcal{R}_\epsilon - 1) = \text{sign}(\omega_0(\mathcal{A} + \mathcal{B}_\epsilon))$, with \mathcal{R}_ϵ defined by (2.8). Let us fix $\epsilon \in (0, 1)$ such that $\omega_0(\mathcal{A} + \mathcal{B}_\epsilon) < 0$. Next, we infer from Theorem 2.2 that there exists $t_0 = t_0(\epsilon, S_0, I_0)$ such that $S(t, \cdot) \leq (1 + \epsilon) \bar{S} = \bar{S}_\epsilon$ for all $t \geq t_0$. Hence, for all $t \geq t_0$

$$\begin{cases} \frac{\partial I(t, x)}{\partial t} \leq \beta(x) \frac{Q(t, x)}{K + Q(t, x)} S^\epsilon(x) - (\gamma(x) + \mu(x)) I(t, x) \\ \frac{\partial Q(t, x)}{\partial t} = A[Q(t, \cdot)](x) + B[I(t, \cdot)](x). \end{cases}$$

Next, consider the following linear equation

$$\begin{cases} \frac{\partial \widehat{I}(t, x)}{\partial t} = \frac{\beta(x) \bar{S}_\epsilon(x)}{K} \widehat{Q}(t, x) - (\gamma(x) + \mu(x)) \widehat{I}(t, x), \quad t > t_0 \\ \frac{\partial \widehat{Q}(t, x)}{\partial t} = A[\widehat{Q}(t, \cdot)](x) + B[\widehat{I}(t, \cdot)](x), \quad t > t_0, \\ \widehat{I}(t_0, \cdot) = I(t_0, \cdot), \quad \widehat{Q}(t_0, \cdot) = \widehat{Q}(t_0, \cdot), \end{cases}$$

that is

$$\begin{pmatrix} \widehat{I}(t, \cdot) \\ \widehat{Q}(t, \cdot) \end{pmatrix} = T_{\mathcal{A}+\mathcal{B}_\epsilon}(t-t_0) \begin{pmatrix} I(t_0, \cdot) \\ Q(t_0, \cdot) \end{pmatrix}, \quad \forall t \geq t_0.$$

Since $t \rightarrow T_{\mathcal{A}+\mathcal{B}_\epsilon}(t)$ is a positive semigroup one can use the comparison principle in [28] to conclude that

$$\begin{pmatrix} I(t, \cdot) \\ Q(t, \cdot) \end{pmatrix} \leq T_{\mathcal{A}+\mathcal{B}_\epsilon}(t-t_0) \begin{pmatrix} I(t_0, \cdot) \\ Q(t_0, \cdot) \end{pmatrix}, \quad \forall t \geq t_0.$$

Because $\omega_0(\mathcal{A} + \mathcal{B}_\epsilon) < 0$, one knows that for each $\eta \in (0, -\omega_0(\mathcal{A} + \mathcal{B}_\epsilon))$ there exists $M = M(\eta) \geq 1$ such that

$$\|T_{\mathcal{A}+\mathcal{B}_\epsilon}(t)\| \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

Since the norm in $C_+(\Omega)^2$ is monotone, we get

$$\|I(t, \cdot)\|_\infty + \|Q(t, \cdot)\|_\infty \leq Me^{-\eta(t-t_0)} (\|I(t_0, \cdot)\|_\infty + \|Q(t_0, \cdot)\|_\infty), \quad \forall t \geq t_0. \quad (6.4)$$

By (6.4), we conclude that $I(t, \cdot)$ and $Q(t, \cdot)$ converge exponentially in $C(\Omega)$ to 0. Consequently, it is easy to obtain that $S(t, \cdot)$ converges to \bar{S} . In fact this follows from the exponential convergence of $I(t, \cdot)$ and $Q(t, \cdot)$ to 0 combined with a variation of constant formula in the S -equation. This ends the proof of Theorem 2.4. \blacksquare

7 Proof of Theorem 2.5

7.1 Proof of Theorem 2.5(i)

Let (S^*, I^*, Q^*) be a stationary solution of (1.1). Then,

$$0 = \Lambda - \beta \frac{Q^*}{K + Q^*} S^* - \mu S^*, \quad (7.1)$$

$$0 = \beta \frac{Q^*}{K + Q^*} S^* - (\gamma + \mu) I^*, \quad (7.2)$$

$$0 = A[Q^*] + B[I^*]. \quad (7.3)$$

By (7.2) and (7.1), we have

$$I^* = \beta \frac{Q^*}{(\gamma + \mu)(K + Q^*)} \left(\mu + \beta \frac{Q^*}{K + Q^*} \right)^{-1} \Lambda.$$

Substituting the last equality into (7.3) it comes,

$$Q^* = \mathcal{G}[Q^*],$$

where $\mathcal{G} : C(\Omega) \rightarrow C(\Omega)$ is a positive operator defined by

$$\mathcal{G}[Q] = (-A)^{-1} B \left[\beta \frac{Q}{(\gamma + \mu)(K + Q)} \left(\mu + \beta \frac{Q}{K + Q} \right)^{-1} \Lambda \right].$$

Since $(-A)^{-1}B$ is a positive operator, we have

$$\|\mathcal{G}[Q]\|_\infty \leq \left\| (-A)^{-1}B \left[\frac{\Lambda}{\gamma + \mu} \right] \right\|_\infty, \quad \forall Q \in C_+(\Omega),$$

from where we can find a finite number $\nu > 0$ such that the set $\{Q \in C_+(\Omega) : \|Q\|_\infty \leq \nu\}$ is invariant with respect to \mathcal{G} . Moreover, since B is a linear operator with $B(0) = 0$, note that the Fréchet derivative \mathcal{G}_0 of the operator \mathcal{G} at the origin is given by

$$\mathcal{G}_0[Q] = (-A)^{-1}B \left[\frac{\beta\Lambda}{K(\gamma + \mu)\mu} Q \right].$$

Since \mathcal{G}_0 is compact and nonsupporting, it has a unique positive eigenvector corresponding to its spectral radius $r(\mathcal{G}_0)$. By the Krasnoselski's fixed point theorem (see p.135, Theorem 4.11, [30]), we conclude that \mathcal{G} has at least one non-zero fixed point in the positive cone of $C_+(\Omega)$ if $r(\mathcal{G}_0) > 1$. On the other hand, we have $r(\mathcal{G}_0) = r((-A)^{-1}\mathcal{H}_0)$, where \mathcal{H}_0 is the operator defined by (2.10). From Lemma 2.3, $r((-A)^{-1}\mathcal{H}_0) = \mathcal{R}_0^2$, hence we conclude that there exists at least one endemic stationary state if $\mathcal{R}_0 > 1$.

7.2 Technical materials and uniform persistence

Before constructing a suitable Lyapunov function to study the stability of the endemic stationary state, let us first collect some properties of the semiflow.

Lemma 7.1 *Let Assumptions 1.1 and 2.1 be satisfied. Let $(S_0, I_0, Q_0) \in C_+(\Omega)$ be given. Then, $S(t, x) > 0$ for all $(t, x) \in (0, \infty) \times \Omega$. If $Q_0(x) > 0$ for some $x \in \Omega$ then, $Q(t, x) > 0$, and $I(t, x) > 0$ for all $(t, x) \in (0, \infty) \times \Omega$.*

Proof. From the S -equation of (1.1) we have for all $t > 0$ and $x \in \Omega$

$$S(t, x) = e^{-\int_0^t (\beta(x) \frac{Q(l, x)}{K + Q(l, x)} + \mu(x)) dl} S(0, x) + \int_0^t e^{-\int_s^t (\beta(x) \frac{Q(l, x)}{K + Q(l, x)} + \mu(x)) dl} \Lambda(x) ds$$

and, the first assertion of the lemma follows. Next, the Q -equation of (1.1) gives, for all $t > 0$ and $x \in \Omega$,

$$Q(t, x) = T_A(t)[Q_0](x) + \int_0^t T_A(t-s) \circ B[I(s, \cdot)](x) ds.$$

Using similar arguments as in the proof of theorem 2.1 in [31] or proposition 2.2 in [32] or lemma 5.3 in [15], one knows that, if $Q_0(x) > 0$ for some $x \in \Omega$ then, $T_A(t)[Q_0](x) > 0$ for all $t > 0$ and $x \in \Omega$. Finally, by the I -equation and the fact that $Q(t, x) > 0$ for all $t > 0$ and $x \in \Omega$ we find that $I(t, x) > 0$ for all $t > 0$ and $x \in \Omega$. Indeed, this follows from the equality

$$I(t, x) = e^{-(\gamma(x) + \mu(x))t} I(0, x) + \int_0^t e^{-(\gamma(x) + \mu(x))(t-s)} \beta(x) \frac{Q(s, x)}{K + Q(s, x)} S(s, x) ds, \quad \forall t \geq 0 \text{ and } x \in \Omega. \quad \blacksquare$$

Lemma 7.2 *Let Assumptions 2.1 and 1.1 be satisfied. If $\mathcal{R}_0 > 1$ then the semiflow generated by System (1.1)-(1.2) is weakly uniformly persistent in the sense that, there exists $\eta > 0$ such that*

$$\limsup_{t \rightarrow +\infty} \sup_{x \in \Omega} Q(t, x) \geq \eta, \quad (7.4)$$

for all nonnegative initial conditions satisfying $Q_0(x) > 0$ for some $x \in \Omega$.

Proof. Assume that $\mathcal{R}_0 > 1$. Then there exists $\epsilon \in (-1, 0)$ such that $\mathcal{R}_\epsilon > 1$, i.e. $r(((-A)^{-1}\mathcal{H}_\epsilon)) > 1$. Set $\mu_0 = \inf_{\Omega} \mu(x)$, and let $\delta > 0$ be small enough such that

$$\frac{\mu_0}{\mu_0 + \delta} > 1 + \epsilon.$$

Assume that the semiflow is not weakly uniformly persistent. Let $\eta \in \left(0, \frac{\delta K}{2\|\beta\|_\infty}\right)$ be given. Then, there exists $(S_0, I_0, Q_0) \in C_+(\Omega)^3$ with $Q_0 \not\equiv 0$ such that

$$\limsup_{t \rightarrow +\infty} \sup_{x \in \Omega} Q(t, x) < \eta. \quad (7.5)$$

Note that we have

$$\limsup_{t \rightarrow +\infty} \sup_{x \in \Omega} \beta(x) \frac{Q(t, x)}{K + Q(t, x)} \leq \frac{\|\beta\|_\infty}{K} \limsup_{t \rightarrow +\infty} \sup_{x \in \Omega} Q(t, x) < \eta \frac{\|\beta\|_\infty}{K} < \frac{\delta}{2}. \quad (7.6)$$

Hence, (7.6) ensures that there exists $t_0 > 0$ large enough such that

$$0 \leq \beta(x) \frac{Q(t, x)}{K + Q(t, x)} < \delta, \quad \forall t \geq t_0, \quad \forall x \in \Omega. \quad (7.7)$$

Using (7.7) and the S -equation of (1.1) we obtain

$$\frac{\partial S(t, x)}{\partial t} \geq \Lambda(x) - (\delta + \mu(x))S(t, x), \quad t \geq t_0, \quad x \in \Omega$$

providing that there exists $t_1 > 0$ large enough such that

$$S(t, x) \geq \frac{\Lambda(x)}{\mu(x) + \delta} = \bar{S}(x) \frac{\mu(x)}{\mu(x) + \delta} \geq \bar{S}(x)(1 + \epsilon) = \bar{S}^\epsilon(x), \quad \forall t \geq t_1, \quad \forall x \in \Omega.$$

As a consequence, we have for all $t \geq 0$ and $x \in \Omega$

$$\begin{cases} \frac{\partial I(t + t_1, x)}{\partial t} \geq \frac{\beta(x)\bar{S}_\epsilon(x)}{K + \eta} Q(t + t_1, x) - (\gamma(x) + \mu(x))I(t + t_1, x) \\ \frac{\partial Q(t + t_1, x)}{\partial t} = d \int_{\Omega} J(x - y)[Q(t + t_1, y) - Q(t + t_1, x)]dy + B[I(t + t_1, \cdot)](x) - \delta(x)Q(t + t_1, x). \end{cases} \quad (7.8)$$

The boundedness of the semiflow allows us to define, for each $\lambda > 0$, the Laplace transform of $I(t + t_1, x)$ and $Q(t + t_1, x)$ with respect to t as

$$\begin{cases} \widehat{I}(\lambda, x) = \int_0^{+\infty} e^{-\lambda t} I(t + t_1, x) dt, \quad \forall x \in \Omega \\ \widehat{Q}(\lambda, x) = \int_0^{+\infty} e^{-\lambda t} Q(t + t_1, x) dt, \quad \forall x \in \Omega. \end{cases}$$

Next, taking the Laplace transform of (7.8) we obtain

$$\begin{cases} \widehat{I}(\lambda, x) \geq \frac{\beta(x)\bar{S}_\epsilon(x)}{(K+\eta)(\lambda+\gamma(x)+\mu(x))}\widehat{Q}(\lambda, x), \quad \forall x \in \Omega, \quad \forall \lambda > 0 \\ \lambda\widehat{Q}(\lambda, x) = d \int_{\Omega} J(x-y)[\widehat{Q}(\lambda, y) - \widehat{Q}(\lambda, x)]dy + B[\widehat{I}(\lambda, \cdot)](x) - \delta(x)\widehat{Q}(\lambda, x), \quad \forall x \in \Omega, \quad \forall \lambda > 0. \end{cases} \quad (7.9)$$

Next, set $\gamma_0 = \inf_{\Omega} \gamma(x)$ and observe that

$$\frac{\beta(x)\bar{S}_\epsilon(x)}{(K+\eta)(\lambda+\gamma(x)+\mu(x))} \geq \frac{K(\gamma_0+\mu_0)}{(K+\eta)(\lambda+\gamma_0+\mu_0)} \frac{\beta(x)\bar{S}_\epsilon(x)}{K(\gamma(x)+\mu(x))}, \quad \forall \lambda > 0, \quad x \in \Omega.$$

Thus, setting $\Gamma(\lambda, \eta) = \frac{K(\gamma_0+\mu_0)}{(K+\eta)(\lambda+\gamma_0+\mu_0)}$, it follows from (7.9) that

$$\lambda\widehat{Q}(\lambda, x) \geq A[\widehat{Q}(\lambda, \cdot)](x) + \Gamma(\lambda, \eta)\mathcal{H}_\epsilon[\widehat{Q}(\lambda, \cdot)](x), \quad \forall x \in \Omega, \quad \forall \lambda > 0.$$

Hence, recalling that $s(A) < 0$ and A is resolvent positive, we get

$$\widehat{Q}(\lambda, x) \geq \Gamma(\lambda, \eta)(\lambda - A)^{-1}\mathcal{H}_\epsilon[\widehat{Q}(\lambda, \cdot)](x), \quad \forall x \in \Omega, \quad \forall \lambda > 0. \quad (7.10)$$

Thanks to Lemma 7.1 we have $\widehat{Q}(\lambda, x) > 0$ for all $x \in \Omega$ and $\lambda > 0$. Therefore, using the same argument in the proof of Proposition 8.1 in [14], we deduce from (7.10) that

$$1 \geq \Gamma(\lambda, \eta)r((\lambda - A)^{-1}\mathcal{H}_\epsilon), \quad \forall \lambda > 0. \quad (7.11)$$

Since $\eta \in \left(0, \frac{\delta K}{2\|\beta\|_\infty}\right)$ is arbitrary, by letting λ and η go to 0 in (7.11) we obtain the contradiction $1 \geq r((-A)^{-1}\mathcal{H}_\epsilon)$. \blacksquare

We now turn to prove the strong uniform persistence of the semiflow $(S(t, \cdot), I(t, \cdot), Q_1(t, \cdot), Q_2(t, \cdot))$ of System (1.1)-(1.2). For that ends, for each $(S_0, I_0, Q_0) \in C_+(\Omega)$, we set

$$\xi(S_0, I_0, Q_0) = \sup_{x \in \Omega} Q_0(x) \quad \text{and} \quad \xi_0(S_0, I_0, Q_0) := \inf_{x \in \Omega} Q_0(x). \quad (7.12)$$

While the semiflow's weakly persistence is satisfied under the general Assumption 1.1, the strong persistence is guaranteed only under Assumption 1.1(ii), *i.e.* under the compactness of the linear operator B . The precise result reads

Proposition 7.3 *Let Assumptions 2.1 and 1.1(ii) be satisfied. Then the semiflow is strongly uniformly persistent in the sense that, there exists $\eta > 0$ such that*

$$\liminf_{t \rightarrow +\infty} \inf_{x \in \Omega} Q(t, x) \geq \eta \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \inf_{x \in \Omega} I(t, x) \geq \eta \quad (7.13)$$

for all nonnegative initial conditions satisfying $Q_0(x) > 0$ for some $x \in \Omega$.

Proof. To prove the our result, we will use Theorem A.34 of [33]. Indeed, the semiflow generated by (1.1)-(1.2) as a compact global attractor. Moreover, by Lemma 7.1 we have

$$\xi(S_0, I_0, Q_0) > 0 \Rightarrow \xi(S(t, \cdot), I(t, \cdot), Q(t, \cdot)) > 0, \quad \forall t > 0.$$

Next, let $\{S(t, \cdot), I(t, \cdot), Q(t, \cdot)\}_{t \in \mathbb{R}}$ be an entire solution of (1.1)-(1.2) such that $\xi(S(t, \cdot), I(t, \cdot), Q(t, \cdot)) > 0$ for some $t < 0$. Then by Lemma 7.1 we have $Q(0, x) > 0$ for all $x \in \Omega$ and by continuity of $Q(0, \cdot)$ we conclude that

$$\xi(S(0, \cdot), I(0, \cdot), Q(0, \cdot)) = \inf_{x \in \Omega} Q(0, x) > 0.$$

Finally, combining the above properties together with the weak uniform persistence in Lemma 7.2, we can apply Theorem A.34 of [33] to conclude that the disease is strongly uniformly persistent in the sense that there exists $\eta_0 > 0$ such that

$$\liminf_{t \rightarrow +\infty} \inf_{x \in \Omega} Q(t, x) \geq \eta_0, \quad (7.14)$$

for all nonnegative initial conditions satisfying $Q_0(x) > 0$ for some $x \in \Omega$. Recalling that the solution of (1.1)-(1.2) are bounded uniformly in $t \geq 0$ and $x \in \Omega$, and setting

$$\begin{cases} S_\infty(x) = \liminf_{t \rightarrow +\infty} S(t, x), \quad \forall x \in \Omega \\ I_\infty(x) = \liminf_{t \rightarrow +\infty} I(t, x), \quad \forall x \in \Omega \\ Q_\infty(x) = \liminf_{t \rightarrow +\infty} Q(t, x), \quad \forall x \in \Omega \end{cases}$$

one can use the method of fluctuations (see Appendix A.3 in [34]) to deduce from (1.1) that

$$\begin{cases} 0 \geq \Lambda(x) - \beta(x)S_\infty(x) - \mu(x)S_\infty(x), \quad \forall x \in \Omega \\ 0 \geq \beta(x)\frac{Q_\infty(x)}{K + Q_\infty(x)}S_\infty(x) - (\gamma(x) + \mu(x))I_\infty(x), \quad \forall x \in \Omega. \end{cases}$$

Solving the first equation, we obtain

$$S_\infty(x) \geq \bar{S}(x)\frac{\mu(x)}{\mu(x) + \beta(x)}, \quad \forall x \in \Omega$$

and the second equation gives

$$I_\infty(x) \geq \beta(x)\frac{\eta_0}{K + \eta_0}\bar{S}(x)\frac{\mu(x)}{\mu(x) + \beta(x)}, \quad \forall x \in \Omega. \quad (7.15)$$

The result follows by using the continuity of the right hand side of (7.15). \blacksquare

7.3 Proof of Theorem 2.5(ii)

Let $E^* = (S^*, I^*, Q^*)$ be the endemic stationary state of (1.1). Let

$$M = \{(S, I, Q) \in C_+(\Omega)^3 : Q(x) > 0 \text{ for some } x \in \Omega\}. \quad (7.16)$$

As a consequence of Proposition 7.3 and Theorem 3.7 in [35] (see also [34]), there exists \mathcal{M}_0 , a compact subset of M , which is a global attractor for the semiflow $\{\mathcal{U}(t)\}_t$ in M . To complete the proof of Theorem 2.5(ii) it remains to prove that $\mathcal{M}_0 = \{E^*\}$. This will be achieved by constructing a suitable Lyapunov functional on \mathcal{M}_0 . Let $\{u(t) = (S(t, \cdot), I(t, \cdot), Q(t, \cdot))\}_t \subset \mathcal{M}_0$ be given. Using Lemma 7.1, let us define the following functional

$$W[u](t) = W_S(t) + W_I(t) + W_Q(t),$$

with

$$\begin{cases} W_S(t) = \int_{\Omega} w_1(x) S^*(x) g\left(\frac{S(t,x)}{S^*(x)}\right) dx, \\ W_I(t) = \int_{\Omega} w_2(x) I^*(x) g\left(\frac{I(t,x)}{I^*(x)}\right) dx, \\ W_Q(t) = \int_{\Omega} Q^*(x)^2 g\left(\frac{Q(t,x)}{Q^*(x)}\right) dx, \end{cases}$$

where

$$w_2(x) = \int_{\Omega} J(x-y) Q^*(y) I^*(x) dy, \quad w_1(x) = \frac{w_2(x)}{N(Q^*(x)) S^*(x)}, \quad g(z) = z - 1 - \ln z,$$

and N defined by (4.1). Then the function $t \mapsto W[u](t)$ is of the class C^1 on \mathbb{R} , and we claim that (see Appendix A for the proof)

Claim 7.4

$$\begin{aligned} \frac{dW[u](t)}{dt} &= - \int_{\Omega} \int_{\Omega} J(x-y) I^*(x) Q^*(y) g\left(\frac{S^*(x)}{S(t,x)}\right) dx dy \\ &+ \int_{\Omega} \int_{\Omega} J(x-y) I^*(x) Q^*(y) \left(g\left(\frac{N(Q(t,x))}{N(Q^*(x))}\right) - g\left(\frac{Q(t,x)}{Q^*(x)}\right) \right) dx dy \\ &- \int_{\Omega} \int_{\Omega} \frac{\mu(x)}{N(Q^*(x)) S^*(x)} \frac{(S(t,x) - S^*(x))^2}{S(t,x)} J(x-y) I^*(x) Q^*(y) dx dy \\ &- \int_{\Omega} \int_{\Omega} K(x-y) I^*(x) Q^*(y) g\left(\frac{N(Q(t,x)) S(t,x)}{N(Q^*(x)) S^*(x)} \frac{I^*(x)}{I(t,x)}\right) dx dy \\ &- d \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) Q^*(y) g\left(\frac{Q^*(x)}{Q(t,x)} \frac{Q(t,y)}{Q^*(y)}\right) dy dx \\ &- \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) I^*(y) \left[g\left(\frac{Q(t,x)}{Q^*(x)}\right) + g\left(\frac{Q^*(x)}{Q(t,x)} \frac{I(t,y)}{I^*(y)}\right) \right] dx dy. \end{aligned} \tag{7.17}$$

Let us complete the proof of Theorem 2.5(ii). By Proposition A.1 of [36], one knows that

$$g\left(\frac{N(Q(t,x))}{N(Q^*(x))}\right) - g\left(\frac{Q(t,x)}{Q^*(x)}\right) \leq 0, \quad \forall x \in \Omega, \quad \forall t \in \mathbb{R}.$$

Consequently, Claim (7.4) leads to

$$\frac{dW[u](t)}{dt} \leq 0, \tag{7.18}$$

so that the map $t \mapsto W[u](t)$ is decreasing along the entire solutions of the semiflow. To conclude, let $\{t_n\}_{n \geq 0}$ be a sequence tending to $-\infty$ as $n \rightarrow \infty$ and consider the sequence of map $u_n(t) = u(t+t_n)$. Note that one has $W[u_n](t) = W[u](t+t_n)$. Up to a subsequence, we assume that $u_n(t) \rightarrow \widehat{u}(t)$ as $n \rightarrow \infty$ locally uniformly for $t \in \mathbb{R}$ where $\{\widehat{u}(t)\}_{t \in \mathbb{R}} \subset \mathcal{M}_0$ is an entire solution of the semiflow. Since W is decreasing, it comes

$$W[\widehat{u}](t) \equiv \lim_{t \rightarrow -\infty} W[u](t) = \sup_{t \in \mathbb{R}} W[u](t).$$

Setting $\widehat{u} = (\widehat{S}, \widehat{I}, \widehat{Q})$, (7.18) gives $(\widehat{S}(t, \cdot), \widehat{I}(t, \cdot), \widehat{Q}(t, \cdot)) \equiv (S^*(\cdot), I^*(\cdot), Q^*(\cdot))$. Therefore, $W[\widehat{u}](t) \equiv 0$, and $0 \leq W[u](t) \leq 0$ for $t \in \mathbb{R}$ and $u(t) \equiv E^*$. This completes the proof of Theorem 2.5(ii).

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A Proof of Claim 7.4

Let $\{u(t) = (S(t, \cdot), I(t, \cdot), Q(t, \cdot))\}_t \subset \mathcal{M}_0$ be the semiflow, where \mathcal{M}_0 is a compact subset of M defined by (7.16). Let (S^*, I^*, Q^*) be an endemic stationary state of Model (1.1). Since J is a symmetric kernel (Assumption 2.1), we have

Lemma A.1

$$\int_{\Omega} \int_{\Omega} J(x-y)Q^*(x)Q^*(y)[w(x) - w(y)]dx dy = 0, \quad \forall w \in L^{\infty}(\Omega).$$

Proof. Using the equality $J(x-y) = J(y-x)$ we have

$$\begin{aligned} \int_{\Omega} \int_{\Omega} J(x-y)Q^*(x)Q^*(y)w(x)dx dy &= \int_{\Omega} \int_{\Omega} J(y-x)Q^*(x)Q^*(y)w(x)dx dy \\ &= \int_{\Omega} \left(\int_{\Omega} J(y-x)Q^*(y)dy \right) w(x)Q^*(x)dx \\ &= \int_{\Omega} \left(\int_{\Omega} J(z-y)Q^*(z)dz \right) w(y)Q^*(y)dy \end{aligned}$$

$$= \int_{\Omega} \left(\int_{\Omega} J(x-y)Q^*(x)dx \right) w(y)Q^*(y)dy$$

and the proof of the lemma is completed. ■

Next, let us set

$$W[u](t) = W_S(t) + W_I(t) + W_Q(t)$$

with

$$\begin{cases} W_S(t) = \int_{\Omega} w_1(x)S^*(x)g\left(\frac{S(t,x)}{S^*(x)}\right) dx \\ W_I(t) = \int_{\Omega} w_2(x)I^*(x)g\left(\frac{I(t,x)}{I^*(x)}\right) dx \\ W_Q(t) = \int_{\Omega} Q^*(x)^2g\left(\frac{Q(t,x)}{Q^*(x)}\right) dx \end{cases}$$

Step 1: We have

$$\begin{aligned} \frac{dW_S(t)}{dt} &= - \int_{\Omega} w_1(x)N(Q^*(x))S^*(x)g\left(\frac{S^*(x)}{S(t,x)}\right) dx \\ &\quad - \int_{\Omega} w_1(x)N(Q^*(x))S^*(x)g\left(\frac{S(t,x)}{S^*(x)}\frac{N(Q(t,x))}{N(Q^*(x))}\right) dx \\ &\quad + \int_{\Omega} w_1(x)N(Q^*(x))S^*(x)g\left(\frac{N(Q(t,x))}{N(Q^*(x))}\right) dx \\ &\quad - \int_{\Omega} \mu w_1(x)\frac{(S(t,x) - S^*(x))^2}{S(t,x)} dx. \end{aligned} \tag{A.1}$$

Indeed,

$$\frac{dW_S(t)}{dt} = \int_{\Omega} w_1(x)\frac{\partial S(t,x)}{\partial x}\frac{S(t,x) - S^*(x)}{S(t,x)} dx. \tag{A.2}$$

Moreover, we have

$$\Lambda(x) = N(Q^*(x))S^*(x) + \mu(x)S^*(x).$$

so that

$$\frac{\partial S(t,x)}{\partial t} = N(Q^*(x))S^*(x) \left[1 - \frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \right] - \mu(x)(S(t,x) - S^*(x)). \tag{A.3}$$

Therefore,

$$\begin{aligned} \left(1 - \frac{S^*(x)}{S(t,x)} \right) \frac{\partial S(t,x)}{\partial t} &= N(Q^*(x))S^*(x) \left[1 - \frac{S^*(x)}{S(t,x)} - \frac{S(t,x)}{S^*(x)} \frac{N(Q(t,x))}{N(Q^*(x))} + \frac{N(Q(t,x))}{N(Q^*(x))} \right] \\ &\quad - \mu(x)\frac{(S(t,x) - S^*(x))^2}{S(t,x)} \end{aligned} \tag{A.4}$$

and by observing that

$$1 - \frac{S^*(x)}{S(t,x)} - \frac{S(t,x)}{S^*(x)} \frac{N(Q(t,x))}{N(Q^*(x))} + \frac{N(Q(t,x))}{N(Q^*(x))} = -g\left(\frac{S^*(x)}{S(t,x)}\right) - g\left(\frac{S(t,x)}{S^*(x)}\frac{N(Q(t,x))}{N(Q^*(x))}\right) + g\left(\frac{N(Q(t,x))}{N(Q^*(x))}\right)$$

we get

$$\begin{aligned} \left(1 - \frac{S^*(x)}{S(t,x)}\right) \frac{\partial S(t,x)}{\partial t} &= -\mu(x) \frac{(S(t,x) - S^*(x))^2}{S(t,x)} \\ &+ N(Q^*(x))S^*(x) \left[-g \left(\frac{S^*(x)}{S(t,x)} \right) - g \left(\frac{S(t,x)}{S^*(x)} \frac{N(Q(t,x))}{N(Q^*(x))} \right) + g \left(\frac{N(Q(t,x))}{N(Q^*(x))} \right) \right] \end{aligned} \quad (\text{A.5})$$

which leads to (A.1) by combining (A.2) and (A.5).

Step 2: We have

$$\begin{aligned} \frac{dW_I(t)}{dt} &= \int_{\Omega} w_2(x) g \left(\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \right) dx \\ &- \int_{\Omega} w_2(x) g \left(\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \frac{I^*(x)}{I(t,x)} \right) dx \\ &- \int_{\Omega} w_2(x) g \left(\frac{I(t,x)}{I^*(x)} \right) dx. \end{aligned} \quad (\text{A.6})$$

Indeed, plugging the equality

$$\gamma(x) + \mu(x) = \frac{N(Q^*(x))S^*(x)}{I^*(x)}$$

into the I -equation we obtain

$$\frac{\partial I(t,x)}{\partial t} = N(Q^*(x))S^*(x) \left[\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} - \frac{I(t,x)}{I^*(x)} \right].$$

Next, we note that

$$\frac{dW_I(t)}{dt} = \int_{\Omega} w_2(x) \frac{\partial I(t,x)}{\partial x} \frac{I(t,x) - I^*(x)}{I(t,x)} dx \quad (\text{A.7})$$

and

$$\begin{aligned} \frac{I(t,x) - I^*(x)}{I(t,x)} \frac{\partial I(t,x)}{\partial t} &= \frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} - 1 \\ &- \left[\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \frac{I^*(x)}{I(t,x)} - 1 \right] - \left[\frac{I(t,x)}{I^*(x)} - 1 \right] \end{aligned}$$

that is

$$\frac{I(t,x) - I^*(x)}{I(t,x)} \frac{\partial I(t,x)}{\partial t} = g \left(\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \right) - g \left(\frac{N(Q(t,x))S(t,x)}{N(Q^*(x))S^*(x)} \frac{I^*(x)}{I(t,x)} \right) - g \left(\frac{I(t,x)}{I^*(x)} \right) \quad (\text{A.8})$$

The equality (A.6) follows by plugging (A.8) into (A.7).

Step 3: We have,

$$\begin{aligned} \frac{dW_Q(t)}{dt} &= -d \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) Q^*(y) g \left(\frac{Q^*(x)}{Q(t,x)} \frac{Q(t,y)}{Q^*(y)} \right) dy dx \\ &+ \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) I^*(y) \left[g \left(\frac{I(t,y)}{I^*(y)} \right) - g \left(\frac{Q(t,x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x)}{Q(t,x)} \frac{I(t,y)}{I^*(y)} \right) \right] dy dx. \end{aligned} \quad (\text{A.9})$$

Indeed, using the equality

$$(d + \delta(x)) = d \int_{\Omega} J(x-y) \frac{Q^*(y)}{Q^*(x)} dy + \int_{\Omega} J(x-y) \frac{I^*(y)}{Q^*(x)} dy$$

we obtain

$$\begin{aligned} \frac{\partial Q(t, x)}{\partial t} &= d \int_{\Omega} J(x-y) Q(t, y) dy - \left[d \int_{\Omega} J(x-y) \frac{Q^*(y)}{Q^*(x)} dy + \int_{\Omega} J(x-y) \frac{I^*(y)}{Q^*(x)} dy \right] Q(t, x) \\ &\quad + \int_{\Omega} J(x-y) I(t, y) dy \\ &= d \int_{\Omega} J(x-y) Q^*(y) \left[\frac{Q(t, y)}{Q^*(y)} - \frac{Q(t, x)}{Q^*(x)} \right] dy \\ &\quad + \int_{\Omega} J(x-y) I^*(y) \left[\frac{I(t, y)}{I^*(y)} - \frac{Q(t, x)}{Q^*(x)} \right] dy \end{aligned}$$

so that

$$\begin{aligned} \left[1 - \frac{Q^*(x)}{Q(t, x)} \right] \frac{\partial Q(t, x)}{\partial t} &= d \int_{\Omega} J(x-y) Q^*(y) \left[\frac{Q(t, y)}{Q^*(y)} - \frac{Q(t, x)}{Q^*(x)} - \frac{Q^*(x)}{Q(t, x)} \frac{Q(t, y)}{Q^*(y)} + 1 \right] dy \\ &\quad + \int_{\Omega} J(x-y) I^*(y) \left[\frac{I(t, y)}{I^*(y)} - \frac{Q(t, x)}{Q^*(x)} - \frac{Q^*(x)}{Q(t, x)} \frac{I(t, y)}{I^*(y)} + 1 \right] dy. \end{aligned}$$

Since we have

$$\frac{Q(t, y)}{Q^*(y)} - \frac{Q(t, x)}{Q^*(x)} - \frac{Q^*(x)}{Q(t, x)} \frac{Q(t, y)}{Q^*(y)} + 1 = g \left(\frac{Q(t, y)}{Q^*(y)} \right) - g \left(\frac{Q(t, x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x)}{Q(t, x)} \frac{Q(t, y)}{Q^*(y)} \right)$$

and

$$\frac{I(t, y)}{I^*(y)} - \frac{Q(t, x)}{Q^*(x)} - \frac{Q^*(x)}{Q(t, x)} \frac{I(t, y)}{I^*(y)} + 1 = g \left(\frac{I(t, y)}{I^*(y)} \right) - g \left(\frac{Q(t, x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x)}{Q(t, x)} \frac{I(t, y)}{I^*(y)} \right)$$

it follows that

$$\begin{aligned} \left[1 - \frac{Q^*(x)}{Q(t, x)} \right] \frac{\partial Q(t, x)}{\partial t} &= d \int_{\Omega} J(x-y) Q^*(y) \left[g \left(\frac{Q(t, y)}{Q^*(y)} \right) - g \left(\frac{Q(t, x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x)}{Q(t, x)} \frac{Q(t, y)}{Q^*(y)} \right) \right] dy \\ &\quad + \int_{\Omega} J(x-y) I^*(y) \left[g \left(\frac{I(t, y)}{I^*(y)} \right) - g \left(\frac{Q(t, x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x)}{Q(t, x)} \frac{I(t, y)}{I^*(y)} \right) \right] dy \end{aligned}$$

Next, using the fact that

$$\frac{dW_Q(t)}{dt} = \int_{\Omega} Q^*(x) \left[1 - \frac{Q^*(x)}{Q(t, x)} \right] \frac{\partial Q(t, x)}{\partial t} dx$$

and, see Lemma A.1, it comes

$$\int_{\Omega} \int_{\Omega} J(x-y) Q^*(y) Q^*(x) \left[g \left(\frac{Q(t, y)}{Q^*(y)} \right) - g \left(\frac{Q(t, x)}{Q^*(x)} \right) \right] dx dy = 0$$

we obtain the equality (A.9).

Step 4: Finally, combining (A.1), (A.6) and (A.9), it follows that :

$$\begin{aligned}
\frac{dW[u](t)}{dt} = & - \int_{\Omega} w_1(x) N(Q^*(x)) S^*(x) g \left(\frac{S^*(x)}{S(t,x)} \right) dx \\
& - \int_{\Omega} w_1(x) N(Q^*(x)) S^*(x) g \left(\frac{S(t,x) N(Q(t,x))}{S^*(x) N(Q^*(x))} \right) dx \\
& + \int_{\Omega} w_1(x) N(Q^*(x)) S^*(x) g \left(\frac{N(Q(t,x))}{N(Q^*(x))} \right) dx \\
& - \int_{\Omega} \mu(x) w_1(x) \frac{(S(t,x) - S^*(x))^2}{S(t,x)} dx \\
& + \int_{\Omega} w_2(x) g \left(\frac{N(Q(t,x)) S(t,x)}{N(Q^*(x)) S^*(x)} \right) dx \\
& - \int_{\Omega} w_2(x) g \left(\frac{N(Q(t,x)) S(t,x) I^*(x)}{N(Q^*(x)) S^*(x) I(t,x)} \right) dx \\
& - \int_{\Omega} w_2(x) g \left(\frac{I(t,x)}{I^*(x)} \right) dx \\
& - d \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) Q^*(y) g \left(\frac{Q^*(x) Q(t,y)}{Q(t,x) Q^*(y)} \right) dy dx \\
& + \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) I^*(y) \left[g \left(\frac{I(t,y)}{I^*(y)} \right) - g \left(\frac{Q(t,x)}{Q^*(x)} \right) - g \left(\frac{Q^*(x) I(t,y)}{Q(t,x) I^*(y)} \right) \right] dy dx.
\end{aligned} \tag{A.10}$$

Setting

$$w_1(x) = \frac{w_2(x)}{N(Q^*(x)) S^*(x)}; \quad w_2(x) = \int_{\Omega} J(x-y) Q^*(y) I^*(x) dy$$

and replacing in (A.10), we obtain :

$$\begin{aligned}
\frac{dW(t)}{dt} = & - \int_{\Omega} \int_{\Omega} J(x-y) I^*(x) Q^*(y) g \left(\frac{S^*(x)}{S(t,x)} \right) dx dy \\
& + \int_{\Omega} \int_{\Omega} J(x-y) I^*(x) Q^*(y) \left(g \left(\frac{N(Q(t,x))}{N(Q^*(x))} \right) - g \left(\frac{Q(t,x)}{Q^*(x)} \right) \right) dx dy \\
& - \int_{\Omega} \int_{\Omega} \frac{\mu(x)}{N(Q^*(x)) S^*(x)} \frac{(S(t,x) - S^*(x))^2}{S(t,x)} J(x-y) I^*(x) Q^*(y) dx dy \\
& - \int_{\Omega} \int_{\Omega} J(x-y) I^*(x) Q^*(y) g \left(\frac{N(Q(t,x)) S(t,x) I^*(x)}{N(Q^*(x)) S^*(x) I(t,x)} \right) dx dy \\
& - d \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) Q^*(y) g \left(\frac{Q^*(x) Q(t,y)}{Q(t,x) Q^*(y)} \right) dy dx \\
& - \int_{\Omega} \int_{\Omega} J(x-y) Q^*(x) I^*(y) \left[g \left(\frac{Q(t,x)}{Q^*(x)} \right) + g \left(\frac{Q^*(x) I(t,y)}{Q(t,x) I^*(y)} \right) \right] dx dy.
\end{aligned} \tag{A.11}$$

From where estimate (7.17) follows, and this ends the proof of Claim 7.4.