Corrigendum to "Belief Functions Contextual Discounting and Canonical Decompositions" [International Journal of Approximate Reasoning 53 (2012) 146–158]

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We hereby correct Proposition 4 and Theorem 1 in [2], which contained erroneous results.

Let us first recall the problem. A source S of information provides to agent Ag a piece of information represented by a mass function m_S^Ω (with $\Omega = \{\omega_1, \ldots, \omega_K\}$), simply denoted by m in this corrigendum. Let $\mathcal{A}$ be a non empty set of subsets of Ω called contexts. Agent Ag owns a meta-knowledge regarding the reliability of S conditionally on each set $A \in \mathcal{A}$. Formally, for all $A \in \mathcal{A}$, we suppose that

$$\begin{cases} m_{Ag}^{\mathcal{R}}[A](\{R\}) & = & 1 - \alpha_A = \beta_A \\ m_{Ag}^{\mathcal{R}}[A](\mathcal{R}) & = & \alpha_A \ , \end{cases} \tag{1}$$

where $\alpha_A \in [0, 1]$ and $\mathcal{R} = \{R, NR\}$ (R meaning the source is reliable, NR otherwise), and the notation $m[\cdot]$ denotes conditioning.

With the same reasoning as in [1] (where $\mathcal{A}$ was supposed to form a partition of Ω), the knowledge m_{Ag}^Ω held by agent Ag on Ω, based on the information m provided by S and his metaknowledge regarding S represented by (1) for all $A \in \mathcal{A}$, can be obtained by the following computation,

$$\left(m^\Omega[\{R\}]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc_{A \in \mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow \Omega \times \mathcal{R}} \right)^{\downarrow \Omega} \ , \tag{2}$$

where symbol $\Uparrow$ and $\downarrow$ denote, respectively, the deconditioning and projection operations, and $m^\Omega[\{R\}] = m$.

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It is stated in [2] that, for $\mathcal{A} = 2^\Omega$ (Proposition 4) and more generally for any set $\mathcal{A}$ of contexts (Theorem 1), Equation (2) is equivalent to

$$m_\bigcirc \left(\bigcirc_{A \in \mathcal{A}} A_{\beta_A} \right) . \tag{3}$$

This statement is incorrect. In the general case, for any non empty $\mathcal{A}$, Equation (2) is equivalent to

$$m_\bigcirc \left(\bigcap_{A \in \mathcal{A}} \overline{A}^{\alpha_A} \right) , \tag{4}$$

as shown by the following proof, which corrects Theorem 1 from [2]. The fact that, in general, (4) is not equivalent to (3) (and particularly when $\mathcal{A} = 2^\Omega$), and therefore (2) is not equivalent in general to (3), is shown below by Example 1.

Proof 1. *Let us denote by A_i, $i \in I = \{1, \ldots, n\}$, the contexts present in $\mathcal{A}$, and let us write β_{A_i} simply by β_i, for all $i \in I$. For all $A_i \in \mathcal{A}$, the deconditioning of $m^{\mathcal{R}}[A_i]$ over $\Omega \times \mathcal{R}$ is given by*

$$m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}}(A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) = \beta_i, \tag{5a}$$
$$m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}}(\Omega \times \mathcal{R}) = \alpha_i. \tag{5b}$$

Moreover, for all $(A_i, A_j) \in \mathcal{A}^2$, such that $j \neq i$,

$$(A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) \cap (A_j \times \{R\} \cup \overline{A_j} \times \mathcal{R})$$
$$= (A_i \cap A_j) \times \{R\} \cup (A_i \cap \overline{A_j}) \times \{R\} \cup (\overline{A_i} \cap A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R}$$
$$= (A_i \cup A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R}.$$

With $\mathcal{A}$ composed of two elements denoted by A_i and A_j, we then have

$$\begin{cases} (m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_j]^{\Uparrow \Omega \times \mathcal{R}})((A_i \cup A_j) \times \{R\} \cup \overline{(A_i \cup A_j)} \times \mathcal{R}) &= \beta_i \beta_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_j]^{\Uparrow \Omega \times \mathcal{R}})(A_i \times \{R\} \cup \overline{A_i} \times \mathcal{R}) &= \beta_i \alpha_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_j]^{\Uparrow \Omega \times \mathcal{R}})(A_j \times \{R\} \cup \overline{A_j} \times \mathcal{R}) &= \alpha_i \beta_j \\ (m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_j]^{\Uparrow \Omega \times \mathcal{R}})(\Omega \times \mathcal{R}) &= \alpha_i \alpha_j \end{cases}.$$

In other words, all the focal elements of $\bigcirc_{A \in \mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow \Omega \times \mathcal{R}}$ are the elements $C \times \{R\} \cup \overline{C} \times \mathcal{R}$ with C composed of a union of elements A_i in $\mathcal{A}$, I' being the set of indices of the A_i's, which means with $C = \cup_{i \in I' \subseteq I} A_i$. Moreover, each focal element has a mass equal to $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$. Let us note that this latter result is also true if $\mathcal{A}$ is composed of one element $A \subseteq \Omega$ (directly from Equations (5)).

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By induction, we can show that this property remains true with $\mathcal{A}$ composed of n contexts A_i, $i \in I = \{1, \ldots, n\}$. Indeed, let us suppose the property true with $\mathcal{A}$ composed of $n-1$ contexts A_i, $i \in I = \{1, \ldots, n-1\}$, we then have for all focal elements $C \times \{R\} \cup \overline{C} \times \mathcal{R}$ of $\bigcirc_{i \in I} m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}}$, with $C = \cup_{i \in I' \subseteq I} A_i$,

$$(\bigcirc_{i \in I} m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_n]^{\Uparrow \Omega \times \mathcal{R}})((C \cup A_n) \times \{R\} \cup \overline{(C \cup A_n)} \times \mathcal{R})$$

$$= \beta_n \prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j = \prod_{i \in I' \cup \{n\}} \beta_i \prod_{j \in (I \cup \{n\}) \setminus (I' \cup \{n\})} \alpha_j \ ,$$

and

$$(\bigcirc_{i \in I} m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_n]^{\Uparrow \Omega \times \mathcal{R}})(C \times \{R\} \cup \overline{C} \times \mathcal{R})$$

$$= \alpha_n \prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j = \prod_{i \in I'} \beta_i \prod_{j \in (I \cup \{n\}) \setminus I'} \alpha_j \ ,$$

which means that focal elements of $\bigcirc_{i \in \{1, \ldots, n-1\}} m^{\mathcal{R}}[A_i]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc m^{\mathcal{R}}[A_n]^{\Uparrow \Omega \times \mathcal{R}}$ are also of the form $C \times \{R\} \cup \overline{C} \times \mathcal{R}$, with $C = \cup_{i \in I' \subseteq I} A_i$, $I = \{1, \ldots, n\}$, $A_i \in \mathcal{A}$, and have for mass: $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$.

Besides, for all $B \subseteq \Omega$,

$$m^{\Omega}[\{R\}]^{\Uparrow \Omega \times \mathcal{R}}(B \times \{R\} \cup \Omega \times \{NR\}) = m(B) \ ,$$

and, for all $B \subseteq \Omega$, for all $C = \cup_{i \in I' \subseteq I} A_i$,

$$(C \times \{R\} \cup \overline{C} \times \mathcal{R}) \cap (B \times \{R\} \cup \Omega \times \{NR\}) = B \times \{R\} \cup \overline{C} \times \{NR\} \ .$$

Therefore, after the projection on Ω, $\left(m^{\Omega}[\{R\}]^{\Uparrow \Omega \times \mathcal{R}} \bigcirc_{A \in \mathcal{A}} m^{\mathcal{R}}[A]^{\Uparrow \Omega \times \mathcal{R}}\right)^{\downarrow \Omega}$ consists in transferring a part $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$ of each mass $m(B)$, $B \subseteq \Omega$, from B to $B \cup \overline{C}$, for all $C = \cup_{i \in I' \subseteq I} A_i$.

On the other hand, $m\textcircled{\cup} \left(\bigcirc_{A \in \mathcal{A}} \overline{A}^{\alpha_A}\right)$ can be written as

$$m\textcircled{\cup} \left(\bigcirc_{i \in I} \overline{A_i}^{\alpha_i}\right) = m\textcircled{\cup} \left(\bigcirc_{i \in I} \left\{ \begin{array}{ccc} \Omega & \mapsto & \alpha_i \\ \hline A_i & \mapsto & \beta_i \end{array} \right. \right) \ .$$

As for all $(i,j) \in I^2$ s.t. $i \neq j$, $\overline{A_i} \cap \overline{A_j} = \overline{A_i \cup A_j}$, it can be shown (with an induction for example) that the focal elements of $\bigcirc_{i \in I} \overline{A_i}^{\alpha_i}$ are the elements $\overline{C}$ with $C = \cup_{i \in I' \subseteq I} A_i$ and have a mass equal to $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$.

Consequently, operation $m\textcircled{\cup} \left(\bigcirc_{i \in I} \overline{A_i}^{\alpha_i}\right)$ also consists in transferring a part $\prod_{i \in I'} \beta_i \prod_{j \in I \setminus I'} \alpha_j$ of each mass $m(B)$, $B \subseteq \Omega$, from B to $B \cup \overline{C}$, for all $C = \cup_{i \in I' \subseteq I} A_i$. We can then conclude that Equations (2) and (4) are equivalent for any non empty set of contexts $\mathcal{A}$.

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$\square$

Example 1. *Let us consider* $\Omega = \{\omega_1, \omega_2\}$ *and* $\mathcal{A} = 2^\Omega$, *and let us denote* $\alpha_{\{\omega_1\}}$ *by* α_1, $\alpha_{\{\omega_2\}}$ *by* α_2, *and* α_Ω *by* α_{12}. *Equation (4) gives*

$$
\begin{aligned}
m \;&\ \textcircled{\cup}\; \left(\textcircled{\cap}_{A \in \mathcal{A}} \overline{A}^{\alpha_A} \right) \\
= m \;&\ \textcircled{\cup}\; \left(\overline{\emptyset}^{\alpha_\emptyset} \textcircled{\cap} \overline{\{\omega_1\}}^{\alpha_1} \textcircled{\cap} \overline{\{\omega_2\}}^{\alpha_2} \textcircled{\cap} \overline{\Omega}^{\alpha_{12}} \right) \\
= m \;&\ \textcircled{\cup}\; \left(\{\omega_2\}^{\alpha_1} \textcircled{\cap} \{\omega_1\}^{\alpha_2} \textcircled{\cap} \emptyset^{\alpha_{12}} \right) \\
= m \;&\ \textcircled{\cup}\; \left(\begin{cases} \{\omega_2\} & \mapsto & \beta_1 \\ \Omega & \mapsto & \alpha_1 \end{cases} \textcircled{\cap} \begin{cases} \{\omega_1\} & \mapsto & \beta_2 \\ \Omega & \mapsto & \alpha_2 \end{cases} \textcircled{\cap} \begin{cases} \emptyset & \mapsto & \beta_{12} \\ \Omega & \mapsto & \alpha_{12} \end{cases} \right) \\
= m \;&\ \textcircled{\cup}\; \begin{cases} \emptyset & \mapsto & \beta_1 \beta_2 \alpha_{12} + \beta_{12} \\ \{\omega_1\} & \mapsto & \alpha_1 \beta_2 \alpha_{12} \\ \{\omega_2\} & \mapsto & \beta_1 \alpha_2 \alpha_{12} \\ \Omega & \mapsto & \alpha_1 \alpha_2 \alpha_{12} \end{cases} \;.
\end{aligned}
$$

In contrast, Equation (3) leads to

$$
\begin{aligned}
m \;&\ \textcircled{\cup}\; \left(\textcircled{\cup}_{A \in \mathcal{A}} A_{\beta_A} \right) \\
= m \;&\ \textcircled{\cup}\; \emptyset_{\beta_\emptyset} \textcircled{\cup} \{\omega_1\}_{\beta_1} \textcircled{\cup} \{\omega_2\}_{\beta_2} \textcircled{\cup} \Omega_{\beta_{12}} \\
= m \;&\ \textcircled{\cup}\; \{\omega_1\}_{\beta_1} \textcircled{\cup} \{\omega_2\}_{\beta_2} \textcircled{\cup} \Omega_{\beta_{12}} \\
= m \;&\ \textcircled{\cup}\; \begin{cases} \emptyset & \mapsto & \beta_1 \\ \{\omega_1\} & \mapsto & \alpha_1 \end{cases} \textcircled{\cup} \begin{cases} \emptyset & \mapsto & \beta_2 \\ \{\omega_2\} & \mapsto & \alpha_2 \end{cases} \textcircled{\cup} \begin{cases} \emptyset & \mapsto & \beta_{12} \\ \Omega & \mapsto & \alpha_{12} \end{cases} \\
= m \;&\ \textcircled{\cup}\; \begin{cases} \emptyset & \mapsto & \beta_1 \beta_2 \beta_{12} \\ \{\omega_1\} & \mapsto & \alpha_1 \beta_2 \beta_{12} \\ \{\omega_2\} & \mapsto & \beta_1 \alpha_2 \beta_{12} \\ \Omega & \mapsto & \alpha_1 \alpha_2 \beta_{12} + \alpha_{12} \end{cases} \;.
\end{aligned}
$$

To summarize, in [1], the equivalence was shown between (2) and (3) when $\mathcal{A}$ forms a partition of Ω. This corrigendum shows that this equivalence does not hold for any $\mathcal{A}$, and that (2) is actually equivalent to (4) for any (non empty) $\mathcal{A}$.

References

References

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[2] D. Mercier, É. Lefèvre and F. Delmotte. Belief Functions Contextual Discounting and Canonical Decompositions. *International Journal of Approximate Reasoning*, volume 53, pages 146–158, 2012.