An optimal control problem for the continuity equation arising in smart charging

Adrien Seguret

▶ **To cite this version:**

Adrien Seguret. An optimal control problem for the continuity equation arising in smart charging. 2022. hal-03353917v3

HAL Id: hal-03353917

https://hal.science/hal-03353917v3

Preprint submitted on 26 Oct 2022 (v3), last revised 24 Jan 2024 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An optimal control problem for the continuity equation arising in smart charging

Adrien Seguret *

Abstract

This paper is focused on the mathematical modeling and solution of the optimal charging of a large population of identical plug-in electric vehicles (PEVs) with mixed state variables (continuous and discrete). A mean field assumption is formulated to describe the evolution interaction of the PEVs population. The optimal control of the resulting continuity equation of the mixed system under state constraints is investigated. We prove the existence of a minimizer. We then characterize the solution as the weak solution of a system of two coupled PDEs: a continuity equation and of a Hamilton-Jacobi equation. We provide regularity results of the optimal feedback control.

Keywords: Optimal control, optimality conditions, mean field control.

1 Introduction

This article studies the optimal control of a first order continuity equation with a reaction term under state constraints. Let us consider a finite time interval $[0,T]$ and a mixed state space equal to the product $[0,1]\times I$, where I is a finite space, the cardinality of which is denoted by $|I|$. The continuity equation is given by:

$$\begin{aligned} &\partial_t m_i(t,s) + \partial_s(m_i(t,s)b_i(s)) = -\sum_{j\neq i}(\alpha_{i,j}(t,s)m_i(t,s) - \alpha_{j,i}(t,s)m_j(t,s)) \quad (i,t,s)\in I\times(0,T)\times(0,1),\\ &m_i(0,s) = m_i^0(s) \qquad\qquad (i,s)\in I\times[0,1], \end{aligned} \tag{1.1}$$

where $m : [0,T] \to \mathcal{P}([0,1]\times I)$ is a curve of probability distribution, $m^0 \in \mathcal{P}([0,1]\times I)$ is the given initial distribution, $b : I\times[0,1]\to\mathbb{R}$ is a velocity field, and the control $\alpha : I\times I\times[0,T]\times[0,1]\to\mathbb{R}_+$ is a jump intensity. The notion of weak solution of (1.1) is specified in Definition 3.1. We further assume that b vanishes at the boundary of $[0,1]$ so that the mass conservation in $I\times[0,1]$ is guaranteed without forcing a boundary condition. The distribution m is subject to the following congestion constraints:

$$m_i(t,[0,1]) \le D_i(t) \quad \forall (i,t)\in I\times[0,T], \tag{1.2}$$

where $D : [0,T]\to\mathbb{R}_+^*$ is given. The objective function J is defined as follows:

$$J(m,\alpha) := \sum_{i\in I}\int_0^T\int_0^1 \big(c_i(t,s) + \sum_{j\in I, j\neq i} L(\alpha_{ij}(t,s))\big)m_i(t,ds)dt + \sum_{i\in I}\int_0^1 g_i(s)m_i(T,ds). \tag{1.3}$$

Our purpose is to study the optimization problem:

$$\inf_{m,\alpha} J(m,\alpha), \text{ where } (m,\alpha) \text{ is a weak solution of (1.1) and satisfies (1.2).} \tag{1.4}$$

Our work is initially motivated by the optimal charging of a population of PEVs controlled by a central planner. The continuous variable $s\in[0,1]$ in (1.1) represents the level of battery of a PEV. The discrete variable $i\in I$ represents the mode of charging (e.g. idling, charging, discharging, etc...). For any $(t,s,i)\in[0,T]\times[0,1]\times I$, the value $m_i(t,s)$ represents the proportion of PEVs at time t at state (s,i). The given velocity field $b_i(s)$ denotes the power of charge or discharge of a PEV in mode i and with battery level s. For any $(t,s,i,j)\in[0,T]\times[0,1]\times I\times I$,

*PSL Research University, Université Paris-Dauphine, CEREMADE, Place de Lattre de Tassigny, F- 75016 Paris, France, OSIRIS department, EDF Lab, Bd Gaspard Monge, 91120 Palaiseau, France, seguret@ceremade.dauphine.fr

the value $\alpha_{i,j}(t,s)$ denotes the jump intensity of PEVs from the state (s,i) to the state (s,j) at time t. The control α is required to be a non-negative measurable function. The congestion constraint (1.2) avoids high demand of energy at each moment over the period. The cost function L penalizes high values of α. It aims at avoiding the synchronization of jumps of the PEVs and the consequent instability of the electrical network. The value $c_i(t,s)$ corresponds to the cost per PEV to pay at time $t \in [0,T)$ at state (s,i); $g_i(s)$ is the final cost per PEV to pay at state (s,i). Numerical results of Problem (1.4) applied to smart charging can be found in [48]. Problem (1.4) can be interpreted heuristically as an approximation of the limit case $N \to \infty$ of an optimal switching problem of N PEVs. Combinatorial techniques as well as optimal control tools may fail to solve problems with large population of PEVs, due to the curse of dimensionality [5]. To overcome these difficulties, a continuum of PEVs can be considered, leading to the techniques of the optimal control of PDE. The connection between the finite population problem and the mean field problem is addressed in a companion paper [47] by the author.

1.1 Contributions, methodology and literature

Contributions This paper makes three main contributions.

First we prove the existence of solutions of Problem (1.4).

Second, we derive optimality conditions of Problem (1.4). More precisely, let H denote the Fenchel conjugate of L and H' denote the derivative of H, we show that, if (m,α) is a solution of (1.4), then there exists a pair (φ,λ) such that, for any $i,j \in I$, $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$, and (φ,λ,m) is a weak solution of the following system:

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \displaystyle\sum_{j\in I, j\neq i} H(\varphi_i - \varphi_j) = 0 & \text{on } (0,T)\times(0,1)\times I \\ \partial_t m_i + \partial_s(m_i b_i) + \displaystyle\sum_{j\neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 & \text{on } (0,T)\times(0,1)\times I \\ m_i(0,s) = m_i^0(s),\ \varphi_i(T,s) = g_i(s) & \text{on } (0,1)\times I \\ \displaystyle\int_0^1 m_i(t,ds) - D_i(t) \le 0,\ \lambda \ge 0 & \text{on } [0,T]\times I \\ \displaystyle\sum_{i\in I} \int_0^T \left(\int_0^1 m_i(t,ds) - D_i(t) \right) \lambda_i(dt) = 0. \end{cases} \tag{1.5}$$

The function φ is the Lagrange multiplier associated to the dynamic constraint (1.1), and the measure λ is associated to the congestion constraint (1.2). The first equation in (1.5) is a backward Hamilton-Jacobi equation. The existence, uniqueness and characterization of weak solutions of the backward Hamilton-Jacobi equation are investigated in the paper. The second equation in (1.5) is a forward continuity equation, where the control α, defined by $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$, is optimal. The measure λ is non-negative and finite. The last equality in (1.5) ensures that the congestion constraint (1.2) is satisfied.

Third, we obtain regularity property for any weak solution (λ,φ,m) of (1.5). We prove that, under suitable assumptions on the data b, g and c, the multiplier φ is in $\mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)$. As a result, the optimal control α is bounded and Lipschitz continuous in space uniformly w.r.t. the time variable and the measure m is in $\mathrm{Lip}([0,1],\mathcal{P}([0,1]\times I))$. We show that if the initial distribution m^0 is absolutely continuous w.r.t. the Lebesgue measure and has a smooth density, then the measure m is absolutely continuous w.r.t. the Lebesgue measure and has Lipschitz continuous density.

Methodology The existence of an optimal solution is established by compactness arguments (see for e.g. [6]). We adopt a duality approach to obtain (1.5). More explicitly, we relax the dynamics (1.1) and the congestion constraint (1.2). The resulting relaxed problem is then expressed as the dual of another convex problem. We show that the system (1.5) is the optimality condition of Problem (1.4).

Literature Solving the optimal control of a Fokker-Planck equation by means of the duality theory has been well known since decades [26, 50]. Our work follows the method developed in the seminal work by Benamou and Brenier [6] for optimal transport problems. In [6], a continuity equation is controlled with initial and final constraints; optimality conditions are obtained as a system of PDEs close to (1.5). Similar method and results also in optimal transport are derived in [15]. More recently, this approach was applied to solve on optimal control problem of a Fokker-Planck equation under state constraints in the Wasserstein space [20, 21], where Lipschitz regularity results of the optimal control are proved.

The optimality conditions (1.5) typically arises in the Mean Field Game (MFG for short) Theory. This class of problems, introduced by Lasry and Lions [34, 35, 36] and Huang, Malhamé and Caines [32, 33], describes the

interaction among a large population of identical and rational agents in competition. Mean Field Control (MFC for short) and MFG theories have been extensively used over the last few years as a mathematical tool in electrical engineering. More specifically, the optimal control of PDEs applied to smart charging can be found in [38, 49], and to the management of a population of thermostatically controlled loads in [27, 42].

Conversely, the duality approach is close to the so-called variational approach used in MFG theory in [16], where the weak solution of the MFG system is characterized as the minimizer of some optimal control of Hamilton-Jacobi and Fokker-Planck equations. This approach allows to use optimization techniques to prove the existence and uniqueness of the solution of MFG and MFC problems. We refer to [1, 8, 13, 17, 43] and the references therein. Besides, the variational approach allows to apply optimization algorithms to numerically solve MFG problems [7, 12, 13].

Note that different optimality conditions for control problems in the space of probability measures can be derived by using a kind of Pontryagin Maximum Principle [10, 11].

A particularity of this paper is to deal with a congestion constraint (1.2) on the measure. Two kinds of congestion effects are explored in the MFG and MFC frameworks. On the one hand, "soft congestion" increases the cost of velocity of the agents in areas with high density. On the other hand, "hard congestion" constraints impose density constraints, e.g. $m \leq \bar{m}$ at any point (t, s). The variational approach yields good results when applied to MFC [1] and MFG with "soft congestion" in a stationary framework [23], as well as to MFG problems dealing with "hard congestion" constraints. This was first investigated in [44] where the density of the population did not exceed a given threshold, then in [40] where stationary second order MFGs were considered. In [18], a price which is imposed on the saturated zone to make the density satisfy the constraints is obtained. In the same vein as the work of Benamou and Brenier [6], "hard congestion" constraints were examined in optimal transport [14]. We highlight that our paper deals with aggregate "hard congestion" constraints on the measure m (1.2), i.e., our constraint is less restrictive than a constraint of the type $m \leq \bar{m}$ a.e..

Besides, we consider a mixed state space with continuous and discrete state variable. This setting has seldomly been investigated in the MFG literature. Indeed, the articles cited in the paragraphs above looked only at continuous state variables. The resulting continuity equation (1.1) contains a term of reaction, indicating mass transfers between states in I. Such PDEs also arised in [3] to model the mean field limit of Piecewise Deterministic Markov Proccesses (PDMP for short). The velocity was controlled in [3]. Here, we control the intensity of the jump α, but the velocity b is given. A MFG problem with discrete time and state space was explored in [29] and applied to socio-economic sciences in [28]. The uniqueness of the solution of a finite state MFG was discussed in [4] and the convergence of the N-player game to the mean field model as $N \to \infty$ was obtained in [30]. Mixed state spaces in a MFG framework were studied in [24, 25], where a major player can switch his state on a finite state space and minor players decide their stopping time. A MFG problem in a finite state space and discrete time settings with "hard congestion" was studied in [9], also by variational methods.

Some of our regularity results, namely that α is Lipschitz continuous w.r.t. s, and the density of m w.r.t. the Lebesgue measure is Lipschitz continuous when m^0 is absolutely continuous w.r.t. the Lebesgue measure, are unusual. We believe that it is mainly due to the linearity of the Hamilton-Jacobi equation w.r.t. $\partial_s \varphi$. These results will be used in a companion work [47] to quantify the mean field limit of the model. The time regularity of φ may not be improved as far as we have no more regularity results on λ. The function φ is discontinuous at each atom of the measure λ. Regularity results about the multiplier of the density constraint can be found in the literature: in [18], the authors showed some BV estimates on the pressure, whereas L^∞ estimates for the price were proved in [21] and [37] in the special case of a quadratic Hamiltonian in a MFG problem. The Sobolev regularity of the solution of a first order MFG was established in [39] and improved in [46]; also see [31].

Organization of the paper The paper is organized as follows. In Section 2, we present our assumptions and main results. The existence of a solution of Problem (1.4) is established in Section 3. In Section 4, which is independent of the other sections, we show the existence, uniqueness and regularity of weak solution of the Hamilton-Jacobi equation in (1.5). In Section 5, we return to Problem (1.4) and develop the duality approach. We formulate its Lagrangian relaxation and show it is the dual problem of another convex problem. We obtain the optimality conditions (1.5) of Problem (1.4) in Section 6. The Lipschitz continuity of the value of Problem (1.4) w.r.t. the data (m^0, D) is proved in Section 7. We recall basic statements about weak solutions of (1.1) in Appendix 8.

2 Assumptions and main results

2.1 Notations and Assumptions

Notations The space of Borel, positive and bounded measures on a space A is denoted by $\mathcal{M}^+(A)$ and the space of Borel probability measures on a space A is denoted by $\mathcal{P}(A)$. For any measure $\mu \in \mathcal{M}([0,T])$ and $0 \leq t_1 < t_2 \leq T$, we set $\int_{t_1}^{t_2} \mu(dt) := \mu([t_1,t_2])$. Given a set $\mathcal{S}$, for any function f defined on $I \times \mathcal{S}$ and any measure $\mu \in \mathcal{M}(\mathcal{S} \times I)$, we use the notations $f_i(x) := f(i,x)$ for any $(i,x) \in I \times \mathcal{S}$ and $\mu_i(S) := \mu(\{i\} \times S)$ for any $(i,S) \in I \times \mathcal{B}(\mathcal{S})$, where $\mathcal{B}(\mathcal{S})$ denotes the Borel algebra. Similarly, for any function g defined on $I \times I \times \mathcal{S}$ and any measure $\nu \in \mathcal{M}(\mathcal{S} \times I \times I)$, we use the notations $g_{i,j}(x) := g(i,j,x)$ for any $(i,j,x) \in I \times I \times \mathcal{S}$ and $\nu_{i,j}(S) := \mu(\{i\} \times \{j\} \times S)$ for any $(i,j,S) \in I^2 \times \mathcal{B}(\mathcal{S})$. If $\mathcal{S}$ is a metric space, let $\mathrm{Lip}(\mathcal{S})$ denote the vector space of bounded and Lipschitz continuous maps $f : \mathcal{S} \to \mathbb{R}$. For any $\mu \in C^0([0,T], \mathcal{P}([0,1]))$, let $L^2_\mu([0,T] \times \mathbb{R}) := \{f : [0,T] \times \mathbb{R} \mapsto \mathbb{R},\ \int_0^T \int_0^1 f(t,s)^2 \mu(t,ds)dt < +\infty\}$. We denote by $\mathcal{W}$ the Wasserstein distance on $\mathcal{P}([0,1] \times I)$, defined by $\mathcal{W}(\mu,\rho) := \sup\{\sum_{i \in I} \int_0^1 \varphi(\mu - \rho) \,|\, \varphi \text{ is } 1 - \text{Lipschitz from } [0,1] \times I \text{ to } \mathbb{R}\}$. We recall that if a function φ is 1-Lipschitz continuous from $[0,1] \times I$ to $\mathbb{R}$, then $|\varphi(x,i) - \varphi(x,j)| \leq 1$ for any $i,j \in I$. The dual of a normed space X is denoted by X^*. We consider the space $\Omega := \mathcal{P}([0,1] \times I) \times C^0(I \times [0,T], \mathbb{R}^*_+)$. For any $\varepsilon > 0$, we define the subspace Ω_ε of Ω, by

$$\Omega_\varepsilon := \Big\{(\mu, f) \in \Omega \,|\, \varepsilon < \inf_{t \in [0,T], i \in I} f_i(t) - \mu_i([0,1])\Big\}. \tag{2.1}$$

We consider the following spaces for any $\delta > 0$:

$$\begin{aligned}
\mathcal{M}^+_\delta([0,T] \times I) &:= \left\{\mu \in \mathcal{M}^+([0,T] \times I) \,|\, \mu([0,T] \times I) \leq \delta\right\}, \\
C^0_\delta([0,T] \times I, \mathbb{R}_+) &:= \Big\{f \in C^0([0,T] \times I, \mathbb{R}_+) \,|\, \int_0^T \sum_{i \in I} f_i(t)dt \leq \delta\Big\}.
\end{aligned}$$

For any $\delta > 0$, the space $C^0_\delta([0,T] \times I, \mathbb{R}_+)$ can be considered as a subspace of $\mathcal{M}^+_\delta([0,T] \times I)$ in the sense that, for any $f \in C^0_\delta([0,T] \times I, \mathbb{R}_+)$, we have $f\mathcal{L} \in \mathcal{M}^+_\delta([0,T] \times I)$, where $\mathcal{L}$ is the Lebesgue measure on $[0,T]$.

Assumptions The following assumptions are in force throughout the paper.

1. For any $i \in I$, $b_i \in C^1(\mathbb{R})$ with $b_i(s) = 0$ for any $s \notin (0,1)$.
2. The initial distribution of (1.1) $m^0 \in \mathcal{P}(\mathbb{R} \times I)$ is such that $\mathrm{supp}(m^0_i) \subset [0,1]$ for any $i \in I$.
3. There exists $\varepsilon^0 > 0$ such that the parameter D of (1.2) and m^0 satisfy $(m^0, D) \in \Omega_{\varepsilon^0}$.
4. For any $i \in I$, it is assumed that $c_i \in C^1([0,T] \times [0,1])$ and $g_i \in C^1([0,1])$.
5. $L : \mathbb{R} \to \bar{\mathbb{R}}$ is a convex function, defined by:

$$L(x) := \begin{cases} l(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $l \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing strongly convex function bounded from above by a quadratic function. More explicitly, there exists $C > 0$ such that for any $x \in \mathbb{R}_+$:

$$\frac{x^2}{C} - C \leq l(x) \leq C(x^2 + 1),$$

where the first inequality is due to the strong convexity of l.

The function H being the Fenchel conjugate of L, by Assumption 5, H is non-decreasing, non-negative, and H' is Lipschitz continuous on $\mathbb{R}$. For any $\lambda \in \mathcal{M}^+([0,T] \times I)$, $\psi \in L^2([0,T] \times [0,1] \times I)$, $(t,\tau,s,i,j) \in [0,T]^2 \times [0,1] \times I^2$, we define H^λ by:

$$H^\lambda(i,j,t,\tau,s,\psi) := H\Big((\psi_i - \psi_j)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_i - \lambda_j)(dr)\Big). \tag{2.2}$$

By Assumption 1, for any $s \in [0,1]$, $i \in I$ and $t \in [0,T]$, there exists a unique $S_i^{t,s}$ satisfying the ODE below

$$\begin{cases} \dfrac{\mathrm{d}S_i^{t,s}(\tau)}{\mathrm{d}\tau} = b_i(S_i^{t,s}(\tau)), \quad \tau \in [0,T], \\ S_i^{t,s}(t) = s. \end{cases} \tag{2.3}$$

Since $b \in C^1([0,1] \times I)$, we recall that the flow S_i satisfies the following equation for any $(i, \tau, t, s) \in I \times (0,T) \times (0,T) \times (0,1)$:

$$\partial_x S_i^{\tau,x}(t) = \exp(\int_\tau^t b_i'(S_i^{\tau,x}(r))dr). \tag{2.4}$$

Remark 2.1. *The main role of Assumptions 1 and 2 is to ensure that the support of the weak solution of* (1.1) *is contained in* $[0,1]$ *over the period* $[0,T]$ *(cf. Lemma 8.1 in Appendix 8). Assumption 3 provides an estimate on* $\lambda([0,T] \times I)$ *for any weak solution* (φ, λ, m) *of* (1.5) *and ensures that the control* $\alpha = 0$ *is an admissible control. Correspondingly, the feasible set of Problem* (1.4) *is not empty. Regularity results of the weak solutions of the system* (1.5) *are derived thanks to the assumptions formulated on* c *and* g *in Assumption 4.*

2.2 Main results

We introduce, for a given $\lambda \in \mathcal{M}^+([0,T] \times I)$, the Hamilton-Jacobi equation on $(0,T) \times (0,1) \times I$:

$$\begin{aligned} &-\partial_t \varphi_i(t,s) - b_i(s)\partial_s \varphi_i(t,s) - c_i(t,s) - \lambda_i(t) + \sum_{j \in I, j \neq i} H((\varphi_j - \varphi_i)(t,s)) = 0 \quad (t,s,i) \in (0,T) \times (0,1) \times I, \\ &\varphi_i(T,s) = g_i(s) \qquad\qquad (s,i) \in [0,1] \times I. \end{aligned} \tag{2.5}$$

We define the function $\tilde{A}$ for any $(\varphi, \lambda) \in \big(\mathrm{Lip}([0,T] \times [0,1] \times I) + BV([0,T] \times I)\big) \times \mathcal{M}^+([0,T] \times I)$ by:

$$\tilde{A}(\varphi, \lambda) := \sum_{i \in I} \int_0^1 -\varphi_i(0^+, s) m_i^0(ds) + \int_0^T D_i(t) \lambda_i(dt). \tag{2.6}$$

The following Theorem summarizes the main results of the paper.

Theorem 2.1. *Problem* (1.4) *has a solution. Furthermore, the minimizers have the following properties:*

1. *If* (m, α) *is a minimizer of Problem* (1.4) *and* $(\varphi, \lambda) \in (\mathrm{Lip}([0,T] \times [0,1] \times I) + BV([0,T] \times I)) \times \mathcal{M}^+([0,T] \times I)$ *is such that* $\tilde{A}(\varphi, \lambda) = -J(m, \alpha)$*, then* (φ, λ, m) *is a weak solution of* (1.5) *in the sense of Definition 6.1, and* $\alpha_{i,j} = H'(\varphi_i - \varphi_j)$ *on* $\{m_i > 0\}$ *for any* $i, j \in I$*.*

2. *Conversely, if* (φ, λ, m) *is a weak solution of* (1.5) *in the sense of Definition 6.1, then there exists* α*, defined for any* $i, j \in I$ *by:* $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ *on* $\{m_i > 0\}$*, such that* (m, α) *is a minimizer of* (1.4) *and* $\tilde{A}(\varphi, \lambda) = -J(m, \alpha)$*.*

3. *If* (m, α) *is a minimizer of Problem* (1.4)*, then for any* $i, j \in I$ $\alpha_{i,j}$ *and* $\partial_s \alpha_{i,j}$ *are both in* $L^\infty([0,T] \times [0,1])$*, and* $m \in \mathrm{Lip}([0,T], \mathcal{P}([0,1] \times I))$*.*

The existence of a solution of Problem (1.4) is stated in Lemma 3.3. The characterization of a solution, i.e., Theorem 2.1.1-2, is given by Theorem 6.1. For this, we adopt a duality approach which is developed in Section 5. In particular, we introduce a convex problem, the dual of which is Problem (1.4) up to a change of variable (cf. Theorem 5.1). The Lipschitz continuity of m stated in Theorem 2.1.3 is deduced from the regularity of φ, which is derived in Section 4, and Proposition 8.1. The proof of Theorem 2.1 is finally given in Section 6.2.

Further regularity results on the solution of Problem (1.4) are obtained with additional conditions on the initial distribution m^0.

Proposition 2.1. *If the initial distribution* m^0 *is absolutely continuous w.r.t. the Lebesgue measure with a density in* $C^1([0,1] \times I)$*, then any solution* (m, α) *of Problem* (1.4) *is such that* m *is absolutely continuous w.r.t. the Lebesgue measure on* $[0,T] \times [0,1]$*, and has a density in* $\mathrm{Lip}([0,T] \times [0,1] \times I)$*.*

The proof of Proposition 2.1 is given in Section 6.2.

3 Existence of an optimal solution

We introduce in this section a convex problem that is equivalent to (1.4). Standard compactness arguments are used to show the existence of an optimal solution (see for e.g. [6]). The definition of a weak solution of (1.1) is specified below.

Definition 3.1. *A pair (m,α) satisfies (1.1) in the weak sense if $t\in[0,T]\mapsto m(t,\cdot)\in\mathcal{P}(\mathbb{R}\times I)$ is continuous, for any $i,j\in I$ with $i\neq j$, it holds that $\alpha_{i,j}\in L^2_{m_i}([0,T]\times\mathbb{R})$ and for any test function $\phi\in C_c^\infty([0,T]\times\mathbb{R}\times I)$, we have:*

$$\begin{aligned}&\sum_{i\in I}\int_{\mathbb{R}}\phi_i(T,s)m_i(T,ds)-\phi_i(0,s)m_i^0(ds)\\&=\int_0^T\int_{\mathbb{R}}\sum_{i\in I}(\partial_t\phi_i(t,s)+b_i(s)\partial_s\phi_i(t,s))m_i(t,ds)+\sum_{j\in I,j\neq i}(\phi_j(t,s)-\phi_i(t,s))\alpha_{i,j}(t,s)m_i(t,ds)dt.\end{aligned}$$

Remark 3.1. *Using Assumptions 1 and 2, Lemma 8.1 in Appendix 8 states that, for any weak solution (m,α) of (1.1) in the sense of Definition 3.1, the measure $m_i(t,\cdot)$ has its support contained in $[0,1]$ for any $(t,i)\in[0,T]\times I$. Thus, we will consider throughout the paper only weak solutions (m,α) of (1.1) satisfying $m(t,\cdot)\in\mathcal{P}([0,1]\times I)$ for any $t\in[0,T]$.*

Problem (1.4) being not convex w.r.t. the variables (m,α), for any $i,j\in I$ with $i\neq j$, we make a change of variables $E_{i,j}:=\alpha_{i,j}m_i$. We now rewrite the continuity equation (1.1):

$$\begin{aligned}&\partial_t m_i(t,s)+\partial_s(m_i(t,s)b_i(s))=-\sum_{j\in I,j\neq i}(E_{i,j}(t,s)-E_{j,i}(t,s))\quad &(i,t,s)\in I\times(0,T)\times(0,1)\\&m_i(0,s)=m_i^0(s)&(i,s)\in I\times[0,1],\end{aligned}\tag{3.1}$$

where $E_{i,j}\in\mathcal{M}^+([0,T]\times[0,1])$ is such that $E_{i,j}\ll m_i$, with $\dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}=\alpha_{i,j}$ and $\dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\in L^2_{m_i}([0,T]\times[0,1])$. For any $(m^0,D)\in\Omega$ satisfying Assumption 3, we introduce the set:

$$\begin{aligned}\mathcal{S}(m^0,D):=\Big\{&(m,E)\text{ such that }E_{i,j}\ll m_i\,\forall i,j\in I,\ (m,\alpha)\text{ satisfies (3.1) in the weak sense, where }\alpha_{i,j}:=\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i},\\&\text{with additional constraints: }\int_0^1 m_i(t,ds)\le D_i(t)\quad\forall(i,t)\in I\times[0,T],\text{ and }\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\ge 0\Big\}.\end{aligned}\tag{3.2}$$

From Assumption 3, the set $\mathcal{S}(m^0,D)$ is not empty. Indeed, denoting by $(m,0)$ a weak solution of (1.1) with control $\alpha\equiv 0$, the distribution m satisfies that, for any $(t,i)\in[0,T]\times I$, $m_i(t,[0,1])=m_i^0([0,1])<D_i(t)$. Thus, $(m,0)\in\mathcal{S}(m^0,D)$. We define the function $\tilde{J}$ on $\mathcal{S}(m^0,D)$ by:

$$\tilde{J}(m,E):=\sum_{i\in I}\int_0^T\int_0^1 c_i(t,s)m_i(t,ds)dt+\int_0^T\int_0^1\sum_{i,j\in I,j\neq i}L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)m_i(t,ds)dt+\sum_{i\in I}\int_0^1 g_i(s)m_i(T,ds),\tag{3.3}$$

where the function L is defined in Assumption 5. Since $m(t)$ is a probability measure for any $t\in[0,T]$, by Assumption 4, the first and last integrals in (3.3) are well defined. Since $\dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\in L^2_{m_i}([0,T]\times[0,1])$ and l is bounded by a quadratic function according to Assumption 5, the second integral in (3.3) is also well defined. Thus, for any $(m,E)\in\mathcal{S}(m^0,D)$, the quantity $\tilde{J}(m,E)$ is finite. The following optimization problem is considered:

$$\inf_{(m,E)\in\mathcal{S}(m^0,D)}\tilde{J}(E,m).\tag{3.4}$$

For any $\gamma>0$, we define the subset $\mathcal{S}_\gamma(m^0,D)$ of $\mathcal{S}(m^0,D)$ by:

$$\mathcal{S}_\gamma(m^0,D):=\left\{(m,E)\in\mathcal{S}(m^0,D)\,\Big|\,\sum_{(i,j)\in I,i\neq j}\int_0^T\int_0^1 L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)m_i(t,ds)dt\le\gamma\right\}.\tag{3.5}$$

For any $(m,E)\in\mathcal{S}_\gamma(m^0,D)$, the next lemma provides a Hölder regularity property on m.

Lemma 3.1. *For any $\gamma>0$, there exists a positive constant C_γ such that, for any $(m,E)\in\mathcal{S}_\gamma(m^0,D)$, m is $\dfrac{1}{2}$-Hölder continuous with constant C_γ from $[0,T]$ to $\mathcal{P}([0,1]\times I)$ (endowed with the Wasserstein distance $\mathcal{W}$).*

Proof. The proof relies on classical arguments for the time regularity of weak solutions of the continuity equation (see [2]) and on the estimate, for any $t, \tau \in [0,T]$ with $t \le \tau$,

$$\begin{aligned}\left(\int_t^\tau \int_0^1 E_{i,j}(t,ds)dt\right)^2 &\le (\tau - t)\int_t^\tau \int_0^1 \sum_{i,j\in I, j\neq i} \left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)^2 m_i(\tau,ds)d\tau \\ &\le C\int_0^T \int_0^1 L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right) m_i(t,ds)dt + C \\ &\le C_\gamma,\end{aligned}$$

where the first inequality is obtained by Cauchy-Schwarz inequality, $C_\gamma := C(\gamma+1)$ and $C > 0$ is the constant defined in Assumption 5. □

The next lemma is useful to show that any minimizing sequence of (3.4) is relatively compact.

Lemma 3.2. *For any $\gamma > 0$, the subset $\mathcal{S}_\gamma(m^0, D)$ is relatively compact in $C([0,T], \mathcal{P}([0,1]\times I)) \times \mathcal{M}^+([0,T]\times[0,1]\times I^2))$, where $\mathcal{M}^+([0,T]\times[0,1]\times I^2))$ is endowed with the weak*-convergence.*

Proof. The compactness is derived from Lemma 3.1 and standard arguments in Optimal Transport [45]. □

Lemma 3.3. *Problem* (3.4) *admits a solution. Consequently, Problem* (1.4) *has a solution.*

Proof. Existence of a solution of (3.4) is a consequence of Lemma 3.2 and of the l.s.c. of $\tilde{J}$ w.r.t. the topology induced by $C([0,T], \mathcal{P}([0,1]\times I)) \times \mathcal{M}([0,T]\times[0,1]\times I^2)$ (where $\mathcal{M}([0,T]\times[0,1]\times I^2)$ is endowed with the weak*-convergence). Since Problem (1.4) is equivalent to Problem (3.4) up to a change of variable, the existence of a solution is straightforward. □

4 Analysis of the Hamilton-Jacobi equation

The purpose of this section is to study weak solutions of the Hamilton-Jacobi equation (2.5) when λ is in $\mathcal{M}^+([0,T]\times I)$. It is divided into three subsections. Section 4.1 is devoted to the analysis of the following equation

$$\begin{aligned}&-\partial_t\psi_i(t,s) - b_i(s)\partial_s\psi_i(t,s) - c_i(t,s) + \sum_{j\in I, j\neq i} H\left((\psi_i - \psi_j)(t,s) + \int_t^T (\lambda_i - \lambda_j)(dr)\right) = 0 \quad \text{on } (0,T)\times(0,1)\times I, \\ &\psi_i(T,\cdot) = g_i \qquad \text{on } [0,1]\times I.\end{aligned} \tag{4.1}$$

for a given $\lambda \in C^0_\delta([0,T]\times I, \mathbb{R}_+)$. Then, equation (4.1) is studied for a given $\lambda \in \mathcal{M}^+_\delta([0,T]\times I)$ in Section 4.2. Finally, we obtain in Section 4.3 the existence and the uniqueness of the solution to (2.5) by using the solution of (4.1).

4.1 The Hamilton-Jacobi equation for continuous valued data

The main result of this subsection is the following.

Proposition 4.1. *For any $\lambda \in C^0_\delta([0,T]\times I, \mathbb{R}_+)$, there exists a unique $\psi^\lambda \in C^1([0,T]\times[0,1]\times I)$ satisfying, for any $(t,s,i) \in [0,T]\times[0,1]\times I$,*

$$\psi^\lambda_i(t,s) = \int_t^T \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\psi^\lambda) + c_i(\tau, S^{t,s}_i(\tau))d\tau + g_i(S^{t,s}_i(T)). \tag{4.2}$$

In addition, ψ^λ is the unique classical solution of (4.1) *on $[0,T]\times[0,1]\times I$.*

We recall that the definition of H^λ is given in (2.2). The next lemma states that a function ψ satisfies (4.2) if and only if it is a classical solution of (4.1).

Lemma 4.1. *For any $\lambda \in C^0([0,T]\times I, \mathbb{R}_+)$ and $t_0 \in [0,T)$, a function $\psi \in C^1((t_0,T]\times[0,1]\times I)$ satisfies equation* (4.2) *on $[t_0,T]\times[0,1]\times I$ if and only if it is a solution of* (4.1) *on $(t_0,T]\times[0,1]\times I$.*

Proof. If $\psi \in C^1((t_0,T]\times[0,1]\times I)$ satisfies (4.2), then $\psi(T,\cdot)=g$ and by computing the partial derivatives in space and time of ψ, one obtains that ψ is a solution of (4.1) on $(t_0,T]\times(0,1)\times I$.

Conversely, if $\psi \in C^1((t_0,T]\times[0,1]\times I)$ is a solution of (4.1) then, by the method of characteristics one deduces that ψ is a solution of (4.2) on $[t_0,T]\times[0,1]\times I$. □

Before proving Proposition (4.1), we need the following estimate on solutions of the Hamilton-Jacobi equation.

Lemma 4.2 (A priori estimate)**.** *For any $\delta>0$, there exists $M>0$ such that, for any $\lambda \in C^0_\delta([0,T]\times I,\mathbb{R}_+)$ and any $t_0\in[0,T)$, if $\psi\in C^1((t_0,T]\times[0,1]\times I)$ satisfies (4.1) on $(t_0,T]\times(0,1)\times I$, then $\|\psi\|_\infty<M$.*

Proof. Let us define $M := \|g\|_\infty + T(\|c\|_\infty + |I|H(\delta)) + 1$ and, for any $(t,s,i)\in[t_0,T]\times[0,1]\times I$,

$$\begin{aligned}\underline{u}_i(t,s) &:= -\|g\|_\infty - (T-t)(\|c\|_\infty + |I|H(\delta)),\\ \bar{u}_i(t,s) &:= \|g\|_\infty + (T-t)(\|c\|_\infty + |I|H(\delta)).\end{aligned}$$

We note that $\|\underline{u}\|_\infty < M$ and $\|\bar{u}\|_\infty < M$. One has, for any $(t,s,i)\in(t_0,T]\times[0,1]\times I$,

$$\begin{aligned}-\partial_t \underline{u}_i(t,s) &\le -\sum_{j\in I, j\neq i} H\left(\int_\tau^T (\lambda_i-\lambda_j)(r)dr\right) + c_i(t,s)\\ -\partial_t \bar{u}_i(t,s) &\ge -\sum_{j\in I, j\neq i} H\left(\int_\tau^T (\lambda_i-\lambda_j)(r)dr\right) + c_i(t,s).\end{aligned} \tag{4.3}$$

We will show that ψ is bounded by $\underline{u}$ and $\bar{u}$. For any $t\in(t_0,T]$, we define:

$$\gamma(t) = \max_{(x,i)\in[0,1]\times I} (\underline{u}_i-\psi_i)(t,x) \quad \text{and} \quad (x_t,i_t)\in \operatorname*{argmax}_{(x,i)\in[0,1]\times I} (\underline{u}_i-\psi_i)(t,x).$$

Since $\underline{u}$ is independent of x and i, we have $(x_t,i_t)\in \operatorname{argmin}_{(x,i)\in[0,1]\times I}\psi_i(t,x)$. On the one hand, if $x_t\in\{0,1\}$, then $b_{i_t}(x_t)=0$. On the other hand, if $x_t\in(0,1)$, then $\partial_s\psi_{i_t}(t,x_t)=0$. Therefore, one has $b_{i_t}(x_t)\partial_s(\underline{u}_{i_t}(t,x_t)-\psi_{i_t}(t,x_t))=0$. Since $\underline{u}$ and ψ are Lipschitz continuous in time uniformly in (s,i), γ is also Lipschitz continuous and thus differentiable a.e. on $(t_0,T]$. Using the Envelop Theorem [41, Theorem 1], γ is absolutely continuous on $(t_0,T]$ and, for a.e. $t\in(t_0,T]$, inequality (4.3) and equality (4.1) give:

$$\gamma'(t)=\partial_t(\underline{u}_{i_t}(t,x_t)-\psi_{i_t}(t,x_t)) \ge \sum_{j\in I, j\neq i_t} H\left(\int_\tau^T(\lambda_i-\lambda_j)(r)dr\right) - H\left((\psi_i-\psi_j)(\tau,S_i^{t,s}(\tau)) + \int_\tau^T(\lambda_i-\lambda_j)(r)dr\right).$$

Since for any $j\in I$ we have $\psi_{i_t}(t,x_t)\le\psi_j(t,x_t)$ and $\underline{u}_{i_t}(t,x_t)=\underline{u}_j(t,x_t)$, the fact that H is non decreasing implies:

$$H\left(\psi_{i_t}(t,x_t)-\psi_j(t,x_t)+\int_\tau^T(\lambda_{i_t}-\lambda_j)(r)dr\right) \le H\left(\int_\tau^T(\lambda_{i_t}-\lambda_j)(r)dr\right),$$

and thus, $\gamma'(t)\ge 0$. Since $\gamma(T)<0$, we deduce that $\psi>\underline{u}$ on $(t_0,T]\times[0,1]\times I$. With similar arguments, one obtains $\bar{u}>\psi$ on $(t_0,T]\times[0,1]\times I$. Therefore, according to the definition of $\underline{u}$ and $\bar{u}$, we have, for any $(t,s,i)\in(t_0,T]\times[0,1]\times I$:

$$-M<\psi_i(t,s)<M. \tag{4.4}$$

□

Proposition 4.1 will be proved by a fixed point argument. For this, we need several lemmas. We fix the constant $M>0$ associated to $\delta>0$ from Lemma 4.2. The constant $\kappa>0$ is defined below and depends only on $M,\|b'\|_\infty,T$ and $|I|$. We consider the space

$$C^{0,1}([0,T]\times[0,1]\times I) := \{f\in C^0([0,T]\times[0,1]\times I)\,|\,\partial_s f_i\in C^0([0,T]\times[0,1]\times I)\},$$

endowed with the norm $\|\cdot\|^\kappa_{0,1}$ defined by:

$$\|f\|^\kappa_{0,1} := \|f\|^\kappa_\infty + \|\partial_s f\|^\kappa_\infty,$$

where, for any $h \in C^0([0,T]\times[0,1]\times I)$,

$$\|h\|_\infty^\kappa := \sup_{(t,s,i)\in[0,T]\times[0,1]\times I} |h_i(t,s)e^{-\kappa(T-t)}|.$$

The space $(C^{0,1}([0,T]\times[0,1]\times I), \|\cdot\|_{0,1}^\kappa)$ is a Banach space. We fix $C_0>0$ and $C_1>0$ to be chosen below, where C_0 depends on $\|\partial_s c\|_\infty, \|g'\|_\infty, \|b'\|_\infty$ and C_1 depends on M,δ,T and $|I|$. We look for a solution in the space Σ defined by:

$$\Sigma := \left\{ f \in C^{0,1}([0,T]\times[0,1]\times I) \,\middle|\, \|f\|_\infty \le M+1 \text{ and } \|\partial_s f(t,\cdot)\|_\infty \le C_0 e^{C_1(T-t)}\ \forall t\in[0,T] \right\}.$$

Remark 4.1. *The set Σ is bounded and closed and therefore complete w.r.t. the topology induced by the norm $\|\cdot\|_{0,1}^\kappa$.*

In this subsection, we are looking for a solution of (4.1) as a fixed point of the map $\Gamma^\lambda := (\Gamma_i^\lambda)_{i\in I}$, which is defined on $C^{0,1}([0,T]\times[0,1]\times I)$ by

$$\Gamma_i^\lambda(\phi)(t,s) := \int_t^T \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\phi) + c_i(\tau, S_i^{t,s}(\tau))d\tau + g_i(S_i^{t,s}(T)). \tag{4.5}$$

Since the function H is not necessarily bounded on $\mathbb{R}$, we need to introduce a smooth truncation $F\in C^1(\mathbb{R},[-M-1,\, M+1])$ to obtain a fixed point in Σ. The function F is defined on $\mathbb{R}$ such that $F'\ge 0$, $\|F'\|_\infty \le 1$ and:

$$F(x) := \begin{cases} -M-1 & \text{if } x \le -(M+2), \\ x & \text{if } -M-1/2 \le x \le M+1/2, \\ M+1 & \text{if } M+2 \le x. \end{cases} \tag{4.6}$$

Finally, we define the function Π^λ by:

$$\forall \phi\in\Sigma,\ \Pi^\lambda(\phi) := (\Pi_1^\lambda(\phi),\dots,\Pi_{|I|}^\lambda(\phi)) \text{ where } \Pi_i^\lambda(\phi) := (F\circ\Gamma_i^\lambda)(\phi) \quad \forall i\in I.$$

The following lemma states that Π^λ maps Σ into itself.

Lemma 4.3. *For a suitable choice of the constants C_0 and C_1, one has $\Pi^\lambda(\phi)\in\Sigma$ for any $\phi\in\Sigma$.*

Proof. Let $\phi\in\Sigma$ and, for any $i\in I$, $\sigma_i := \Gamma_i^\lambda(\phi)$. We have $\sigma\in C^{0,1}([0,T]\times[0,1]\times I)$. We need to show that, for any $t\in[0,T]$, $\|\partial_s\sigma(t,\cdot)\|_\infty \le C_0 e^{C_1(T-t)}$. According to the definition of Γ_i^λ, we have, for any $(i,t)\in I\times[0,T]$,

$$\begin{aligned} \|\partial_s\sigma_i(t,\cdot)\|_\infty &\le \|\partial_s S\|_\infty K\int_t^T \sum_{i,j\in I} \|\partial_s(\phi_j-\phi_i)(t,\cdot)\|_\infty dt + T\|\partial_s S\|_\infty\|\partial_s c\|_\infty + \|\partial_s S\|_\infty\|g'\|_\infty \\ &\le \frac{C_0 C}{C_1} e^{C_1(T-t)} + C, \end{aligned}$$

where $K := \sup_{x\in[-2M-\delta,2M+\delta]} |H'(x)|$ and $C>0$ is a positive constant which depends on $\|\partial_s S\|_\infty, \|\partial_s c\|_\infty, \|g'\|_\infty$, $|I|, T$ and K. Choosing carefully C_0 and C_1 depending on C, one obtains that, for any $(i,t)\in I\times[0,T]$,

$$\|\partial_s\sigma_i(t,\cdot)\|_\infty \le C_0 e^{C_1(T-t)}.$$

Finally, from the definition of F in (4.6), the function $\Pi(\phi) = (F(\sigma_1),\dots,F(\sigma_{|I|}))$ is in Σ. □

The existence of a fixed point of Π^λ in Σ is established in the following lemma.

Lemma 4.4. *For a suitable choice of the constant κ, the function Π^λ admits a unique fixed point $\psi^\lambda\in\Sigma$. In addition, we have $\psi^\lambda\in C^1([0,T]\times[0,1]\times I)$.*

Proof. Let $\phi^1,\phi^2\in\Sigma$. Since ϕ^1 and ϕ^2 are bounded by $M+1$, one has, for any $(t,s,i)\in[0,T]\times[0,1]\times I$,

$$\begin{aligned} |\Gamma_i^\lambda(\phi^1)(t,s) - \Gamma_i^\lambda(\phi^2)(t,s)| &\le \sum_{j\in I, j\neq i}\int_t^T |H^\lambda(i,j,t,\tau,s,\phi^1) - H^\lambda(i,j,t,\tau,s,\phi^2)|\, d\tau \\ &\le C\sum_{j\in I}\int_t^T |\phi_j^1(\tau,S_i^{t,s}(\tau)) - \phi_j^2(\tau,S_i^{t,s}(\tau))|\, d\tau \\ &\le \|\phi^1-\phi^2\|_\infty^\kappa \frac{Ce^{\kappa(T-t)}}{\kappa}, \end{aligned} \tag{4.7}$$

where $C > 0$ is a constant depending on $\delta, |I|$ and $\sup_{x\in[-2(M+1)-\delta,2(M+1)+\delta]} |H'(x)|$. Since H' is Lipschitz continuous on $[-2(M+1),\, 2(M+1)]$, one has, for any $(t,s,i) \in [0,T]\times[0,1]\times I$,

$$\begin{aligned} |\partial_s(\Gamma_i^\lambda(\phi^1)(t,s) - \Gamma_i^\lambda(\phi^2)(t,s))| &\leq \sum_{j\in I, j\neq i} \int_t^T |\partial_s H^\lambda(i,j,t,\tau,s,\phi^1) - \partial_s H^\lambda(i,j,t,\tau,s,\phi^2)|\, d\tau \\ &\leq \|\phi^1 - \phi^2\|_{0,1}^\kappa \frac{Ke^{\kappa(T-t)}}{\kappa}, \end{aligned} \tag{4.8}$$

where $K > 0$ is a constant that depends on $|I|, T, \|b'\|_\infty, M, C_0, C_1$ and on the bound and the Lipschitz constant of H' on $[-2(M+1)-\delta,\, 2(M+1)+\delta]$. From (4.7) and (4.8), one obtains that:

$$\|\Gamma^\lambda(\phi^1) - \Gamma^\lambda(\phi^2)\|_{0,1}^\kappa \leq \max(C_0/\kappa, K/\kappa)\|\phi^1 - \phi^2\|_{0,1}^\kappa.$$

Choosing $\kappa > \max(C_0, K)$ and the function F being non expensive, one deduces that the function Π^λ is a contraction on Σ and that it admits a unique fixed point $\psi^\lambda \in \Sigma$. By the definitions of Π^λ and Σ, it is straightforward that $\psi^\lambda \in C^1([0,T]\times[0,1]\times I)$. □

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $t_0 \in [0,T]$ be the minimum time such that $\|\psi^\lambda(t,\cdot)\|_\infty \leq M + 1/2$ for any $t \in [t_0, T]$. Since $\|\psi_i^\lambda(T,\cdot)\|_\infty = \|g_i\|_\infty < M$ for any $i \in I$ and ψ^λ is continuous, the time t_0 is smaller than T. The function ψ^λ is a fixed point of Γ^λ on $[t_0, T]$ and thus, ψ^λ is a solution of (4.2) on $[t_0,T]\times[0,1]\times I$. By Lemma 4.1, we deduce that ψ^λ satisfies (4.1) on $(t_0,T]\times[0,1]\times I$. If $t_0 = 0$, then the conclusion follows. If $t_0 > 0$, then $\|\psi^\lambda(t_0,\cdot)\|_\infty = M + 1/2$. By Lemma 4.2, one also has $\|\psi^\lambda(t_0,\cdot)\|_\infty \leq M$. Hence, there is a contradiction. Therefore, ψ^λ is a solution of (4.2) and a classical solution of (4.1) on $[0,T]\times[0,1]\times I$. □

Remark 4.2. *One can show that $\|\partial_t\psi^\lambda\|_\infty + \|\partial_s\psi^\lambda\|_\infty$ is bounded by a function that is non-decreasing w.r.t. the variable δ. Indeed, by Lemma 4.3 and its proof, $\|\partial_s\psi^\lambda\|_\infty$ is bounded by $C_0 e^{C_1 T}$ where C_0 and C_1 depend on M and on δ by the definition of M in Lemma 4.2. By Lemma 4.1, ψ^λ is a classical solution of* (4.1)*. Therefore,*

$$\|\partial_t\psi^\lambda\|_\infty \leq \|b\|_\infty\|\partial_s\psi^\lambda\|_\infty + \|c\|_\infty + \sup_{x\in[-2M-\delta,2M+\delta]} |H(x)|.$$

Thus, $\|\partial_t\psi^\lambda\|_\infty$ is bounded by a non-decreasing function of δ.

4.2 The Hamilton-Jacobi equation for measure valued data

In this subsection, we prove

Proposition 4.2. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$, there exists a unique $\psi^\lambda \in \mathrm{Lip}([0,T]\times[0,1]\times I)$ satisfying that, for any $(t,s,i) \in [0,T]\times[0,1]\times I$,*

$$\psi_i^\lambda(t,s) = \int_t^T \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\psi^\lambda) + c_i(\tau, S_i^{t,s}(\tau))d\tau + g_i(S_i^{t,s}(T)). \tag{4.9}$$

In addition, the map $\lambda \mapsto \psi^\lambda$ is continuous from $\mathcal{M}^+([0,T]\times I)$, endowed with the weak topology, to $C^0([0,T]\times[0,1]\times I)$, endowed with the norm $\|\cdot\|_\infty$.

We recall that the definition of H^λ, with $\lambda \in \mathcal{M}^+([0,T]\times I)$, is given in (2.2). The proof of Proposition 4.2 relies on the results of the previous subsection. We define the map $\Theta : C_\delta^0([0,T]\times I, \mathbb{R}_+) \to C^1([0,T]\times[0,1]\times I)$ by:

$$\Theta : \lambda \mapsto \psi^\lambda, \tag{4.10}$$

where ψ^λ is given by Proposition 4.1. We know from Proposition 4.1 that Θ is well defined on $C^0([0,T]\times I, \mathbb{R}_+)$. We want to show that Θ can be continuously extended to a function defined on $\mathcal{M}_\delta^+([0,T]\times I)$ with values in $\mathrm{Lip}([0,T]\times[0,1]\times I)$. We define the distance $\mathcal{D}$ on $\mathcal{M}^+([0,T]\times I)$ by

$$\mathcal{D}(\lambda,\mu) := \int_0^T \sum_{i\in I} \left| \int_t^T (\lambda_i - \mu_i)(d\tau) \right| dt + \sum_{i\in I} \left| \int_0^T (\lambda_i - \mu_i)(dt) \right|. \tag{4.11}$$

Remark 4.3. *If a sequence $\{\lambda^n\}_n$ in $\mathcal{M}^+_\delta([0,T]\times I)$ converges w.r.t. the weak topology in $\mathcal{M}^+([0,T]\times I)$ to $\lambda\in\mathcal{M}^+_\delta([0,T]\times I)$, then $\lim_{n\to\infty}\mathcal{D}(\lambda^n,\lambda)=0$. Indeed, since $\{\lambda^n\}_n$ weakly converges to λ, then $\lim_{n\to\infty}(\lambda^n_i-\lambda_i)([t,T])=0$ for any $i\in I$, a.e. $t\in[0,T]$ and $t=0$. Applying the dominated convergence theorem, the conclusion follows.*

The next remark will be useful to extend the domain of Θ.

Remark 4.4. *The space $C^0_\delta([0,T]\times I,\mathbb{R}_+)$ is dense in $\mathcal{M}^+_\delta([0,T]\times I)$ w.r.t. the topology induced by $\mathcal{D}$. More precisely, for any $\lambda\in\mathcal{M}^+_\delta([0,T]\times I)$, there exists a sequence $\{\lambda^n\}_n$ in $C^0_\delta([0,T]\times I,\mathbb{R}_+)$, such that $\lim_{n\to\infty}\mathcal{D}(\lambda,\lambda^n\mathcal{L})=0$ (where $\mathcal{L}$ is the Lebesgue measure on $[0,T]$).*

Lemma 4.5. *The map Θ can be extended to a Lipschitz continuous map from $\mathcal{M}^+_\delta([0,T]\times I)$, endowed with distance $\mathcal{D}$, to $C^0([0,T]\times[0,1]\times I)$, endowed with the norm $\|\cdot\|_\infty$. In addition we have $\Theta(\lambda)\in\mathrm{Lip}([0,T]\times[0,1]\times I)$ for any $\lambda\in\mathcal{M}^+_\delta([0,T]\times I)$.*

Proof. We need to show that there exists a constant $C>0$ such that, for any $\lambda^1,\lambda^2\in C^0_\delta([0,T]\times I,\mathbb{R}_+)$, we have: $\|\Theta(\lambda^1)-\Theta(\lambda^2)\|_\infty\le C\mathcal{D}(\lambda^1,\lambda^2)$. Since H is locally Lipschitz, there exists a constant $K>0$ such that, for any $(t,s,i)\in[0,T]\times[0,1]\times I$,

$$\begin{aligned}|\Theta(\lambda^1)_i(t,s)-\Theta(\lambda^2)_i(t,s)| &= |\psi_i^{\lambda^1}(t,s)-\psi_i^{\lambda^2}(t,s)|\\ &= \left|\sum_{j\in I,j\neq i}\int_t^T H^{\lambda^1}(i,j,t,\tau,s,\psi^{\lambda^1})-H^{\lambda^2}(i,j,t,\tau,s,\psi^{\lambda^2})d\tau\right|\\ &\le |I|\sum_{j\in I}K\int_t^T\left(|\psi_j^{\lambda^1}(\tau,S_i^{t,s}(\tau))-\psi_j^{\lambda^2}(\tau,S_i^{t,s}(\tau))|+|\int_\tau^T\lambda_j^1(r)dr-\int_\tau^T\lambda_j^2(r)dr|\right)d\tau.\end{aligned}$$

Taking the supremum over $I\times[0,1]$ yields:

$$\begin{aligned}&\|\psi^{\lambda^1}(t,\cdot)-\psi^{\lambda^2}(t,\cdot)\|_\infty\\ &\le |I|^2\int_t^T K\|\psi^{\lambda^1}(\tau,\cdot)-\psi^{\lambda^2}(\tau,\cdot)\|_\infty d\tau+|I|^2K\sup_{i\in I}\int_t^T|\int_\tau^T\lambda_i^1(r)dr-\int_\tau^T\lambda_i^2(r)dr|d\tau\\ &\le |I|^2\int_t^T K\|\psi^{\lambda^1}(\tau,\cdot)-\psi^{\lambda^2}(\tau,\cdot)\|_\infty d\tau+|I|^2K\mathcal{D}(\lambda^1,\lambda^2).\end{aligned}$$

Then, by applying Gronwall Lemma to $t\mapsto\|\psi^{\lambda^1}(t,\cdot)-\psi^{\lambda^2}(t,\cdot)\|_\infty$, one has, for any $t\in[0,T]$,

$$\|\psi^{\lambda^1}(t,\cdot)-\psi^{\lambda^2}(t,\cdot)\|_\infty\le C\mathcal{D}(\lambda^1,\lambda^2),\tag{4.12}$$

where the constant $C>0$ depends on δ,K,T and $|I|$. Therefore,

$$\|\Theta(\lambda^1)-\Theta(\lambda^2)\|_\infty=\|\psi^{\lambda^1}-\psi^{\lambda^2}\|_\infty\le C\mathcal{D}(\lambda^1,\lambda^2).$$

From the previous inequality and Remark 4.4, the map Θ can be continuously extended to a Lipschitz continuous map from $\mathcal{M}^+_\delta([0,T]\times I)$ to $C^0([0,T]\times[0,1]\times I)$.

Finally, we approximate $\lambda\in\mathcal{M}^+_\delta([0,T]\times I)$ by a sequence in $\{\lambda^n\}_n$ in $C^0_\delta([0,T]\times I,\mathbb{R}_+)$ w.r.t. the distance $\mathcal{D}$. One has that $\{\Theta(\lambda^n)\}_n$ is uniformly Lipschitz continuous on $[0,T]\times[0,1]\times I$ according to Remark 4.2. Thus, $\Theta(\lambda)$ is in $\mathrm{Lip}([0,T]\times[0,1]\times I)$. □

Proof of Proposition 4.2. The proposition is a direct consequence of Lemma 4.5. □

Remark 4.5. *By Remark 4.2 and Lemma 4.5, one can show that for any $\lambda\in\mathcal{M}^+([0,T]\times I)$, the solution ψ^λ of (4.9) is such that, $\max(\|\psi^\lambda\|_\infty,\|\partial_s\psi^\lambda\|_\infty)$ depends only on H,b,c,g and $\lambda([0,T]\times I)$.*

4.3 Analysis of weak solution of the Hamilton-Jacobi equation (2.5)

For any $\lambda\in\mathcal{M}^+([0,T]\times I)$, we consider the function $L^\lambda\in BV([0,T]\times I)$ defined for any $i\in I$ and any $t\in[0,T]$ by:

$$L_i^\lambda(t):=\lambda_i([t,T]).\tag{4.13}$$

We introduce the notion of weak solution for equation (2.5).

Definition 4.1. *For a given $\lambda \in \mathcal{M}^+([0,T]\times I)$, a function φ defined from $[0,T]\times[0,1]\times I$ to $\mathbb{R}$ is a weak solution of equation (2.5) if $\varphi - L^\lambda$ is in $\mathrm{Lip}([0,T]\times[0,1]\times I)$ and if, for any test function $f \in C^1([0,T]\times[0,1]\times I)$,*

$$\begin{aligned}&\int_0^1 \varphi_i(0,s)f_i(0,s)ds - \int_0^1 g_i(s)f_i(T,s)ds + \int_0^T\int_0^1 \left(\partial_t f_i(t,s) + \partial_s(f_i(t,s)b_i(s))\right)\varphi_i(t,s)dsdt\\ &+ \int_0^T\int_0^1 \left(\sum_{j\in I, j\neq i} H(\varphi_i(t,s)-\varphi_j(t,s)) - c_i(t,s)\right) f_i(t,s)dtds - \sum_{i\in I}\int_0^T\int_0^1 f_i(t,s)ds\lambda_i(dt)\\ &= 0,\end{aligned} \tag{4.14}$$

where $\varphi_i(0,\cdot)$ is understood in the sense of trace.

Remark 4.6. *1. There is no boundary condition in (4.14). This is due to the fact that $b(0) = b(1) = 0$, involving a null incoming flow in the domain $[0,1]$.*

2. Since φ is in $BV([0,T]\times I) + \mathrm{Lip}([0,T]\times[0,1]\times I)$, $\varphi(0,\cdot)$ exists in the sense of trace. In addition, φ is bounded and thus, it belongs to $L^2([0,T]\times[0,1]\times I)$ so that the integrals in (4.14) exist.

3. The final condition in (2.5) is misleading. Indeed, any weak solution φ of (2.5) in the sense of Definition 4.1 satisfies $\varphi_i(T,\cdot) = g_i(\cdot) + L_i^\lambda(T)$, where $\varphi_i(T,\cdot)$ and $L_i^\lambda(T)$ are in the sense of trace. Thus, $\varphi_i(T,\cdot) \geq g_i(\cdot)$.

The main result of this subsection is the following.

Theorem 4.1. *Let $\lambda \in \mathcal{M}^+([0,T]\times I)$. A function $\varphi \in \mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)$ is the unique weak solution of (2.5) in the sense of Definition 4.1, if and only if it satisfies that, for any $(t,s,i) \in [0,T]\times[0,1]\times I$,*

$$\varphi_i(t,s) = \int_t^T \sum_{j\in I, j\neq i} -H((\varphi_i-\varphi_j)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau))d\tau + L_i^\lambda(t) + g_i(S_i^{t,s}(T)). \tag{4.15}$$

Theorem 4.1 is a consequence of Lemmas 4.7 and 4.8. Lemma 4.7 shows that a function satisfying (4.15) is a weak solution of (2.5). Lemma 4.8 shows the converse. In what follows, for any $\lambda \in \mathcal{M}^+([0,T]\times I)$, we define ψ^λ as in Proposition 4.2 and define $\varphi^\lambda \in \mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)$ by:

$$\varphi^\lambda := \psi^\lambda + L^\lambda. \tag{4.16}$$

Remark 4.7. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$, φ^λ is a solution of (4.15) on $[0,T]\times[0,1]\times I$ if and only if ψ^λ is a solution of (4.9) on $[0,T]\times[0,1]\times I$. According to Remark 4.5, the quantity $\max(\|\varphi^\lambda\|_\infty, \|\partial_s\varphi^\lambda\|_\infty)$ depends on $\lambda([0,T]\times I)$.*

Lemma 4.6. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$, there exists a sequence $\{(\lambda^n,\varphi^n)\}_n$ such that*

(i) $\{\lambda^n\}_n$ is in $C^0([0,T]\times I,\mathbb{R}_+)$ and converges to λ w.r.t. the weak topology in $\mathcal{M}^+([0,T]\times I)$,

(ii) for any $n\in\mathbb{N}$, φ^n is a classical solution of (2.5) associated to λ^n on $(0,T)\times(0,1)\times I$

(iii) we have $\lim_{n\to\infty}\|\varphi^\lambda(t,\cdot)-\varphi^n(t,\cdot)\|_\infty = 0$ for a.e. $t\in[0,T]$ and $\lim_{n\to\infty}\|\varphi^\lambda(0,\cdot)-\varphi^n(0,\cdot)\|_\infty = 0$.

Proof. We consider a sequence $\{\lambda^n\}_n$ in $C^0([0,T]\times I,\mathbb{R}_+)$ that weakly converges to λ. We set $\psi^n := \psi^{\lambda^n}$, where ψ^{λ^n} is defined by Proposition 4.2, and $\varphi^n := \psi^n + L^{\lambda^n}$. By Lemma 4.1, for any $n\in\mathrm{N}$, ψ^n is a classical solution of (4.1). Then, for any $n\in\mathrm{N}$, φ^n is a classical solution of (2.5) associated to λ^n on $(0,T)\times(0,1)\times I$. The weak convergence of $\{\lambda^n\}_n$ implies:

$$\lim_{n\to\infty} L^{\lambda^n}(t) = L^\lambda(t) \text{ for a.e. } t\in[0,T] \quad \text{and} \quad \lim_{n\to\infty} L^{\lambda^n}(0) = L^\lambda(0).$$

By Proposition 4.2, $\{\psi^n\}_n$ converges to ψ^λ w.r.t. the norm $\|\cdot\|_\infty$. Then, the two previous equalities imply that:

$$\lim_{n\to\infty}\|\varphi^\lambda(t,\cdot)-\varphi^n(t,\cdot)\|_\infty = 0 \text{ for a.e. } t\in[0,T] \quad \text{and} \quad \lim_{n\to\infty}\|\varphi^\lambda(0,\cdot)-\varphi^n(0,\cdot)\|_\infty = 0.$$

□

Lemma 4.7. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$, φ^λ is a weak solution of (2.5) in the sense of Definition 4.1.*

Proof. We consider a sequence $\{\lambda^n, \varphi^n\}_n$ defined as in Lemma 4.6. For any $n \in \mathbb{N}$, φ^n is a classical solution of (2.5). Thus, for any test function $f \in C^1([0,T] \times [0,T] \times I)$:

$$\begin{aligned}
&\int_0^1 \varphi_i^n(0,s) f_i(0,s) ds - \int_0^1 g_i(s) f_i(T,s) ds + \int_0^T \int_0^1 \left(\partial_t f_i(t,s) + \partial_s(f_i(t,s) b_i(s))\right) \varphi_i^n(t,s) ds dt \\
&+ \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i^n(t,s) - \varphi_j^n(t,s)) - c_i(t,s) \right) f_i(t,s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 f_i(t,s) ds \lambda_i^n(t) dt \\
&= 0.
\end{aligned} \tag{4.17}$$

The conclusion follows by using Lemma 4.6, the continuity of H, the limit of (4.17) when n tends to infinity and by applying the dominated convergence theorem. □

The next lemma states the converse of Lemma 4.7.

Lemma 4.8. *For any $(\lambda, \varphi) \in \mathcal{M}^+([0,T] \times I) \times (\mathrm{Lip}([0,T] \times [0,1] \times I) + BV([0,T] \times I))$, if φ is a weak solution of (2.5) associated to λ in the sense of Definition 4.1, then φ satisfies (4.15) for any $(t,s,i) \in [0,T] \times [0,1] \times I$.*

Proof. Let φ be a weak solution of (2.5) associated to λ. Let $\beta \in C^1([0,T] \times [0,1] \times I, \mathbb{R})$ be a test function. Then,

$$\begin{aligned}
&\int_0^1 \varphi_i(0,s) \beta_i(0,s) ds - \int_0^1 g_i(s) \beta_i(T,s) ds + \int_0^T \int_0^1 \left(\partial_t \beta_i(t,s) + \partial_s(\beta_i(t,s) b_i(s))\right) \varphi_i(t,s) ds dt \\
&+ \int_0^T \int_0^1 \left(\sum_{j \in I, j \neq i} H(\varphi_i(t,s) - \varphi_j(t,s)) - c_i(t,s) \right) \beta_i(t,s) dt ds - \sum_{i \in I} \int_0^T \int_0^1 \beta_i(t,s) ds \lambda_i(dt) \\
&= 0.
\end{aligned} \tag{4.18}$$

We choose the function β such that there exist $\theta \in C^\infty([0,T] \times [0,1] \times I)$ and $\xi \in C^\infty([0,1] \times I)$ satisfying:

$$\begin{array}{ll}
\partial_t \beta_i(t,s) + \partial_s(\beta_i(t,s) b_i(s)) = \theta_i(t,s) & \text{for any } (t,s,i) \in (0,T) \times (0,1) \times I, \\
\beta_i(0,\cdot) = \xi_i(\cdot) & \text{for any } (s,i) \in [0,1] \times I.
\end{array} \tag{4.19}$$

The function β is given by for any $(t,s,i) \in [0,T] \times [0,1] \times I$,

$$\beta_i(t,s) = \int_0^t \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(- \int_\tau^t b_i'(S_i^{t,s}(r)) dr \right) d\tau + \xi_i(S_i^{t,s}(0)) \exp\left(- \int_0^t b_i'(S_i^{t,s}(\tau)) d\tau \right).$$

To simplify (4.18), we introduce, for any $i \in I$ the following functions ν_i and π_i, satisfying $\beta_i = \nu_i + \pi_i$: for any $(t,s) \in [0,T] \times [0,1]$,

$$\nu_i(t,s) := \int_0^t \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(- \int_\tau^t b_i'(S_i^{t,s}(r)) dr \right) d\tau \quad \text{and} \quad \pi_i(t,s) := \xi_i(S_i^{t,s}(0)) \exp\left(- \int_0^t b_i'(S_i^{t,s}(\tau)) d\tau \right).$$

Setting $h(t,s) := \sum_{j \in I, j \neq i} H(\varphi_i(t,s) - \varphi_j(t,s)) - c_i(t,s)$ for any $(t,s,i) \in [0,T] \times [0,1] \times I$, we have $h \in L^1((0,T) \times (0,1))$. By switching the order of integration, and applying the change of variable $x = S_i^{t,s}(\tau)$ and equality (2.4), one has:

$$\begin{aligned}
\int_0^T \int_0^1 h(t,s) \nu_i(t,s) dt ds &= \int_0^T \int_0^1 \int_0^t h(t,s) \theta_i(\tau, S_i^{t,s}(\tau)) \exp\left(- \int_\tau^t b_i'(S_i^{t,s}(r)) dr \right) d\tau ds\, dt \\
&= \int_0^T \int_0^1 \theta_i(\tau, x) \int_\tau^T h(t, S_i^{\tau,x}(t)) \exp\left(- \int_\tau^t b_i'(S_i^{\tau,x}(r)) dr \right) \partial_x S_i^{\tau,x}(t) dt\, dx\, d\tau, \\
&= \int_0^T \int_0^1 \theta_i(\tau, x) \int_\tau^T h(t, S_i^{\tau,x}(t)) dt\, dx\, d\tau.
\end{aligned} \tag{4.20}$$

By the same computations, for any $i \in I$, one has:

$$\int_0^1 g_i(s) \nu_i(T,s) ds = \int_0^T \int_0^1 g_i(S_i^{\tau,x}(T)) \theta_i(\tau,x) ds d\tau, \tag{4.21}$$

$$\int_0^T \int_0^1 h(t,s)\pi_i(t,s)dtds = \int_0^1 \xi_i(x) \int_0^T h(t, S_i^{0,x}(t))dtdx, \tag{4.22}$$

$$\int_0^1 g_i(s)\pi_i(T,s)ds = \int_0^1 \xi_i(x) g_i(S_i^{0,x}(T))dx, \tag{4.23}$$

and:

$$\int_0^T \int_0^1 \beta_i(t,s)ds\lambda_i(dt) = \int_0^T \int_0^1 \xi_i(\tau,x)L_i^\lambda(\tau)dxd\tau + \int_0^1 \xi_i(x)L_i^\lambda(0)dx. \tag{4.24}$$

Using (4.20), (4.22), (4.23), (4.24) and (4.19), equality (4.18) becomes:

$$\begin{aligned}
&\int_0^1 \xi_i(s) \left(\varphi_i(0,s) + \int_0^T \sum_{j\in I, j\neq i} H((\varphi_i - \varphi_j)(\tau, S_i^{0,s}(\tau))) - c_i(\tau, S_i^{0,s}(\tau))d\tau - L_i^\lambda(0) - g_i(S_i^{0,s}(T)) \right) ds \\
&+ \int_0^T \int_0^1 \theta_i(t,s) \left(\varphi_i(t,s) + \int_t^T \sum_{j\in I, j\neq i} H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) - c_i(\tau, S_i^{t,s}(\tau))d\tau - L_i^\lambda(t) - g_i(S_i^{t,s}(T)) \right) dsdt \\
&= 0.
\end{aligned}$$

This equality holds for any test functions θ_i and ξ_i. Then, one has for any $s \in [0,1]$:

$$\varphi_i(0,s) = \int_0^T \sum_{j\in I, j\neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{0,s}(\tau))) + c_i(\tau, S_i^{0,s}(\tau))d\tau + L_i^\lambda(0) + g_i(S_i^{0,s}(T)),$$

and forn any $(t,s) \in [0,T] \times [0,1]$:

$$\varphi_i(t,s) = \int_t^T \sum_{j\in I, j\neq i} -H((\varphi_i - \varphi_j)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau))d\tau + L_i^\lambda(t) + g_i(S_i^{t,s}(T)).$$

□

With the above lemmas, the proof of Theorem 4.1 is straightforward.

Proof of Theorem 4.1. The proof is a direct consequence of Lemmas 4.7 and 4.8. The uniqueness of a weak solution is deduced by Remark 4.7. Indeed, since ψ^λ is the unique solution of (4.9), φ^λ is the unique solution of (4.15) and thus, the unique weak solution of (2.5) in the sense of Definition 4.1. □

5 The dual problem

In this section, an optimization problem (5.3) is introduced. Using tools from convex analysis [22], we show that this problem is in duality with (3.4). We consider the following spaces:

$$E_0 = C^1([0,T] \times [0,1] \times I) \times C^0([0,T] \times I) \text{ and } E_1 := C^0([0,T] \times [0,1] \times I) \times C^0([0,T] \times [0,1] \times I^2).$$

Given $(\varphi, \lambda) \in E_0$, we consider the following inequality:

$$\begin{aligned}
&-\partial_t\varphi_i(t,s) - b_i(s)\partial_s\varphi_i(t,s) - c_i(t,s) - \lambda(t) + \sum_{j\in I, j\neq i} H((\varphi_i - \varphi_j)(t,s)) \leq 0 \quad \text{on } (0,T)\times(0,1)\times I, \\
&\varphi_i(T,\cdot) \leq g_i \qquad\qquad \text{on } (0,1)\times I.
\end{aligned} \tag{5.1}$$

The set $\mathcal{K}_0$ is defined by: $\mathcal{K}_0 := \{(\varphi,\lambda) \in E_0,\ \lambda \geq 0$ and φ satisfies (5.1) associated to $\lambda\}$. We introduce the function A, defined on $\mathcal{K}_0$ by :

$$A(\varphi,\lambda) := \sum_{i\in I} \int_0^1 -\varphi_i(0,s)m_i^0(ds) + \int_0^T \lambda_i(t)D_i(t)dt, \tag{5.2}$$

and the following problem is considered:

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda). \tag{5.3}$$

Lemma 5.1. $\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda)$ *is finite.*

Proof. Let $(\varphi,\lambda)\in\mathcal{K}_0$. Since φ is in $C^1([0,T]\times[0,1]\times I)$ and satisfies (5.1), using that H is non negative, we have, for any $(i,s)\in I\times[0,1]$, $\varphi_i(0,s)\le T\|c\|_\infty+\int_0^T\lambda_i(\tau)d\tau+\|g\|_\infty$. Setting $Q:=-|I|(T\|c\|_\infty+\|g\|_\infty)$, one has:

$$Q+\sum_{i\in I}\int_0^T\lambda_i(t)\left(D_i(t)-\int_0^1 m_i^0(ds)\right)dt\le A(\varphi,\lambda).$$

Since $\lambda\ge 0$, we deduce from the Assumption 3 and previous inequality that $Q\le\inf_{(\varphi,\lambda)\in\mathcal{K}_0}A(\varphi,\lambda)$. □

We consider the linear and bounded function $\Lambda:E_0\to E_1$ defined by: $\Lambda(\varphi,\lambda):=(\partial_t\varphi+b\partial_s\varphi+\tilde\lambda,\Delta\varphi)$, where $\partial_t\varphi+b\partial_s\varphi:=(\partial_t\varphi_i+b_i\partial_s\varphi_i)_{i\in I}$, $\Delta\varphi:=(\Delta\varphi_{i,j})_{(i,j)\in\tilde I}$ with $\Delta\varphi_{i,j}=\varphi_j-\varphi_i$ and, for any $(s,i)\in[0,1]\times I$, $\tilde\lambda_i(\cdot,s):=\lambda_i(\cdot)$. The linear function $\Lambda^*:E_1^*\to E_0^*$ is the adjoint operator of Λ. The functional $\mathcal{F}$ is defined, for any $(\varphi,\lambda)\in E_0$, by

$$\mathcal{F}(\varphi,\lambda):=\begin{cases}\displaystyle\sum_{i\in I}\int_0^1-\varphi_i(0,s)m_i^0(ds)+\int_0^T D_i(t)\lambda_i(t)dt & \text{if } \varphi_i(T,\cdot)\le g_i \text{ and } \lambda_i\ge 0\quad\forall i\in I,\\ +\infty & \text{otherwise.}\end{cases}$$

Using that:

$$\begin{aligned}&\langle(m,E),\Lambda(\varphi,\lambda)\rangle_{E_1^*,E_1}\\&=\sum_{i\in I}\int_0^1\int_0^T(\partial_t\varphi_i(t,s)+b_i(s)\partial_s\varphi_i(t,s))m_i(ds,t)+\sum_{j\in I,j\neq i}(\varphi_j(t,s)-\varphi_i(t,s))E_{i,j}(t,ds)dt\\&+\sum_{i\in I}\int_0^T\int_0^1 m_i(t,ds)\tilde\lambda_i(t,s)dt,\end{aligned}$$

defining $\mathcal{F}^*$ as the Fenchel conjugate of $\mathcal{F}$, we have:

$$\mathcal{F}^*\left(\Lambda^*(m,E)\right):=\begin{cases}\displaystyle\int_0^1\sum_{i\in I}g_i(s)m_i(T,ds) & \text{if } (m,E) \text{ is a weak solution of (3.1)}\\ & \text{and } \displaystyle\int_0^1 m_i(t,ds)\le D_i(t)\quad\forall(t,i)\in[0,T]\times I,\\ +\infty & \text{otherwise.}\end{cases}$$

For any $(x,y)\in E_1$, the functional $\mathcal{G}$ is defined by:

$$\mathcal{G}(x,y):=\begin{cases}0 & \text{if } -c_i(t,s)-x_i(t,s)+\displaystyle\sum_{j\in I,j\neq i}H(-y_{i,j}(t,s))\le 0\quad\forall(t,s,i)\in(0,T)\times(0,1)\times I,\\ +\infty & \text{otherwise.}\end{cases}$$

Then, for any $(\varphi,\lambda)\in E_0$, it holds:

$$\mathcal{G}(\Lambda(\varphi,\lambda)):=\begin{cases}0 & \text{if } -c_i(t,s)-\partial_t\varphi_i(t,s)-b_i(t,s)\partial_s\varphi_i(t,s)-\tilde\lambda_i(t,s)+\displaystyle\sum_{j\in I,j\neq i}H(-\Delta\varphi_{i,j}(t,s))\le 0\\ & \forall(t,s,i)\in(0,T)\times(0,1)\times I,\\ +\infty & \text{otherwise.}\end{cases}$$

Using that $L(x)=H^*(x)$, one can show, as in [6] for the quadratic case, that, for any $(v,w)\in\mathbb{R}^2$,

$$\sup_{a,b\in\mathbb{R}}\{av+bw;\ a+H(b)\le 0\}=\begin{cases}L(\frac{w}{v})v & \text{if } v>0 \text{ and } w\ge 0,\\ 0 & \text{if } v=0 \text{ and } w=0,\\ +\infty & \text{otherwise.}\end{cases}$$

Then, with similar computations as in [16, Lemma 4.3], for any $(m, E) \in E_1'$, we have:

$$
\begin{aligned}
&\mathcal{G}^*(-(m,E)) \\
&= \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 -x_i(t,s)m_i(t,ds)dt - \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(x,y) \\
&= \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 (-x_i(t,s) - c_i(t,s) + c_i(t,s))m_i(t,ds) - \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(x,y) \\
&= \sum_{i\in I} \int_0^T \int_0^1 c_i(t,s)m_i(t,ds)dt + \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 x_i(t,s)m_i(t,ds) + \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(-x-c,-y) \\
&= \begin{cases} \displaystyle\int_0^T \int_0^1 \sum_{i\in I} c_i(t,s)m_i(t,ds) + \sum_{j\neq i} L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right) m_i(t,ds)dt & \text{if } m>0, E\geq 0 \text{ and } E \ll m, \\ 0 & \text{if } m=0 \text{ and } E=0, \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}
\tag{5.4}
$$

The following lemma shows the constraint qualification for Problem (5.3).

Lemma 5.2. *There exists $(\varphi, \lambda) \in E_0$ such that $\mathcal{F}(\varphi,\lambda) < \infty$ and $\mathcal{G}$ is continuous at $\Lambda(\varphi,\lambda)$.*

Proof. Let φ and λ be such that, for any $(t,s,i) \in [0,T]\times[0,1]\times I$,

$$\varphi_i(t,s) = -\max_{i\in I}(\|g_i\|_\infty) - 1,$$

and

$$\lambda_i(t) := \|c_i\|_\infty + 1,$$

Functions φ and λ being constant, it holds that $(\varphi,\lambda) \in E_0$ and $\mathcal{F}(\varphi,\lambda) < \infty$. Also, from the choice of φ and λ, it follows that, for any $i \in I$, $s \in [0,1]$ and $t \in [0,T]$,

$$-c_i(t,s) - \partial_t\varphi_i(t,s) - b_i(t,s)\partial_s\varphi_i(t,s) - \lambda_i(t,s) + \sum_{j\in I, j\neq i} H(\varphi_i(t,s) - \varphi_j(t,s)) < 0.$$

Thus, $\mathcal{G}$ is continuous at $\Lambda(\varphi,\lambda)$. □

The main result on the duality of this section is the following.

Theorem 5.1. *We have:*

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) = -\inf_{(m,E)\in\mathcal{S}(m^0,D)} \tilde{J}(m,E)$$

Proof. On can observe that:

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) = \inf_{(\varphi,\lambda)\in E_0} \mathcal{F}(\varphi,\lambda) + \mathcal{G}(\Lambda(\varphi,\lambda)),$$

and

$$\inf_{(m,E)\in\mathcal{S}(m^0,D)} \tilde{J}(m,E) = \inf_{(m,E)\in E_1^*} \mathcal{F}(\Lambda^*(m,E)) + \mathcal{G}^*(-(m,E)),$$

Using Lemmas 5.2 and 5.1, the conclusion follows by applying the Fenchel-Rockafellar duality theorem [22]. □

For any $(\varphi,\lambda) \in (\mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)) \times \mathcal{M}^+([0,T]\times I)$, we define the function

$$\tilde{A}(\varphi,\lambda) := \sum_{i\in I} \int_0^1 -\varphi_i(0,s)m_i^0(ds) + \int_0^T D_i(t)\lambda_i(dt). \tag{5.5}$$

Proposition 5.1. *There exists $\lambda \in \mathcal{M}^+([0,T]\times I)$ such that:*

$$\tilde{A}(\lambda,\varphi^\lambda) = \inf_{(\phi,\mu)\in\mathcal{K}_0} A(\phi,\mu),$$

where φ^λ is defined in (4.16).

Before proving this result, we need the following lemma.

Lemma 5.3. *For any $K > 0$, there exists a constant $C > 0$ such that, for any $(\varphi, \lambda) \in \mathcal{K}_0$ satisfying $A(\varphi, \lambda) \leq K$, we have:*

$$\sum_{i \in I} \int_0^T \lambda_i(t) dt \leq C.$$

Proof. Let $K \in \mathbb{R}_+^*$ and $(\varphi, \lambda) \in \mathcal{K}_0$ be such that $A(\varphi, \lambda) \leq K$. Since $(\varphi, \lambda) \in \mathcal{K}_0$, one can show, as in the proof of Lemma 5.1, that, for any $(i, s) \in I \times [0, 1]$,

$$\varphi_i(0, s) \leq T\|c\|_\infty + \|g\|_\infty + \sum_{i \in I} \int_0^T \lambda_i(t) dt.$$

Therefore, recalling that $A(\varphi, \lambda) \leq K$:

$$\sum_{i \in I} \int_0^T \lambda_i(t)(D_i(t) - m_i([0, 1])) dt \leq K + T\|c\|_\infty + \|g\|_\infty.$$

From Assumption 3, there exists $\varepsilon^0 > 0$ such that $D_i(t) - m_i([0, 1]) > \varepsilon^0$ for any $(t, i) \in [0, T] \times I$. Setting $C := (K + T\|c\|_\infty + \|g\|_\infty)/\varepsilon^0$, we have:

$$\sum_{i \in I} \int_0^T \lambda_i(t) dt \leq C.$$

□

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Let $\{(\phi^n, \lambda^n)\}_n$ be a minimizing sequence of (5.3). For any $n \in \mathbb{N}$, we consider φ^n the classical solution of (2.5) associated to λ^n. Since φ^n is a classical solution and ϕ^n satisfies (5.1) on $[0, T] \times [0, 1] \times I$, we can easily check that the following comparison holds: $\varphi^n \leq \phi^n$. Thus, $A(\varphi^n, \lambda^n) \leq A(\phi^n, \lambda^n)$ for any $n \in \mathbb{N}$. Therefore, $\{(\varphi^n, \lambda^n)\}_n$ is also a minimizing sequence and, there exist $K > 0$ and $n^0 > 0$ such that for any $n \geq n^0$ one has $A(\varphi^n, \lambda^n) \leq K$. From Lemma 5.3, the sequence $\{\sum_{i \in I} \int_0^T \lambda_i^n(t) dt\}_n$ is uniformly bounded. Thus, a subsequence of $\{\lambda^n\}_n$ weakly converges to a measure $\lambda \in \mathcal{M}^+([0, T] \times I)$ w.r.t. the weak topology in $\mathcal{M}^+([0, T] \times I)$. We set $\varphi^\lambda := \Theta(\lambda) + L^\lambda$ and $\varphi^n := \Theta(\lambda^n) + L^{\lambda^n}$ for any $n \in \mathbb{N}$, where Θ is defined in (4.10). By the same arguments as in the proof of Lemma 4.6, one has $\lim_{n \to \infty} \|\varphi^\lambda(0, \cdot) - \varphi^n(0, \cdot)\|_\infty = 0$ up to a subsequence of $\{\varphi^n\}_n$ and, therefore, we have:

$$\lim_{n \to \infty} A(\varphi^n, \lambda^n) = \tilde{A}(\varphi^\lambda, \lambda),$$

and the conclusion follows. □

6 Characterization of the minimizers

The purpose of this section is to define and characterize the solutions of Problem (3.4). We show that the following system gives optimality conditions for (3.4):

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_i - \varphi_j) = 0 & \text{on } (0, T) \times (0, 1) \times I, \\ \partial_t m_i + \partial_s (m_i b_i) + \sum_{j \neq i} H'(\varphi_i - \varphi_j) m_i - H'(\varphi_j - \varphi_i) m_j = 0 & \text{on } (0, T) \times (0, 1) \times I, \\ m_i(0, s) = m_i^0(s),\ \varphi_i(T, s) = g_i(s) & \text{on } (0, 1) \times I, \\ \int_0^1 m_i(t, ds) - D_i(t) \leq 0,\ \lambda \geq 0 & \text{on } [0, T] \times I, \\ \sum_{i \in I} \int_0^T \left(\int_0^1 m_i(t, ds) - D_i(t) \right) \lambda_i(dt) = 0. \end{cases} \tag{6.1}$$

The notion of weak solutions of system (6.1) is given in the following definition.

Definition 6.1. *A triplet* $(\varphi, \lambda, m) \in (\mathrm{Lip}([0,T]\times[0,1]\times I)+BV([0,T]\times I))\times\mathcal{M}^+([0,T]\times I)\times C^0([0,T],\mathcal{P}([0,1]\times I))$ *is called a weak solution of* (6.1) *if it satisfies the following conditions:*

1. *The function* φ *is a weak solution of* (2.5)*, associated to* λ *in the sense of Definition 4.1;*
2. m *satisfies the continuity equation:*

$$\partial_t m_i + \partial_s(m_i b_i) + \sum_{j\neq i} H'(\varphi_i - \varphi_j)m_i - H'(\varphi_j - \varphi_i)m_j = 0, \quad m_i(0,\cdot) = m_i^0,$$

in the sense of Definition 3.1, with $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$*;*

3. *it holds that, for any* $t \in [0,T]$*,*

$$\int_0^1 m_i(t,ds) - D_i(t) \leq 0 \quad \text{and} \quad \sum_{i\in I}\int_0^T \left(\int_0^1 m_i(t,ds) - D_i(t)\right)\lambda_i(dt) = 0.$$

Remark 6.1. *Since* $\varphi \in \mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)$ *and* H' *is Lipschitz continuous, the control* $\alpha_{i,j} := H'(\varphi_i - \varphi_j)$ *is bounded on* $[0,T]\times[0,1]\times I$ *and* $\partial_s\alpha_{i,j} \in L^\infty([0,T]\times[0,1])$*. Thus,* $\alpha_{i,j}$ *is in* $L^2_{m_i}([0,T]\times[0,1])$ *and the forward equation in* (6.1) *makes sense.*

The following theorem states the optimality conditions of Problem (3.4).

Theorem 6.1. *1. If* $(m,E) \in \mathcal{S}(m^0,D)$ *is a minimizer of Problem* (3.4)*, and* φ *a weak solution of* (2.5) *in the sense of Definition 4.1 associated to* λ *satisfying* $\tilde{A}(\varphi,\lambda) = \inf_{(\phi,\mu)\in\mathcal{K}_0} A(\phi,\mu)$*, then* (φ,λ,m) *is a weak solution of* (6.1) *and* $\dfrac{dE_{i,j}}{dm_i} = H'(\varphi_i - \varphi_j)$ *on* $\{m_i > 0\}$ *for any* $i,j \in I$*.*

2. Conversely, if (φ,λ,m) *is a weak solution of* (6.1)*, then* $\tilde{A}(\varphi,\lambda) = \inf_{(\phi,\mu)\in\mathcal{K}_0} A(\phi,\mu)$ *and there exists* E*, defined for any* $i,j \in I$ *by* $\dfrac{dE_{i,j}}{dm_i} := H'(\varphi_i - \varphi_j)$*, such that* $(m,E) \in \mathcal{S}(m^0,D)$ *is a minimizer of* (3.4)*.*

Remark 6.2. *If* (φ,λ,m) *is a weak solution of* (6.1)*, then* (φ,λ) *is a minimizer of a relaxed version of Problem* (5.3)*, i.e.* (φ,λ) *is the minimum of* $\tilde{A}$ *over the space* $(\mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I))\times\mathcal{M}^+([0,T]\times I)$*.*

6.1 Proof of Theorem 6.1

Before the proof of Theorem 6.1, we make the following remark.

Remark 6.3. *For any* $\lambda \in \mathcal{M}^+([0,T]\times I)$ *and any* $(m,E) \in \mathcal{S}(m^0,D)$ *one has* $-\tilde{J}(m,E) \leq \tilde{A}(\varphi^\lambda,\lambda)$*. Indeed, considering a sequence* $\{(\lambda^n,\varphi^n)\}_n$ *defined as in Lemma 4.6 and using the proof of Theorem 5.1, we get:* $-\tilde{J}(m,E) \leq A(\varphi^n,\lambda^n)$ *and, therefore,* $-\tilde{J}(m,E) \leq \tilde{A}(\varphi^\lambda,\lambda)$*.*

Proof of Theorem 6.1. 1. By Theorem 5.1, one has:

$$\inf_{(\hat{\varphi},\hat{\lambda})\in\mathcal{K}_0} A(\hat{\varphi},\hat{\lambda}) = -\inf_{(\hat{m},\hat{E})\in\mathcal{S}(m^0,D)} \tilde{J}(\hat{m},\hat{E}),$$

and thus

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \int_0^T D_i\lambda_i + \int_0^T\int_0^1 \left(c_i + \sum_{j\neq i} L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)\right) m_i = 0. \tag{6.2}$$

We want to show that $E_{i,j} = H'(\varphi_i - \varphi_j)m_i$. We consider a sequence $\{(\lambda^n,\varphi^n)\}_n$ defined as in Lemma 4.6. For any $n \in \mathbb{N}$, φ^n is smooth enough to be a test function for the weak formulation of (1.1) satisfied by m. According

to Lemma 4.6 and the fact that $D \in C^0([0,T]\times I)$, it holds that, for any $n\in\mathbb{N}$ and $i\in I$,

$$\begin{aligned}
&\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \int_0^T \lambda_i D_i\\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i^n(0)m_i^0 + \int_0^T D_i\lambda_i^n\\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1 \left(\partial_t\varphi_i^n + b_i\partial_s(\varphi_i^n) + \sum_{j\in I, j\neq i}(\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) m_i + \int_0^T \lambda_i^n D_i\\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1 \left(-c_i + \sum_{j\in I, j\neq i} H(\varphi_i^n-\varphi_j^n) + \sum_{j\in I, j\neq i}(\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) m_i + \int_0^T \lambda_i^n\left(D_i - \int_0^1 m_i\right).
\end{aligned}$$

By the previous equality and (6.2),

$$\lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1 \left(\sum_{j\in I, j\neq i} H(\varphi_i^n-\varphi_j^n) + L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) + (\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) m_i + \int_0^T \lambda_i^n\left(D_i - \int_0^1 m_i\right) = 0.$$

According to Lemma 4.6, for a.e. $t\in[0,T]$, the sequence $\{\varphi(t,\cdot)\}_n$ converges uniformly to φ. By the continuity of H and the dominated convergence theorem, we get $\int_0^T\int_0^1 m_i H(\varphi_i^n-\varphi_j^n)$ converges to $\int_0^T\int_0^1 m_i H(\varphi_i-\varphi_j)$ for any $i,j\in I$. Since (m,E) is a solution of (3.4), $\tilde{J}(m,E)$ is finite. Then one can show that, for any $i,j\in I$, $\int_0^T\int_0^1 E_{i,j}<\infty$. Applying dominated convergence theorem, we can show that: $\int_0^T\int_0^1(\varphi_j^n-\varphi_i^n)E_{i,j}$ converges to $\int_0^T\int_0^1(\varphi_j-\varphi_i)E_{i,j}$. Since, for any $i\in I$, the map $t\mapsto D_i(t)-\int_0^1 m_i(t,ds)$ is continuous, the weak convergence of λ^n to λ in $\mathcal{M}^+([0,T]\times I)$ gives:

$$\lim_{n\to\infty}\sum_{i\in I}\int_0^T \lambda_i^n\left(D_i-\int_0^1 m_i\right) = \sum_{i\in I}\int_0^T \lambda_i\left(D_i-\int_0^1 m_i\right).$$

Thus,

$$\int_0^T\int_0^1 \left(\sum_{j\in I, j\neq i} H(\varphi_i-\varphi_j) + L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) + (\varphi_j-\varphi_i)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) m_i + \int_0^T \lambda_i\left(D_i-\int_0^1 m_i\right) = 0. \tag{6.3}$$

Since $\lambda\geq 0$ and $\int_0^1 m_i(t,ds)\leq D_i(t)$ for any $t\in[0,T]$, one has, for any $i\in I$ and $t\in[0,T]$,

$$0\leq \int_0^T\left(D_i(t)-\int_0^1 m_i(t,ds)\right)\lambda_i(dt). \tag{6.4}$$

Recalling that $L^*(p)=H(p)$, we have $L(p)+H(q)-pq\geq 0$ for any $p,q\in\mathbb{R}$. Thus, by inequality (6.4) and equality (6.3), one deduces

$$\sum_{j\in I, j\neq i} H(\varphi_i-\varphi_j) + L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right) + (\varphi_j-\varphi_i)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i} = 0 \quad m-\text{a.e.}.$$

Therefore,

$$\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s) = H'(\varphi_i(t,s)-\varphi_j(t,s)) \quad m-\text{a.e. }, \tag{6.5}$$

and inequality (6.4) becomes an equality. Thus, $\int_0^T\left(D_i-\int_0^1 m_i(t,ds)\right)\lambda_i(dt)=0$. By equality (6.5), the properties of H and the fact that $\varphi\in \mathrm{Lip}([0,T]\times[0,1]\times I)+BV([0,T]\times I)$, one has $\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\in L^\infty([0,T]\times[0,1])$ and $\partial_s\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)\in L^\infty((0,T)\times(0,1))$. Thus, by Proposition 8.1 in Appendix, we deduce that $m\in\mathrm{Lip}([0,T],\mathcal{P}([0,1]\times I))$.

2. We assume now that (λ,φ,m) is a weak solution of (6.1). Since φ is in $\mathrm{Lip}([0,T]\times[0,1]\times I)+BV([0,T]\times I)$ and λ is a finite measure, the quantity $\tilde{A}(\varphi,\lambda)$ is well defined. We want to show that $\tilde{A}(\varphi,\lambda)+\tilde{J}(m,E)=0$. We

approximate (λ, φ) by the sequence $\{(\lambda^n, \varphi^n)\}_n$ defined in Lemma 4.6.(1). For any n, φ^n is smooth enough to be considered as a test function for the equation (1.1) satisfied in the weak sense by m. We have, for any $i \in I$,

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i^n(0)m_i^0 + \sum_{i\in I}\int_0^T\int_0^1 -m_i b_i \partial_s\varphi_i^n - m_i\partial_t\varphi_i^n + \sum_{j\in I, j\neq i}(\varphi_i^n - \varphi_j^n)H'(\varphi_i - \varphi_j)m_i = 0. \tag{6.6}$$

For any $i \in I$, φ_i^n is a classical solution of (2.5) associated to λ^n. Multiplying (2.5) by m_i, summing over I and integrating over $[0,T]\times[0,1]$, we have:

$$\sum_{i\in I}\int_0^T\int_0^1 -m_i\partial_t\varphi_i^n - m_i b_i\partial_s\varphi_i^n - m_i c_i - m_i\lambda^n + \sum_{j\in I, j\neq i} H(\varphi_i^n - \varphi_j^n)m_i = 0. \tag{6.7}$$

Combining (6.6) and (6.7) yields

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i^n(0)m_i^0 + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i + \lambda_i^n m_i + m_i\left(\sum_{j\in I, j\neq i} H'(\varphi_i-\varphi_j)(\varphi_i^n-\varphi_j^n) - H(\varphi_i^n - \varphi_j^n)\right) = 0.$$

Since (φ, λ, m) is a weak solution of (6.1), by Lemma 4.6, and letting n tend to infinity, one deduces:

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \sum_{i\in I}\int_0^1 D_i\lambda_i + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i + m_i\left(\sum_{j\in I, j\neq i} H'(\varphi_i-\varphi_j)(\varphi_i-\varphi_j) - H(\varphi_i-\varphi_j)\right) = 0.$$

By the definition of L and H, we have:

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \sum_{i\in I}\int_0^1 D_i\lambda_i + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i + m_i\left(\sum_{j\in I, j\neq i} L(H'(\varphi_i-\varphi_j))\right) = 0.$$

By the definition of $\tilde{A}$ in (5.5) and $\tilde{J}$ in (3.3), we have $\tilde{A}(\varphi,\lambda) + \tilde{J}(m,E) = 0$. Finally, by Remark 6.3, one deduces that (m, E) is a minimizer of (3.4). □

6.2 Proof of Theorem 2.1

We are now ready to prove our main theorem by using Theorem 1 and applying the change of variable $\alpha_{i,j} := \dfrac{\mathrm{d}E_{i,j}}{m_i}$.

Proof of Theorem 2.1. The existence of a solution to Problem (1.4) is given by Lemma 3.3.

1 This statement is proved by Theorem 6.1.1

2 This point is given by Theorem 6.1.2.

3 The uniform bound on α and $\partial_s\alpha$ are deduced by Theorem 2.1.1, using the fact that H has a globally Lipschitz continuous gradient and that φ is in $\mathrm{Lip}([0,T]\times[0,1]\times I) + BV([0,T]\times I)$. The time regularity of m is obtained by Proposition 8.2 in Appendix 8. □

We now prove Proposition 2.1.

Proof of Proposition 2.1. Let (m,α) be a solution of Problem (1.4) and μ^0 be the density of m^0 w.r.t. the Lebesgue measure. By a fixed point argument, it is easy to check that there exists a unique solution $\mu \in \mathrm{Lip}([0,T]\times[0,1]\times I)$ of the following equation on $[0,T]\times[0,1]\times I$:

$$\begin{aligned}\mu_i(t,s) = \quad & \mu_i^0(S_i^{t,s}(0)) + \int_0^t \mu_i(\tau, S_i^{t,s}(\tau))\partial_s b_i(S_i^{t,s}(\tau))d\tau \\ & + \int_0^t \sum_{j\in I, j\neq i} -\alpha_{i,j}(\tau, S_i^{t,s}(\tau))\mu_i(\tau, S_i^{t,s}(\tau)) + \alpha_{j,i}(\tau, S_i^{t,s}(\tau))\mu_j(\tau, S_i^{t,s}(\tau))d\tau.\end{aligned} \tag{6.8}$$

Denote by $\mathcal{L}$ the Lebesgue measure on $[0,T]\times[0,1]$. The conclusion follows by proving that $\mu\mathcal{L}$ is the unique weak solution of (1.1). □

7 Sensitivity analysis of the value of the optimization problem w.r.t. the data

In this section we study how the value of Problem (1.4) depends on the initial distribution $m^0 \in \mathcal{P}([0,1]\times I)$ and on the parameter $D \in C^0([0,T]\times I, \mathbb{R}_+^*)$ of the constraint (1.2). We endow the space $\Omega := \mathcal{P}([0,1]\times I)\times C^0(I\times[0,T],\mathbb{R}_+^*)$ with the distance $\mathcal{D}_\Omega$ defined by:

$$\mathcal{D}_\Omega((m^0,D),(\bar m^0,\bar D)) := \mathcal{W}(m^0,\bar m^0) + \|D-\bar D\|_\infty,$$

where $\mathcal{W}$ is the Wasserstein distance on $\mathcal{P}([0,1]\times I)$. We recall that the definition of Ω_ε is given in (2.1). For any $\varepsilon>0$ we consider the function $\mathcal{V}:\Omega_\varepsilon\to\mathbb{R}$ defined by:

$$\mathcal{V}(m^0,D) := \inf_{(m,E)\in\mathcal{S}(m^0,D)} \tilde J(m,E), \tag{7.1}$$

where the set $\mathcal{S}(m^0,D)$ is defined in (3.2).

The main result of this section is the following proposition, which shows the Lipschitz continuity of the value of the problem (1.4) w.r.t. the initial distribution and the congestion constraint (1.2).

Proposition 7.1. *For any $\varepsilon>0$, $\mathcal{V}$ is Lipschitz continuous on Ω_ε w.r.t. the distance $\mathcal{D}_\Omega$.*

To prove the proposition, we need to introduce some lemmas. For any $(m^0,D)\in\Omega_\varepsilon$, we consider the function $A[m^0,D]: C^0([0,T]\times[0,1]\times I)\times C^0([0,T]\times I,\mathbb{R}_+)\to\mathbb{R}$ defined by:

$$A[m^0,D](\varphi,\lambda) := \sum_{i\in I}\int_0^1 -\varphi_i(0,s)m_i^0(ds) + \int_0^T \lambda_i(t)D_i(t)dt.$$

The following result gives some properties of the function $\mathcal{V}$.

Lemma 7.1. *For any $\varepsilon>0$, the function $\mathcal{V}$ is bounded independently of ε, convex and l.s.c. on Ω_ε.*

Proof. Let $\varepsilon>0$ and $C := (T\|c\|_\infty + \|g\|_\infty)$. By the definition of $\mathcal{V}$ in (7.1), one can show, for any $(m^0,D)\in\Omega_\varepsilon$, that $|\mathcal{V}(m^0,D)|\le C$. For any $(m^0,D)\in\Omega_\varepsilon$, the duality result in Theorem 5.1 gives

$$\mathcal{V}(m^0,D) = \sup_{(\varphi,\lambda)\in\mathcal{K}_0} -A[m^0,D](\varphi,\lambda). \tag{7.2}$$

Since $\mathcal{V}$ is the supremum of continuous and linear functions, we deduce that $\mathcal{V}$ is convex and l.s.c. on Ω_ε. □

For any $\varepsilon>0$ and $(m^0,D)\in\Omega_\varepsilon$, we know by Proposition 5.1 that there exists $(\varphi^{m^0,D},\lambda^{m^0,D})\in(\mathrm{Lip}([0,T]\times[0,1]\times I)+BV([0,T]\times I))\times\mathcal{M}^+([0,T]\times I)$ such that $\varphi^{m^0,D}$ is a weak solution of (2.5), in the sense of Definition 4.1, associated to $\lambda^{m^0,D}$ and such that $(\varphi^{m^0,D},\lambda^{m^0,D})$ satisfies:

$$\sum_{i\in I}\int_0^1 \varphi_i^{m^0,D}(0,s)m_i^0(ds) - \int_0^T D_i(t)\lambda_i^{m^0,D}(dt) = -\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A[m^0,D](\varphi,\lambda) = \mathcal{V}(m^0,D).$$

The next lemma provides an estimate on $\varphi^{m^0,D}$ and $\lambda^{m^0,D}$ for any $\varepsilon>0$ and $(m^0,\lambda)\in\Omega_\varepsilon$.

Lemma 7.2. *For any $\varepsilon>0$, there exists $C>0$ such that, for any $(m^0,D)\in\Omega_\varepsilon$,*

$$\max\left(\|\varphi^{m^0,D}\|_\infty,\ \|\partial_s\varphi^{m^0,D}\|_\infty,\ \lambda^{m^0,D}(I\times[0,T])\right)\le C.$$

Proof. Let $\varepsilon>0$ and $(m^0,D)\in\Omega_\varepsilon$. According to Lemma 7.1, there exists a constant $K>0$, independent of ε, such that $\sum_{i\in I}\int_0^1\varphi_i^{m^0,D}(0,s)m_i^0(ds) - \int_0^T D_i(t)\lambda_i^{m^0,D}(dt) < K$. Thus, by using same arguments as in the proof of Lemma 5.3 and setting $\tilde K := (K+T\|c\|_\infty+\|g\|_\infty)/\varepsilon$, one obtains:

$$\lambda^{m^0,D}(I\times[0,T])\le\tilde K. \tag{7.3}$$

By Remark 4.7 and the previous inequality, there exists a constant $\tilde C$, which depends on $\tilde K$, such that $\|\varphi^{m^0,D}\|_\infty$ and $\|\partial_s\varphi^{m^0,D}\|_\infty$ are bounded by $\tilde C$. The conclusion follows by setting $C := \max(\tilde C,\tilde K)$. □

We are now ready to prove the Lipschitz regularity of $\mathcal{V}$.

Proof of Proposition 7.1. Let $\varepsilon > 0$ and $(m^0, D), (\bar{m}^0, \bar{D}) \in \Omega_\varepsilon$. By the definition of $\mathcal{V}$ in (7.1) and Lemma 7.2, one has:

$$\begin{aligned}\mathcal{V}(m^0, D) &\leq \sum_{i\in I} \int_0^1 \varphi_i^{m^0,D} \bar{m}_i^0(ds) - \sum_{i\in I} \int_0^T \bar{D}_i(t)\lambda_i^{m^0,D}(dt) + \|\partial_s \varphi^{m^0,D}\|_\infty \mathcal{W}(m^0, \bar{m}^0) + \lambda^{m^0,D}(I\times[0,T])\|D - \bar{D}\|_\infty \\ &\leq \mathcal{V}(\bar{m}^0, \bar{D}) + C\mathcal{D}_\Omega((m^0, D), (\bar{m}^0, \bar{D})),\end{aligned}$$

where $C > 0$ is a constant defined in Lemma 7.2. Similarly, we have:

$$\mathcal{V}(\bar{m}^0, \bar{D}) \leq \mathcal{V}(m^0, D) + C\mathcal{D}_\Omega((m^0, D), (\bar{m}^0, \bar{D})).$$

The conclusion follows. □

8 Appendix

Some properties of the weak solution of the continuity equation (1.1) are derived in this subsection. Assumptions in Section 1 are in force in the Appendix. A first result on the support of the solution is established in Lemma 8.1.

Lemma 8.1. *For any weak solution (α, m) of (1.1) in the sense of Definition 3.1, $m(t)$ has a support contained in $[0,1]\times I$ for any $t \in [0,T]$.*

Proof. Let $\varepsilon > 0$ and $\varphi^\varepsilon \in C_c^\infty([0,T]\times\mathbb{R}\times I)$ such that, for any $t \in [0,T]$ and $i \in I$,

$$\varphi_i^\varepsilon(t,s) \in [0,1], \quad \forall s \in \mathbb{R}; \quad \varphi_i^\varepsilon(t,s) = 0, \quad \forall s \in \mathbb{R}\setminus(-1-\varepsilon, 2+\varepsilon); \quad \text{and} \quad \varphi_i^\varepsilon(t,s) = 1 \quad \forall s \in [-1,2].$$

Since m is a weak solution of (1.1), b satisfies Assumption 1, and $\varphi_i = \varphi_j$ for any $i, j \in I$, we deduce that, for any $t \in (0,T)$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \sum_{i\in I} \varphi_i^\varepsilon(t,s) m_i(t,ds) &= \int_{\mathbb{R}} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \\ &= \int_{-1-\varepsilon}^{-1} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) + \int_2^{2+\varepsilon} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \\ &= 0.\end{aligned} \tag{8.1}$$

By (8.1) and the continuity of m, we deduce that $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} \varphi_i^\varepsilon(t,s) m_i(t,ds)$ is constant on $[0,T]$. Let ε tend to $+\infty$, it holds that $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds)$ is constant over $[0,T]$. Then, we have for any $t \in (0,T)$, $\int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds) = \int_{\mathbb{R}} \sum_{i\in I} m_i^0(ds) = 1$. Now let us show that $\int_0^1 \sum_{i\in I} m_i(t,ds) = 1$. Let $\varepsilon > 0$ and ψ^ε be another test function in $C_c^\infty([0,T]\times\mathbb{R}\times I)$ such that, for any $t \in [0,T]$ and $i \in I$,

$$\begin{aligned}&\psi_i^\varepsilon(t,s) = 0, \quad \forall s \in \mathbb{R}\setminus(-\varepsilon, 1+\varepsilon); \quad \partial_s \psi_i^\varepsilon(t,s) \geq 0, \quad \forall s \in (-\varepsilon, 0); \quad \partial_s \psi_i^\varepsilon(t,s) \leq 0, \quad \forall s \in (1, \varepsilon); \\ &\quad \text{and } \psi_i^\varepsilon(t,s) = 1, \quad \forall s \in [0,1].\end{aligned}$$

By the same computation as in (8.1) and Assumption 1, one has, for any $t \in (0,T)$,

$$\frac{d}{dt}\int_{\mathbb{R}} \sum_{i\in I} \psi_i^\varepsilon(t,s) m_i(t,ds) = \int_{-\varepsilon}^0 \sum_{i\in I} \partial_s \psi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) + \int_1^{1+\varepsilon} \sum_{i\in I} \partial_s \psi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \geq 0.$$

Thus, $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} \psi_i^\varepsilon(t,s) m_i(t,ds)$ is non-decreasing on $[0,T]$. Taking the limit $\varepsilon \to 0$, the map $t \mapsto \int_0^1 \sum_{i\in I} m_i(t,ds)$ is also non-decreasing on $[0,T]$. Finally, for any $t \in [0,T]$: $1 = \int_0^1 \sum_{i\in I} m_i(0,ds) \leq \int_0^1 \sum_{i\in I} m_i(t,ds) \leq \int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds) = 1$. □

For any pair of weak solution (m, α) of (1.1), the next lemma provides some regularity on m if α and $\partial_s \alpha$ are bounded.

Lemma 8.2. *Let $m^0 \in \mathcal{P}([0,1] \times I)$ and (m, α) be a weak solution of (1.1) in the sense of Definition 3.1, with $\alpha \in L^\infty([0,T] \times I \times I \times [0,1], \mathbb{R}_+)$. Then, m belongs to $\mathrm{Lip}([0,T], \mathcal{P}([0,1] \times I))$ with a Lipschitz constant independent of m^0.*

Proof. As in Lemma 3.1, following standards arguments and computations from Optimal Transport [2], one can show that, for any $t, \tilde{t} \in [0,T]$,

$$\mathcal{W}(m(t,\cdot), m(\tilde{t},\cdot)) \leq |t - \tilde{t}|(|I|\|b\|_\infty + \|\alpha\|_\infty).$$

The conclusion follows. □

Finally, the next Proposition states that, for any α, the existence and uniqueness of an m such that (m, α) is a weak solution of (1.1).

Proposition 8.1. *Let $m^0 \in \mathcal{P}([0,1] \times I)$ and $\alpha \in L^\infty([0,T] \times I \times I \times [0,1])$ satisfy $\partial_s \alpha \in L^\infty([0,T] \times I \times I \times [0,1])$. Then, there exists a unique $m \in \mathrm{Lip}([0,T], \mathcal{P}([0,1] \times I)$ such that (m, α) is a weak solution of (1.1) in the sense of Definition 3.1.*

Proof. The existence and uniqueness of a weak solution are proved in [19] for controls α that are continuous in space and time independent. The extension of this result to bounded controls that are measurable in time is straightforward. □

Acknowledgement

This research benefited from the support of the FiME Lab (Institut Europlace de Finance) and the support of the FMJH Program Gaspard Monge for optimization and operations research and their interactions with data science. I would like to thank my advisor, Pierre Cardaliaguet, for suggesting the problem and many helpful conversations.

References

[1] Yves Achdou and Mathieu Laurière. Mean field type control with congestion. *Applied Mathematics & Optimization*, 73(3):393–418, 2016.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.

[3] Mario Annunziato, Alfio Borzì, Fabio Nobile, and Raul Tempone. On the connection between the Hamilton-Jacobi-Bellman and the Fokker-Planck control frameworks. *Appl. Math.*, 5:2476–2484, 2014.

[4] Erhan Bayraktar, Alekos Cecchin, Asaf Cohen, and Francois Delarue. Finite state mean field games with Wright-Fisher common noise. *Journal de Mathématiques Pures et Appliquées*, 147:98–162, 2021.

[5] Richard E Bellman. *Adaptive control processes: a guided tour*, volume 2045. Princeton university press, 2015.

[6] Jean-David Benamou and Yann Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.

[7] Jean-David Benamou and Guillaume Carlier. Augmented Lagrangian methods for transport optimization, mean field games and degenerate elliptic equations. *Journal of Optimization Theory and Applications*, 167(1):1–26, 2015.

[8] Jean-David Benamou, Guillaume Carlier, and Filippo Santambrogio. Variational mean field games. In *Active Particles, Volume 1*, pages 141–171. Springer, 2017.

[9] J Frédéric Bonnans, Pierre Lavigne, and Laurent Pfeiffer. Discrete potential mean field games. *arXiv preprint arXiv:2106.07463*, 2021.

[10] Benoît Bonnet and Hélène Frankowska. Necessary optimality conditions for optimal control problems in Wasserstein spaces. *Applied Mathematics & Optimization*, 84(2):1281–1330, 2021.

[11] Benoît Bonnet and Francesco Rossi. The Pontryagin maximum principle in the Wasserstein space. *Calculus of Variations and Partial Differential Equations*, 58(1):1–36, 2019.

[12] Luis Briceño-Arias, Dante Kalise, Ziad Kobeissi, Mathieu Laurière, A Mateos González, and Francisco José Silva. On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings. *ESAIM: Proceedings and Surveys*, 65:330–348, 2019.

[13] Luis M Briceno-Arias, Dante Kalise, and Francisco J Silva. Proximal methods for stationary mean field games with local couplings. *SIAM Journal on Control and Optimization*, 56(2):801–836, 2018.

[14] Giuseppe Buttazzo, Chloé Jimenez, and Edouard Oudet. An optimization problem for mass transportation with congested dynamics. *SIAM Journal on Control and Optimization*, 48(3):1961–1976, 2009.

[15] Pierre Cardaliaguet, Guillaume Carlier, and Bruno Nazaret. Geodesics for a class of distances in the space of probability measures. *Calculus of Variations and Partial Differential Equations*, 48(3):395–420, 2013.

[16] Pierre Cardaliaguet and P Jameson Graber. Mean field games systems of first order. *ESAIM: Control, Optimisation and Calculus of Variations*, 21(3):690–722, 2015.

[17] Pierre Cardaliaguet, P Jameson Graber, Alessio Porretta, and Daniela Tonon. Second order mean field games with degenerate diffusion and local coupling. *Nonlinear Differential Equations and Applications NoDEA*, 22(5):1287–1317, 2015.

[18] Pierre Cardaliaguet, Alpár R Mészáros, and Filippo Santambrogio. First order mean field games with density constraints: pressure equals price. *SIAM Journal on Control and Optimization*, 54(5):2672–2709, 2016.

[19] Christiane Cocozza-Thivent and Robert Eymard. Approximation of the marginal distributions of a semi-Markov process using a finite volume scheme. *ESAIM: Mathematical Modelling and Numerical Analysis*, 38(5):853–875, 2004.

[20] Samuel Daudin. Optimal control of diffusion processes with terminal constraint in law. *arXiv preprint arXiv:2012.10707*, 2020.

[21] Samuel Daudin. Optimal control of the Fokker-Planck equation under state constraints in the Wasserstein space. *arXiv preprint arXiv:2109.14978*, 2021.

[22] Ivar Ekeland and Roger Temam. *Convex analysis and variational problems*. SIAM, 1999.

[23] David Evangelista, Rita Ferreira, Diogo A Gomes, Levon Nurbekyan, and Vardan Voskanyan. First-order, stationary mean-field games with congestion. *Nonlinear Analysis*, 173:37–74, 2018.

[24] Dena Firoozi, Ali Pakniyat, and Peter E Caines. A mean field game-hybrid systems approach to optimal execution problems in finance with stopping times. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 3144–3151. IEEE, 2017.

[25] Dena Firoozi, Ali Pakniyat, and Peter E Caines. A class of hybrid lqg mean field games with state-invariant switching and stopping strategies. *Automatica*, 141:110244, 2022.

[26] Wendell H Fleming and Domokos Vermes. Convex duality approach to the optimal control of diffusions. *SIAM journal on control and optimization*, 27(5):1136–1155, 1989.

[27] Azad Ghaffari, Scott Moura, and Miroslav Krstić. Modeling, control, and stability analysis of heterogeneous thermostatically controlled load populations using partial differential equations. *Journal of Dynamic Systems, Measurement, and Control*, 137(10):101009, 2015.

[28] Diogo Gomes, Roberto M Velho, and Marie-Therese Wolfram. Socio-economic applications of finite state mean field games. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20130405, 2014.

[29] Diogo A Gomes, Joana Mohr, and Rafael Rigao Souza. Discrete time, finite state space mean field games. *Journal de mathématiques pures et appliquées*, 93(3):308–328, 2010.

[30] Diogo A Gomes, Joana Mohr, and Rafael Rigao Souza. Continuous time finite state mean field games. *Applied Mathematics & Optimization*, 68(1):99–143, 2013.

[31] P Jameson Graber and Alpár R Mészáros. Sobolev regularity for first order mean field games. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 35, pages 1557–1576. Elsevier, 2018.

[32] Minyi Huang, Peter E Caines, and Roland P Malhamé. Large-population cost-coupled lqg problems with nonuniform agents: individual-mass behavior and decentralized ε-nash equilibria. *IEEE transactions on automatic control*, 52(9):1560–1571, 2007.

[33] Minyi Huang, Roland P Malhamé, Peter E Caines, et al. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[34] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. i–le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619–625, 2006.

[35] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. ii–horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, 343(10):679–684, 2006.

[36] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[37] Hugo Lavenant and Filippo Santambrogio. New estimates on the regularity of the pressure in density-constrained mean field games. *Journal of the London Mathematical Society*, 100(2):644–667, 2019.

[38] Caroline Le Floch, Florent Di Meglio, and Scott Moura. Optimal charging of vehicle-to-grid fleets via PDE aggregation techniques. In *2015 American Control Conference (ACC)*, pages 3285–3291. IEEE, 2015.

[39] Pierre-Louis Lions. Théorie des jeux de champ moyen et applications (mean field games). *Cours du College de France. http://www. college-de-france. fr/default/EN/all/equ der/audio video. jsp*, 2009, 2007.

[40] Alpár Richárd Mészáros and Francisco J Silva. A variational approach to second order mean field games with density constraints: the stationary case. *Journal de Mathématiques Pures et Appliquées*, 104(6):1135–1159, 2015.

[41] Paul Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.

[42] Scott Moura, Victor Ruiz, and Jan Bendsten. Modeling heterogeneous populations of thermostatically controlled loads using diffusion-advection pdes. In *Dynamic Systems and Control Conference*, volume 56130, page V002T23A001. American Society of Mechanical Engineers, 2013.

[43] Carlo Orrieri, Alessio Porretta, and Giuseppe Savaré. A variational approach to the mean field planning problem. *Journal of Functional Analysis*, 277(6):1868–1957, 2019.

[44] Filippo Santambrogio. A modest proposal for mfg with density constraints. *arXiv preprint arXiv:1111.0652*, 2011.

[45] Filippo Santambrogio. Optimal transport for applied mathematicians. *Birkäuser, NY*, 55(58-63):94, 2015.

[46] Filippo Santambrogio. Regularity via duality in calculus of variations and degenerate elliptic pdes. *Journal of Mathematical Analysis and Applications*, 457(2):1649–1674, 2018.

[47] Adrien Seguret. Mean field approximation of an optimal control problem for the continuity equation arising in smart charging. Unpublished Manuscript, 2022.

[48] Adrien Seguret, Cheng Wan, and Clemence Alasseur. A mean field control approach for smart charging with aggregate power demand constraints. *accepted in IEEE PES Innovative Smart Grid Technologies Europe (ISGT Europe)*, oct 2021.

[49] Colin Sheppard, Laurel N Dunn, Sangjae Bae, and Max Gardner. Optimal dispatch of electrified autonomous mobility on demand vehicles during power outages. In *2017 IEEE Power & Energy Society General Meeting*, pages 1–5. IEEE, 2017.

[50] Richard Vinter. Convex duality and nonlinear optimal control. *SIAM journal on control and optimization*, 31(2):518–538, 1993.