Constrained optimal control of a first order Fokker-Planck with reaction term

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Abstract

We consider a constrained optimal control of an advection-reaction partial differential equation (PDE). We prove the existence of a minimizer and we characterize the solution as the weak solution of a system of two coupled PDEs. This system is composed of a Fokker-Planck equation and of a Hamilton-Jacobi equation, similarly to systems obtained in Mean Field Games (MFG). We provide regularity results of the solutions.

Keywords: Optimal control, optimality conditions, mean field control.

1 Introduction

We study in this article the optimal control of a first order Fokker-Planck equation with a reaction term, under congestion constraints. This kind of equation typically arises to model the evolution of a probability measure of a large population of agents. In this paper, the state of each agent is composed of a continuous variable and of a discrete variable. The optimal control problem we study can be interpreted heuristically as an approximation of the limit case $n \to \infty$ of an optimal switching problem of n agents. Our work is motivated by the optimal switching control for a large population of agents, and more precisely to the smart charging in electrical engineering [43]. Each agent can represent a plug-in electric vehicle (PEV) aiming at charging its battery. The overall population of PEVs is controlled by a central planner. The continuous variable represents the level of battery of the PEV and the discrete variable the mode of charging (e.g. not charging, charging, discharging, etc...). Finally, the congestion constraint avoids high demand of energy over the period. Combinatorial techniques as well as optimal control tools fail to solve problems with large population of PEVs, due to the curse of dimensionality [5]. To overcome these difficulties, a continuum of PEVs can be considered, leading to optimal control of PDE techniques. Optimal control of a Fokker-Planck applied to smart charging can be found in [33, 44], and applied to the management of a population of thermostatically controlled loads in [23, 37].

Through the article, we consider a finite horizon $[0,T]$ and a mixed state space equal to the product $[0,1] \times I$, where I is a finite space, whose cardinality is denoted by $|I|$. We consider the uncontrolled velocity field b, which describes how agents move on the segment $[0,1]$. We consider the function α, which is the control determining the jump intensity of the agents between the different modes in I. For any $(t,s,i,j) \in [0,T] \times [0,1] \times I \times I$, the value $\alpha_{i,j}(t,s)$ denotes the jump intensity of agents from state (s,i) to the state (s,j), at time t. The control α is required to be a non negative measurable function and to satisfy $\alpha_{i,i} = 0$, meaning that no agents can jump from state (s,i) to state (s,i). The function α is determined by an aggregator. We highlight that the agents are controlled by the same function α. We define m such that for any $(t,s,i) \in [0,T] \times [0,1] \times I$, the value $m_i(t,s)$ represents the proportion of agents at time t at state (s,i). The pair (α, m) is the weak solution, in the sense of Definition 2.1, of the continuity equation on $[0,T] \times [0,1] \times I$:

$$\begin{aligned} &\partial_t m_i(t,s) + \partial_s(m_i(t,s)b_i(s)) = -\sum_{j \neq i}(\alpha_{i,j}(t,s)m_i(t,s) - \alpha_{j,i}(t,s)m_j(t,s)) \quad && (i,t,s) \in I \times (0,T) \times (0,1), \\ &m_i(0,s) = m_i^0(s) && (i,s) \in I \times [0,1], \end{aligned} \tag{1.1}$$

where the initial distribution m^0 is given. This equation is a first order Fokker-Planck equation, where the right-hand side is a reaction term. As mentioned above, we consider congestion constraints on the total mass per mode i

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in I at any $t \in [0,T]$. This is used to avoid synchronization effects and to limit the proportion of agents per mode $i \in I$ at any time. Note that this constraint introduces interactions between agents in our model. In the limit case $n \to \infty$, the constraint is of the form:

$$m_i(t,[0,1]) \leq D_i(t) \quad \forall (i,t) \in I \times [0,T], \tag{1.2}$$

where $D_i > 0$ is given. The objective function J is defined as followed:

$$J(m,\alpha) := \sum_{i\in I} \int_0^T \int_0^1 \Big(\sum_{j\in I, j\neq i} L(\alpha_{ij}(t,s)) + c_i(t,s) \Big) m_i(t,ds)dt + \sum_{i\in I} \int_0^1 g_i(s) m_i(T,ds), \tag{1.3}$$

where the function $L : \mathbb{R} \mapsto \bar{\mathbb{R}}_+$ is defined by:

$$L(x) := \begin{cases} \dfrac{x^2}{2} & \text{if } x \geq 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

The cost function L penalizes high values of α. It aims at avoiding multiple jumps of agents between the elements of the set I. The value $c_i(t,s)$ corresponds to the cost per agent to be at time $t \in [0,T)$ at state (s,i), while $g_i(s)$ is a final cost per agent to be at state (s,i). Regularity assumptions on c and g will be introduced later. Our purpose is to study the optimization problem:

$$\begin{array}{l} \inf\limits_{m,\alpha} J(m,\alpha) \\ (\alpha, m) \text{ satisfies (1.1) and (1.2)} \end{array} \tag{1.5}$$

1.1 Motivations

We briefly present an optimal switching problem of n agents in this subsection. Problem (1.5) can be interpreted heuristically as an approximation of the limit case $n \to \infty$ of this problem.

We consider n controlled agents $x^1, \dots, x^n$ over the period $[0,T]$. The state of the k^{th} agent at time t is denoted by $x^k(t) := (q^k(t), z^k(t))$ and is composed of a continuous variable $q^k(t) \in [0,1]$ and a discrete one $z^k(t) \in I$. A strategy is a couple (τ, ι) where τ is composed of n sequences (one sequence per agent) of stopping times in $[0,T]$ and ι is composed of n sequences (also one sequence per agent) with values I. The dynamic of the state of the k^{th} agent is controlled as follows:

$$\frac{\mathrm{d}q^k(t)}{\mathrm{d}t} = b_{z^k(t)}(q^k(t)) \text{ and } z_k(t) = \sum_{h=0}^{\infty} \iota_h^k 1_{[\tau_h^k, \tau_{h+1}^k)}(t).$$

The stopping time $\tau_h^k \in [0,T]$ is the h^{th} jump of the k^{th} agent and ι_h^k its jump destination. The cost of a strategy is estimated by:

$$J^n(\tau,\iota) := \frac{1}{n} \sum_{k=0}^{n} \left(\int_0^T c_{z^k(t)}(t, q^k(t))dt + g_{z^k(T)}(q^k(T)) \right) + \frac{1}{n} P(\iota, \tau)$$

in which P is the cost of switching between nodes, aiming at avoiding large number of jumps and synchronization effects between agents. This function will be determined in a later work. Functions c and g have already been defined in the definition of J at (1.3). The goal of the switching problem is to solve:

$$\begin{array}{l} \inf\limits_{\tau,\iota} J^n(\tau,\iota) \\ \text{s.c. } \dfrac{1}{n} \sum\limits_{n} 1_i(z^k(t)) \leq D_i(t) \text{ for any } t \in [0,T]. \end{array} \tag{1.6}$$

The reader can refer to [3, Section 4.4] for an introduction to optimal switching. Heuristically, Problem (1.5) can be interpreted as a formulation, when n tends to infinity, of (1.6). The connection between the two problems will be addressed in a later work. Note that the mean field behaviour of interacting and controlled processes has been investigated, in deterministic and stochastic settings, in [18, 28] and the references therein. As precised above, this problem is motivated by its application in smart charging. Each agent represents an electric vehicle aiming at charging its battery. The continuous variable q represents the level of battery of the electric vehicle and the discrete variable z the mode of charging (e.g. not charging, charging, discharging, etc...). The transfers from one mode of charging to another one are penalized through the cost P in order to avoid multiple switches, synchronization effects and battery aging. The goal of the final cost g is to penalize small battery level at the end of the period, while the function c can represent a cost of electricity for power consumption. Finally, the constraint (1.6) avoids high demand of energy over the period.

1.2 Contributions and literature

One of our main results states the existence of solutions of (1.5) and gives optimality conditions, by using classic tools of optimization and convex duality theory [19]. More precisely we show that, if (m, α) is a solution to (1.5), then there exists a pair (ϕ, λ) such that for any $i, j \in I$, $\alpha_{i,j} = (\varphi_i - \varphi_j)^+$ and (ϕ, λ, m) is a weak solution of the following system:

$$\begin{cases} -\partial_t \varphi_i - b_i \partial_s \varphi_i - c_i - \lambda_i + \sum_{j \in I, j \neq i} H(\varphi_j - \varphi_i) = 0 & \text{on } (0,T) \times (0,1) \times I \\ \partial_t m_i + \partial_s (m_i b_i) + \sum_{j \neq i} ((\varphi_i - \varphi_j)^+ m_i - (\varphi_j - \varphi_i)^+ m_j) = 0 & \text{on } (0,T) \times (0,1) \times I \\ m_i(0,s) = m_i^0(s),\ \varphi_i(T,s) = g_i(s) & \text{on } (0,1) \times I \\ \int_0^1 m_i(t,s) ds - D(t) \leq 0,\ \lambda \geq 0 & \text{on } [0,T] \times I \\ \sum_{i \in I} \int_0^T \int_0^1 m_i(t, ds) \lambda_i(dt) - \int_0^T D(t) \lambda(dt) = 0 \end{cases} \tag{1.7}$$

The function φ is the multiplier associated to the dynamic constraint (1.1), and λ is associated to the congestion constraint (1.2). The first equation is a backward Hamilton-Jacobi equation, where H is defined by $H(y) := (y^-)^2$ for any $y \in \mathbb{R}$. Existence, uniqueness and characterization of weak solution of the backward Hamilton-Jacobi equation are investigated in the paper. The second equation is a forward Fokker-Planck equation, similar to (1.1), where the control α, defined by $\alpha_{i,j} = (\varphi_i - \varphi_j)^+$, is optimal. The measure λ is non negative and finite and the last equality in (1.7) ensures that that the congestion constraint (1.2) is satisfied. Moreover, one of our main contributions is a regularity property for any weak solution (λ, φ, m) of (1.7). We prove, that under suitable assumptions on the data m^0, b, g and c, the function φ is continuous for a.e. t in $[0,T]$ and $\partial_s \varphi \in L^\infty((0,T) \times I, C^0([0,1]))$. In addition, for any $i \in I$ the measure m_i is a Lipschitz continuous function on $[0,T] \times [0,1]$.

This kind of system (1.7) typically arises in the Mean Field Game Theory (MFG for short). This class of problem, introduced by Lasry and Lions [29, 30, 31] and Huang, Malhamé and Caines [26, 27], describes the interaction between a large population of identical and rational agents in competition.

The duality approach adopted to obtain (1.7) consists in relaxing the dynamic (1.1) and congestion constraint (1.2). The resulting relaxed problem is then expressed as the dual of an other convex problem. We show that the system (1.7) is an optimality condition of both problems. Solving optimal control of a Fokker-Planck equation by means of duality theory is well known since few decades [22, 45]. Our work follows the method developed in the seminal work of Benamou and Brenier [6], for optimal transport problems. In [6], a Fokker-Planck equation is controlled with initial and final constraint, optimality conditions are obtained as a system of PDEs close to (1.7). Similar method and results still in optimal transport are derived in [14]. The duality approach adopted in the paper is close to method used in MFG theory as in [15], where existence and uniqueness of the weak solution of the MFG system are proved, and the solution is characterized as the minimizer of some optimal control of Hamilton-Jacobi and Fokker-Planck equations. This approach enables to use optimization techniques, to prove existence and uniqueness of the solution of the MFG system as well as for Mean Field Control (MFC for short) problem. We refer to [1, 8, 12, 16, 38] and the references therein. The variational approach allows besides to apply optimization algorithms to solve numerically MFG problems [7, 11, 12]. Note that different optimality conditions, for control problems in the space of probability measures, can be derived by using a kind of Pontryagin Maximum Principle [10].

The paper deals with a congestion constraint (1.2) on the measure. Two kinds of congestions effects have been explored in the MFG and MFC frameworks. On the one hand, "soft constraints" which increase the cost of velocity of the agents in areas with high density. On the other hand, "hard congestion" which imposes density constraints, e.g. $m \leq \bar{m}$ at any point (t,s). The variational approach shows good results when applied to MFC [1] and MFG with "soft congestion" in a stationary framework [20], as well as to MFG problems dealing with "hard congestion" constraints. This has been first investigated in [40] where the density of the population does not exceed a given threshold, then in [35] where stationary second order MFG are considered. In [17] a price, imposed on the saturated zone to make the density satisfy the constraints, is obtained. In the same vein as the work of Benamou and Brenier [6], "hard congestion" constraints are also examined in optimal transport [13]. We highlight that our paper deals with aggregated "hard congestion" constraints on the measure m (1.2), i.e. our constraint is less restrictive than a constraint of the type $m \leq \bar{m}$ a.e..

We consider a mixed state space, with continuous and discrete state variables. To the best of our knowledge, these settings have been barely investigated in the MFG literature, e.g. articles cited above look only at continuous

state variables. The resulting Fokker-Planck equation (1.1) contains a term of reaction, indicating mass transfers between states on I. Such PDE arises also in [2], to model the mean field limit of Piecewise Deterministic Markov Proccesses (PDMP for short). The velocity is controlled in [2] while we control the intensity of the jump α (the velocity b is given). A discrete time and state space MFG problem is explored in [24]. The uniqueness of the solution of a finite state MFG is discussed in [4]. Mixed state space in a MFG framework can be found in [21], where a major player can switch his state on a finite state space and minor players decide their stopping time. A MFG problem in a finite state space and discrete time settings with "hard congestion", has been studied in [9], using also variational methods.

Concerning the regularity results, let us point out that the regularity of solutions of (1.7) is not usual, and we believe that it is mainly due to the linearity of the Hamilton-Jacobi equation w.r.t. $\partial_s\varphi$, and to the regularity assumptions on b, allowing to use the characteristic method to solve the PDEs. These results will be useful in a later work to quantify the mean field limit assumption of the model. The time regularity of m and φ may not be improved as far as we have no more regularity results on λ. The function φ is discontinuous at each atom of the measure λ. Regularity results, about the multiplier of the density constraint can be found in the literature: in [17] the authors show some BV estimates on the pressure, whereas L^∞ estimates for the price have been proved in [32], in the special case of a quadratic Hamiltonian. Sobolev regularity, for the solution of a first order MFG, has been established in [34], and improved in [42]; see also [25].

The paper is organized as follows. In the rest of Section 1 we present our assumptions and main results. Problem (1.5) is analyzed in Section 2 and we give some regularity results on the solution m of (1.1). We study in detail the Hamilton-Jacobi equation of the system (1.7) in Section 3. In Section 4, the variational approach of Problem (1.5) is developed. We prove our main result in Section 5. Finally, we recall basic statements about weak solutions in the Appendix A.

1.3 Assumptions

The following assumptions are in force throughout the paper.

1. For any $i \in I$, $b_i \in C^2(\mathbb{R})$ with $b_i(s) = 0$ for any $s \notin (0,1)$.
2. m^0 is a probability measure on $[0,1]$, absolutely continuous w.r.t. the Lebesgue measure, with a density denoted in the same way. We assume that for any $i \in I$: $m_i^0 \in C^1(\mathbb{R})$, with $\mathrm{supp}(m_i^0) \subset [0,1]$.
3. For any $i \in I$, $D_i \in C^0([0,T])$ and there exists $\varepsilon^0 > 0$ such that for any $i \in I$ and $t \in [0,T]$:
$$\varepsilon^0 < D_i(t) - \int_0^1 m_i^0(s)ds. \tag{1.8}$$
4. For any $i \in I$, it is assumed that $c_i \in C^1([0,T] \times [0,1])$ and $g_i \in C^1([0,1])$.

The main role of assumptions 1 and 2 is to ensure that the population of agents remains concentrated on $[0,1]$. Inequality (1.8) of Assumption 3 is used to show the existence of a solution of the optimization problem, whose optimality condition is the system (1.7). Finally, the regularity results of the weak solutions of the system (1.7) are derived thanks to the assumptions formulated on c and g in Assumption (4).

1.4 Notations

The space of positive and bounded measures on a space A is denoted by $\mathcal{M}^+(A)$ and the space of probability measure $\mathcal{P}(A)$. For any measure $\mu \in \mathcal{M}([0,T])$ and $0 \le t_1 < t_2 \le T$, we set $\int_{t_1}^{t_2} \mu(dt) := \mu([t_1,t_2))$. Given a finite vector space $\mathcal{S}$, for any function f defined on $I \times \mathcal{S}$ we use the notation $f_i(x) := f(i,x)$ for any $(i,x) \in I \times \mathcal{S}$. Similarly, for any function g defined on $I \times I \times \mathcal{S}$ we consider the notation $f_{i,j}(x) := f(i,j,x)$ for any $(i,j,x) \in I \times I \times \mathcal{S}$. The Wasserstein distance on $\mathcal{P}([0,1] \times I)$ is denoted by W_1. Given a finite vector space $\mathcal{S}$, let $\mathrm{Lip}(\mathcal{S})$ denote the vector space of bounded functions $f : \mathcal{S} \to \mathbb{R}$ such that the Lipschitz constant of f defined by $\sup\{|f(x)-f(y)|/\|x-y\| \,|x,y \in \mathcal{S}, x \neq y\}$ is finite. Let $L^\infty((0,T)\times I\times I, \mathrm{Lip}([0,1])$ be the vector space of measurable maps $f : (0,T) \times (0,1) \times I \times I \to \mathbb{R}$ such that there exists a finite constant $c_f > 0$ where for a.e. $t \in (0,T)$ and any $i,j \in I$, $\|f_{i,j}(t,\cdot)\|_\infty \le c_f$ and such that $f_{i,j}(t,\cdot)$ is Lipschitz continuous on $[0,1]$ with Lipschitz constant c_f. For any $\mu \in C^0([0,T],\mathcal{P}(\mathbb{R}))$, we define the set: $L^2_\mu([0,T]\times\mathbb{R}) := \{f : [0,T]\times\mathbb{R} \mapsto \mathbb{R}, \int_0^T \int_0^1 f(t,s)^2 \mu(t,ds)dt < +\infty\}$.

The norm $\|\cdot\|_1$ is defined for any $f \in L^1([0,T], \times[0,1] \times I)$ by $\|f\|_1 = \sum_{i\in I} \int_0^T \int_0^1 |f_i|$. The dual of a normed space X is denoted by X^*. We denote for any $s \in [0,1]$, $i \in I$ and $t \in [0,T]$ by $S_i^{t,s}$ the unique solution on $[0,T]$ of:

$$S_i^{t,s}(t) = s, \quad \frac{\mathrm{d}S_i^{t,s}(\tau)}{\mathrm{d}\tau} = b_i(S_i^{t,s}(\tau)) \quad \tau \in [0,T]. \tag{1.9}$$

1.5 Main results

We introduce, for a given $\lambda \in \mathcal{M}^+([0,T] \times I)$, the Hamilton-Jacobi equation on $(0,T) \times (0,1) \times I$:

$$\begin{aligned} &-\partial_t \varphi_i(t,s) - b_i(s)\partial_s \varphi_i(t,s) - c_i(t,s) - \lambda_i(t) + \sum_{j\in I, j\neq i} H((\varphi_j - \varphi_i)(t,s)) = 0 \quad (t,s,i) \in (0,T)\times(0,1)\times I, \\ &\varphi_i(T,\cdot) = g_i \qquad\qquad (s,i) \in [0,1]\times I, \end{aligned} \tag{1.10}$$

We denote by $\mathcal{R}_0$ the set of weak solutions of (1.10), in the sense of Definition 3.1, and consider the following problem

$$\inf_{(\varphi,\lambda)\in\mathcal{R}_0} \tilde{A}(\varphi,\lambda) \tag{1.11}$$

where:

$$\tilde{A}(\varphi,\lambda) := \sum_{i\in I} \int_0^1 -\varphi_i(0,s) m_i^0(ds) + \int_0^T D_i(t)\lambda_i(dt). \tag{1.12}$$

We can now state our main result.

Theorem 1.1. *Problem* (1.5) *has a solution* (m,α)*, and Problem* (1.11) *has also a solution* (φ,λ)*, where* $\varphi \in L^\infty([0,T]\times[0,1]\times I)$ *and* $\partial_s\varphi \in L^\infty((0,T)\times I, C^0(0,1))$*. In addition, we have the following characterization of the minimizers:*

1. *If* (m,α) *is a minimizer of Problem* (1.5) *and* $(\varphi,\lambda) \in \mathcal{R}_0$ *a minimizer of Problem* (1.11)*, then* (φ,λ,m) *is a weak solution of* (1.7)*, in the sense of Definition 5.1, and* $\alpha_{i,j} = (\varphi_i - \varphi_j)^+$ *on* $\{m_i > 0\}$ *for any* $i,j \in I$*.*
2. *Conversely, if* (φ,λ,m) *is a weak solution of* (1.7) *in the sense of Definition 5.1, then* $(\varphi,\lambda) \in \mathcal{R}_0$ *is a minimizer of Problem* (1.11) *and there exists* α*, defined for any* $i,j \in I$ *by:* $\alpha_{i,j} := (\varphi_i - \varphi_j)^+$ *on* $\{m_i > 0\}$*, such that* (m,α) *is a minimizer of* (1.5)*.*
3. *If* (m,α) *is a minimizer of Problem* (1.5)*, then for any* $i,j \in I$ $\alpha_{i,j} \in L^\infty((0,T), \mathrm{Lip}([0,1]))$*, and* $m \in \mathrm{Lip}([0,T]\times[0,1]\times I)$

Remark 1.1. *For the sake of simplicity, we have defined* L *in* (1.4)*. However, results in this paper still hold for any function* $L : \mathbb{R} \to \mathbb{R}_+$*, satisfying:*

1. L *is a lower semi continuous and convex function and* $\mathrm{dom}\ L = \mathbb{R}_+$*.*
2. *The function* H*, defined by* $H(x) := L^*(-x)$ *is non decreasing and differentiable, and is such that its Fenchel conjugate* H^**, defined by* $H^*(x) := L(-x)$*, is essentially strictly convex [39, Theorems 26.1, 26.3].*
3. *There exist* $r > 1$ *and* $C > 0$*, such that for any* $x \in \mathbb{R}_+$ *we have:*

$$\frac{x^r}{rC} - C \leq L(x) \leq \frac{C}{r}x^r + C.$$

The existence of a solution of (1.5) is stated by Lemma 2.3 in Section 2. In section 4 Theorem 4.2 proves the existence of a solution of (1.11). These results are obtained by classical techniques in convex optimization. The characterization of these solutions are given by Theorem 5.1 in Section 5. A variational approach is used to deduce this characterization. We introduce a convex problem, whose dual is, up to a change of variable, Problem (1.5) (from Theorem 4.1) and Problem (1.11) is a relaxed version of this problem. The Lipschitz continuity of m is deduced from the regularity of φ, derived in Section 3, and Lemma 2.4.

2 Variational problem

Definition 2.1. *A pair (α, m) satisfies (1.1) in the weak sense if $t \in [0,T] \mapsto m(t,\cdot) \in \mathcal{P}(\mathbb{R}\times I)$ is continuous, for any $i,j \in I$ with $i \neq j$ it holds $\alpha_{i,j} \in L^2_{m_i}([0,T]\times\mathbb{R})$ and for any test function $\phi \in C_c^\infty([0,T]\times\mathbb{R}\times I)$ we have:*

$$\begin{aligned}&\sum_{i\in I}\int_{\mathbb{R}} \phi_i(T,s)m_i(T,ds) - \phi_i(0,s)m_i^0(ds)\\ &= \int_0^T \int_{\mathbb{R}} \sum_{i\in I}(\partial_t\phi_i(t,s) + b_i(s)\partial_s\phi_i(t,s))m_i(t,ds) + \sum_{j\in I, j\neq i}(\phi_j(t,s)-\phi_i(t,s))\alpha_{i,j}(t,s)m_i(t,ds)dt,\end{aligned}$$

Remark 2.1. *Recalling Assumptions 1 and 2, Lemma A.3 in Appendix A.2 states that for any weak solution (α, m) of (1.1), in the sense of Definition 2.1, the measure $m_i(t,\cdot)$ has its support included in $[0,1]$ for any $(t,i) \in [0,T]\times I$. Thus, we will consider throughout the paper, only weak solutions (α, m) of (1.1) where for any $t \in [0,T]$ we have $m(t,\cdot) \in \mathcal{P}([0,1]\times I)$.*

Problem (1.5) being not convex w.r.t. the variables (m,α), we make a change of variables $E := \alpha m$. We now rewrite the continuity equation (1.1) for any $i \in I$:

$$\begin{aligned}&\partial_t m_i(t,s) + \partial_s(m_i(t,s)b_i(s)) = -\sum_{i,j\in I, j\neq i}(E_{i,j}(t,s) - E_{j,i}(t,s)) \quad (i,t,s) \in I\times(0,T)\times(0,1)\\ &m_i(0,s) = m_i^0(s) \qquad\qquad (i,s) \in I\times[0,1],\end{aligned} \tag{2.1}$$

where $E_{i,j} \in \mathcal{M}^+([0,T]\times[0,1])$, with a first marginal equals to the Lebesgue measure on $[0,T]$ and such that $E_{i,j}(t,\cdot) \ll m_i(t,\cdot)$ with $\dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i} = \alpha_{i,j}$ and $\dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i} \in L^2_{m_i}([0,T]\times\mathbb{R})$. For any initial distribution, absolutely continuous w.r.t. the Lebesgue measure, with density satisfying $m^0 \in C^1([0,1]\times I)$ and any $D \in C^0([0,T]\times I)$, we introduce the set:

$$\begin{aligned}CE(m^0,D) := \Big\{&(m,E) \text{ such that } (m,\alpha) \text{ satisfies (2.1) in the weak sense, where } \alpha_{i,j} := \frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i},\\ &\text{with additional constraints: } \int_0^1 m_i(t,ds) \le D_i(t) \quad \forall (i,t)\in I\times[0,T], \text{ and } \frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i} \ge 0\Big\}.\end{aligned} \tag{2.2}$$

The function ρ denotes, throughout the paper, the function such that $(\rho, 0)$ is the weak solution of (1.1). One can easily show that for any $i \in I$ it holds $\rho_i \in C^1([0,T]\times[0,1])$ and for any $t \in [0,T]$: $\displaystyle\int_0^1 \rho_i(t,s)ds = \int_0^1 m_i^0(s)ds < D_i(t)$. Then, it follows that $(\rho,0) \in CE(m^0,D)$. We define the function $\tilde{B}$ for any $(m,E) \in CE(m^0,D)$ by:

$$\tilde{B}(E,m) := \sum_{i\in I}\int_0^T\int_0^1 c_i(t,s)m_i(t,ds) + \sum_{nI, j\neq i} L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)m_i(t,ds)dt + \sum_{i\in I}\int_0^1 g_i(s)m_i(T,ds), \tag{2.3}$$

where the function L is defined in (1.4). The following optimization problem is considered:

$$\inf_{(m,E)\in CE(m^0,D)} \tilde{B}(E,m) \tag{2.4}$$

From Assumption 4, we deduce that the quantity $\tilde{B}(E,m)$ is finite for any $(m,E) \in CE(m^0,D)$. For any $\gamma > 0$, we denote by $CE_\gamma(m^0,D)$ the subset of $CE(m^0,D)$ whose elements (m,E) satisfy:

$$\sum_{(i,j)\in I, i\neq j}\int_0^T\int_0^1 L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)m_i(t,ds)dt = \sum_{(i,j)\in I, i\neq j}\int_0^T\int_0^1 \frac{1}{2}\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)^2 m_i(t,ds)dt \le \gamma. \tag{2.5}$$

For any $(m,E) \in CE_\gamma(m^0,D)$, the next Lemma provides a Hölder regularity property on m.

Lemma 2.1. *For any $\gamma > 0$, there exists a positive constant C_γ such that, for any $(m,E) \in CE_\gamma(m^0,D)$, m is $\dfrac{1}{2}$-Hölder continuous of constant C_γ.*

Proof. Let $\varphi : \mathbb{R} \times I \to \mathbb{R}$ be globally $1-$Lipschitz continuous and C^1 w.r.t. the first variable. We show that the function: $t \to \frac{d}{dt} \int_{\mathbb{R}} \sum_{i=0}^{d} \varphi_i(s) m_i(t, ds)$ is uniformly bounded on $[0, T]$:

$$\frac{d}{dt} \int_{\mathbb{R}} \sum_{i=1}^{d} \varphi_i(s) m_i(t, ds) = \int_{\mathbb{R}} \sum_{i \in I} b_i(s) \partial_s \varphi_i(s) m_i(t, ds) + \int_{\mathbb{R}} \sum_{i,j \in I, j \neq i} (\varphi_j(s) - \varphi_i(s)) \alpha_{i,j}(t, s) m_i(t, ds),$$

where we used Assumption 1 and that (m, E) is a weak solution of (2.1) with $\alpha_{i,j} := \frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}$. Since b_i is bounded by $\|b\|_\infty := \max_i(\|b_i\|_\infty)$ and $\|\partial_s \varphi_i\|$ is bounded by 1 for any $i \in I$, Lemma A.3 in the Appendix A implies that:

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}} \sum_{i \in I} \varphi_i(s) m_i(t, ds) \right| &\leq |I| \|b\|_\infty + \int_0^1 \sum_{i,j \in I, j \neq i} |\varphi_j(s) - \varphi_i(s)| \alpha_{i,j}(t, s) m_i(t, ds) \\ &\leq |I| \|b\|_\infty + \int_0^1 \sum_{i,j \in I, j \neq i} \alpha_{i,j}(t, s) m_i(t, ds), \end{aligned}$$

where we used that $s \mapsto (\varphi_j(s) - \varphi_i(s))$ is bounded by 1 for any pair (i, j). Let $t, \tilde{t} \in (0, T)$, with $t > \tilde{t}$, using Cauchy-Schwarz inequality, it holds:

$$\begin{aligned} &\left| \int_{\mathbb{R}} \sum_{i=1}^{d} \varphi_i(s) m_i(t, ds) - \int_{\mathbb{R}} \sum_{i=1}^{d} \varphi_i(s) m_i(\tilde{t}, ds) \right| \\ &\leq (t - \tilde{t}) |I| \|b\|_\infty + \left(\int_{\tilde{t}}^t \int_0^1 \sum_{i,j \in I, j \neq i} m_i(\tau, ds) d\tau \right)^{\frac{1}{2}} \left(\int_{\tilde{t}}^t \int_0^1 \sum_{i,j \in I, j \neq i} (\alpha_{i,j}(\tau, s))^2 m_i(\tau, ds) d\tau \right)^{\frac{1}{2}} \\ &\leq |t - \tilde{t}|^{\frac{1}{2}} \left(T^{\frac{1}{2}} |I| \|b\|_\infty + \sqrt{2} \gamma^{\frac{1}{2}} \right). \end{aligned} \tag{2.6}$$

From (2.6) it holds:

$$W_1(m(t, \cdot), m(\tilde{t}, \cdot)) \leq |t - \tilde{t}|^{\frac{1}{2}} \left(T^{\frac{1}{2}} |I| \|b\|_\infty + \sqrt{2} \gamma^{\frac{1}{2}} \right).$$

□

The next Lemma is usefull to show that any minimizing sequence of (2.4) is relatively compact.

Lemma 2.2. *For any $\gamma > 0$, the subset $CE_\gamma(m^0, D)$ is relatively compact.*

Proof. For any $(m, E) \in CE_\gamma(m^0, D)$, using Lemma A.3, it holds that m is tight. In addition for any $i, j \in I$ with $i \neq j$, using that $E_{i,j}$ is a positive measure and Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \int_0^T \int_0^1 E_{i,j}(t, ds) dt = \int_0^T \int_0^1 \frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t, s) m_i(t, ds) dt &\leq \left(\int_0^T \int_0^1 \left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t, s) \right)^2 m_i(t, ds) dt \right)^{\frac{1}{2}} \left(\int_0^T \int_0^1 m_i(t, ds) dt \right)^{\frac{1}{2}} \\ &\leq (2\gamma T)^{\frac{1}{2}}, \end{aligned}$$

thus the mass of $E_{i,j}$ is bounded on $[0, 1] \times I$ by $(\gamma T)^{\frac{1}{2}}$. Since $E_{i,j} \ll m_i$, it holds that E is also tight. Thus, for any sequence $\{(m^n, E^n)\}_n$ in $CE_\gamma(m^0, D)$, there exists a subsequence $\{(m^{\theta_n}, E^{\theta_n})\}_n$ converging weakly to $(\tilde{m}, \tilde{E})$. Using Lemma 2.1, $\{m^{\theta_n}\}_n$ converges uniformly on $[0, T]$ to a $\tilde{m}$ and it holds $\tilde{m} \in C([0, T], \mathcal{P}(\mathbb{R} \times I))$.

We want to show that $\tilde{E}$ is absolutely continuous w.r.t. $\tilde{m}$. We define the functional Θ by:

$$\Theta(m, E) : \begin{cases} \displaystyle\int_0^T \int_0^1 \sum_{i,j \in I, i \neq j} L(\alpha_{i,j}(t, s)) m_i(t, ds) dt & \text{if } \forall i, j,\ E_{i,j} \ll m_i \text{ and } \alpha_{i,j} := \frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i} \text{ with } \alpha_{i,j} \geq 0, \\ +\infty & \text{otherwise} \end{cases}$$

The functional Θ being w.l.s.c. [41, Proposition 5.18], and $\Theta(m^n, E^n)$ being bounded by γ for any n, we deduce that $\Theta(\tilde{m}, \tilde{E}) \leq \gamma$ and $\tilde{E} \ll \tilde{m}$. Finally, using the definition of weak convergence, it is easy to check that $(\tilde{m}, \tilde{E})$ is a weak solution of (2.1) and $\tilde{m}$ satisfies for any $(i, t) \in I \times [0, T]$: $\displaystyle\int_0^1 \tilde{m}_i(t, ds) \leq D_i(t)$. □

Lemma 2.3. *Problem* (2.4) *admits a solution.*

Proof. We consider $(\bar{m}, \bar{E}) \in CE(m^0, D)$ and we define γ by:

$$\gamma := \tilde{B}(\bar{m}, \bar{E}) + |I|(T\|c\|_\infty + \|g\|_\infty) + 1,$$

where $\|c\|_\infty := \max_{i\in I}(\|c_i\|_\infty)$, $\|g\|_\infty := \max_{i\in I}(\|g_i\|_\infty)$. For any pair $(m, E) \in CE(m^0, D)$, if $\tilde{B}(m, E) \leq \tilde{B}(\bar{m}, \bar{E})$, then:

$$\sum_{(i,j)\in I, i\neq j} \int_0^T \int_0^1 \frac{1}{2}\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)^2 m_i(t,ds)dt \leq \gamma.$$

We deduce that $(\bar{m}, \bar{E}) \in CE_\gamma(m^0, D)$. Taking a minimizing sequence $\{(m^n, E^n)\}_n$ of Problem (2.4), there exists $\tilde{n} \in \mathbb{N}$ such that $(m^n, E^n) \in CE_\gamma(m^0, D)$ for all $n \geq \tilde{n}$. From Lemma 2.2, a subsequence of $\{(m^n, E^n)\}_n$ weakly converges to a certain $(m^*, E^*) \in CE_\gamma(m^0, D)$. Since $\tilde{B}$ is weakly lower semi continuous on $CE(m^0, D)$, (m^*, E^*) minimizes $\tilde{B}$. □

We end up this section giving the following lemma, which provides Lipschitz continuity results on m. This Lemma will be useful in section 5.

Lemma 2.4. *Suppose $\alpha \in L^\infty((0,T)\times I \times I, \mathrm{Lip}([0,1]))$, then there exists a unique solution m of* (1.1) *associated to α. In addition we have $m \in \mathrm{Lip}([0,T]\times[0,1]\times I)$.*

To prove Lemma 2.4, we need to rewrite the equation (1.1) in $\mathbb{R}^{|I|}$, in the following form:

$$\partial_t m(t,s) + b(s)\partial_s m(t,s) = G(t,s)m(t,s), \tag{2.7}$$

where $m(t,s) := (m_0(t,s), \ldots, m_{|I|}(t,s))$, $b(s) := \mathrm{diag}(b_0(s), \ldots, b_{|I|}(s))$ and $G(t,s)$ is a square matrix of size $|I|$, such that the i^{th} coordinate of the vector $G(t,s)m(t,s)$ is equal to:

$$(G(t,s)m(t,s))_i := -m_i(t,s)\partial_s b_i(s) - \sum_{j\neq i}(\alpha_{i,j}(t,s)m_i(t,s) - \alpha_{j,i}(t,s)m_j(t,s)), \tag{2.8}$$

with the initial constraint: $m(0,\cdot) = m^0(\cdot)$ on $[0,1]$. Note that if $\alpha \in L^\infty((0,T)\times I\times I, \mathrm{Lip}([0,1]))$, then $G \in L^\infty((0,T)\times I\times I, \mathrm{Lip}([0,1]))$.

Proof of Lemma 2.4. This Lemma is a direct application of Proposition 1 in Appendix A. □

3 Analysis of the HJB solutions

This section is devoted to the analysis of the equation:

$$\begin{aligned} &-\partial_t\varphi_i(t,s) - b_i(s)\partial_s\varphi_i(t,s) - c_i(t,s) - \lambda_i(t) + \sum_{j\in I, j\neq i} H((\varphi_j - \varphi_i)(t,s)) = 0 \quad \text{on } (0,T)\times(0,1)\times I, \\ &\varphi_i(T,\cdot) = g_i \qquad\qquad \text{on } (0,1)\times I, \end{aligned} \tag{3.1}$$

where $\lambda \in \mathcal{M}^+([0,T]\times I)$ is given and $H(y) := \frac{1}{2}(y^-)^2$ for any $y \in \mathbb{R}$. We introduce the notion of weak solution for this equation.

Definition 3.1. *For a given $\lambda \in \mathcal{M}^+([0,T]\times I)$, φ is called a weak solution of equation* (3.1) *if for any $i \in I$ $\varphi_i \in BV((0,T)\times(0,1))$ and for any test function $\psi \in C^1((0,T)\times(0,1)\times I)$, it satisfies:*

$$\begin{aligned} &\int_0^1 \varphi_i(0,s)\psi_i(0,s)ds - \int_0^1 g_i(s)\psi_i(T,s)ds + \int_0^T\int_0^1 \left(\partial_t\psi_i(t,s) + \partial_s(\psi_i(t,s)b_i(s))\right)\varphi_i(t,s)dsdt \\ &+ \int_0^T\int_0^1 \left(\sum_{j\in I, j\neq i} H((\varphi_j - \varphi_i)(t,s)) - c_i(t,s)\right)\psi_i(t,s)dtds - \int_0^T\int_0^1 \psi_i(t,s)\lambda_i(dt)ds \\ &= 0, \end{aligned} \tag{3.2}$$

$\phi_i(0,\cdot)$ is understood in the sense of trace.

Remark 3.1. *Observe that there is no boundary condition in* (3.2)*. This is due to the Assumption 1, involving a null incoming flow in the domain* $[0,1]$*.*

In order to analyze (3.1) we now introduce several notations. Let $\delta > 0$ (to be chosen below) and assume that:

$$\lambda([0,T] \times I) < \delta. \tag{3.3}$$

Let M^δ and $\bar{M}^\delta$ be such that:

$$\begin{array}{l} M^\delta := \max\limits_{i\in I} \|g_i\|_\infty + T(\max\limits_{i\in I} \|c_i\|_\infty + |I|H(\delta)) + 2. \\ \bar{M}^\delta := M^\delta + \delta. \end{array} \tag{3.4}$$

From Assumptions 1 and 4, there exists a positive constant K such that for any $i \in I$ and $t \in [0,T]$, functions g_i, $c_i(t,\cdot)$ and b_i are Lipschitz continuous on $[0,1]$ with Lipschitz constant K. Using the definition of H, introduced in equation (3.1), there exists a positive constant K^δ such that H and H' are Lipschitz continuous on $[-2\bar{M}^\delta, 2\bar{M}^\delta]$ with Lipschitz constant K^δ. For any $i \in I$, let $S_i^{t,s} : [0,T] \to \mathbb{R}$ be the maximal solution of the ODE (1.9) with condition: $S_i^{t,s}(t) = s$. The function b_i being C^1, the map:

$$S_i : (\tau, t, s) \to S_i^{t,s}(\tau)$$

is C^1 on $[0,T] \times [0,T] \times [0,1]$. The flow $S_i^{t,\cdot}(\tau)$ is a diffeomorphism on $[0,1]$ whose inverse function is $S_i^{\tau,\cdot}(t)$ with derivative w.r.t. space variable $\partial_s S_i^{\tau,\cdot}(t)$. For any $i \in I$ $\partial_s S_i$ being continuous on $[0,T] \times [0,T] \times [0,1]$ we define: $\|\partial_s S\|_\infty := \max\limits_{i\in I}(\|\partial_s S_i\|_\infty)$. Let $k^\delta \in \mathbb{R}^+$ be such that:

$$k^\delta := K + l^\delta \text{ where } l^\delta := 4(|I| - 1)K^\delta + 1, \tag{3.5}$$

We consider the space $L^1((0,T) \times I, C^1([0,1]))$ endowed with the norm $\|\cdot\|_1^\delta$, which is defined for every $v \in L^1((0,T) \times I, C^1(0,1))$ by :

$$\|v\|_1^\delta := \sum_{i\in I} \int_0^T \|v_i(t,\cdot)\|_{C^1} e^{-\kappa^\delta (T-t)} dt, \tag{3.6}$$

where $\|v_i(t,\cdot)\|_{C^1} := \|v_i(t,\cdot)\|_\infty + \|\partial_s v_i(t,\cdot)\|_\infty$ and the constant κ is defined by:

$$\kappa^\delta := |I|^2 K^\delta (\|\partial_s S\|_\infty + 1) + 1. \tag{3.7}$$

The space $(L^1((0,T) \times I, C^1(0,1)), \|\cdot\|_1^\delta)$ is a Banach space. The constants defined in (3.5) and (3.7) are determined to build a contracting map in a subspace of $(L^1((0,T) \times I, C^1(0,1)), \|\cdot\|_1^\delta)$.

In this section we are looking for a solution of (3.1) in an integral form, i.e. a function φ, defined on $[0,T] \times [0,1] \times I$, satisfying:

$$\begin{array}{ll} \varphi_i(t,s) = \displaystyle\int_t^T \sum_{j\in I, j\neq i} -H((\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau))d\tau + \int_t^T \lambda_i(d\tau) + g_i(S_i^{t,s}(T)) & \text{on } (0,T) \times [0,1] \times I \\ \varphi_i(T,s) = g_i(s) & \text{on } [0,1] \times I \end{array} \tag{3.8}$$

One can observe that φ is a solution of (3.8) if and only if the function ν, defined by:

$$\nu_i(t,s) := \varphi_i(t,s) - \int_t^T \lambda_i(d\tau)$$

is a solution of:

$$\begin{array}{ll} \nu_i(t,s) = \displaystyle\int_t^T \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\nu) + c_i(\tau, S_i^{t,s}(\tau))d\tau + g_i(S_i^{t,s}(T)) & \text{a.e. on } (0,T) \times [0,1] \times I \\ \nu_i(T,s) = g_i(s) & \text{a.e. on } [0,1] \times I \end{array} \tag{3.9}$$

where H^λ is defined on $I \times I \times [0,T] \times [0,T] \times [0,1] \times L^1((0,T) \times (0,1) \times I)$ by:

$$H^\lambda(i,j,t,\tau,s,\nu) := H\left((\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_j - \lambda_i)(dr)\right).$$

Our aim is to build a solution to (3.9) (and thus to (3.1)) by a fixed point argument. Let for any $i \in I$, Γ_i^λ be the map defined for any $\varphi \in L^1((0,T)\times I, C^1([0,1]))$ by:

$$\Gamma_i^\lambda(\varphi)(t,s) := \int_t^T \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\varphi) + c_i(\tau, S_i^{t,s}(\tau))d\tau + g_i(S_i^{t,s}(T)). \tag{3.10}$$

Let Σ^λ be the set of functions $f : [0,T]\times[0,1]\times I \to \mathbb{R}$ such that for any $i \in I$, $(t,s) \mapsto f_i(t,s)$ is measurable on $[0,T]\times[0,1]$ and for a.e. $t \in (0,T)$: $f_i(t,\cdot) \in C^1([0,1])$, $\|f_i(t,\cdot)\|_\infty \leq M^\delta$ and $\|\partial_s f_i(t,\cdot)\|_\infty \leq 2Ke^{k(T-t)}$. We want to apply a fixed point theorem in the space Σ^λ. To do so, we need to define a function on Σ^λ with values in Σ^λ. For a given $v \in \Sigma^\lambda$, the value $\|\Gamma_i^\lambda(v)\|_\infty$ may be larger than M^δ. Thus, we introduce the smooth truncation $F_\delta \in C^1(\mathbb{R}, [-M^\delta + 1/2,\, M^\delta - 1/2])$, satisfying $F_\delta' \geq 0$, $|F_\delta'(x)| \leq 1$ for any $x \in \mathbb{R}$ and:

$$F_\delta(x) : \begin{cases} -M^\delta + \dfrac{1}{2} & \text{if } x < -M^\delta, \\ x & \text{if } -(M^\delta - 1) \leq x \leq M^\delta - 1, \\ M^\delta - \dfrac{1}{2} & \text{if } M^\delta \leq x. \end{cases} \tag{3.11}$$

Finally we define the function Π^λ by:

$$\forall \varphi \in \Sigma^\lambda,\ \Pi^\lambda(\varphi) := (\Pi_1^\lambda(\varphi), \ldots, \Pi_{|I|}^\lambda(\varphi)) \text{ where } \Pi_i^\lambda(\varphi) := (F_M \circ \Gamma_i^\lambda)(\varphi) \quad \forall i \in I$$

Remark 3.2. *The set Σ^λ is bounded and closed w.r.t. the topology induced by the norm $\|\cdot\|_1^\delta$, defined in* (3.6).

3.1 Existence of a fixed point of Π^λ on Σ^λ

The following lemma states that Π^λ maps Σ^λ to itself.

Lemma 3.1. *For any $\varphi \in \Sigma^\lambda$, it holds $\Pi^\lambda(\varphi) \in \Sigma^\lambda$.*

Proof. For any $i \in I$, let φ_i be a function in Σ^λ and ψ_i be such that: $\psi_i := \Gamma_i^\lambda(\varphi)$. From equation (3.10), it holds that for any $i \in I$, $(t,s) \mapsto \psi_i(t,s)$ is measurable on $[0,T]\times[0,1]$. We need to show that for all $i \in I$ and a.e. $t \in [0,T]$, the function $s \mapsto \psi_i(t,s)$ is in $C^1([0,1])$ and that $\|\partial_s\psi_i(t,s)\|_\infty$ is bounded by $2Ke^{k(T-t)}$. From Assumption 4, it is clear that $\psi_i(T,\cdot) \in C^1([0,1])$ for any $i \in I$ and that $\|\partial_s\psi_i(T,\cdot)\|_\infty \leq 2K$. For any $(i,s,t) \in I\times[0,1]\times(0,T)$ and a.e $\tau \in [0,T]$, using the chain rule it holds:

$$\partial_s H^\lambda(i,j,t,\tau,s,\varphi) = \partial_s S_i^{t,s}(\tau)\partial_s(\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau))H'\left((\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_j - \lambda_i)(dr)\right). \tag{3.12}$$

Since H' is bounded by K^δ on $[-2\bar{M}^\delta,\, 2\bar{M}^\delta]$, it comes:

$$|\partial_s H^\lambda(i,j,t,\tau,s,\varphi)| \leq 4\|\partial_s S\|_\infty K K^\delta e^{k(T-\tau)}.$$

Therefore for any $t \in [0,T]$, the function: $s \mapsto \displaystyle\int_t^T -\sum_{j\in I, j\neq i} H^\lambda(i,j,t,\tau,s,\varphi)d\tau$ is differentiable on $[0,1]$ and ψ_i satisfies:

$$\begin{aligned}\partial_s\psi_i(t,s) = &\int_t^T -\sum_{j\in I, j\neq i} \partial_s S_i^{t,s}(\tau)\partial_s(\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau))H'\left((\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_j - \lambda_i)(dr)\right)d\tau \\ &+ \int_\tau^T \partial_s S_i^{t,s}(\tau)\partial_s c_i(\tau, S_i^{t,s}(\tau))d\tau + \partial_s S_i^{t,s}(T) g_i'(S_i^{t,s}(T)).\end{aligned} \tag{3.13}$$

From equality (3.12), it holds that for any $s \in [0,1]$ and any $t \in [0,T]$, the function $\tau \mapsto \partial_s H^\lambda(i,j,t,\tau,s,\varphi)$ is measurable on $[0,T]$. In addition for a.e. $\tau, t \in [0,T]$ the function $s \mapsto \partial_s H^\lambda(i,j,t,\tau,s,\varphi)$ is continuous on $[0,1]$. Therefore it comes for a.e. $t \in [0,T]$ that $\psi_i(t,\cdot) \in C^1([0,1])$. We need now to show that for a.e. $t \in [0,T]$, it holds $\|\partial_s\psi(t,\cdot)\|_\infty \leq 2Ke^{k(T-t)}$. From Lemma A.2 and the Lipschitz continuity of b_i, it holds for any $t \in [0,T]$:

$\|\partial_s S_i^{t,s}(\tau)\|_\infty \leq e^{K(\tau-t)}$. Using the Lipschitz continuity of c_i and g_i, the bound of H' and the bound on $\|\partial_s\varphi(t,\cdot)\|_\infty$, from equation (3.13) it holds for a.e. $t \in [0,T]$ and any $s \in [0,1]$:

$$\|\partial_s\psi_i(t,\cdot)\|_\infty \leq \int_t^T \left(4(|I|-1)KK^\delta e^{K(\tau-t)}e^{k^\delta(T-\tau)} + Ke^{K(\tau-t)}\right)d\tau + Ke^{K(T-t)} \tag{3.14}$$

Using that $k^\delta = K + l^\delta$ and that $\displaystyle\int_t^T e^{K(\tau-t)}d\tau \leq \frac{K}{l^\delta}e^{k^\delta(T-t)}$, inequality(3.14) becomes:

$$\|\partial_s\psi_i(t,\cdot)\|_\infty \leq K\left(\frac{4(|I|-1)K^\delta+1}{l^\delta}+1\right)e^{k^\delta(T-t)}.$$

Using the definition of l at equation (3.5), it comes for a.e. $t \in [0,T]$: $\|\partial_s\psi_i(t,\cdot)\|_\infty \leq 2Ke^{k^\delta(T-t)}$. From the definition of F_δ in equation (3.11), it is clear that $F_\delta(\psi_i)$ is bounded by M^δ. Finally the composition by F_δ preserves the continuity and differentiability. Since $|F_\delta'| \leq 1$ and $\|\partial_s\psi_i(t,\cdot)\|_\infty \leq 2Ke^{k^\delta(T-t)}$ a.e. on $[0,T]$, it comes that $\|\partial_s F_\delta(\psi_i(t,\cdot))\|_\infty \leq 2Ke^{k^\delta(T-t)}$ a.e. on $[0,T]$. □

Lemma 3.2. *The map Π^λ is a contraction on Σ^λ.*

Proof. Let $\varphi, \theta \in \Sigma^\lambda$, using that φ and θ are bounded by M^δ, it holds for any $(t,s,i) \in (0,T)\times[0,1]\times I$:

$$\begin{aligned}|\Gamma_i^\lambda(\varphi)(t,s) - \Gamma_i^\lambda(\theta)(t,s)| &\leq \sum_{j\in I, j\neq i}\int_t^T |H^\lambda(i,j,t,\tau,s,\theta) - H^\lambda(i,j,t,\tau,s,\varphi)|\,d\tau \\ &\leq |I|K^\delta\sum_{j\in I}\int_t^T |\theta_j(\tau,S_i^{t,s}(\tau)) - \varphi_j(\tau,S_i^{t,s}(\tau))|\,d\tau.\end{aligned}$$

Then it holds:

$$\begin{aligned}\int_0^T\sum_{i\in I}\|\Gamma_i^\lambda(\varphi)(t,\cdot) - \Gamma_i^\lambda(\theta)(t,\cdot)\|_\infty e^{-\kappa^\delta(T-t)}dt &\leq |I|^2K^\delta\int_0^T\int_t^T\sum_{i\in I}\|\theta_i(\tau,\cdot)-\varphi_i(\tau,\cdot)\|_\infty e^{-\kappa^\delta(T-t)}\,d\tau dt \\ &\leq \frac{|I|^2K^\delta}{\kappa^\delta}\int_0^T\sum_{i\in I}\|\theta_i(\tau,\cdot)-\varphi_i(\tau,\cdot)\|_\infty e^{-\kappa^\delta(T-\tau)}\,d\tau.\end{aligned} \tag{3.15}$$

Now consider: for any $(s,i) \in [0,1]\times I$ and a.e. $(t,\tau) \in (0,T)\times(0,T)$:

$$|\partial_s(\Gamma_i^\lambda(\varphi)(t,s) - \Gamma_i^\lambda(\theta)(t,s))| \leq \sum_{j\in I, j\neq i}\int_t^T |\partial_s H^\lambda(i,j,t,\tau,s,\theta) - \partial_s H^\lambda(i,j,t,\tau,s,\varphi)|\,d\tau \tag{3.16}$$

Using (3.12) and that H' is bounded by K^δ on $[-2\bar{M}^\delta, 2\bar{M}^\delta]$, it comes:

$$|\partial_s H^\lambda(i,j,t,\tau,s,\theta) - \partial_s H^\lambda(i,j,t,\tau,s,\varphi)| \leq \|\partial_s S\|_\infty K^\delta\|\partial_s(\varphi_i-\varphi_j-\theta_i+\theta_j)(\tau,\cdot)\|_\infty \tag{3.17}$$

From inequalities (3.16) and (3.17), it holds:

$$|\partial_s(\Gamma_i^\lambda(\varphi)(t,s) - \Gamma_i^\lambda(\theta)(t,s))| \leq |I|\sum_{j\in I}\int_t^T \|\partial_s S\|_\infty K^\delta\|\partial_s(\varphi_j-\theta_j)(\tau,\cdot)\|_\infty\,d\tau. \tag{3.18}$$

Integrating over $[0,T]$ inequality (3.18), one has:

$$\begin{aligned}&\int_0^T\sum_{i\in I}\|\partial_s(\Gamma_i^\lambda(\varphi)-\Gamma_i^\lambda(\theta))(t,\cdot)\|_\infty e^{-\kappa^\delta(T-t)}dt \\ &\leq |I|^2\|\partial_s S\|_\infty K^\delta\int_0^T\int_t^T\sum_{i\in I}\|\partial_s(\varphi_i-\theta_i)(\tau,\cdot)\|_\infty e^{-\kappa^\delta(T-t)}d\tau dt \\ &\leq \frac{|I|^2}{\kappa^\delta}\|\partial_s S\|_\infty K^\delta\int_0^T\sum_{i\in I}\|\partial_s(\varphi_i-\theta_i)(\tau,\cdot)\|_\infty e^{-\kappa^\delta(T-\tau)}d\tau\end{aligned} \tag{3.19}$$

From equations (3.15) and (3.19) and the definition of the norm $\|\cdot\|_1^\delta$ in (3.6), we deduce:

$$\|\Gamma_i^\lambda(\varphi) - \Gamma_i^\lambda(\theta)\|_1^\delta \leq \frac{|I|^2 K^\delta(\|\partial_s S\|_\infty + 1)}{\kappa^\delta}\|\varphi - \theta\|_1^\delta$$

Using the definition of κ at (3.7), it follows that Γ is a contraction. The function F_M being also non expensive, the conclusion follows. □

Lemma 3.3. *The function Π^λ admits a fixed point $\nu^\lambda \in \Sigma^\lambda$.*

Proof. This is a direct consequence of Lemma 3.2. □

For any $\lambda \in \mathcal{M}^+([0,T]\times I)$ satisfying inequality (3.3), the subset E_λ of $(0,T)$ denotes the set of points where for any i, the function $\lambda \mapsto \int_t^T \lambda_i(d\tau)$ is differentiable. One has $[0,T]\setminus E_\lambda$ is negligible w.r.t. the Lebesgue measure. The next lemma provides usefull regularity properties of the fixed point ν^λ.

Lemma 3.4. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$ the associated fixed point ν^λ of Π^λ is Lipschitz continuous w.r.t. the time variable, differentiable at any $(t,s,i) \in E_\lambda \times [0,1]\times I$, and for any $(t,i) \in E_\lambda \times I$, $s \mapsto \partial_s \nu_i(t,s)$ is continuous on $[0,1]$. In addition, if $\lambda \in C^0([0,T]\times I, \mathbb{R}^+)$ then it holds $\nu^\lambda \in C^1((0,T)\times[0,1]\times I)$.*

Proof. From the definition of Γ^λ in equation (3.10), the continuity of g and of the flow S, it holds that for any $(s,i) \in [0,1]\times I$, the function $\Gamma_i^\lambda(\nu^\lambda)(\cdot,s)$ is continuous on $[0,T]$. For any $(s,i)\in[0,1]\times I$, the function $t \mapsto g_i(S_i^{t,s}(T))$ being Lipschitz on $[0,T]$ and the function $\tau \mapsto \sum_{j\in I, j\neq i} -H^\lambda(i,j,t,\tau,s,\nu^\lambda) + c_i(\tau, S_i^{t,s}(\tau))$ bounded by $\sup_{x\in[-2\bar{M}^\delta, 2\bar{M}^\delta]} |H(x)| + \|c_i\|_\infty$ on $[0,T]$, the function $\Gamma_i^\lambda(\nu^\lambda)(\cdot,s)$ is Lipschitz continuous on $[0,T]$. Since F_δ is a non expensive map, by composition the function $\nu_i^\lambda(\cdot,s)$ is Lipschitz continuous for any $(i,s)\in I\times[0,1]$ on $[0,T]$. For any $(i,s,t)\in I\times[0,1]\times(0,T)$ and $\tau \in E_\lambda$, using the chain rule we have:

$$\partial_t H^\lambda(i,j,t,\tau,s,\nu^\lambda) = \partial_t(S_i^{t,s}(\tau))\partial_s(\nu_j^\lambda - \nu_i^\lambda)(\tau, S_i^{t,s}(\tau))H'\left((\nu_j^\lambda - \nu_i^\lambda)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_j - \lambda_i)(dr)\right). \quad (3.20)$$

Thanks to the Lemma A.1 in Appendix, we have the bound: $|\partial_t S_i^{t,s}(\tau)| \leq \|b_i\|_\infty \|\partial_s S\|_\infty$. Thus, one deduces that: $|\partial_t H^\lambda(i,j,t,\tau,s,\nu^\lambda)| \leq 4\|b_i\|_\infty\|\partial_s S\|_\infty K K^\delta e^{k(T-t)}$. Therefore for any $s\in[0,1]$, the function

$$t \mapsto \int_t^T - \sum_{j\in I, j\neq i} H^\lambda(i,j,t,\tau,s,\nu^\lambda)d\tau$$

is differentiable on E_λ. It has been shown in the proof of Lemma 3.1, that $|\partial_s H^\lambda(i,j,t,\tau,s,\varphi)| \leq 4\|\partial_s S\|_\infty K K^\delta e^{k^\delta(T-\tau)}$, therefore for any $t\in E_\lambda$, the function $s \mapsto \int_t^T - \sum_{j\in I, j\neq i} H^\lambda(i,j,t,\tau,s,\nu^\lambda)d\tau$ is differentiable on $[0,1]$. From Assumption 4, it holds for any $i\in I$ that $c_i \in C^1((0,T)\times[0,1])$ and $g_i \in C^1([0,1])$. Then, for any $s\in[0,1]$ the function $t\mapsto \Gamma_i(\nu^\lambda)(t,s)$ is differentiable on E_λ. Since the function F_δ belongs to C^1, ν^λ is also differentiable on $E_\lambda\times[0,1]\times I$. Now suppose $\lambda \in C^0([0,T]\times I, \mathbb{R}^+)$. It comes $E_\lambda = [0,T]$ and the conclusion follows. □

Lemma 3.5. *Let $\lambda \in \mathcal{M}^+([0,T]\times I)$ satisfies inequality (3.3). Let $t_0 \in [0,T)$ be such that for any $t\in[t_0,T]$ it holds for any $i\in I$ $\|\nu_i^\lambda(t,\cdot)\|_\infty \leq M^\delta - 1$. Then for any $(t,s,i)\in(E_\lambda\cap[t_0,T])\times[0,1]\times I$, we have:*

$$-\partial_t \nu_i^\lambda(t,s) - b_i(s)\partial_s\nu_i^\lambda(t,s) - c_i(t,s) + \sum_{j\in I, j\neq i} H^\lambda(i,j,t,t,s,\varphi) = 0 \quad (3.21)$$

Proof. From Lemma 3.4, for any $i\in I$ the function ν_i^λ is differentiable on $E_\lambda\times[0,1]$. Since ν^λ satisfies (3.9) on $(E_\lambda\cap(t_0,T))\times[0,1]\times I$, we have:

$$\begin{aligned}\partial_t\nu_i^\lambda(t,s) = \; & -\int_t^T \partial_t S_i^{t,s}(\tau) \sum_{j\in I, j\neq i} \partial_s(\nu_j^\lambda - \nu_i^\lambda)(\tau, S_i^{t,s}(\tau))H'\left((\nu_j^\lambda - \nu_i^\lambda)(\tau, S_i^{t,s}(\tau)) + \int_\tau^T (\lambda_j - \lambda_i)(dr)\right)d\tau \\ & + \int_t^T \partial_t S_i^{t,s}(\tau)\partial_s c_i(\tau, S_i^{t,s}(\tau))d\tau + \sum_{j\in I, j\neq i} H^\lambda(i,j,t,t,s,\nu^\lambda) - c_i(t,s) + \partial_t(S_i^{t,s}(T))g_i'(S_i^{t,s}(T)),\end{aligned} \quad (3.22)$$

and

$$b_i(s)\partial_s\nu_i^\lambda(t,s) = \int_t^T -b_i(s)\partial_s S_i^{t,s}(\tau)\sum_{j\in I,j\neq i}\partial_s(\nu_j^\lambda-\nu_i^\lambda)(\tau,S_i^{t,s}(\tau))H'\left((\nu_j^\lambda-\nu_i^\lambda)(\tau,S_i^{t,s}(\tau))+\int_\tau^T(\lambda_j-\lambda_i)(dr)\right)d\tau$$
$$+\int_t^T b_i(s)\partial_s S_i^{t,s}(\tau)\partial_s c_i(\tau,S_i^{t,s}(\tau))d\tau + b_i(s)\partial_s(S_i^{t,s}(T))g_i'(S_i^{t,s}(T)). \tag{3.23}$$

Adding (3.22) and (3.23) and using Lemma A.1, it holds on $(E_\lambda\cap[t_0,T])\times[0,1]$:

$$\partial_t\nu_i^\lambda(t,s)+b_i(s)\partial_s\nu_i^\lambda(t,s)=\sum_{j\in I,j\neq i}H^\lambda(i,j,t,t,s,\nu^\lambda)-c_i(t,s). \tag{3.24}$$

□

For any $\lambda\in\mathcal{M}^+([0,T]\times I)$, let ϕ^λ be defined on $I\times[0,T]\times[0,1]$ by:

$$\phi_i^\lambda(t,s):=\nu_i^\lambda(t,s)+\int_t^T\lambda_i(d\tau), \tag{3.25}$$

where ν^λ is the fixed point of Π^λ, whose existence is established in Lemma 3.3. We want to prove that ϕ^λ is a solution of (3.8). To obtain this result, it suffices to show that ϕ^λ is bounded independently of $\bar{M}^\delta-1/2$.

3.2 Comparison principle

Definition 3.2. *Let $\lambda\in\mathcal{M}^+([0,T]\times I)$ and $t_0\in[0,T)$. A function $\underline{u}\in L^1((0,T)\times I,C^1([0,1]))$ (resp. $\bar{u}\in L^1((0,T)\times I,C^1([0,1]))$), is a weak subsolution (resp. a weak supersolution) of (3.1) if the function $\underline{\nu}$ (resp. $\bar{\nu}$), defined on $(t_0,T]\times[0,1]\times I$ by:*

$$\underline{\nu}_i(t,s)=\underline{u}_i(t,s)-\int_t^T\lambda_i(d\tau) \tag{3.26}$$

(and resp. $\bar{\nu}_i(t,s)=\bar{u}_i(t,s)-\int_t^T\lambda_i(d\tau)$) is Lipschitz continuous in time, differentiable on $E_\lambda\times[0,1]\times I$, and satisfies for any $(t,s,i)\in(E_\lambda\cap(t_0,T))\times[0,1]\times I$:

$$-\partial_t\underline{\nu}_i(t,s)-b_i(s)\partial_s\underline{\nu}_i(t,s)\leq-\sum_{j\in I,j\neq i}H^\lambda(i,j,t,t,s,\underline{\nu})+c_i(t,s)$$

and for any $(s,i)\in(0,1)\times I$:

$$\underline{\nu}_i(T,s)\leq g_i(s),$$

and resp. $\bar{\nu}$ is Lipschitz continuous in time, differentiable on $E_\lambda\times(0,1)\times I$, and satisfies for any $(t,s,i)\in(E_\lambda\cap(t_0,T))\times[0,1]\times I$:

$$-\partial_t\bar{\nu}_i(t,s)-b_i(s)\partial_s\bar{\nu}_i(t,s)\geq-\sum_{j\in I,j\neq i}H^\lambda(i,j,t,t,s,\bar{\nu})+c_i(t,s)$$

and for any $(s,i)\in(0,1)\times I$:

$$\bar{\nu}_i(T,s)\geq g_i(s).$$

Lemma 3.6 (Comparison principle)**.** *Let $\underline{u}$ and $\bar{u}$ be respectively weak subsolution and supersolution of (3.1) on $(t_0,T)\times[0,1]\times I$. Then one has $\underline{u}_i\leq\bar{u}_i$ on $(t_0,T)\times[0,1]\times I$.*

Proof. Let γ be defined on $(t_0,T]$ by: $\gamma(t):=\sup_{j\in I,s\in[0,1]}(\underline{u}_j(t,s)-\bar{u}_j(t,s))$. From (3.26), it comes: $\gamma(t):=\sup_{j\in I,s\in[0,1]}(\underline{\nu}_j(t,s)-\bar{\nu}_j(t,s))$. For any $t\in(t_0,T)$, $\underline{\nu}(t,\cdot)$ and $\bar{\nu}(t,\cdot)$ are continuous on $[0,1]$ thus, γ is well defined. Since $\underline{\nu}$ and $\bar{\nu}$ are Lipschitz continuous in time, γ is also Lipschitz continuous and thus, differentiable a.e. on $[0,T]$. Using the envelop theorems [36, Theorem 1], γ is absolutely continuous on $(t_0,T]$ and for a.e. $t\in(t_0,T]$ there exists maximum point $(i(t),x(t))\in I\times[0,1]$ such that:

$$\gamma'(t)=\partial_t(\underline{\nu}_{i(t)}(t,x(t))-\bar{\nu}_{i(t)}(t,x(t))).$$

Since $\bar{\nu}_{i(t)}$ and $\underline{\nu}_{i(t)}$ are respectively weak supersolution and subsolution:

$$-\gamma'(t) - b_{i(t)}(x(t))\partial_s(\underline{\nu}_{i(t)}(t,x(t)) - \bar{\nu}_{i(t)}(t,x(t))) \leq \sum_{j\neq i(t)} H((\bar{u}_j - \bar{u}_{i(t)})(t,x(t))) - H((\underline{u}_j - \underline{u}_{i(t)})(t,x(t))).$$

From Assumption 1 and definition of $i(t)$ and $x(t)$, if $x(t) \in \{0,1\}$ then $b_{i(t)}(x(t)) = 0$, while if $x(t) \in (0,1)$, then $\partial_s((\underline{u}_{i(t)}(t,x(t)) - \bar{u}_i(t,x(t))) = 0$. It comes: $b_{i(t)}(x(t))\partial_s((\underline{u}_{i(t)}(t,x(t)) - \bar{u}_i(t,x(t))) = 0$. Thus:

$$-\gamma'(t) \leq \sum_{j\neq i(t)} H((\bar{u}_j - \bar{u}_{i(t)})(t,x(t))) - H((\underline{u}_j - \underline{u}_{i(t)})(t,x(t))).$$

The function H being convex and differentiable, it holds:

$$-\gamma'(t) \leq \sum_{j\neq i(t)} H'((\bar{u}_j - \bar{u}_{i(t)})(t,x(t)))(\bar{u}_j - \bar{u}_{i(t)} - (\underline{u}_j - \underline{u}_{i(t)}))(t,x(t))$$

Using that H is non increasing and that at time t it holds for any $j \neq i(t)$: $(\underline{u}_{i(t)} - \bar{u}_{i(t)})(t,x(t)) \geq (\underline{u}_j - \bar{u}_j)(t,x(t))$ it comes that γ is increasing over $(t_0,T]$. Since $\gamma(T) \leq 0$, the conclusion follows. □

Lemma 3.7. *Let $\lambda \in \mathcal{M}^+([0,T]\times I)$ verifying (3.3) and u be a function satisfying (3.8) a.e. on $(t_0,T)\times[0,1]\times I$. Then for any $i \in I$, u_i is bounded a.e. on $(t_0,T)\times[0,1]$ by P^δ, where*

$$P^\delta := \max_{i\in I}\|g_i\|_\infty + T(\max_{i\in I}\|c_i\|_\infty + |I|H(\delta)) + \delta = \bar{M}^\delta - 2 \tag{3.27}$$

Proof. From Lemma 3.5, it holds that u is both a weak sub-solution and a super solution of (3.1) on $(t_0,T)\times[0,1]$. Let $\underline{u}$ be such that for any $(t,s,i) \in (t_0,T]\times[0,1]\times I$:

$$\underline{u}_i(t,s) := -\max_{i\in I}\|g_i\|_\infty - (T-t)(\max_{i\in I}\|c_i\|_\infty + |I|H(\delta)).$$

The function $\underline{u}$ is a weak sub-solution of (3.8). Let $\bar{u}$ be such that for any $(t,s,i) \in (t_0,T]\times[0,1]\times I$:

$$\bar{u}_i(t,s) := \max_{i\in I}\|g_i\|_\infty + (T-t)\max_{i\in I}\|c_i\|_\infty + \int_t^T \lambda_i(d\tau).$$

The function $\bar{u}$ is a weak super-solution of (3.8). Thus from comparison principle in Lemma 3.6, it holds that for any $(t,s,i) \in (t_0,T)\times[0,1]\times I$:

$$-\max_{i\in I}\|g_i\|_\infty - (T-t)(\max_{i\in I}\|c_i\|_\infty + |I|H(\delta)) \leq u_i(t,s) \leq \max_{i\in I}\|g_i\|_\infty + (T-t)\max_{i\in I}\|c_i\|_\infty + \int_t^T \lambda_i(d\tau)$$

Using the definition of P^δ in (3.27), the conclusion follows. □

Lemma 3.8. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$ satisfying inequality (3.3), the function ϕ^λ defined in (3.25) is bounded independently of $\bar{M}^\delta - 1$ and is a solution of (3.8) a.e. on $[0,T]\times[0,1]\times I$.*

Proof. To show that ϕ^λ is a solution a.e. on $[0,T]\times[0,1]\times I$ of (3.8), one needs to prove that ν^λ is bounded independently of $M^\delta - 1$. To do so, we only need to show that ϕ^λ is bounded independently of $\bar{M}^\delta - 1$. Let $t_0 \in [0,T)$ be the minimum time such that ϕ^λ is a solution (3.8) a.e. on $(t_0,T]$. The time t_0 is less than T. Indeed, for any $i \in I$ it holds $\|\nu_i^\lambda(T,\cdot)\|_\infty = \|g_i\|_\infty < M^\delta - 1$ and thus $\|\phi_i^\lambda(T,\cdot)\|_\infty < \bar{M}^\delta - 1$. From the continuity of c and H, the boundedness of ϕ^λ in Lemma 3.7 and the definition of $\bar{M}^\delta$ in (3.4), there exists $\varepsilon > 0$ such that for any $s \in [0,1]$ and $i \in I$:

$$\varepsilon\left(\|c_i\|_\infty + |I|\sup_{x\in[-2\bar{M}^\delta, 2\bar{M}^\delta]}|H(x)|\right) + \int_{T-\varepsilon}^T \lambda_i(d\tau) + \|g_i\|_\infty < \bar{M}^\delta - 1.$$

Therefore it holds $t_0 \leq T - \varepsilon$ and for all $t \in (t_0,T]$ and any $i \in I$ it holds $\|\nu_i^\lambda(t,\cdot)\|_\infty \leq M^\delta - 1$. From Lemma 3.7, for any i the function ϕ_i^λ is bounded by P^δ, defined in 3.27, a.e. on $(t_0,T]\times[0,1]\times I$. We deduce that a.e. on

$(t_0, T] \times [0,1] \times I$, $|\nu_i^\lambda(t,s)| \leq M^\delta - 2$. Applying the same argument as previously, there exists $\varepsilon' > 0$ such that for a.e. $(t's, i) \in (t_0 - \varepsilon', t_0] \times [0,1] \times I$:

$$|\nu_i^\lambda(t', s)| \leq \|\nu_i^\lambda(t_0, \cdot)\|_\infty + \varepsilon' \left(\|c_i\|_\infty + |I| \sup_{x \in [-2\bar{M}^\delta, 2\bar{M}^\delta]} |H(x)| \right) < M^\delta - 1,$$

and thus for a.e. $(t's, i) \in (t_0 - \varepsilon', t_0] \times [0,1] \times I$ $|\phi_i^\lambda(t,s)| \leq \bar{M}^\delta - 1$. Therefore ϕ^λ is a solution of (3.8) a.e. on $(t_0 - \varepsilon', T]$ and the contradiction holds. Therefore ϕ^λ is a solution of (3.8) a.e. on $[0,T] \times [0,1] \times I$. □

Lemma 3.9. *For any $\lambda \in \mathcal{M}^+([0,T] \times I)$ satisfying inequality* (3.3)*, the function ϕ^λ defined in* (3.25)*, is the unique solution of* (3.8)*.*

Proof. From Lemma 3.8 it holds that ϕ^λ is a solution of (3.8). Uniqueness is a direct consequence of the comparison principle in Lemma 3.6. □

The following Lemma gives an important continuity property of the mapping $\lambda \mapsto \nu^\lambda$.

Lemma 3.10. *Let $\lambda \in \mathcal{M}^+([0,T] \times I)$ satisfying inequality* (3.3) *and $\{\lambda^n\}_n$ be a sequence $\{\lambda^n\}_n$, where for any $n \in \mathbb{N}$ $\lambda^n \in C^0([0,T] \times I, \mathbb{R}_+)$, weakly converging to λ. Then we have for any $i \in I$ and for a.e. $t \in [0,T]$:*

$$\lim_{n \to \infty} \|\phi_i^n(t, \cdot) - \phi_i^\lambda(t, \cdot)\|_\infty = 0 \tag{3.28}$$

where ϕ^λ (resp. ϕ^n) is the solution to (3.8) *associated to λ (resp. λ^n).*

Proof. There exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, λ^n satisfies inequality (3.3). Using that ϕ^λ and ϕ^n are fixed points of Π^λ and Π^{λ^n} and that F_δ is a contraction, it comes for all $(s,i) \in [0,1] \times I$ and a.e. $t \in (0,T)$:

$$\begin{aligned} &|\phi_i^\lambda(t,s) - \phi_i^n(t,s)| \\ &= |\Pi_i^\lambda(\phi^\lambda)(t,s) - \Pi_i^{\lambda^n}(\phi^n)(t,s)| \\ &\leq \left| \int_t^T \sum_{j \in I, j \neq i} H((\phi_j^\lambda - \phi_i^\lambda)(\tau, S_i^{t,s}(\tau))) - H((\phi_j^n - \phi_i^n)(\tau, S_i^{t,s}(\tau)))d\tau + \int_t^T (\lambda^n - \lambda)(d\tau) \right|. \end{aligned} \tag{3.29}$$

Recalling that functions ϕ^λ and ϕ^n are bounded a.e. on $(0,T) \times [0,1] \times I$ by $\bar{M}^\delta$, and that H is Lipschitz continuous on $[-2M^\delta, 2M^\delta]$ with Lipschitz constant K, from inequality (3.29) it comes:

$$|\phi_i^\lambda(t,s) - \phi_i^n(t,s)| \leq \int_t^T K^\delta \sum_{j \in I, j \neq i} |(\phi_i^\lambda - \phi_i^n)(\tau, S_i^{t,s}(\tau))| + |(\phi_j^\lambda - \phi_j^n)(\tau, S_i^{t,s}(\tau))| d\tau + \left| \int_t^T (\lambda_i^n - \lambda_i)(d\tau) \right|$$

Taking the supremum over $I \times [0,1]$ and applying Gronwall Lemma to $t \mapsto \sup_{i \in I, s \in [0,1]} |\phi_i^\lambda(t,s) - \phi_i^n(t,s)|$ on $[0,T]$, it comes for a.e. $t \in [0,T]$:

$$\begin{aligned} \sup_{i,s} |\phi_i^\lambda(t,s) - \phi_i^n(t,s)| &\leq 2K^\delta |I| \int_t^T \sup_{i,s} |\phi_i^\lambda(t,s) - \phi_i^n(t,s)| d\tau + \sup_i \left| \int_t^T (\lambda_i^n - \lambda_i)(d\tau) \right| \\ &\leq \sup_i \left| \int_t^T (\lambda_i^n - \lambda_i)(d\tau) \right| + 2K^\delta |I| e^{2TK^\delta |I|} \int_0^T \sup_i \left| \int_t^T (\lambda_i^n - \lambda_i)(d\tau) \right| dt \end{aligned} \tag{3.30}$$

Since for any $t \in E_\lambda$ we have $\lim_{n \to \infty} \left| \int_t^T (\lambda_i^n - \lambda_i)(d\tau) \right| = 0$, the result follows. □

3.3 Link between weak solution (3.2) and fixed point solution (3.8)

We start to show the connection between the solutions of (3.2) and (3.8) when $\lambda \in C^0([0,T] \times I, \mathbb{R}^+)$. Let $\lambda \in C^0([0,T] \times I)$ satisfy (3.3).

Lemma 3.11. *For any $\lambda \in C^0([0,T] \times I, \mathbb{R}^+)$ satisfying inequality 3.3, the solution ϕ^λ of* (3.8) *is a classical solution of* (3.1) *and a weak solution in the sense of Definition* (3.1)*.*

Proof. From Lemma 3.4, the function ν^λ is in C^1 and from (3.25) it holds that ϕ^λ is also in C^1. Applying Lemma 3.5 with $t_0 = 0$ it holds on $(0,T)\times(0,1)$:

$$\partial_t \phi_i^\lambda(t,s) + b_i(s)\partial_s \phi_i^\lambda(t,s) = \sum_{j\in I, j\neq i} H((\phi_j^\lambda - \phi_i^\lambda)(t,s)) - c_i(t,s) - \lambda_i(t). \tag{3.31}$$

For any $\psi \in C^\infty(0,T)\times(0,1)\times I)$, integrating by part over $(0,T)\times(0,1)$ and using that $\phi_i^\lambda(T,\cdot) = g_i$ on $[0,1]$, it comes for any $i \in I$:

$$\int_0^1 \phi^\lambda(0,s)\psi_i(0,s)ds - \int_0^1 g_i(s)\psi_i(T,s)ds = -\int_0^T\int_0^1 (\partial_t\psi_i(t,s))\phi_i^\lambda(t,s)dsdt - \int_0^T\int_0^1 \psi_i(t,s)\partial_t\phi_i^\lambda(dt,s)ds, \tag{3.32}$$

Using (3.31), equality (3.32) becomes:

$$\begin{aligned}\int_0^1 \phi^\lambda(0,s)\psi_i(0,s)ds - \int_0^1 g_i(s)\psi_i(T,s)ds \quad &= -\int_0^T\int_0^1 (\partial_t\psi_i(t,s))\phi_i^\lambda(t,s) + \psi_i(t,s)b_i(s)\partial_s\phi_i^\lambda(t,s)dsdt \\ &+ \int_0^T\int_0^1 \psi_i(t,s)\left(\sum_{j\in I, j\neq i} -H((\phi_j^\lambda - \phi_i^\lambda)(t,s)) + c_i(t,s) + \lambda_i(t)\right)dsdt\end{aligned}$$

Integrating by part $\psi_i b_i \partial_s \phi_i^\lambda$ knowing that $b_i(0) = b_i(1) = 0$ and the result follows. □

The previous Lemma is then extended for any $\lambda \in \mathcal{M}^+([0,T]\times I)$, satisfying $t \mapsto \lambda([t,T))$ is continuous at 0. This continuity assumption is motivated by the following remark.

Remark 3.3. *Let $\lambda \in \mathcal{M}^+([0,T]\times I)$ and $\{\lambda^n\}_n$, a sequence in $C^\infty([0,T]\times I, \mathbb{R}_+)$ converging weakly to λ. If for any $i \in I$ the function $t \mapsto \lambda_i([t,T))$ is continuous at 0, then for any $\psi \in C^0(0,1)$, we have for any $i \in I$:*

$$\int_0^1 \phi_i^n(0,s)\psi(s)ds \xrightarrow[n\to\infty]{} \int_0^1 \phi^\lambda(0,s)\psi_i(s)ds, \tag{3.33}$$

where ϕ^n is a solution of (3.8) associated to λ^n. Indeed, the continuity of $t \mapsto \lambda_i([t,T))$ at 0 implies $\lim_{n\to\infty}\left|\int_0^T (\lambda_i^n - \lambda_i)(d\tau)\right| = 0$. Applying the same arguments as in the proof of Lemma 3.10, the result is then deduced from inequality (3.30) at time $t = 0$.

Lemma 3.12. *For any $\lambda \in \mathcal{M}^+([0,T]\times I)$, such that $t \mapsto \lambda([t,T))$ is continuous at 0, the solution ϕ^λ of (3.8) is a weak solution of (3.1) in the sense of Definition 3.1.*

Proof. Let $\tilde{\lambda} \in \mathcal{M}^+(\mathbb{R}\times I)$ be an extension of λ to $\mathbb{R}\times I$, defined for any $i \in I$ by $\tilde{\lambda}_i(B) = \lambda_i(B \cap [0,T])$ for any $B \in \mathcal{B}(\mathbb{R})$. Let ξ be a standard convolution kernel on $\mathbb{R}_+$ such that $\xi > 0$. Let $\xi^n(t) := \xi(t/\varepsilon_n)/\varepsilon_n$ with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$. For any $n \in \mathbb{N}$, let the function λ^n be defined by:

$$\lambda^n := \xi^n * \tilde{\lambda}, \tag{3.34}$$

where $*$ stands for the convolution product. Then, $\lambda^n \in C^\infty([0,T]\times I, \mathbb{R}_+)$ and the sequence $\{\lambda^n\}_n$ weakly converges to λ in $\mathcal{M}^+([0,T]\times I)$. From Lemmas 3.8 and 3.4, we know that for any λ^n, there exists a function $\phi^n \in C^1([0,T]\times[0,1]\times I)$ such that ϕ^n is a solution of (3.8). From Lemma 3.10 it comes that the sequence ϕ^n converges to ϕ^λ w.r.t. the norm $\|\cdot\|_1$. Since for any $i \in I$ and $n \in \mathbb{N}$ $\lambda_i^n \in C^\infty((0,T),\mathbb{R}_+)$, Lemma 3.11 gives that for any $\psi_i \in C^\infty((0,T)\times(0,1))$ we have:

$$\begin{aligned}&\int_0^1 \phi_i^n(0,s)\psi_i(0,s)ds - \int_0^1 g_i(s)\psi_i(T,s)ds + \int_0^T\int_0^1 \left(\partial_t\psi_i(t,s) + \partial_s(\psi_i(t,s)b_i(s))\right)\phi_i^n(t,s)dsdt \\ &+ \int_0^T\int_0^1 \left(\sum_{j\in I, j\neq i} H((\phi_j^n - \phi_i^n)(t,s)) - c_i(t,s)\right)\psi_i(t,s)dtds - \int_0^T\int_0^1 \psi_i(t,s)\lambda_i^n(dt)ds \\ &= 0,\end{aligned} \tag{3.35}$$

Taking a subsequence of $\{\phi^{n_k}\}_k$ converging a.e. on $[0,T]\times[0,1]\times I$ to ϕ^λ and using Remark 3.3, letting k tend to infinity in equality (3.35) gives the result. □

Finally, the next lemma states the converse of the previous lemma.

Lemma 3.13. *For any $(\lambda,\varphi)\in\mathcal{M}^+([0,T]\times I)\times BV((0,T)\times(0,1)\times I)$, if φ is a weak solution of (3.1), in the sense of Definition 3.1, associated to λ, then φ satisfies a.e. on $[0,T]\times[0,1]\times I$ equality (3.8).*

Proof. Let $\theta\in C^\infty((0,T)\times(0,1)\times I)$, $\Theta\in C^\infty((0,1)\times I)$ and $\psi\in C^1((0,T)\times(0,1)\times I,\mathbb{R})$ such that it holds on $(0,T)\times(0,1)\times I$:

$$\begin{array}{ll}\partial_t\psi_i(t,s)+\partial_s(\psi_i(t,s)b_i(s))=\theta_i(t,s) & \text{on } (0,T)\times(0,1)\times I\\ \psi_i(0,\cdot)=\Theta_i(\cdot) & \text{on}(0,1)\times I.\end{array}\tag{3.36}$$

One has for any $(i,t,s)\in[0,T]\times[0,1]\times I]$:

$$\psi_i(t,s)=\int_0^t\theta_i(\tau,S_i^{t,s}(\tau))\exp\left(-\int_\tau^t b_i'(S_i^{t,s}(r))dr\right)d\tau+\Theta_i(S_i^{t,s}(0))\exp\left(-\int_0^t b_i'(S_i^{t,s}(\tau))d\tau\right)$$

For any $i\in I$, let ν_i and π_i be defined for any $(t,s)\in[0,T]\times[0,1]$ by:

$$\nu_i(t,s):=\int_0^t\theta_i(\tau,S_i^{t,s}(\tau))\exp\left(-\int_\tau^t b_i'(S_i^{t,s}(r))dr\right)d\tau\quad\text{ and }\quad\pi_i(t,s):=\Theta_i(S_i^{t,s}(0))\exp\left(-\int_0^t b_i'(S_i^{t,s}(\tau))d\tau\right).$$

One can observe: $\psi_i=\nu_i+\pi_i$. For any function $f\in L^1((0,T)\times(0,1))$, by switching the order of integration, applying the change of variable $x=S_i^{t,s}(\tau)$ and Lemma A.2, it holds:

$$\begin{aligned}\int_0^T\int_0^1 f(t,s)\nu_i(t,s)dtds&=\int_0^T\int_0^1\int_0^t f(t,s)\theta_i(\tau,S_i^{t,s}(\tau))\exp\left(-\int_\tau^t b_i'(S_i^{t,s}(r))dr\right)d\tau dsdt\\&=\int_0^T\int_0^1\theta_i(\tau,x)\int_\tau^T f(t,S_i^{\tau,x}(t))\exp\left(-\int_\tau^t b_i'(S_i^{\tau,x}(r))dr\right)\partial_x S_i^{t,x}(\tau)dtdxd\tau,\\&=\int_0^T\int_0^1\theta_i(\tau,x)\int_\tau^T f(t,S_i^{\tau,x}(t))dtdxd\tau.\end{aligned}\tag{3.37}$$

Applying same calculus, for any $i\in I$ one has:

$$\int_0^1 g_i(s)\nu_i(T,s)ds=\int_0^T\int_0^1 g_i(S_i^{\tau,x}(T))\theta_i(\tau,x)dsd\tau,\tag{3.38}$$

$$\int_0^T\int_0^1 f(t,s)\pi_i(t,s)dtds=\int_0^1\Theta_i(x)\int_0^T f(t,S_i^{0,x}(t))dtdx,\tag{3.39}$$

and:

$$\int_0^1 g_i(s)\pi_i(T,s)ds=\int_0^1\Theta_i(x)g_i(S_i^{0,x}(T))dx\tag{3.40}$$

Let $\lambda_i\in\mathcal{M}^+([0,T])$, applying same computation gives:

$$\int_0^T\int_0^1\psi_i(t,s)ds\lambda_i(dt)=\int_0^T\int_0^1\theta_i(\tau,x)\left(\int_\tau^T\lambda_i(dt)\right)dx+\int_0^1\Theta_i(x)\left(\int_0^T\lambda_i(dt)\right)dx\tag{3.41}$$

Taking $f(t,s)=\sum_{j\in I,j\neq i}H((\varphi_j-\varphi_i)(t,s))+c_i(t,s)$, using (3.37), (3.39), (3.40) and (3.41) and (3.36) satisfied by ψ_i, for any $i\in I$ equation (3.2) becomes:

$$\begin{aligned}&\int_0^1\Theta_i(s)\left(\varphi_i(0,s)+\int_0^T\sum_{j\in I,j\neq i}H((\varphi_j-\varphi_i)(\tau,S_i^{0,s}(\tau)))-c_i(\tau,S_i^{0,s}(\tau))dt-\int_0^T\lambda_i(d\tau)-g_i(S_i^{0,s}(T))\right)ds\\&+\int_0^T\int_0^1\theta_i(t,s)\left(\varphi_i(t,s)+\int_t^T\sum_{j\in I,j\neq i}H((\varphi_j-\varphi_i)(\tau,S_i^{t,s}(\tau)))-c_i(\tau,S_i^{t,s}(\tau))d\tau-\int_t^T\lambda_i(d\tau)-g_i(S_i^{t,s}(T))\right)dsdt\\&=0\end{aligned}$$

Since θ_i and Θ_i are test functions, for a.e. $s \in [0,1]$ one has:

$$\varphi_i(0,s) = \int_0^T \sum_{j\in I, j\neq i} -H((\varphi_j - \varphi_i)(\tau, S_i^{0,s}(\tau))) + c_i(\tau, S_i^{0,s}(\tau))dt + \int_0^T \lambda_i(d\tau) + g_i(S_i^{0,s}(T)) \tag{3.42}$$

and a.e. on $(0,T)\times(0,1)$:

$$\varphi_i(t,s) = \int_t^T \sum_{j\in I, j\neq i} -H((\varphi_j - \varphi_i)(\tau, S_i^{t,s}(\tau))) + c_i(\tau, S_i^{t,s}(\tau))d\tau + \int_t^T \lambda_i(d\tau) + g_i(S_i^{t,s}(T)).$$

From (3.42), and using that $\varphi_i \in BV((0,T)\times(0,1)$, it holds in the sense of trace: $\varphi_i(T,\cdot) = g_i$ on $(0,1)$. □

4 Dual problem

In this section, an optimization problem (4.3) is introduced. Using tools from convex analysis [19], we show that this problem is in duality with (2.4).

We consider the set $\tilde{I} := \{(i,j) \in I^2;\ i \neq j\}$ and the following spaces:

$$E_0 = C^1([0,T]\times[0,1]\times I) \times C^0([0,T]\times I) \text{ and } E_1 := C^0([0,1]\times[0,T]\times I) \times C^1([0,T]\times[0,1]\times \tilde{I})$$

We consider the following inequality:

$$\begin{aligned} &-\partial_t\bar{\varphi}_i(t,s) - b_i(s)\partial_s\bar{\varphi}_i(t,s) - c_i(t,s) - \lambda(t) + \sum_{j\in I, j\neq i} H((\bar{\varphi}_j - \bar{\varphi}_i)(t,s)) \leq 0 \quad \text{on } (0,T)\times(0,1)\times I, \\ &\bar{\varphi}_i(T,\cdot) \leq g_i \qquad\qquad \text{on } (0,1)\times I. \end{aligned} \tag{4.1}$$

The set $\mathcal{K}_0$ is defined by: $\mathcal{K}_0 := \{(\varphi,\lambda) \in E_0;\ \varphi \text{ solution of (4.1) associated to } \lambda\}$. We introduce the function A, defined on $\mathcal{K}_0$ by :

$$A(\varphi,\lambda) := \sum_{i\in I} \int_0^1 -\varphi_i(0,s)m_i^0(ds) + \int_0^T \lambda_i(t)D_i(t)dt, \tag{4.2}$$

and the following problem is considered:

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) \tag{4.3}$$

Lemma 4.1. $\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda)$ *is finite.*

Proof. We consider $(\varphi,\lambda) \in \mathcal{K}_0$ and $\bar{\varphi}$ a classical solution of the PDE (3.1) associated to λ, where the inequality is replaced by an equality. From the comparison principle (see Lemma 3.6), it holds $\varphi \leq \bar{\varphi}$ on $[0,T]\times\mathbb{R}\times I$. Thus, we have:

$$A(\bar{\varphi},\lambda) \leq A(\varphi,\lambda). \tag{4.4}$$

The set $\mathcal{L}_0$ is defined by: $\mathcal{L}_0 := \{(\varphi,\lambda) \in E_0; (\varphi,\lambda) \text{ solution of (3.1) and } \lambda \geq 0\}$. From (4.4), we obtain:

$$\inf_{(\varphi,\lambda)\in\mathcal{L}_0} A(\varphi,\lambda) = \inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda). \tag{4.5}$$

Let $(\bar{\varphi},\lambda) \in \mathcal{L}_0$. From Lemma 3.13 $\bar{\varphi}$ satisfies (3.8). Then, taking $t = 0$, we have for any $(i,s) \in I\times[0,1]$: $\varphi_i(0,s) \leq \int_0^T c_i(\tau, S_i^{0,s}(\tau)) + \lambda_i(\tau)d\tau + g_i(S_i^{0,s}(T))$, where S_i is the flow defined at equation (1.9). Setting $Q := -\sum_{i\in I}\int_0^1 \big(g_i(S_i^{0,s}(T) + \int_0^T c_i(t, S_i^{0,s}(t))dt\big)m_i^0(ds)$, one has:

$$Q + \sum_{i\in I} \int_0^T \lambda_i(t)\left(D_i(t) - \int_0^1 m_i^0(ds)\right)dt \leq A(\bar{\varphi},\lambda) \tag{4.6}$$

Using that $\lambda \geq 0$, we deduce from Assumption 3 and (4.6):

$$Q \leq \inf_{(\varphi,\lambda)\in\mathcal{L}_0} A(\varphi,\lambda). \tag{4.7}$$

Combining (4.7) and (4.5), the conclusion follows. □

We consider the linear and bounded function $\Lambda : E_0 \to E_1$ defined by: $\Lambda(\varphi, \lambda) := (\partial_t \varphi + b\partial_s \varphi + \tilde{\lambda}, \Delta\varphi)$, where $\partial_t \varphi + b\partial_s \varphi := (\partial_t \varphi_i + b_i \partial_s \varphi_i)_{i\in I}$, $\Delta\varphi := (\Delta\varphi_{i,j})_{(i,j)\in \tilde{I}}$ with $\Delta\varphi_{i,j} = \varphi_j - \varphi_i$ and for any $(s, i) \in [0, 1] \times I$, $\tilde{\lambda}_i(\cdot, s) := \lambda_i(\cdot)$. The linear function $\Lambda^* : E_1^* \to E_0^*$ is the adjoint operator of Λ. The functional $\mathcal{F}$ is defined by:

$$\mathcal{F}(\varphi, \lambda) := \begin{cases} \displaystyle\sum_{i\in I} \int_0^1 -\varphi_i(0, s) m_i^0(ds) + \int_0^T D_i(t)\lambda_i(t)dt & \text{if } \varphi_i(T, \cdot) \le g_i \text{ and } \lambda_i \ge 0 \quad \forall i \in I, \\ +\infty & \text{otherwise.} \end{cases}$$

Using that:

$$\begin{aligned} &\langle (m, E), \Lambda(\varphi, \lambda)\rangle_{E_1^*, E_1} \\ &= \sum_{i\in I} \int_0^1 \int_0^T (\partial_t \varphi_i(t, s) + b_i(s)\partial_s \varphi_i(t, s)) m_i(ds, t) + \sum_{j\in I, j\neq i} (\varphi_j(t, s) - \varphi_i(t, s)) E_{i,j}(t, ds)dt \\ &+ \sum_{i\in I} \int_0^T \int_0^1 m_i(t, ds)\tilde{\lambda}_i(t, s)dt, \end{aligned}$$

defining $\mathcal{F}^*$ as the Fenchel conjugate of $\mathcal{F}$, we have:

$$\mathcal{F}^*\left(\Lambda^*(m, E)\right) := \begin{cases} \displaystyle\int_0^1 \sum_{i\in I} g_i(s) m_i(T, ds) & \text{if } (m, E) \text{ weak solution of (2.1)} \\ & \text{and } \displaystyle\int_0^1 m_i(t, ds) \le D_i(t) \quad \forall (t, i) \in [0, T] \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $(x, y) \in E_1$, the functional $\mathcal{G}$ is defined by:

$$\mathcal{G}(x, y) := \begin{cases} 0 & \text{if } -c_i(t, s) - x_i(t, s) + \displaystyle\sum_{j\in I, j\neq i} \frac{(y_{i,j}(t, s)^-)^2}{2} \le 0 \quad \forall (t, s, i) \in (0, T) \times (0, 1) \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

Then for any $(\varphi, \lambda) \in E_0$ it holds:

$$\mathcal{G}(\Lambda(\varphi, \lambda)) := \begin{cases} 0 & \text{if } -c_i(t, s) - \partial_t \varphi_i(t, s) - b_i(t, s)\partial_s \varphi_i(t, s) - \tilde{\lambda}_i(t, s) + \displaystyle\sum_{j\in I, j\neq i} \frac{((\Delta\varphi_{i,j}(t, s))^-)^2}{2} \le 0 \\ & \forall (t, s, i) \in (0, T) \times (0, 1) \times I, \\ +\infty & \text{otherwise.} \end{cases}$$

Observing, from [6], that for any $(\rho, w) \in \mathbb{R}^2$:

$$\sup_{a,b\in\mathbb{R}} \{a\rho + bw;\ a + \frac{(b^+)^2}{2} \le 0\} = \begin{cases} \dfrac{1}{2}\dfrac{w^2}{\rho} & \text{if } \rho > 0 \text{ and } w \ge 0, \\ 0 & \text{if } \rho = 0 \text{ and } w = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

then, applying similar computations as in [15][Lemma 4.3], for any $(m, E) \in E_1'$, we have:

$$
\begin{aligned}
&\mathcal{G}^*(-(m,E)) \\
&= \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 -x_i(t,s)m_i(t,ds)dt - \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(x,y) \\
&= \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 (-x_i(t,s) - c_i(t,s) + c_i(t,s))m_i(t,ds) - \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(x,y) \\
&= \sum_{i\in I} \int_0^T \int_0^1 c_i(t,s)m_i(t,ds)dt + \sup_{(x,y)\in E_1} \sum_{i\in I} \int_0^T \int_0^1 x_i(t,s)m_i(t,ds) + \sum_{j\neq i} y_{i,j}(t,s)E_{i,j}(t,ds)dt - \mathcal{G}(-x-c,-y) \\
&= \begin{cases} \displaystyle\int_0^T \int_0^1 \sum_{i\in I} c_i(t,s)m_i(t,ds) + \sum_{j\neq i} \frac{1}{2}\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)\right)^2 m_i(t,ds)dt & \text{if } m>0, E\geq 0 \text{ and } E \ll m, \\ 0 & \text{if } m=0 \text{ and } E=0, \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}
\tag{4.8}
$$

The following lemma is useful to show the constraint qualification for Problem (4.3).

Lemma 4.2. *There exists $(\varphi, \lambda) \in E_0$ such that $\mathcal{F}(\varphi, \lambda) < \infty$ and $\mathcal{G}$ is continuous at $\Lambda(\varphi, \lambda)$.*

Proof. Let φ and λ be such that for any $i \in I$, $s \in [0,1]$ and $t \in [0,T]$:

$$\varphi_i(t,s) = -\max_{i\in I}(\|g_i\|_\infty) - 1,$$

and

$$\lambda_i(t) := \|c_i\|_\infty + 1,$$

Functions φ and λ being constant, it holds that $(\varphi, \lambda) \in E_0$ and $\mathcal{F}(\varphi, \lambda) < \infty$. Also, from the choice of φ and λ, it follows that for any $i \in I$, $s \in (0,1)$ and $t \in (0,T)$:

$$-c_i(t,s) - \partial_t\varphi_i(t,s) - b_i(t,s)\partial_s\varphi_i(t,s) - \lambda_i(t,s) + \sum_{j\in I, j\neq i} \frac{((\Delta\varphi_{i,j}(t,s))^-)^2}{2} < -\frac{1}{2}.$$

Thus, $\mathcal{G}$ is continuous at $\Lambda(\varphi, \lambda)$. □

Theorem 4.1. *We have:*

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) = -\inf_{(m,E)\in CE(m^0,D)} \tilde{B}(E,m)$$

Proof. On can observe that:

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) = \inf_{(\varphi,\lambda)\in E_0} \mathcal{F}(\varphi,\lambda) + \mathcal{G}(\Lambda(\varphi,\lambda)),$$

and

$$\inf_{(m,E)\in CE(m^0,D)} \tilde{B}(m,E) = \inf_{(m,E)\in E_1'} \mathcal{F}(\Lambda^*(m,E)) + \mathcal{G}^*(-(m,E)),$$

Using Lemmas 4.2 and 4.1, the conclusion follows by applying the Fenchel-Rockafellar duality theorem [19]. □

4.1 Relaxed problem of (4.3)

The problem defined at (4.3) might not have a solution. A relaxed problem is introduced and the existence of a solution is proved. We define $\mathcal{R}_0$ by:

$$\mathcal{R}_0 := \{(\varphi,\lambda) \,|\, \lambda \in \mathcal{M}^+([0,T]\times I) \text{ and } \varphi \text{ solution of (3.1), in the sense of definition 3.1, associated to } \lambda\}.$$

The following relaxed problem is considered:

$$\inf_{(\varphi,\lambda)\in\mathcal{R}_0} \tilde{A}(\varphi,\lambda) \tag{4.9}$$

where:

$$\tilde{A}(\varphi,\lambda) := \sum_{i\in I} \int_0^1 -\varphi_i(0,s)m_i^0(ds) + \int_0^T D_i(t)\lambda_i(dt) \tag{4.10}$$

4.2 Existence of solution of the relaxed problem (4.9)

In order to prove the existence of a solution of (4.9), we need the following estimate on λ.

Lemma 4.3. *Let $A > 0$, there exists a constant $K_A > 0$ such that for any $(\varphi, \lambda) \in \mathcal{R}_0$ satisfying $\tilde{A}(\varphi, \lambda) \leq A$, we have:*

$$\sum_{i \in I} \int_0^T \lambda_i(dt) \leq K_A,$$

Proof. Let $A \in \mathbb{R}$ and $(\varphi, \lambda) \in \mathcal{R}_0$ be such that $\tilde{A}(\varphi, \lambda) \leq A$. Since $(\rho, 0) \in CE(m^0, D)$ (where ρ is defined in Section 2) and using Assumptions 1 and 2, it comes $\rho \in C^1((0,T) \times (0,1) \times I)$. From the definition of a weak solution 3.2, taking ρ as a test function, we have for any $i \in I$:

$$\begin{aligned}
&\int_0^1 \left(\varphi_i(0,s) m_i^0(s) - g_i(s)\rho_i(T,s)\right) ds + \int_0^T \int_0^1 \sum_{j \in I, j \neq i} H((\varphi_j - \varphi_i)(t,s))\rho_i(t,s) ds dt \\
&= \int_0^T \int_0^1 c_i(t,s)\rho_i(t,s) ds dt + \int_0^T \int_0^1 \rho_i(t,s) ds \lambda_i(dt).
\end{aligned}$$

Since that for any $(t,s) \in [0,T] \times [0,1]$ and $(i,j) \in \tilde{I}$ it holds $H((\varphi_j - \varphi_i)(t,s))\rho_i(t,s) \geq 0$, it comes:

$$\int_0^1 \varphi_i(0,s) m_i^0(ds) \leq \int_0^T \int_0^1 \rho_i(t,s) ds \lambda_i(dt) + K_i,$$

where: $K_i := \displaystyle\int_0^T g_i(s)\rho_i(T,s) ds + \int_0^T \int_0^1 c_i(t,s)\rho_i(t,s) ds dt$. From the definition $\tilde{A}$ in (4.2), we deduce:

$$-\sum_{i \in I} K_i + \int_0^T \left(D_i(t) - \int_0^1 \rho_i(t,s) ds \right) \lambda_i(dt) \leq \tilde{A}(\varphi, \lambda) \leq A.$$

Using Assumption 3, there exists $\varepsilon^0 > 0$ such that for any $i \in I$ and $t \in [0,T]$, it holds $D_i(t) - \displaystyle\int_0^1 \rho_i(t,s) ds = D_i(t) - \int_0^1 m_i^0(s) ds > \varepsilon^0$. Thus, we get $\lambda([0,T] \times I) dt \leq K_A$, where $K_A := \dfrac{A + \sum_{i \in I} K_i}{\varepsilon^0}$. □

Next lemma is useful to show that for any minimizing sequence $\{(\varphi^n, \lambda^n)\}_n$ of 4.9, $\{\tilde{A}(\varphi^n, \lambda^n)\}_n$ converges up to a subsequence.

Lemma 4.4. *Let $(\phi, \lambda) \in \mathcal{R}_0$ and a sequence $\{(\phi^n, \lambda^n)\}_n \in \mathcal{K}_0^{\mathbb{N}}$ be such that $\{\phi^n\}_n$ converges to ϕ in $L^1((0,T) \times (0,1) \times I)$ and $\{\lambda^n\}_n$ weakly converges to λ in $\mathcal{M}^+(I \times [0,T])$. Then, up to a subsequence of $\{(\phi^n, \lambda^n)\}_n$ it holds for any $i \in I$:*

$$\lim_{n \to \infty} \int_0^1 \sum_{i \in I} (\phi_i^n(0,s) - \phi_i(0,s)) m_i^0(s) ds = 0.$$

Proof. Let ρ be defined as in Section 2. Since $(\phi, \lambda) \in \mathcal{R}_0$ and that for any $n \in \mathbb{N}$ $(\phi^n, \lambda^n) \in \mathcal{K}_0$, we obtain for any $i \in I$:

$$\begin{aligned}
&\int_0^1 (\phi_i(0,s) - \phi_i^n(0,s)) m_i^0(s) ds + \int_0^T \int_0^1 \sum_{j \in I, j \neq i} \left(H((\phi_j - \phi_i)(t,s)) - H((\phi_j^n - \phi_i^n)(t,s)) \right) \rho_i(t,s) ds dt \\
&= \int_0^T \int_0^1 \rho_i(t,s) ds (\lambda_i - \lambda_i^n)(dt).
\end{aligned} \tag{4.11}$$

Since $\phi^n \underset{n \to \infty}{\to} \phi$ in L^1, there exists a subsequence of $\{(\phi^n, \lambda^n)\}_n$ such that $\phi^n \underset{n \to \infty}{\to} \phi$ a.e. on $[0,T] \times [0,1] \times I$. From Lemma 3.8 and the weak convergence of $\{\lambda^n\}_n$, the sequence $\{\phi^n\}_n$ is uniformly bounded. Then, the continuity of H and the dominated convergence theorem give:

$$\lim_{n \to \infty} \int_0^T \int_0^1 \sum_{j \in I, j \neq i} \left(H((\phi_j - \phi_i)(t,s)) - H((\phi_j^n - \phi_i^n)(t,s)) \right) \rho_i(t,s) ds dt = 0. \tag{4.12}$$

Since for any $i \in I$ it holds $\rho_i \in C^1((0,T)\times(0,1))$. Then from the weak* convergence of $\{\lambda_i^n\}_n$ to λ_i, we get:

$$\lim_{n\to\infty} \int_0^T \int_0^1 \rho_i(t,s)ds(\lambda_i - \lambda_i^n)(dt) = 0. \tag{4.13}$$

Taking the limit in (4.11) and using (4.12) and (4.13), the result follows. □

Theorem 4.2. *The relaxed problem* (4.9) *has a solution.*

Proof. Taking a minimizing sequence $\{(\varphi^n, \lambda^n)\}_n$ of (4.9), according to Lemmas 4.3, the sequence $\{\lambda^n\}_n$ is bounded. Thus there exists a subsequence $\{\lambda^{\theta_n}\}_n$ of $\{\lambda^n\}_n$ weakly converging to λ^* in $\mathcal{M}^+([0,T]\times I)$. From Lemma 3.8, there exists a weak solution φ^* of (3.2) associated to λ^*. From Lemma 3.10, it holds $\{\varphi^{\theta_n}\}_n$ converges to φ^* w.r.t. the norm $\|\cdot\|_1$. Thus, Lemma 4.4 gives up to a subsequence of $\{\varphi^{\theta_n}\}_n$:

$$\lim_{n\to\infty} \sum_{i\in I} \int_0^1 (\varphi^n(0,s) - \varphi^*(0,s))m_i^0(s)ds = 0. \tag{4.14}$$

From Assumption 3, we have for any $i \in I$ $D_i \in C^0(0,T)$, then from the weak* convergence of $\{\lambda_i^{\theta_n}\}_n$ to λ_i^*, one has:

$$\lim_{n\to\infty} \sum_{i\in I} \left| \int_0^T D_i(t)\lambda_i^*(dt) - \int_0^T D_i(t)\lambda_i^n(t)dt \right| = 0. \tag{4.15}$$

Thus, (φ^*, λ^*) minimizes $\tilde{A}$. □

The next Theorem shows that Problem (4.3) and the relaxed problem (4.9) have the same value.

Theorem 4.3. *It holds:*

$$\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) = \inf_{(\varphi,\lambda)\in\mathcal{R}_0} \tilde{A}(\varphi,\lambda) \tag{4.16}$$

Proof. From Theorem 4.2 we get that (4.9) has a solution, $(\varphi^*, \lambda^*) \in \mathcal{R}_0$. Since $\mathcal{K}_0 \subset \mathcal{R}_0$, it is clear that

$$\tilde{A}(\varphi^*, \lambda^*) \leq \inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda). \tag{4.17}$$

Let ξ be a standard convolution kernel on $\mathbb{R}_+$ such that $\xi > 0$ and the functions $\{\lambda^n\}_n$ be defined as in (3.34) to approximate λ^*. For any $n \in \mathbb{N}$, let ϕ^n be the solution of (3.8) associated to λ^n. From Lemma 3.10, we have that the sequence ϕ^n converges to ϕ^* w.r.t. the norm $\|\cdot\|_1$. From Lemma 3.4, it holds for any $n \in \mathbb{N}$ that $\varphi^n \in C^1((0,T)\times(0,1)\times I)$ and thus, $(\varphi^n, \lambda^n) \in \mathcal{K}_0$. Using similar arguments as in the proof of Theorem 4.2, one obtains: $\inf_{(\varphi,\lambda)\in\mathcal{K}_0} A(\varphi,\lambda) \leq \inf_{(\varphi,\lambda)\in\mathcal{R}_0} \tilde{A}(\varphi,\lambda)$. □

5 Characterization of minimizers

The purpose of this section is to define and characterize the solutions of Problem (2.4). We show that the following system gives optimality conditions for (2.4):

$$\begin{cases} -\partial_t\varphi_i - b_i\partial_s\varphi_i - c_i - \lambda_i + \sum_{j\in I, j\neq i} H((\varphi_j - \varphi_i)) = 0 & \text{on } (0,T)\times(0,1)\times I \\ \partial_t m_i + \partial_s(m_i b_i) + \sum_{j\neq i}((\varphi_i - \varphi_j)^+ m_i - (\varphi_j - \varphi_i)^+ m_j) = 0 & \text{on } (0,T)\times(0,1)\times I \\ m_i(0,s) = m_i^0(s),\ \varphi_i(T,s) = g_i(s) & \text{on } (0,1)\times I \\ \int_0^1 m_i(t,s)ds - D(t) \leq 0,\ \lambda \geq 0 & \text{on } [0,T]\times I \\ \sum_{i\in I} \int_0^T \int_0^1 m_i(t,ds)\lambda_i(dt) - \int_0^T D(t)\lambda(dt) = 0 & \end{cases} \tag{5.1}$$

The notion of weak solutions of the system (5.1) is detailed in the following definition.

Definition 5.1. *A triplet* $(\varphi, \lambda, m) \in BV((0,T)\times(0,1)\times I)\times\mathcal{M}^+([0,T]\times I)\times \mathrm{Lip}([0,T]\times[0,1]\times I)$ *is called a weak solution of* (5.1) *if it satisfies the following conditions.*

1. *The function φ is a weak solution of* (3.1)*, associated to λ in the sense of Definition 3.1 and $\varphi(T,\cdot)=g_i$ in the sense of trace.*

2. *m satisfies the continuity equation:*

$$\partial_t m_i + \partial_s(m_i b_i) + \sum_{j\neq i}((\varphi_i-\varphi_j)^+ m_i - (\varphi_j-\varphi_i)^+ m_j) = 0, \quad m_i(0,\cdot)=m_i^0$$

in the sense of Definition 2.1, with $\alpha_{i,j}=(\varphi_i-\varphi_j)^+$.

3. *It holds for any $t\in[0,T]$:*

$$\int_0^1 m_i(t,s)ds - D(t) \leq 0 \quad \text{and} \quad \sum_{i\in I}\int_0^T\int_0^1 m_i(t,ds)\lambda_i(dt) - \int_0^T D(t)\lambda(dt) = 0.$$

Remark 5.1. *From Lemmas 3.4 and 3.8, it holds that for any $i\in I$, φ_i is bounded and for a.e. $t\in[0,T]$ $\varphi(t,\cdot)$ is continuous on $[0,1]$. Thus, the forward equation in* (5.1) *makes sense.*

Theorem 5.1. 1. *If $(m,E)\in CE(m^0,D)$ is a minimizer of Problem* (2.4) *and $(\varphi,\lambda)\in\mathcal{R}_0$ a minimizer of Problem* (4.9)*, then (φ,λ,m) is a weak solution of* (5.1) *and $\frac{dE_{i,j}}{dm_i}=(\varphi_i-\varphi_j)^+$ on $\{m_i>0\}$ for any $i,j\in I$.*

2. *Conversely, if (φ,λ,m) is a weak solution of* (5.1)*, then $(\varphi,\lambda)\in\mathcal{R}_0$ is a minimizer of Problem* (4.9) *and there exists E, defined for any $i,j\in I$ by: $\frac{dE_{i,j}}{dm_i}:=(\varphi_i-\varphi_j)^+$, such that $(m,E)\in CE(m^0,D)$ is a minimizer of* (2.4)*.*

5.1 Proof of Theorem (5.1)

Before to start the proof of Theorem 5.1, we make the following remark.

Remark 5.2. *Suppose $(\varphi,\lambda)\in\mathcal{R}_0$ and $(m,E)\in CE(m^0,D)$ with $m\in L^\infty([0,T]\times[0,1]\times I)$ and $E\in L^\infty([0,T]\times[0,1]\times I\times I,\mathbb{R}_+)$, then one has: $-\tilde{B}(m,E)\leq\tilde{A}(\varphi,\lambda)$. Indeed, let $\{\lambda_n\}_n$ be a sequence of smooth functions, approximating λ, obtained by convolution as in* (3.34)*. For any $n\in\mathbb{N}$, $\varphi^n\in C^1((0,T)\times(0,1)\times I)$ denotes the solution of* (3.8) *associated to λ^n. From Lemma 4.4, one has : $\lim_{n\to\infty}A(\varphi^n,\lambda^n)=\tilde{A}(\varphi,\lambda)$. Observing that for any $n\in\mathbb{N}$ we have $(\varphi^n,\lambda^n)\in\mathcal{R}_0$, and using the proof of Theorem 4.1, we get: $A(\varphi^n,\lambda^n)\geq-\tilde{B}(m,E)$ and thus $\tilde{A}(\varphi,\lambda)\geq-\tilde{B}(m,E)$.*

Proof of Theorem 5.1.(1). From Theorems 4.3 and 4.1, it comes:

$$\inf_{(\varphi,\lambda)\in\mathcal{R}_0}\tilde{A}(\varphi,\lambda) = -\inf_{(m,E)\in CE(m^0,D)}\tilde{B}(E,m),$$

and thus

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \int_0^T D_i\lambda_i + \int_0^T\int_0^1\left(c_i+\sum_{j\neq i}L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)\right)m_i = 0. \tag{5.2}$$

We want to show that $E_{i,j}=m_i(\varphi_i-\varphi_j)^+$. Let $\{\lambda_n\}_n$ be a sequence of smooth functions, approximating λ, obtained by convolution as in (3.34). For any $n\in\mathbb{N}$, $\varphi^n\in C^1((0,T)\times(0,1)\times I)$ denotes the solution of (3.8) associated to λ^n. For any $n\in\mathbb{N}$, φ^n is smooth enough to be a test function for the weak formulation of (3.1) satified by m. Using Lemma 4.4 and that $D\in C^0([0,T]\times I)$, it holds for any $n\in\mathbb{N}$ and $i\in I$:

$$\begin{aligned}
&\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \int_0^T\lambda_i D_i \\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i^n(0)m_i^0 + \int_0^T D_i\lambda_i^n \\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1\left(\partial_t\varphi_i^n + b_i\partial_s(\varphi_i^n) + \sum_{j\in I,j\neq i}(\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)m_i + \int_0^T\lambda_i^n D_i \\
&= \lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1\left(-c_i + \sum_{j\in I,j\neq i}H(\varphi_j^n-\varphi_i^n) + \sum_{j\in I,j\neq i}(\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)m_i + \int_0^T\lambda_i^n\left(D_i - \int_0^1 m_i\right)
\end{aligned}$$

Using previous equality and (5.2), it comes:

$$\lim_{n\to\infty}\sum_{i\in I}\int_0^T\int_0^1\left(\sum_{j\in I,j\neq i}H(\varphi_j^n-\varphi_i^n)+L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)+(\varphi_j^n-\varphi_i^n)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)m_i+\int_0^T\lambda_i^n\left(D_i-\int_0^1 m_i\right)=0$$

From Lemma 3.10, for a.e. $t\in[0,T]$, the sequence $\{\varphi(t,\cdot)\}_n$ converges uniformly to φ. From Lemma 3.8 we have that φ is bounded, consequently the sequence $\{\varphi(t,\cdot)\}_n$ is uniformly bounded. Using dominated convergence theorem, we get $\int_0^T\int_0^1 m_iH(\varphi_j^n-\varphi_i^n)$ converges to $\int_0^T\int_0^1 m_iH(\varphi_j-\varphi_i)$ for any $i,j\in I$ and a.e. $t\in[0,T]$. Since (m,E) is a solution of (2.4), $\tilde{B}(m,E)$ is finite and from Lemma 2.2 one has for any $i,j\in I$: $\int_0^T\int_0^1 E_{i,j}<\infty$. Applying dominated convergence theorem, we get: $\int_0^T\int_0^1(\varphi_j^n-\varphi_i^n)E_{i,j}$ converges to $\int_0^T\int_0^1(\varphi_j-\varphi_i)E_{i,j}$. In addition, for any $i\in I$, the map $t\mapsto\int_0^1 m_i(t,s)ds$ is continuous, the weak convergence of λ^n to λ in $\mathcal{M}^+([0,T]\times I)$ gives:

$$\lim_{n\to\infty}\sum_{i\in I}\int_0^T\lambda_i^n\left(D_i-\int_0^1 m_i\right)=\sum_{i\in I}\int_0^T\lambda_i\left(D_i-\int_0^1 m_i\right).$$

Thus, we have:

$$\int_0^T\int_0^1\left(\sum_{j\in I,j\neq i}H(\varphi_j-\varphi_i)+L\left(\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)+(\varphi_j-\varphi_i)\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}\right)m_i+\int_0^T\lambda_i\left(D_i-\int_0^1 m_i\right)=0 \tag{5.3}$$

Since $\lambda\geq 0$ and for any $t\in[0,T]$ $\int_0^1 m_i(t,s)ds\leq D_i(t)$, one has for any $i\in I$ and $t\in[0,T]$:

$$0\leq\int_0^T\lambda_i(t)\left(D_i-\int_0^1 m_i(t,s)ds\right). \tag{5.4}$$

Recalling the definition of L and H:

$$L(p):=\begin{cases}\dfrac{p^2}{2} & \text{if } p\geq 0\\ +\infty & \text{otherwise}\end{cases}\quad\text{and}\quad H(q)=\frac{(q^-)^2}{2},$$

we have: $L^*(p)=H(-p)$. One can observe that:

$$\forall q\in\mathbb{R}_-\quad H(q)+L(p)+pq=\frac{(p+q)^2}{2}\geq 0\quad\text{ and }\quad\forall q\in\mathbb{R}_+\quad H(q)+L(p)+pq\geq\frac{p^2}{2}\geq 0.$$

Using the two previous equations and (5.4), equality (5.3) gives:

$$\frac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}(t,s)=\begin{cases}\varphi_i(t,s)-\varphi_j(t,s) & \text{if } \varphi_i(t,s)-\varphi_j(t,s)\geq 0\\ 0 & \text{otherwise}\end{cases}\quad m-\text{a.e.} \tag{5.5}$$

and inequality (5.4) becomes an equality. Thus, it also comes: $\int_0^T\left(D_i-\int_0^1 m_i(t,s)ds\right)\lambda_i(dt)=0$. From equality (5.5) and Lemma 2.4, we deduce that $m\in\mathrm{Lip}([0,T]\times[0,1]\times I)$.

(2) We assume now that (λ,φ,m) is a weak solution of (5.1). Since φ is in $BV((0,T)\times(0,1)\times I)$ and λ is a finite measure, the quantity $\tilde{A}(\varphi,\lambda)$ is well defined. Thus (φ,λ) belongs to $\mathcal{R}_0$. We want to show that $\tilde{A}(\varphi,\lambda)+\tilde{B}(m,E)=0$. We approximate (λ,φ) with the same sequence $\{(\varphi^n,\lambda^n)\}_n$ as in the proof of Theorem 5.1. (1). For any n, φ^n is smooth enough to be considered as a test function for m and we have for any $i\in I$:

$$\sum_{i\in I}\int_0^1 g_im_i(T)-\varphi_i^n(0)m_i^0-\sum_{i\in I}\int_0^T\int_0^1 m_ib_i\partial_s\varphi_i^n+m_i\partial_t\varphi_i^n+\varphi_i^n\sum_{j\in I,j\neq i}(\varphi_i-\varphi_j)^+m_i-(\varphi_j-\varphi_i)^+m_j=0. \tag{5.6}$$

For any $i\in I$, φ_i^n is a classical solution of (3.1) associated to λ^n. Multiplying (3.1) by m_i, summing over I and integrating over $[0,T]\times[0,1]$, we have:

$$\sum_{i\in I}\int_0^T\int_0^1-m_i\partial_t\varphi_i^n-m_ib_i\partial_s\varphi_i^n-m_ic_i-m_i\lambda^n+\sum_{j\in I,j\neq i}H(\varphi_j^n-\varphi_i^n)m_i=0. \tag{5.7}$$

Combining (5.6) and (5.7):

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i^n(0)m_i^0 + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i + \lambda_i m_i - m_i\left(\sum_{j\in I, j\neq i}(\varphi_i-\varphi_j)^+(\varphi_i^n-\varphi_j^n) + H(\varphi_j^n-\varphi_i^n)\right) = 0.$$

Since (φ, λ, m) is a weak solution of (5.1), using Lemmas 4.4 and 3.10, and letting n tend to infinity one deduces:

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \sum_{i\in I}\int_0^1 D_i\lambda_i + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i - m_i\left(\sum_{j\in I, j\neq i}(\varphi_i-\varphi_j)^+(\varphi_i-\varphi_j) + H(\varphi_j-\varphi_i)\right) = 0.$$

Using the definition of L and H, we have:

$$\sum_{i\in I}\int_0^1 g_i m_i(T) - \varphi_i(0)m_i^0 + \sum_{i\in I}\int_0^1 D_i\lambda_i + \sum_{i\in I}\int_0^T\int_0^1 c_i m_i + m_i\left(\sum_{j\in I, j\neq i} L((\varphi_i-\varphi_j)^+)\right) = 0.$$

Using the definition of $\tilde{A}$ in (4.10) and $\tilde{B}$ in (2.3) and we have $\tilde{A}(\varphi,\lambda) + \tilde{B}(m,E) = 0$. The conclusion follows from Remark 5.2 □

5.2 Proof of Theorem (1.1)

Using Theorem 5.1, we are now ready to prove our main Theorem, applying the change of variable $\alpha_{i,j} = \dfrac{\mathrm{d}E_{i,j}}{\mathrm{d}m_i}$.
1.1.

Proof of Theorem 1.1. (1) This statement is proved by Theorem 5.1.1

(2) This point is given by Theorem 5.1.2.

(3) Using Theorem 1.1.(1), there exists (φ,λ) such that (φ,λ,m) is a weak solution of 5.1 and for any $i,j\in I$ $\alpha_{i,j} = (\varphi_i-\varphi_j)^+$. Since $\partial_s\varphi \in L^\infty((0,T)\times I, C^0([0,1]))$, then one deduces $\alpha_{i,j}\in L^\infty((0,T),\mathrm{Lip}(0,1))$. Applying Lemma 2.4, we deduce that $m\in \mathrm{Lip}([0,T]\times[0,1]\times I)$. □

A Appendix

Assumptions in subsection 1.3 are in forced in the Appendix.

A.1 On the flow defined by b

We recall the following basic properties on the flow S_i.

Lemma A.1. *For any $i\in I$, the flow S_i satisfies the equation for any $(\tau,t,s)\in(0,T)\times(0,T)\times(0,1)$:*

$$\partial_t S_i^{t,s}(\tau) + b_i(s)\partial_s S_i^{t,s}(\tau) = 0. \tag{A.1}$$

Proof. Using the definition of the flow, we have for any $(t,\tau,r,x,i)\in[0,T]\times[0,T]\times[0,T]\times[0,1]\times I$,:

$$S_i^{t,S_i^{\tau,x}(t)}(\tau) = S_i^{r,x}(\tau).$$

Deriving w.r.t. to t previous equality, applying the change of variable $s = S_i^{r,x}(t)$, and the result follows. □

Lemma A.2. *For any $(t,\tau,x,i)\in[0,T]\times[0,T]\times[0,1]\times I$,it holds:*

$$\partial_x S_i^{\tau,x}(t) = \exp\Big(\int_\tau^t b_i'(S_i^{\tau,x}(r))dr\Big) \tag{A.2}$$

Proof. From its definition, S_i satisfies for any $(t,\tau,x)\in[0,T]\times[0,T]\times[0,1]$:

$$\partial_t S_i^{\tau,x}(t) = b_i(S_i^{\tau,x}(t)), \tag{A.3}$$

deriving both terms in (A.3) w.r.t. x, it comes:

$$\partial_t\left(\partial_x S_i^{\tau,x}(t)\right) = \partial_x S_i^{\tau,x}(t) b_i'(S_i^{\tau,x}(t)). \tag{A.4}$$

Since $\partial_x S_i^{\tau,x}(\tau) = 1$, solving (A.4) gives the result (A.2). □

A.2 Analysis of weak solutions of (1.1) and (2.7)

Lemma A.3. *For any weak solution (α, m) of (1.1), in the sense of Definition 2.1, it holds for any $i \in I$ and $t \in [0,T]$: $\mathrm{supp}(m_i(t,\cdot)) \subset [0,1]$.*

Proof. Let $\varepsilon > 0$ and φ^ε be a test function in $C_c^\infty([0,T] \times \mathbb{R} \times I)$ with for any $t \in [0,T]$ and $i \in I$:

$$\varphi_i^\varepsilon(t,s) \in [0,1] \quad \forall s \in \mathbb{R}, \quad \varphi_i^\varepsilon(t,s) = 0 \quad \forall s \in \mathbb{R} \setminus (-1-\varepsilon, 2+\varepsilon) \quad \text{and} \quad \varphi_i^\varepsilon(t,s) = 1 \quad \forall s \in [-1,2],$$

Since m is a weak solution of (1.1) and b satisfies Assumption 1, we have for any $t \in (0,T)$:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \sum_{i\in I} \varphi_i^\varepsilon(t,s) m_i(t,ds) &= \int_{\mathbb{R}} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \\ &= \int_{-1-\varepsilon}^{-1} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) + \int_2^{2+\varepsilon} \sum_{i\in I} \partial_s \varphi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \\ &= 0. \end{aligned} \tag{A.5}$$

From (A.5) and continuity of m we deduce that $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} \varphi_i^\varepsilon(t,s) m_i(t,ds)$ is constant on $[0,T]$. Letting $\varepsilon \to +\infty$, it holds that $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds)$ is constant over $[0,T]$ and we have for any $t \in (0,T)$: $\int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds) = \int_{\mathbb{R}} \sum_{i\in I} m_i^0(ds) = 1$. Now we show that $\int_0^1 \sum_{i\in I} m_i(t,ds) = 1$. Let $\varepsilon > 0$ and ψ^ε be another test function in $C_c^\infty([0,T] \times \mathbb{R} \times I)$ with for any $t \in [0,T]$ and $i \in I$:

$$\begin{aligned} &\psi_i^\varepsilon(t,s) = 0 \quad \forall s \in \mathbb{R} \setminus (-\varepsilon, 1+\varepsilon), \quad \partial_s \psi_i^\varepsilon(t,s) \geq 0 \quad \forall s \in (-\varepsilon, 0), \quad \partial_s \psi_i^\varepsilon(t,s) \leq 0 \quad \forall s \in (1, \varepsilon), \\ &\quad \text{and } \psi_i^\varepsilon(t,s) = 1 \quad \forall s \in [0,1]. \end{aligned}$$

Using same calculus as in (A.5) and Assumption 1 have for any $t \in (0,T)$:

$$\frac{d}{dt}\int_{\mathbb{R}} \sum_{i\in I} \psi_i^\varepsilon(t,s) m_i(t,ds) = \int_{-\varepsilon}^0 \sum_{i\in I} \partial_s \psi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) + \int_1^{1+\varepsilon} \sum_{i\in I} \partial_s \psi_i^\varepsilon(t,s) b_i(s) m_i(t,ds) \geq 0.$$

Thus $t \mapsto \int_{\mathbb{R}} \sum_{i\in I} \psi_i^\varepsilon(t,s) m_i(t,ds)$ is non-decreasing on $[0,T]$. Taking the limit $\varepsilon \to 0$, $t \mapsto \int_0^1 \sum_{i\in I} m_i(t,ds)$ is also non-decreasing on $[0,T]$. Finally we have for any $t \in [0,T]$: $1 = \int_0^1 \sum_{i\in I} m_i(0,ds) \leq \int_0^1 \sum_{i\in I} m_i(t,ds) \leq \int_{\mathbb{R}} \sum_{i\in I} m_i(t,ds) = 1$. □

Lemma A.4. *If the pairs (α, m) and (α, μ) are weak solutions of (1.1) in the sense of Definition 2.1 and $\alpha \in L^\infty((0,T) \times I \times I, \mathrm{Lip}([0,1]))$, then $m_i(t,\cdot) = \mu_i(t,\cdot)$ for any $(t,i) \in [0,T] \times I$.*

Proof. Let χ be a standard convolution kernel on $\mathbb{R}_+$ such that $\chi > 0$. Let $\chi^n(t) := \chi(t/\varepsilon_n)/\varepsilon_n$ with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$. Fix $\theta \in C_c^\infty((0,T) \times (0,1))$ and for any $n \in \mathbb{N}$, let the function α^n be defined by:

$$\alpha^n := \chi^n * \alpha,$$

where $*$ stands for the convolution product. Then, $\alpha^n \in C^0((0,T) \times (0,1), \mathbb{R}^{|I|\times|I|})$ and for any $s \in [0,1]$, the sequence $\{\alpha^n(s,\cdot)\}_n$ converges a.e. on $[0,T]$ to α. For any $n \in \mathbb{N}$ let ξ^n be the classical solution on $(0,t) \times (0,1) \times I$ of:

$$\partial_t \xi_i^n + b_i \partial_x \xi_i^n + \sum_{j\in I, j\neq i} (\xi_j^n - \xi_i^n) \alpha_{i,j}^n = \theta \quad \text{with} \quad \xi_i^n(T,\cdot) = 0.$$

Using the method of characteristics to solve the previous PDE and the Gronwall's Lemma, one can show that $\{\xi_n\}_n$ is uniformly bounded. Since (α, m) and (α, μ) are weak solutions of (1.1), we deduce that $(\alpha, m-\mu)$ is a weak

solution of (1.1) associated to $m_0 = 0$. For any $n \in \mathbb{N}$, The function ξ^n is smooth enough to be a test function for equation (1.1) associated to $(\alpha, m - \mu)$:

$$\begin{aligned} 0 &= \sum_{i\in I} \int_0^T \int_0^1 (\partial_t \xi_i^n + b_i \partial_s \xi_i^n)(m_i - \mu_i) + \sum_{j\in I, j\neq i} (\xi_j^n - \xi_i^n)\alpha_{i,j}(m_i - \mu_i) \\ &= \sum_{i\in I} \int_0^T \int_0^1 \theta_i(m_i - \mu_i) + \sum_{j\in I, j\neq i} \left((\xi_j^n - \xi_i^n)\alpha_{i,j} - (\xi_j^n - \xi_i^n)\alpha_{i,j}^n\right)(m_i - \mu_i) \end{aligned}$$

Using the uniform bound of $\{\xi_n\}_n$ and the convergence for any $s \in [0,1]$ of $\{\alpha^n(\cdot, s)\}_n$ to $\alpha(\cdot, s)$ a.e. on $[0,T]$, letting n tends to infinity in the previous equality and one has:

$$\sum_{i\in I} \int_0^T \int_0^1 \theta_i(m_i - \mu_i) = 0$$

Since θ is arbitrary, the proof is complete. □

We are looking for regularity properties of solutions of (2.7). The following equation is introduced on $(0,T) \times (0,1) \times I$:

$$m_i(t,s) = m_i^0(S_i^{t,s}(0)) + \int_0^t (G(\tau, S_i^{t,s}(\tau))m(\tau, S_i^{t,s}(\tau)))_i d\tau, \tag{A.6}$$

where $G(t,s)$ is a square matrix of size $|I|$ and, using an abuse of notation, $m(t,s)$ is considered as a vector of size $|I|$. The quantity $(G(\tau, S_i^{t,s}(\tau))m(\tau, S_i^{t,s}(\tau)))_i$ is the i^{th} coordinate of the vector $G(t,s)m(t,s)$.

We introduce following assumptions.

a) The function G satisfies $G \in L^\infty((0,T), \mathrm{Lip}([0,1], \mathbb{R}^{|I|\times|I|}))$.

b) The function α satisfies $\alpha \in L^\infty((0,T) \times I \times I, \mathrm{Lip}([0,1]))$.

The two next lemmas state the existence of a solution m of (A.6).

Lemma A.5. *Let G satisfies Assumption* a)*, then there exists a solution m of* (A.6)*, such that for any $i \in I$ $m_i \in \mathrm{Lip}((0,T) \times (0,1))$.*

Proof. The proof of existence is based on a fixed point argument, following the steps of the proof of Lemmas 3.1, 3.2 and 3.3. The regularity of m is obtained by applying the fixed point argument on a subspace of $\mathrm{Lip}((0,T) \times I \times (0,1))$ of functions uniformly bounded with the same Lipschitz constant. □

Remark A.1. *For any $G \in C^1((0,T) \times (0,1), \mathbb{R}^{|I|\times|I|})$, there exists a unique solution $m \in C^1((0,T) \times (0,1) \times I)$ of* (A.6) *which is also a classical solution of* (2.7)*. It holds:*

$$\|m\|_\infty \leq \|m^0\|_\infty e^{|I|T\|G\|_\infty}. \tag{A.7}$$

The existence is also proved by a fixed point argument. The regularity of m is obtained by applying the fixed point argument on the space $C^1((0,T) \times (0,1) \times I)$. For any $t \in [0,T]$, using Gronwall Lemma, we have: $\sup_{i\in I} \|m_i(t,\cdot)\|_\infty \leq \sup_{i\in I} \|m_i^0\|_\infty + |I|\|G\|_\infty \int_0^t \sup_{i\in I} \|m_i(\tau,\cdot)\|_\infty d\tau \leq \|m^0\|_\infty e^{T|I|\|G\|_\infty}$, which gives uniqueness.

Lemma A.6. *Let G satisfies Assumption* a)*, m be the solution of* (A.6) *associated to G, $\{G^n\}_n$ a sequence in $C^1((0,T) \times (0,1), \mathbb{R}^{|I|\times|I|})$ and $\{m^n\}_n$ the sequence of solutions of* (A.6) *associated respectively to $\{G^n\}_n$. If $\{G^n\}_n$ converges to G w.r.t. to the L^1 norm $\|\cdot\|_1$, then $\{m^n\}_n$ converges to m w.r.t. the norm $\|\cdot\|_1$ in $L^1((0,T) \times (0,1) \times I)$ and up to a subsequence, for any $t \in [0,T]$, $\{m^n(t,\cdot)\}_n$ converges to $m(t,\cdot)$ w.r.t. the metric W_1 in $\mathcal{P}([0,1] \times I)$.*

Proof. Since m and m^n are solutions of (A.6), associated respectively to G and G^n, one has for any $(i,t,s) \in I \times [0,T] \times [0,1]$:

$$|m_i^n(t,s) - m_i(t,s)| \leq \int_0^t |(G^n(\tau, S_i^{t,s}(\tau))m^n(\tau, S_i^{t,s}(\tau)))_i - (G(\tau, S_i^{t,s}(\tau))m(\tau, S_i^{t,s}(\tau)))_i| d\tau$$

Applying the triangle inequality, integrating over $[0,1]$ and summing over I, applying the change of variable $x = S_i^{t,s}(\tau)$, using that G is bounded and that from Remark A.1 $\{\|m_n\|_\infty\}_n$ is bounded by a constant $C_G > 0$, we have:

$$\begin{aligned}\sum_{i\in I}\int_0^1 |m_i^n(t,s) - m_i(t,s)|ds \leq \quad & |I|\|\partial_x S\|_\infty \|G\|_\infty \int_0^t \int_0^1 \sum_{j\in I} |m_j(\tau,x) - m_j^n(\tau,x)|ds d\tau \\ & + |I| C_G \int_0^t \int_0^1 \sum_{i,j} |G_{i,j}(\tau,x) - G_{i,j}^n(\tau,x)|ds d\tau.\end{aligned}$$

Applying Gronwall lemma and integrating over $[0,T]$, we have: $\|m^n - m\|_1 \leq T|I|C_G\|G^n - G\|_1 e^{T\|\partial_x S\|_\infty \|G\|_\infty}$. Using the convergence of $\{G^n\}_n$ to G w.r.t. to the norm $\|\cdot\|$, convergence of $\{m^n\}_n$ to m w.r.t. to the norm $\|\cdot\|_1$ is obtained. Using the continuity of m and the convergence a.e. of a subsequence of $\{m^n\}_n$ to m, the convergence of $\{m^n(t,\cdot)\}$, up to a subsequence, to $m(t,\cdot)$ w.r.t. W_1 in $\mathcal{P}([0,1]\times I)$ is deduced. □

Lemma A.7. *Let α satisfies Assumption* b)*, G be defined from α as in* (2.8) *and m be the solution of* (A.6) *associated to G. Then, m is a weak solution of* (1.1)*, in the sense of Definition 2.1.*

Proof. Let the sequence $\{\alpha^n\}_n$ in $C^1((0,T)\times(0,1)\times|I|\times|I|)$ converge to α w.r.t. the norm $\|\cdot\|_1$ and be uniformly bounded by $\|\alpha\|_\infty$. Let the sequence $\{G^n\}_n$ in $C^1((0,T)\times(0,1),\mathbb{R}^{|I|\times|I|})$ be defined as in (2.8) associated with $\{\alpha^n\}_n$. Then $\{G^n\}_n$ converges to G w.r.t. the norm $\|\cdot\|_1$. For any $n\in\mathbb{N}$, m^n denotes the solution of (1.1) associated to G^n. From Lemma A.6, for any test function $\psi \in C_c^\infty([0,T]\times\mathbb{R}\times I)$, it holds:

$$\sum_{i\in I}\int_0^1 \psi_i(T)m_i^n(T) - \psi_i(0)m_i^0 \quad = \int_0^T \int_0^1 \sum_{i\in I}(\partial_t\psi_i + b_i\partial_s\psi)m_i^n + \sum_{j\in I, j\neq i}(\psi_j - \psi_i)\alpha_{i,j}^n m_i^n, \tag{A.8}$$

where α^n is linked to G^n by definition (2.8). Since $\{\alpha^n\}_n$ converges weakly*, up to a subsequence, to α in $L^\infty((0,T)\times(0,1)\times|I|\times|I|)$, the result is obtained by applying Lemma A.6 and taking the limit $n\to\infty$ in equation (A.8). □

Lemmas A.4, A.5 and A.7 can be sum up by the following proposition.

Proposition 1. *Let α satisfies Assumption* b) *and G be defined from α as in* (2.8)*. Then there exists a unique solution m of* (A.6)*, which is also the unique weak solution of* (1.1)*, in the sense of Definition 2.1, associated to α. In addition, m is bounded by $\|m^0\|_\infty e^{|I|T\|G\|_\infty}$, and is such that for any $i\in I$, $m_i \in \mathrm{Lip}([0,T]\times[0,1])$.*

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