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Min-Plus realizable control design for partially observable timed event graphs under marking constraints

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Abstract This paper deals with a control problem of discrete event systems subject to capacity constraints. Combined use of timed event graphs and Min-Plus algebra is a well-known approach and efficient for handling timed behavior and mathematical modelling of discrete event systems. **However, in current literature, many of control approaches assume that system states are fully observable, which is not the case in our study. Hence, we propose in this paper a feedback control method to guarantee the respect of marking constraints imposed for some paths of partially observable timed event graphs. We demonstrate that if each loop of the considered TEG contains at last one observable transition, we can derive a realisable control law satisfying a set of constraints.**

Keywords Discrete Event Systems, Timed event graphs, Min-Plus algebra, Marking constraints, Realizable control laws.

1 Introduction

In many industrial contexts, we find different critical specifications that must be met to ensure the proper functioning of the systems. These functional constraints can take several forms and they are often encountered in various real applications like flexible manufacturing workshops including thermal or chemical treatments, real-time systems, urban or rail transport and networked automation systems. In this work we treat a class of deterministic controlled

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discrete event systems subject to strict constraints. In the literature, some authors focused in their research on the respect of durations in some sensitive areas [1], [2], [16], [30] where the objective is to satisfy temporal constraints imposed on the behavior of systems. Others considered specifications related to capacities which are described by marking constraints or forbidden states in Petri nets [13], [3], [35], [33]. In our study, we are specifically interested to the problem of control under constraints for discrete event systems represented by timed event graphs. **It's well known that dioids are an algebraic structure that can be used to formally generate linear models for a class of timed discrete-event systems [4].** Combined use of TEG and Max-Plus (or Min-Plus) is a well-known approach in the literature initiated back in the 1960's [10] and it is still a very active field of research. **This framework has been successfully applied to solve diverse control problems, such as dynamic scheduling [7], [11], [17], synchronization of switching models [21], [22], trajectory tracking [8], disturbance decoupling problem [29], or just-in-time control [14], [19]. Both open loop [29] and feedback [20], [24], [23] approaches have been considered to solve those problems. Significant efforts have also been made to apply model predictive control techniques to Max-Plus linear systems [11], [32]. Other methods are also developed to design a controller to guarantee that a Max-Plus system evolves without violating time restrictions imposed on the state [24], or even characterized by a semimodule [23]. The feedback control problem of Max-Plus linear systems under temporal constraints has been addressed using a temporal approach based on daters like in [1], [25]. The objective of calculated control laws is to ensure maximum time limits of the sojourn for tokens in some places of a TEG. In [30], authors proposed a closed-loop control to satisfy a critical response time for networked automation systems based on timed event graphs and Max-Plus algebra. Others approaches deal with the constraint control of max-plus linear systems by using its transformed version based on power series [9], [15] which is similar to Z transform for conventional linear system theory. An alternative method was proposed in [18], where the constraints satisfaction is expressed as the (A, B)invariance of a semimodule. The control problem is solved by computing the maximal of such semimodules, included in a domain defined by constraints.**

In this paper, we address the feedback control problem for TEGs under marking constraints by means of Min-Plus algebra formalisms. Historically, the question is originally treated by Ramadge and Wonham [28], [27], using finite automata and formal languages. The synthesized controller (supervisor) has the role to inhibit controlled events in order to satisfy the specifications. Other approaches are based on untimed Petri nets tools [26], where the constraints are formulated as a problem of linear marking constraints. **The role of the controller is to limit the number of tokens in the some places,** which can represent the bounded stock in the industrial context: the buffer memory in communication systems or stations in transportation networks. In [13], [34], [33], authors express various specifications to be respected by linear inequalities on markings. Monitor places are synthesized using the P-invariants property of Petri nets. Another approach to solve the forbidden state problem in discrete event

systems modeled by Petri nets with unobservable transitions is proposed in [12]. This method is based on a linear constraint transformation, **authors deal with the problem of forbidden states** in safe Petri nets and they establish a maximally permissive controller without the time. Time is not taken into account in the system model and control synthesis in existing contribution on capacity constraints. Practically for the dynamic study of systems, time parameter is very important and must be integrated into modelling and control laws. Some studies based on timed Petri nets models dealing with the control under marking constraints appeared during these last years. We can cite for example the approach [31] based on the TEG and the Min-Plus algebra proposed for the control of TEGs under linear marking constraints. Authors considered the case where all transitions of the TEG are not controllable but all are observable. The problem posed in the case of TEGs with some non observable transitions is that the calculated controller is not always realizable. For this, it is necessary to assure that established control laws are given in function of observable transitions.

The present paper deals with the design of feedback control laws for timed discrete event systems subject to capacity constraints using Min-Plus models. This study directly extends and improves the methods recently developed in [31], which have a strong limitation that all transitions of the TEG are observable and measurable. In this work, we propose a new control method of partially observable TEGs subject to linear marking constraints. **The objective is to provide realizable control laws to guarantee different specifications required on markings of TEGs.** The calculated laws are expressed only in terms of observable transition states.

The remainder of this paper is organized as follows. Section 2 recalls some basic notions of TEGs and their linear Min-Plus algebra representations well as so some definitions and hypothesis that are needed for the comprehension of the sequel of document. Problem of limited capacities and linear marking constraints is presented in section 3 where the specifications are expressed in terms of linear inequalities in Min-Plus algebra. Section 4 is dedicated to the description of the proposed approach. Finally, section 5 concludes this paper with some perspectives for future work.

2 PRELIMINARIES

2.1 Min-Plus algebra

A monoid is a set, say D , endowed with an internal law, noted \oplus which is associative, commutative and has a neutral element denoted ε (i.e., $\forall a \in D, a \oplus \varepsilon = \varepsilon \oplus a = a$), \otimes is associative, admits a unit element denoted $e = 0$, and distributes over \oplus ; and ε is absorbing for the multiplicative law ($\forall a \in D, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). The semiring is commutative when \otimes is commutative. A dioid (D, \oplus, \otimes) is a semiring with an idempotent internal law (i.e., $\forall a \in D, a \oplus a = a$). The dioid is said to be commutative if the second law \otimes is

commutative. The Min-Plus algebra is defined as $(R \cup -\infty, +\infty, \min, +)$. This semiring, denoted \bar{R}_{min} , is a commutative dioid, the law \oplus is the operator \min with neutral element $\varepsilon = +\infty$, and the second law \otimes is the usual addition, with neutral element $e = 0$. \otimes is abbreviated by \cdot (dot) for convenience. $(+, \times)$ refers to the usual addition and multiplication.

As in classical algebra, the binary operations can be extended to the matrix case in dioid. For a matrix $A \in \bar{R}_{min}^{(m \times n)}$, the element of the i^{th} row and j^{th} column of the matrix A is denoted by A_{ij} . For any three matrix defined by $A, B \in \bar{R}_{min}^{(m \times n)}$ and $C \in \bar{R}_{min}^{(n \times p)}$ in Min-Plus algebra \bar{R}_{min} , we define: Addition $A \oplus B$:

$$[A \oplus B]_{ij} = A_{ij} \oplus B_{ij} = \min(A_{ij}, B_{ij})$$

Multiplication $A \otimes C$:

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^n A_{ik} \otimes C_{kj} = \min_{k=1, \dots, n} (A_{ik} + C_{kj}).$$

Theorem 1 [4] *The implicit equation $x = A \cdot x \oplus B$ defined over a complete dioid D admits $x = A^* \cdot B$ as a least solution with A^* is the Kleene star of the matrix A , such as:*

$$\forall A \in D, A^* = \bigoplus_{i>0} A^i$$

where

$$A^i = A^{i-1} \otimes A \text{ and } A^0 = I$$

I is the unit matrix, with entries equal to e on the diagonal, and ε else.

The reader is invited to consult [4] and [5] for a complete presentation of the following theoretical recalls.

2.2 Timed Event Graphs (TEGs) and linear Min-Plus models

2.2.1 Timed Event Graphs (TEGs)

An event graph is a Petri net where each place has exactly one upstream and one downstream transition. A timed event graph (TEG) is an event graph with delays associated to places, the holding times, and/or transitions, the firing times. An event is the firing of a transition. In the present work, we use the TEG, where the delays are associated only with the places and we adopted the following notations:

- P will denote the set of places of the considered graph and T is a set of transitions.
- The internal transitions (transition having at least one upstream place) are denoted by T_i (with $\|T_i\| = N$), and the source transitions or the input transitions (transition having no upstream place) are represented by T_{uc} (with $\|T_{uc}\| = \mu$).
- t_j denotes the j^{th} transition of the graph,
- T_ω denote the set of the observable transitions and T_β denote the set of the unobservable transitions, where $\|T_\omega\| + \|T_\beta\| = N$,
- p_{ij} denotes the place linking t_j to t_i when it exists,

- An elementary path between two internal transitions t_j and t_i of a graph is an oriented alternating sequence of transitions and places successively connected by an arc, of the form $(t_j, p_{1j}, t_1, p_{21}, t_2, p_{32}, \dots, t_i)$. This path is denoted by ρ . The cumulated token in this path is given by m_ρ and their timing is denoted by τ_ρ where this timing is given by the sum of the places of the path timing.
- Place of a TEG is a particularly case of the path with a longer is 1.
- An elementary path between the input transition t_u and the internal transitions t_j is an oriented alternating sequence of transitions and places successively connected by an arc. This path is denoted by α . The cumulated tokens in this path is given by m_α and its timing is denoted by τ_α .
- For a number m_{ij} of tokens, several m_{ij} -token paths connecting t_j to t_i can exist in general.
- $t_j \xrightarrow{m_{ij}, \tau_{ij}} t_i$ denotes the m_{ij} -token path with the maximal delay, which is denoted by τ_{ij} ,
- An empty path contains no token,
- A loop in Petri net is a path whose start node coincides with the arrival node.
- We note that the transitions without downstream places are outputs and those without upstream places are inputs. Others are simply called internal transitions. Inputs are also referred to as controls.

2.2.2 Linear Min-Plus models

A TEG being a directed graph, its dynamic behavior can be described by the linear equations thanks to counter functions $x_i(t)$ associated for each transition t_i of the graph. The counter functions $u(t)$ attributed to the input transition t_u , with the function $x_i(t)$ represents the firing number of the transition t_i at time T. Generally, the behavior of the TEG is represented by a following equation:

$$x(t) = \bigoplus_{\tau=0}^{\tau^{max}} (A_\tau \cdot x(t-\tau) \oplus B_\tau \cdot u(t-\tau)) \quad (1)$$

where $A_\tau \in \bar{\mathbb{R}}_{min}^{N \times N}$ is the matrix whose term $A_{\tau,ij}$ it is equal to m_{ij} which is the number of the initial marking of the place p_{ij} if it exists and ε else. $B_\tau \in \bar{\mathbb{R}}_{min}^{N \times \mu}$ corresponds to the initial markings of the exit places of the source transitions.

It is known that, for a live TEG, by decomposing each place into several places which are timed with values less than or equal to 1, the state vector can be extended and rewriting under the following explicit form:

$$x(t) = A \cdot x(t-1) \oplus B \cdot u(t) \quad (2)$$

where $A = A_0^* \cdot A_1$ and $B = A_0^* \cdot B_0$.

The detailed procedure for the Min-Plus equations representation of timed

event graphs is illustrated in the application example in section 4.3 of the page 13.

Finally, for any integer τ such that $\tau \geq 1$, by doing τ substitutions in (2), one obtains:

$$x(t) = A^\tau \cdot x(t - \tau) \bigoplus \bigoplus_{k=0}^{\tau-1} A^k \cdot B \cdot u(t - k) \quad (3)$$

Property 1 : *In the TEG with the assumptions that each loop of this graph contains at last one observable transition, for any unobservable transition $t_i \in T_\beta$, there exists a path δ connecting this transition to an observable transition $t_j \in T_\omega$. The counter associated to the unobservable transition t_i can be expressed in function of the observable transition t_j by the following expression:*

$$x_i(t) \leq m_\delta \cdot x_j(t - \tau_\delta), \quad (4)$$

where m_δ represents the number of tokens in the path δ , and τ_δ corresponds to the timing of this path.

Definition 1 (Observability). A transition t_i is observable if it is either a control or it is linked directly to an output transition. A TEG is fully observable if all his internal transitions t_i are observable.

Definition 2 (Partially observable TEG)

A graph G is said to be partially observable if it has at least one unobservable transition.

Definition 3 (Structural controllability)

An event graph is structurally controllable if every internal transition can be reached by a path from at least one input transition.

Definition 4 (Valid control)

The control law $u(t)$ is said to be a validated controller if it grantees the constraints imposed to the system without taking into account any other system specification.

Definition 5 (Realizable control)

A control law is said to be realizable if it is a valid control expressed by a causal and observable expression.

3 Problem of limited capacities and linear marking constraints

3.1 Limited capacities problem

To simplify the explanation and show the motivations of the work, we present a simple information transmission example with two tasks A and B in series (see Fig.1). We consider that the execution time of the task A is fixed to 1 milliseconds (ms), the execution time of the task B is fixed to 3 ms, and the transmission delay of i_1 , is 2 ms.

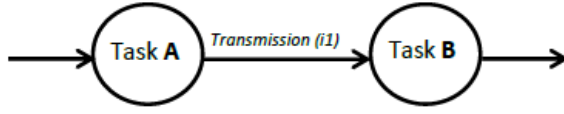


Fig. 1 Example of an information transmission network

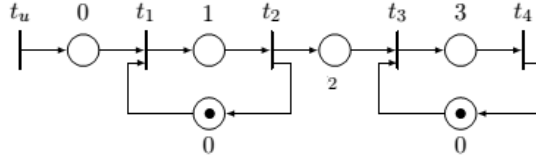


Fig. 2 Model TEG of the information transmitting system

The considered system can be modeled by the TEG shown in the Fig.2. Let us presume that the transitions (t_u, t_1, t_3) are the only observable transitions of this graph and we consider that the transmission line cannot transfer more than one information at a time which implies that the number of tokens in the place p_{32} cannot exceed 1 token.

The cycle time of the task B is greater than the cycle time of both the task A and the delay of transmission. This leads to exceeding a maximum number of the information transmitted in the transmission line. For the case where the internal transitions of the graph are controllable, the solution of this problem is intuitive; it is enough to add a place p_{23} marked with 1 tokens for example, to ensure that the number of the tokens in the place p_{32} does not exceed 1 tokens. The resolution of the problem in the case where the input transition of the place under constraint is uncontrollable is not obvious. Therefore, it is necessary, to have other techniques to guarantee these marking constraints. The objective of this paper is to synthesise control laws to satisfy the marking constraints imposed on the some path of the partial observable timed event graph with uncontrollable transitions.

3.2 Linear marking constraints

We consider a discrete event system modeled by a timed event graph. We assume that the path ρ of this graph is subject to a marking constraint, where the path ρ cannot contain more than b tokens throughout the system operation. We denote by t_j and t_i respectively the input and the output transition of the constrained path, and by $M_\rho(t)$ the number of the tokens in this path at time

t . The marking constraint can be described by the following inequality:

$$M_\rho(t) \leq b \quad (5)$$

In order to express this constraint by a linear inequality in the Min-Plus algebra, we associate with each transition of the graph a counter function and we denote by $x_i(t)$ and $x_j(t)$ respectively the counter associated to the transitions t_i and t_j . By definition, the counter $x_j(t)$ represents the number of firing of the transition t_j at time t . Then, in Min-Plus algebra $M_\rho(t)$ can be represented by $((x_j(t) - x_i(t)) + M_{0\rho})$, such that $M_{0\rho}$ is the initial marking of the path ρ .

The inequality (5) can be written as:

$$x_j(t) - x_i(t) + M_{0\rho} \leq b \quad (6)$$

This marking constraint can be expressed by a linear inequality in Min-Plus algebra as:

$$x_j(t) \leq x_i(t) + (b - M_{0\rho}) \quad (7)$$

4 Control synthesis

4.1 Realizable feedback controls

In this section, we propose a formal approach for the control of the TEG subject to one or several marking constraints. As expressed previously, the proposed approach is based on the use of the TEG and Min-Plus algebra for synthesizing a realizable control law on the form $u(t) = F \cdot x(t - 1)$. The role of the calculated controller is to ensure the satisfaction of the marking constraints imposed on the considered partially observable TEG. We consider that the firing of the transitions of the TEG is performed as soon as possible. The transitions of this considered TEG are not all controllable i.e only the input transitions are controllable. First, we consider the case where the system has a single input and then we extend the results to the case where the system has several inputs. In this work we consider that the marking constraints concerns the places situated on the same path and we consider the following Hypothesis:

Hypothesis We assume in the following that each loop of the considered timed event graph contains at last one observable transition.

4.1.1 Single control

a. Only one marking constraint

In this first case, we consider a timed event graph with the state equation of this system is given by the equation (2). The path ρ connecting the transition t_j to the transition t_i of this graph is subject to a marking constraint. This constraint is given by the inequality (7). We assume that there exists at least a path α from the control transition t_u to the transition t_j , where t_j is the

upstream transition of the constrained path ρ , and we denote by m_α and τ_α respectively the marking and the timing of this path.

From the definition of t_u , we have:

$$x_j(t) \leq (A^{\tau_\alpha} \cdot B)_j \cdot u(t - \tau_\alpha) \quad (8)$$

Taking $\tau = \phi$ in the equation (3), $x_i(t)$ can be expressed by:

$$x_i(t) = \left[\bigoplus_{r=1}^N A_{ir}^\phi \cdot x_r(t - \phi) \right] \oplus \left[\bigoplus_{k=0}^{\phi-1} (A^k \cdot B)_i \cdot u(t - k) \right], \quad (9)$$

where $\phi \geq 1$, $A \in \bar{\mathbb{R}}_{min}^{N \times N}$, N represents the number of the internal transition of the TEG and A_{ir} indicates the i^{th} row of the matrix A^ϕ .

For every integer $\phi \geq 1$, we take into account the Property 1 and we have the following result.

Theorem 2 Taking $\phi = \tau_\alpha + 1$, the equation:

$$u(t) = F_1 \cdot x_r(t - \tau_{\delta_z} - 1) \oplus F_2 \cdot x_r(t - 1) \quad (10)$$

with, $F_1 = \bigoplus_{z=1}^{\beta} \bigoplus_{r \in \omega} [(b + m_{\delta_z} - M_{0\rho}) - (A^{\tau_\alpha} \cdot B)]_j \cdot A_{ir}^{(\tau_\alpha+1)}$ and $F_2 = \bigoplus_{r \in \omega} [b - M_{0\rho} - (A^{\tau_\alpha} \cdot B)]_j \cdot A_{ir}^{(\tau_\alpha+1)}$

defines realizable controls which guarantee that the constraint (7) is satisfied if the following condition is verified:

$$(A^{\tau_\alpha} \cdot B)_j \leq (b - M_{0\rho})(A^k \cdot B)_i \quad \text{for } k = 0 \text{ to } \tau_\alpha \quad (11)$$

Proof Let G be a timed event graph and ρ a path under marking constraint with

$$M_\rho(t) \leq b$$

As we saw previously this constraint can be expressed by the following inequality:

$$x_j(t) \leq x_i(t) + (b - M_{0\rho}) \quad (12)$$

We consider control laws defined by the following inequality (13):

$$(A^{\tau_\alpha} \cdot B)_j \cdot u(t - \tau_\alpha) \leq x_i(t) + (b - M_{0\rho}) \quad (13)$$

It is clear that the combination of inequalities (13) and (8) implies the satisfaction of the marking constraint (12). In other words, if we choose control laws that verify the inequality (13) and as the expression (8) that describes the path between two transitions t_u and t_j is always true, then by the transitivity relation, we conclude that the marking constraint (12) is guaranteed.

The next step of this demonstration consists to replace the counter function $x_i(t)$ in the inequality (13) by its expression given by the equation (9). After

this change, the inequation (13) is divided into both conditions (14) and (15) which are as follows:

$$(B^{\tau_\alpha} \cdot B)_j \cdot u(t - \tau_\alpha) \leq (b - M_{0\rho}) + [\oplus_{r=1}^N A_{ir}^\phi \cdot x_r(t - \phi)] \quad (14)$$

$$(A^{\tau_\alpha} \cdot B)_j \cdot u(t - \tau_\alpha) \leq (b - M_{0\rho}) + [\oplus_{k=0}^{\phi-1} (A^k \cdot B)_i \cdot u(t - k)] \quad (15)$$

We can easily see that in order to check the satisfaction of inequality (13), it is sufficient to ensure that the two inequalities (14) and (15) if they are simultaneously true.

The purpose is to synthesize a realizable control law in the form $u(t) = F \cdot x(t - 1)$. In order to obtain the controller in this form, we take $\phi = \tau_\alpha + 1$, and the fact that $u(t) \geq u(t - 1)$, the inequality (15) can be given as:

$$(A^{\tau_\alpha} \cdot B)_j \leq (b - M_{0\rho}) + (A^k \cdot B)_i$$

This inequality represents the condition of the Theorem 2. The control inequality described in the Theorem 2 derived from the inequality (15), represents the control laws which satisfy the marking constraint (12), if the condition is (14) is satisfied.

The formula:

$$u(t) = \oplus_{r=1}^N (b - M_{0\rho} - (A^{\tau_\alpha} \cdot B)_j) \cdot A_{ir}^{(\tau_\alpha+1)} \cdot x_r(t - 1)$$

describes a valid controller for a partially observable TEG, but it is not a realizable controller, since the formula of this controller is given in function of the internal transitions of the TEG which contains the unobservable transitions. Since each loop of the considered TEG contains at last one observable transition, this controller can be transformed into a realizable controller by applying the result of the property 1. For this, we divide the controller formula into two parts, one part according to the observable transitions and the other part according to the unobservable transitions as described by the inequality (17):

$$u(t) = [\oplus_{r \in \beta} (b - M_{0\rho} - (A^{\tau_\alpha} \cdot B)_j) \cdot A_{ir}^{(\tau_\alpha+1)} \cdot x_r(t - 1)] \oplus \quad (16) \\ [\oplus_{r \in \omega} (b - M_{0\rho}) (A^{\tau_\alpha} \cdot B)_j \cdot A_{ir}^{(\tau_\alpha+1)}] \cdot x_r(t - 1).$$

We take into account the results described in the Property 1, where each counter function associated to the unobservable transition can be expressed in function of the counter function associated with observable transition. The equation that describes the calculated control law in function of the observable transitions is as follows:

$$u(t) = \oplus_{z=1}^\beta [\oplus_{r \in \omega} [b + m_{\delta z} - M_{0\rho}] - (A^{\tau_\alpha} \cdot B)_j \cdot A_{ir}^{(\tau_\alpha+1)} \cdot x_r(t - \tau_{\delta z} - 1)] \oplus \\ [\oplus_{r \in \omega} [(b - M_{0\rho} - (A^{\tau_\alpha} \cdot B)_j) \cdot A_{ir}^{(\tau_\alpha+1)}] \cdot x_r(t - 1)]$$

b. Several constraints

We assume now a timed event graph with one input transition and subject to S marking constraints. These constraints are imposed on S paths of this graph denoted ρ_s for $s = 1$ to S . Let $M_{0\rho_s}$ and τ_{ρ_s} respectively represent the initial marking and the timing of the path ρ_s . We denote by t_d and $t_{d'}$ respectively the input and the output transition of the path ρ_s . x_d and $x_{d'}$ indicate respectively the corresponding counters.

We take into account the fact that the marking constraints concern only the places located on the same path and we assume that for each s , with $s = 1$ to S , there is at least one path denoted α_s . This path connects the input transition t_u to the transition t_d , which represents the input transition of the constrained path ρ_s . We denote by τ_{α_s} the timing of the path α_s . The different marking constraints imposed in this graph are expressed by the following inequality:

$$x_d(t) \leq (b_s - M_{0\rho_s}) \cdot x_{d'}(t) \quad \text{for } s = 1 \text{ to } S \quad (17)$$

Theorem 3 *The following control law:*

$$u(t) = \oplus_{s \geq 1} u_s(t), \quad (18)$$

with

$$u_s(t) = F_{1s} \cdot x_r(t - \tau_{\delta_z} - 1) \oplus F_{2s} \cdot x_r(t - 1) \quad (19)$$

where

$$F_{1s} = \bigoplus_{z=1}^{\beta} [\oplus_{r \in \omega} [(b_s + m_{\delta_z} - M_{0\rho_s}) - (A^{\tau_{\alpha_s}} \cdot B)_d \cdot A_{ir}^{(\tau_{\alpha_s} + 1)}]] \text{ and}$$

$$F_{2s} = [\oplus_{r \in \omega} [b_s - M_{0\rho_s} - (A^{\tau_{\alpha_s}} \cdot B)_d] \cdot A_{d'r}^{(\tau_{\alpha_s} + 1)}].$$

ensures the satisfaction of marking constraints (17) if the condition below:

$$(A^{\tau_{\alpha_s}} \cdot B)_d \leq (b_s - M_{0\rho_s})(A^k \cdot B)_{d'} \quad (20)$$

is verified for each $s = 1$ to S .

Proof According to the results obtained in the section 4.1, the controller $u(t)$ described in the Theorem 2 guarantees the satisfaction of the marking constraint (7) if the condition (11) is verified. Then, for S marking constraints the control law $u_s(t)$ prevents the violation of the S^{th} constraint if the condition (20) holds. The control laws which guarantee the satisfaction of the S marking constraints (17) are given by the following equation: $u(t) = \oplus_{s \geq 1} u_s(t)$, for $s = 1$ to S .

4.2 Several controls

We now generalize the previous approach to solve the problem of marking constraints in TEGs with several input transitions. The behavior of the considered TEG is described by the state equation (2), with $u(t) \in \bar{R}_{min}^{\mu}$ and $B \in \bar{R}_{min}^{N \times \mu}$, μ represents the number of the input transition, with $\mu \geq 1$. We associate with each source transition a counter noted $u_l(t)$, for $l = 1$ to μ . We assume that

this graph is subject to marking constraints related for the same path. First, we consider the case where this graph is subject to one constraint, and then we extend the results to the case where several constraints are imposed to this graph.

a. Single constraint

We suppose in this sub-section a TEG with μ input transitions, and subject to one marking constraint. This constraint is imposed on a single path of this graph denoted ρ . We denote by t_j and t_i respectively the input and the output transition of the path ρ . $M_{0\rho}$ and τ_ρ represent respectively initial marking and the timing of the path ρ . This constraint can be described by the following inequality:

$$M_\rho(t) \leq b \quad (21)$$

The aim of the approach is to compute the components of the vector $u(t)$ which will satisfy the marking constraint (21). We always assume that the input transition t_j of the constrained path ρ is connected to at least one input transition t_{u_c} of the considered TEG. This path is noted α_c with $c = 1$ to μ . The timing of this path is denoted by τ_{α_c} .

Theorem 4 *Let G be a partially observable TEG with μ input transitions, whose dynamics is modeled by the equation (2). This graph is subject to a marking constraint represented in (21).*

The marking constraint is satisfied if there exists an index c such as

$$u_c(t) = F_{1c} \cdot x_r(t - \tau_{\delta_z} - 1) \oplus F_{2c} \cdot x_r(t - 1). \quad (22)$$

where

$$F_{1c} = \bigoplus_{z=1}^{\beta} [\bigoplus_{r \in \omega} [(b + m_{\delta_z} - M_{0\rho}) - (A^{\tau_\alpha} \cdot B)_{j_c}] \cdot A_{ir}^{\tau_\alpha + 1}]$$

$$F_{2c} = \bigoplus_{r \in \omega} [(b - M_{0\rho} - (A^{\tau_\alpha} \cdot B)_{j_c})] \cdot A_{ir}^{(\tau_\alpha + 1)}$$

With $u_l(t) = \varepsilon$ for $l \neq c$ and if the condition given by

$$(A^{\tau_{\alpha_s}} \cdot B)_{dc} \leq (b - M_{0\rho_0})(A^k \cdot B)_{d'l} \quad (23)$$

for $l = 1$ to μ is satisfied.

Proof The procedure to calculate the control laws $u_c(t)$ for a given $c \in \{1, \dots, \mu\}$, is similar that the controller synthesis $u(t)$ proposed in the Theorem 2. In this case, the formula (22) of the control law $u_c(t)$ is given if the condition (21) is satisfied for an index c and then the other controls can be forced to ε .

b. Multiple constraints

In this section, we extend the work to deal with the general case, where a partially observable TEG has μ input transitions, and is subject to s marking constraints with $s = 1$ to S . In this case as well we assume that each path

ρ_s is connected to at least one input transition by a path α_s where ρ_s represents the constrained path. The marking constraints are given by the following inequality:

$$M_{\rho_s}(t) \leq b_s, \quad s = 1 \text{ to } S. \quad (24)$$

The following Theorem gives the results for this study.

Theorem 5 *Let G be a partially observable TEG with μ input transitions and is subject to S constraints of the form (24). The following control law:*

$$u(t) = \bigoplus_{s=1}^S u_{c_s}(t) \quad (25)$$

requires compliance with the constraints (24) if the condition holds true for each $s = 1$ to S

Proof The control law (25) provided from the Theorem 5, ensures the respect of one constraint. For the case where we have several constraints, the controller represented by the equation (22) ensures the satisfaction of all considered constraints (25) if the condition (23) is verified for each constraint.

4.3 Example

We consider the example of the timed event graph described in the section 3. We suppose that the place p_{32} subject to a marking constraint, as it was represented in the section 3, where the number of tokens in the place p_{32} cannot exceed 1 token. This constraint can be reformulated in the form of the following inequality:

$$x_2(t) \leq x_3(t) + 1 \quad (26)$$

We associate for each transition of the TEG, represented by the Fig.2, a counter function ($\theta_i(t)$) for $i = 1$ to $i = 4$. The behavior of this graph can be described by following expressions:

$$\begin{cases} \theta_1(t) = \min(0 + u(t), 1 + \theta_2(t), \\ \theta_2(t) = 0 + \theta_1(t - 1), \\ \theta_3(t) = \min(0 + \theta_2(t - 2)), 1 + \theta_4(t), \\ \theta_4(t) = 0 + \theta_3(t - 3), \end{cases}$$

Using Min-Plus algebra operator (\oplus) for the min operator and (\cdot) for the classical sum $+$, these equations can be written as:

$$\begin{cases} \theta_1(t) = u(t) \oplus 1 \cdot \theta_2(t), \\ \theta_2(t) = \theta_1(t - 1), \\ \theta_3(t) = \theta_2(t - 2) \oplus 1 \cdot \theta_4(t), \\ \theta_4(t) = \theta_3(t - 3), \end{cases}$$

We finally get to the following matrix notation:

$$\theta(t) = A_0 \cdot \theta(t) \oplus A_1 \cdot \theta(t-1) \oplus A_2(t-2) \oplus A_3(t-3) \oplus B_0 \cdot u(t) \quad (27)$$

with:

$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \end{bmatrix}, A_0 = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, A_1 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, A_4 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \end{bmatrix}, B_0 = \begin{bmatrix} e \\ e \\ e \\ e \end{bmatrix}$$

Using the Theorem 1, the implicit equation (27) can be replaced by the following explicit solution,

$$\theta(t) = A_0^* \cdot A_1 \cdot \theta(t-1) \oplus A_0^* \cdot A_2 \cdot \theta(t-2) \oplus A_0^* \cdot A_3 \cdot \theta(t-3) \oplus A_0^* \cdot B_0 \cdot u(t)$$

where A_0^* is the Kleene star of A_0 .

Our objective is to calculate a realizable controller which satisfies this marking constraint. For this we apply Theorem 2. In order to obtain the state equation of this system, we decompose each place of the TEG shown in Fig.2 with time greater than 1 time unit to several places timed to 1 time unit. **The initial graph of the information transmission network (see the Fig.2) has been extended to the new timed event graph represented by the Fig.3).** The place p_{32} and the place p_{43} timed respectively with 2 and 3 time units has been split respectively to 2 and 3 places timed to 1 time unit, and the intermediate transition t_5 , t_6 and t_7 has been added.

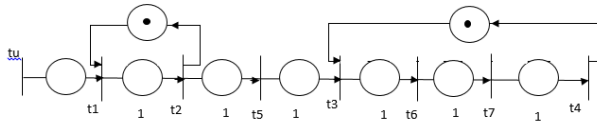


Fig. 3 The extended TEG

We apply for this example the Min-Plus modeling approach of TEG described in section 2.2.2. The resulting state representation of the extended

graph is given below:

$$x(t) = \begin{bmatrix} 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \cdot x(t-1) \oplus \begin{bmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix} \cdot u(t). \quad (28)$$

We check the compliance of required conditions by the Theorem 2 for the existence of the control laws which guarantee the constraint formalized by the inequality 26. There exists a path α from transition t_u to transition t_2 , with timing $\tau_\alpha = 1$. Then $\phi = \tau_\alpha + 1 = 2$. We have also, $(A^k \cdot B)_i = [e, \varepsilon]$, for $k = 0$ to 1, where $(A^{\tau_\alpha} \cdot B)_j = 1$, where the condition of the Theorem 2 is satisfied. We have also each loop of the considered TEG containing at last one observable transition. According to the Theorem 2, there exists a realizable control law which satisfies the marking constraint. For this example, we have $A_{3r}^2 = [\varepsilon \ e \ \varepsilon \ \varepsilon \ \varepsilon \ 1 \ \varepsilon]$. When take into account the equation (10), the control law is written as follows:

$u(t) = (1 - 1) \cdot [e \cdot x_1(t - 1 - 1) \oplus 1 \cdot x_3(t - 2 - 1)] = 1 \cdot x_1(t - 2) \oplus 1 \cdot x_3(t - 3)$
Using the propriety of the Min-Plus algebra, this controller can be simplified to:

$$u(t) = 1 \cdot x_1(t - 2)$$

This feedback control is represented by a marked and a timed monitor place in Fig.4 (place represented by a double circle)

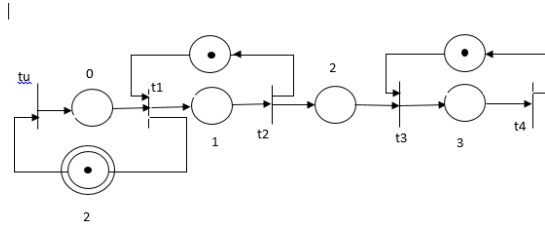


Fig. 4 The controlled TEG of the example

5 Conclusion

This paper presents a new approach to design a controller which satisfies marking constraints for partially observable timed event graphs (TEGs). In these studies, the behavior of TEGs is described by linear Min-Plus equations and the different marking constraints are translated by inequalities in Min-Plus algebra. We have established a sufficient condition for the existence of

control laws which guarantee the respect of these tokens capacity constraints in some paths in TEGs. To develop this control strategy, we have taken into account the hypothesis where for each loop of the TEG considered, there exists always a path connecting the unobservable transition to the observable transition. For this, we have calculated realizable feedback controls expressed in function of observable counters with the aim of ensuring that marking constraints are met. We first considered the case where the problem is to limit the number of tokens in one place of TEG. The approach is then extended to the case where the control law imposes a upper bound of tokens in a sequence of places. The results in this paper include the single control and the multi-variable control. For future work, it is important to relax the hypothesis of the cyclicity for TEG and it would be interesting to calculate the optimal control under a given criterion. We believe that this contribution could be valuable in various applications as well notably in flexible manufacturing workshops, communication processes and transportation networks.

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