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Total variation distance between a jump-equation and its Gaussian approximation

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Abstract

We deal with stochastic differential equations with jumps. In order to obtain an accurate approximation scheme, it is usual to replace the "small jumps" by a Brownian motion. In this paper, we prove that for every fixed time t , the approximate random variable X_t^ε converges to the original random variable X_t in total variation distance and we estimate the error. We also give an estimate of the distance between the densities of the laws of the two random variables. These are done by using some integration by parts techniques in Malliavin calculus.

Key words: Stochastic differential equations with jumps, Malliavin calculus, Total variation distance, Gaussian approximation

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1 Introduction

In this paper we consider the stochastic differential equation with jumps

$$X_t = x + \int_0^t \int_{(0,1]} c(s, z, X_{s-}) N_\mu(ds, dz)$$

where N_μ is a Poisson random measure on $(0, 1]$ with compensator $\mu(dz)ds$ and c is a coefficient which verifies strong regularity hypotheses (see **Hypotheses 2.1-2.4** in Section 2.1). The typical example that we have in mind is $\mu(dz) = \frac{dz}{z^{1+\rho}} 1_{\{z \in (0,1]\}}$, with $\rho \in [0, 1)$, so this is a truncated stable process - however, throughout the paper, we keep the general framework in which μ is a measure which has infinite mass around zero. Our aim is to replace the "small jumps" by a space-time Brownian motion:

$$\begin{aligned} X_t^\varepsilon &= x + \int_0^t \int_{\{z > \varepsilon\}} c(s, z, X_{s-}^\varepsilon) N_\mu(ds, dz) \\ &+ \int_0^t b_\varepsilon(s, X_s^\varepsilon) ds + \int_0^t \int_{(0, \varepsilon]} c(s, z, X_s^\varepsilon) W_\mu(ds, dz), \end{aligned} \quad (1)$$

where $W_\mu(ds, dz)$ is a space-time Brownian motion (in the sense of Walsh [36]) with covariance $\mu(dz)ds$, and the coefficient b_ε is defined by

$$b_\varepsilon(s, x) = \int_{(0, \varepsilon]} c(s, z, x) \mu(dz).$$

The interest of such approximations appears in various frameworks.

Our main motivation comes from numerical computations. If $\mu(E) < \infty$ then there are a finite number of jumps in any compact interval of time, so X_t may be represented by means of a compound Poisson process which may be explicitly simulated. But if $\mu(E) = \infty$ this is not possible anymore (except in very particular situations - see Talay and Protter [34] for example), and the "small jumps" should be truncated to revert to the case of a finite measure. This procedure is rather rough and gives large errors. In order to improve the approximation scheme, one may replace the "small jumps", namely those smaller than ε , by a stochastic integral with respect to $W_\mu(ds, dz)$. Note that the Poisson measure dN_μ is not compensated, which is why the drift corresponding to b_ε appears. This idea goes back to Asmussen and Rosinski [3]. In the case of *SDE's* driven by a Lévy process, Fournier [16] gives a precise estimate of the error and compares the approximation obtained just by truncating the small jumps to the one obtained by adding a Gaussian noise as in (1). An enlightening discussion on the complexity of the two methods is also given. However, in that paper, the strong error is considered, while in our paper we discuss the weak error.

A second motivation comes from modelization problems in chemistry and biology: we are concerned by reactions which are naturally modelled by means of jump processes containing two regimes: one is very rapid but the jumps are small, and another is much slower and the jumps are larger – see for example [1], [2], [4], [13], [28], [29]. In this case the regime corresponding to the rapid scale may be modelled by a stochastic integral with respect to a Gaussian process and the slow regime by a compound Poisson process. It may also be reasonable to consider an intermediary regime and this would be modelled by a drift term.

A third motivation is given by a class of statistical problems (see [11], [15] and references therein), where a stochastic process is observed at various times and it should be decided whether its increments are due to small jumps or to a Gaussian component. In this framework it is important to estimate the error in total variation sense. The authors explain that, if the error in total variation between the laws of X_t and of X_t^ε goes to zero, then there is no way to construct a test which decides if the noise comes from small jumps or from the Brownian motion. So, asymptotically, the two models contain the same information.

Let us now discuss briefly our results and the relation to previously available estimates. If L_t (respectively L_t^ε) represents the infinitesimal operator of X_t (respectively of X_t^ε) then a development in Taylor series of order two gives

$$\|(L_t - L_t^\varepsilon)f\|_\infty \leq C \|f\|_{3, \infty} \int_{(0, \varepsilon]} |\hat{c}(z)|^3 \mu(dz),$$

where $\hat{c}(z) := \sup_{s \leq T} \sup_{x \in \mathbb{R}} |c(s, z, x)|$ and $\|f\|_{3, \infty} := \sum_{0 \leq i \leq 3} \|\phi^{(i)}\|_{\infty}$. Then a Trotter-Kato type argument yields

$$\sup_{s \leq T} \|(P_t - P_t^\epsilon)f\|_{\infty} \leq C \|f\|_{3, \infty} \int_{(0, \epsilon]} |\hat{c}(z)|^3 \mu(dz)$$

where P_t (respectively of P_t^ϵ) represents the semigroup of X_t (respectively of X_t^ϵ).

The drawback of the above estimate is that the bound on the error involves $\|f\|_{3, \infty}$, so it only applies to smooth test functions. The main contribution of our paper is to replace $\|f\|_{3, \infty}$ by $\|f\|_{\infty}$, so as to prove convergence in total variation distance. This is done under non-degeneracy and regularity assumptions on the coefficient c . Moreover, under these hypotheses, we prove that $\mathbb{P}(X_t(x) \in dy) = p_t(x, y)dy$ and $\mathbb{P}(X_t^\epsilon(x) \in dy) = p_t^\epsilon(x, y)dy$ with smooth densities $y \rightarrow p_t(x, y)$ and $y \rightarrow p_t^\epsilon(x, y)$. And, for every k and every $\delta > 0$, we obtain

$$\|\partial_y^k p_t - \partial_y^k p_t^\epsilon\|_{\infty} \leq C_{k, \delta} \left(\int_{(0, \epsilon]} |\hat{c}(z)|^3 \mu(dz) \right)^{1-\delta}.$$

This proves that p_t^ϵ converges to p_t in distribution norms as $\epsilon \rightarrow 0$.

Our approach uses a strategy based on integration by parts (an abstract Malliavin calculus) developed in [6].

The paper is organized as follows. In Section 2, we give the main results and in Section 3, we recall the integration by parts technique introduced in [6] and used here. In Section 4, we use these results in the framework of stochastic equations with jumps and we prove the main result (**Theorem 2.2**). The Appendix contains technical estimates concerning Sobolev norms in Malliavin sense.

2 Main results

2.1 The basic equation and the hypotheses

A time horizon $T > 0$ will be fixed throughout the paper. As already mentioned, we deal with the one-dimensional jump equation

$$X_t = x + \int_0^t \int_{(0, 1]} c(s, z, X_{s-}) N_\mu(ds, dz), \quad (2)$$

where N_μ is a Poisson point measure with intensity $\widehat{N}_\mu(ds, dz) = \mu(dz)ds$, and μ is a positive σ -finite measure on $(0, 1]$, $t \in [0, T]$.

For technical reasons which will be discussed in Section 4, we introduce the following change of variables. Let $\theta : (0, 1] \rightarrow [1, \infty)$ be the function defined by $\theta(z) = \frac{1}{z}$, and let $\nu(dz) := \mu \circ \theta^{-1}(dz)$. Then ν is a positive σ -finite measure on $[1, \infty)$. Consider a Poisson point measure $N_\nu(ds, dz)$ with intensity $\widehat{N}_\nu(ds, dz) = \nu(dz)ds$. One may then check that for every $t \in [0, T]$, X_t has the same law as \widehat{X}_t , with $(\widehat{X}_t)_{t \in [0, T]}$ the solution of

$$\widehat{X}_t = x + \int_0^t \int_{[1, \infty)} \tilde{c}(s, z, \widehat{X}_{s-}) N_\nu(ds, dz), \quad (3)$$

where $\tilde{c}(s, z, x) := c(s, \frac{1}{z}, x)$.

Since all the results in this paper concern the laws of the solution to (2), it is equivalent to consider the equation (3). Consequently, formulate our hypotheses in terms of \tilde{c} and ν (instead of c and μ).

Hypothesis 2.1 (Regularity with parameter q^*) The map $(s, z) \mapsto \tilde{c}(s, z, x)$ is continuous, and there exists a non-negative and decreasing function $\bar{c} : [1, \infty) \rightarrow \mathbb{R}_+$ and a constant $q^* \in \mathbb{N}$ such that for every indices β_1, β_2 , with $\beta_1 \leq q^*$ and $\beta_2 \leq q^*$, we have

$$\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} (|\tilde{c}(s, z, x)| + |\partial_z^{\beta_2} \partial_x^{\beta_1} \tilde{c}(s, z, x)|) \leq \bar{c}(z), \quad \forall z \in [1, \infty),$$

with

$$\int_{[1,\infty)} |\bar{c}(z)|^p \nu(dz) =: \bar{c}_p < \infty, \quad \forall p \geq 1. \quad (4)$$

Remark. We will use several times the following consequence of (4) and of Burkholder's inequality (see for example the Theorem 2.11 in [26]): We assume that $\Phi(s, z, \omega)$ and $\varphi(s, \omega)$ are two non-negative functions such that

$$|\Phi(s, z, \omega)| \leq \bar{c}(z)\varphi(s, \omega).$$

Then for any $p \geq 2$,

$$\mathbb{E} \left| \int_0^T \int_{[1,\infty)} \Phi(s, z, \omega) N_\nu(ds, dz) \right|^p \leq C \times C_p \mathbb{E} \int_0^T |\varphi(s, \omega)|^p ds, \quad (5)$$

where $C_p = \max\{(\bar{c}_2)^{\frac{p}{2}}, \bar{c}_p, (\bar{c}_1)^p\}$ and C is a constant depending on p and T .

Proof. By compensating N_ν , using Burkholder's inequality and (4), we have

$$\begin{aligned} \mathbb{E} \left| \int_0^T \int_{[1,\infty)} \Phi(s, z, \omega) N_\nu(ds, dz) \right|^p &\leq C \left(\mathbb{E} \left| \int_0^T \int_{[1,\infty)} \Phi(s, z, \omega) \widetilde{N}_\nu(ds, dz) \right|^p + \mathbb{E} \left| \int_0^T \int_{[1,\infty)} \Phi(s, z, \omega) \nu(dz) ds \right|^p \right) \\ &\leq C \left(\mathbb{E} \int_0^T \left(\int_{[1,\infty)} |\Phi(s, z, \omega)|^2 \nu(dz) \right)^{\frac{p}{2}} ds + \mathbb{E} \int_0^T \int_{[1,\infty)} |\Phi(s, z, \omega)|^p \nu(dz) ds + \mathbb{E} \int_0^T \left| \int_{[1,\infty)} \Phi(s, z, \omega) \nu(dz) \right|^p ds \right) \\ &\leq C \times C_p \mathbb{E} \int_0^T |\varphi(s, \omega)|^p ds. \end{aligned}$$

□

Hypothesis 2.2 There exists a non-negative function $\check{c} : [1, \infty) \rightarrow \mathbb{R}_+$ such that $\int_{[1,\infty)} |\check{c}(z)|^p \nu(dz) =: \check{c}_p < \infty$, $\forall p \geq 1$, and

$$\left| \frac{\partial_x \tilde{c}(s, z, x)}{1 + \partial_x \tilde{c}(s, z, x)} \right| \leq \check{c}(z), \quad \forall s \in [0, T], x \in \mathbb{R}, z \in [1, \infty).$$

To avoid overburdening notation, since both hypotheses 2.1 and 2.2 apply, we will take $\check{c}(z) = \bar{c}(z)$.

Hypothesis 2.3 (Ellipticity) There exists a non-negative function $\underline{c} : [1, \infty) \rightarrow \mathbb{R}_+$ such that for every $s \in [0, T], x \in \mathbb{R}, z \in [1, \infty)$,

$$|\partial_z \tilde{c}(s, z, x)|^2 \geq \underline{c}(z) \quad \text{and} \quad |\tilde{c}(s, z, x)|^2 \geq \underline{c}(z).$$

Hypothesis 2.4 (Sector condition) This is a supplementary hypothesis concerning the measure ν . Two version of this hypothesis will be used; we state them separately below. Let $I_k = \{z \in [1, \infty) : k \leq z < k+1\}$, $k \in \mathbb{N}$ and $m_k = \nu(I_k)$.

(a) **Strong sector condition:** We say that the strong sector condition is satisfied if there exist constants $\varepsilon_* > 0$ and $\alpha_1 \geq \alpha_0 > \alpha_2 > 0$, such that

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \frac{\varepsilon_*}{z^{1-\alpha_1}} dz \quad \text{for all } k \in \mathbb{N}, \quad (6)$$

$$\underline{c}(z) \geq e^{-z^{\alpha_2}} \quad \text{for all } z \geq 1 \text{ and,}$$

$$\int_1^\infty \frac{|\bar{c}(z)|^p}{z^{1-\alpha_0}} dz < \infty \quad \text{for all } p \geq 1. \quad (7)$$

Notice that if (6) is true for some α_1 , then it is also true for any $\alpha \leq \alpha_1$. So (6) also reads

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k dz, \quad \text{with} \quad \varepsilon_k = \frac{\varepsilon_*}{(k+1)^{1-\alpha}},$$

for any $\alpha \leq \alpha_1$.

(b) **Weak sector condition:** We say that the weak sector condition holds if there exist constants $\varepsilon_* > 0$ and $\alpha > 0$, such that for every $k \in \mathbb{N}$, we have

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \frac{\varepsilon_*}{z} dz \quad \text{for all } k \in \mathbb{N}, \quad (8)$$

$$\underline{c}(z) \geq \frac{1}{z^\alpha} \quad \text{for all } z \geq 1 \text{ and,}$$

$$\int_1^\infty \frac{|\bar{c}(z)|^p}{z} dz < \infty \quad \text{for all } p \geq 1. \quad (9)$$

We notice that (8) also reads $1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k dz$ with $\varepsilon_k = \frac{\varepsilon_*}{k+1}$.

Remark. Notice that hypotheses 2.1, 2.2 and 2.3 are analogous to those in [10](2-7,2-26,2-24).

Henceforth, we will suppose that hypotheses 2.1-2.3 hold, as well as either 2.4(a) or 2.4(b).

2.2 Approximation

We come back now to equation (2). The goal of this paper is to replace the small jumps in (2) by a drift and a Brownian motion. In equation (2), the Poisson point measure N_μ is not compensated, so the first step is to introduce a drift (see b_ε below) which represents the compensator. Afterwards, we introduce a space-time Brownian motion W_μ in order to replace the "compensated small jumps":

$$\begin{aligned} X_t^\varepsilon &= x + \int_0^t \int_{\{z > \varepsilon\}} c(s, z, X_{s-}^\varepsilon) N_\mu(ds, dz) \\ &\quad + \int_0^t b_\varepsilon(s, X_s^\varepsilon) ds + \int_0^t \int_{(0, \varepsilon]} c(s, z, X_s^\varepsilon) W_\mu(dz, ds), \end{aligned} \quad (10)$$

where

$$b_\varepsilon(s, x) = \int_{(0, \varepsilon]} c(s, z, x) \mu(dz)$$

and W_μ is a space-time Brownian motion with covariance measure $\mu(dz)ds$, which is independent of N_μ .

Let us discuss this equation. We notice that we keep the "big jumps" with $z > \varepsilon$ but we eliminate the "small jumps" with $z \leq \varepsilon$. We replace the "small jumps" by the drift with coefficient b_ε and by the stochastic integral with coefficient c . This stochastic integral is driven by the so called space-time Brownian motion W_μ , as introduced by Walsh in [36]. The existence and uniqueness of the solution to this equation (10) are also given by Kunita (see [26], [27])

We recall that we work on a fixed interval of time $[0, T]$. We now precise the filtration that we consider. Let

$$\begin{aligned} \mathcal{F}_t^W &= \sigma(W_\mu(\varphi 1_{[0, t]}) : \varphi \in L^2((0, 1] \times [0, T], \mu \times Leb)), \\ \mathcal{F}_t^N &= \sigma(N_\mu(\varphi 1_{[0, t]}) : \varphi \in L^1((0, 1] \times [0, T], \mu \times Leb)), \\ \mathcal{F}_t &= \mathcal{F}_t^W \vee \mathcal{F}_t^N, \end{aligned} \quad (11)$$

where Leb denotes the Lebesgue measure and

$$W_\mu(\varphi) = \int_0^T \int_{(0, 1]} \varphi(s, z) W_\mu(ds, dz), \quad N_\mu(\varphi) = \int_0^T \int_{(0, 1]} \varphi(s, z) N_\mu(ds, dz).$$

So, X_t^ε is \mathcal{F}_t -measurable and X_t is \mathcal{F}_t^N -measurable.

We denote

$$L^2(W) = \{F \in \mathcal{F}_T^W : \mathbb{E}|F|^2 < \infty\}, \quad L^2(N) = \{G \in \mathcal{F}_T^N : \mathbb{E}|G|^2 < \infty\}. \quad (12)$$

Remark. Let $(\Phi(s, z, \omega))_{s \in [0, T]}$ be a process which is progressively measurable with respect to $(\mathcal{F}_s)_{s \in [0, T]}$, is piecewise constant with respect to s, z , and

$$\mathbb{E} \int_0^T \int_{(0,1]} |\Phi(s, z, \omega)|^2 \mu(dz) ds < \infty.$$

Then for every $G \in L^2(N)$, we have

$$\mathbb{E} \left[G \times \int_0^T \int_{(0,1]} \Phi(s, z, \omega) W_\mu(ds, dz) \right] = 0. \quad (13)$$

Now we write the infinitesimal operator of X_s and X_s^ε , respectively: For $\phi \in C_b^3(\mathbb{R})$ (the space of functions with continuous and bounded derivatives up to order 3),

$$\begin{aligned} L_s \phi(x) &= \int_{(0,1]} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz) \quad \text{and} \\ L_s^\varepsilon \phi(x) &= \int_{\{z > \varepsilon\}} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz) + \phi'(x) b_\varepsilon(s, x) + \frac{1}{2} \phi''(x) a_\varepsilon(s, x), \end{aligned} \quad (14)$$

where

$$a_\varepsilon(s, x) = \int_{(0, \varepsilon]} |c(s, z, x)|^2 \mu(dz).$$

Using Taylor's formula of order 2, we find

$$L_s \phi(x) = \int_{\{z > \varepsilon\}} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz) + \phi'(x) b_\varepsilon(s, x) + \frac{1}{2} \phi''(x) a_\varepsilon(s, x) + R_s(x),$$

where

$$|R_s(x)| \leq \frac{1}{6} \|\phi\|_{3, \infty} \int_{(0, \varepsilon]} |c(s, z, x)|^3 \mu(dz),$$

with $\|\phi\|_{l, \infty} := \sum_{0 \leq i \leq l} \|\phi^{(i)}\|_\infty$, the sum of all the uniform norms of the derivatives of function ϕ up to order l . In conclusion, we find

$$\|(L_s - L_s^\varepsilon) \phi\|_\infty = \|R_s\|_\infty \leq \frac{1}{6} \|\phi\|_{3, \infty} \eta_3(\varepsilon), \quad (15)$$

with

$$\eta_p(\varepsilon) = \int_{(0, \varepsilon]} |\bar{c}(1/z)|^p \mu(dz) = \int_{[\varepsilon^{-1}, \infty)} |\bar{c}(z)|^p \nu(dz), \quad p \geq 1. \quad (16)$$

Then, we can give an estimate of the distance between the semigroups. We use the standard semigroup notation, which we remind below. Let $[X_t(s, x)]_{t \geq s}$ and $[X_t^\varepsilon(s, x)]_{t \geq s}$ be the solutions to (2) and (10), respectively, starting at time s from point x . Denote by $P_{s,t} \phi(x) = \mathbb{E} \phi(X_t(s, x))$ and $P_{s,t}^\varepsilon \phi(x) = \mathbb{E} \phi(X_t^\varepsilon(s, x))$. Also, set $P_t := P_{0,t}$ and $P_t^\varepsilon := P_{0,t}^\varepsilon$.

Lemma 2.1. *There exists a constant C depending on T such that for $\phi \in C_b^3(\mathbb{R})$ and $0 \leq t \leq T$, we have*

$$\|P_t \phi - P_t^\varepsilon \phi\|_\infty \leq C \|\phi\|_{3, \infty} \eta_3(\varepsilon). \quad (17)$$

Proof. Step 1 Trotter-Kato method: We know from Kunita[27](Theorem 4.5.1) that we have the Kolmogorov forward and backward equations:

$$\partial_t P_{s,t} \phi(x) = P_{s,t} L_t \phi(x), \quad \partial_t P_{s,t}^\varepsilon \phi(x) = P_{s,t}^\varepsilon L_t^\varepsilon \phi(x); \quad (18)$$

$$\partial_s P_{s,t} \phi(x) = -L_s P_{s,t} \phi(x), \quad \partial_s P_{s,t}^\varepsilon \phi(x) = -L_s^\varepsilon P_{s,t}^\varepsilon \phi(x). \quad (19)$$

Then using Newton-Leibniz's formula and (18),(19),

$$P_t^\varepsilon \phi(x) - P_t \phi(x) = \int_0^t \partial_s(P_{0,s}^\varepsilon P_{s,t}) \phi(x) ds = \int_0^t (P_{0,s}^\varepsilon (L_s^\varepsilon - L_s) P_{s,t}) \phi(x) ds.$$

It follows that

$$\begin{aligned} \|P_t \phi - P_t^\varepsilon \phi\|_\infty &\leq \int_0^t \|P_{0,s}^\varepsilon (L_s^\varepsilon - L_s) P_{s,t} \phi\|_\infty ds \\ &\leq \int_0^t \|(L_s^\varepsilon - L_s) P_{s,t} \phi\|_\infty ds \\ &\leq \frac{1}{6} \eta_3(\varepsilon) \int_0^t \|P_{s,t} \phi\|_{3,\infty} ds. \end{aligned} \quad (20)$$

Step 2 (propagation of regularity) In [27], Kunita has shown in Theorem 3.4.1 and Theorem 3.4.2 the regularity of the flow associated with the jump-diffusion. So in our case, we have

$$\begin{aligned} \|P_{s,t} \phi\|_{3,\infty} &\leq \sup_{x \in \mathbb{R}} (\mathbb{E}|\phi(X_t(s,x))| + \mathbb{E}|\partial_x \phi(X_t(s,x))| + \mathbb{E}|\partial_x^2 \phi(X_t(s,x))| + \mathbb{E}|\partial_x^3 \phi(X_t(s,x))|) \\ &\leq \|\phi\|_{3,\infty} \sup_{x \in \mathbb{R}} \mathbb{E}[1 + 3|\partial_x X_t(s,x)| + 3|\partial_x^2 X_t(s,x)| + |\partial_x^3 X_t(s,x)|] \leq C \|\phi\|_{3,\infty}. \end{aligned} \quad (21)$$

Substituting (21) into (20), we obtain (17). \square

Remark. A similar result has been obtained in [21](Theorem 4.7). Besides, one may also consider an approximate equation obtained just by discarding the small jumps:

$$\widetilde{X}_t^\varepsilon = x + \int_0^t \int_{\{z > \varepsilon\}} c(s, z, \widetilde{X}_{s-}^\varepsilon) N_\mu(ds, dz).$$

Then, if $\widetilde{L}_s^\varepsilon$ is the infinitesimal operator of $\widetilde{X}_s^\varepsilon$, we have $\|(L_s - \widetilde{L}_s^\varepsilon) \phi\|_\infty \leq \|\phi\|_{1,\infty} \eta_1(\varepsilon)$. So the same reasoning as above gives

$$\|P_t \phi - \widetilde{P}_t^\varepsilon \phi\|_\infty \leq C \|\phi\|_{1,\infty} \times \eta_1(\varepsilon) \rightarrow 0. \quad (22)$$

The gain in (17) is that we have $\eta_3(\varepsilon)$ instead of $\eta_1(\varepsilon)$ in (22), which means that we have a faster speed of convergence.

2.3 The main theorem

We are finally ready to state the main results of this paper. Denote by $d_{TV}(F, G)$ the total variation distance between the laws of two random variables F and G .

Theorem 2.2. *Assume that **Hypotheses 2.1, 2.2. and 2.3** hold with $q^* \geq \frac{3}{\delta} + 1$ for some $\delta > 0$.*

(a) *If in addition we assume **Hypothesis 2.4 (a)**, then there exists a constant C depending on δ and T such that*

$$d_{TV}(X_t, X_t^\varepsilon) \leq C \eta_3(\varepsilon)^{1-\delta}. \quad (23)$$

Under the above hypotheses, the laws of X_t and X_t^ε are absolutely continuous with respect to the Lebesgue measure, with smooth densities $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$. Moreover, if l is an index such that $q^ \geq \frac{3+l}{\delta} + 1$, then there exists a constant C depending on δ , T and l such that*

$$\|p_{X_t} - p_{X_t^\varepsilon}\|_{l,\infty} \leq C \eta_3(\varepsilon)^{1-\delta}. \quad (24)$$

(b) *If in addition we assume **Hypothesis 2.4 (b)**, then there exists a constant C depending on δ and T such that for every $t \in [0, T]$ with $t > \frac{8\alpha(\frac{3}{\delta}-1)}{\varepsilon_*}$ (with ε_* and α given in **Hypothesis 2.4 (b)**), we have*

$$d_{TV}(X_t, X_t^\varepsilon) \leq C \eta_3(\varepsilon)^{1-\delta}. \quad (25)$$

For any index l and for $t > \frac{8\alpha(3l+2)}{\varepsilon_*}$, both the laws of X_t and X_t^ε have l -times differentiable densities $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$. Assume moreover that $q^* \geq \frac{3+l}{\delta} + 1$. Then there exists a constant C depending on δ, T and l such that for $t > \max\{\frac{8\alpha}{\varepsilon_*}(\frac{3+l}{\delta} - 1), \frac{8\alpha(3l+2)}{\varepsilon_*}\}$, we have

$$\|p_{X_t} - p_{X_t^\varepsilon}\|_{l,\infty} \leq C\eta_3(\varepsilon)^{1-\delta}. \quad (26)$$

The proof of this theorem is left to Section 4.4.

Remark. Some recent results concerning the weak approximation of the SDE with jumps are also given in [14], [22], [23] for example. But they do not concern the convergence in total variation distance.

2.4 A typical example

For $t \in [0, T]$, we consider the following SDE driven by a Lévy process:

$$X_t = x + \int_0^t \sigma(X_{s-}) dZ_s, \quad (27)$$

where $(Z_t)_{t \in [0, T]}$ is a Lévy process of Lévy triplet $(0, 0, \mu)$, with $\mu(dz) = 1_{(0,1]}(z) \frac{dz}{z^{1+\rho}}$, $0 \leq \rho < 1$.

We approximate (27) by

$$X_t^\varepsilon = x + \int_0^t \sigma(X_{s-}^\varepsilon) dZ_s^\varepsilon + b(\varepsilon) \int_0^t \sigma(X_s^\varepsilon) ds + c(\varepsilon) \int_0^t \sigma(X_s^\varepsilon) dB_s, \quad (28)$$

where $(Z_t^\varepsilon)_{t \in [0, T]}$ is a Lévy process of Lévy triplet $(0, 0, 1_{\{z > \varepsilon\}} \mu(dz))$, $(B_t)_{t \in [0, T]}$ is a standard Brownian motion independent of $(Z_t^\varepsilon)_{t \in [0, T]}$, and

$$b(\varepsilon) = \int_{(0, \varepsilon]} z \mu(dz), \quad c(\varepsilon) = \sqrt{\int_{(0, \varepsilon]} z^2 \mu(dz)}.$$

Then we have the following theorem.

Theorem 2.3. *We assume that $\sigma \in C_b^\infty(\mathbb{R})$, $0 < \underline{\sigma} \leq \sigma(x) \leq \bar{\sigma}$ and $-1 < a \leq \sigma'(x) \leq \bar{\sigma}$, $\forall x \in \mathbb{R}$, for some universal constants $\bar{\sigma}, \underline{\sigma}, a$, where σ' is the differential of σ in x . Then for any $\delta > 0$, there is a constant $C > 0$ such that for any $t \in [0, T]$,*

$$d_{TV}(X_t, X_t^\varepsilon) \leq C\varepsilon^{3-\rho-\delta}.$$

Moreover, the laws of X_t and X_t^ε have smooth densities $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$ respectively. And for any index l and any $\delta > 0$, there exists a constant $C > 0$ such that

$$\|p_{X_t} - p_{X_t^\varepsilon}\|_{l,\infty} \leq C\varepsilon^{3-\rho-\delta}.$$

Proof. We notice that

$$Z_t = \int_0^t \int_{(0,1]} z N_\mu(ds, dz),$$

where N_μ is a Poisson point measure with intensity $\mu(dz)ds$. Then (27) coincides with (2) with $c(s, z, x) = \sigma(x)z$, and (28) coincides with (10) with $c(s, z, x) = \sigma(x)z$, $b_\varepsilon(s, x) = b(\varepsilon)\sigma(x)$, and $\int_{\{z \leq \varepsilon\}} z W_\mu(ds, dz) = c(\varepsilon)dB_s$.

Let $\theta : (0, 1] \rightarrow [1, \infty)$ be a function defined by $\theta(z) = \frac{1}{z}$. By a change of variables,

$$\tilde{c}(s, z, x) = c(s, \frac{1}{z}, x) = \sigma(x) \times \frac{1}{z}, \quad \nu(dz) = \mu \circ \theta^{-1}(dz) = 1_{[1, \infty)}(z) \frac{dz}{z^{1-\rho}}.$$

One can easily check that **Hypothesis 2.1** is verified (for every $q^* \in \mathbb{N}$) with $\bar{c}(z) = \bar{\sigma} \times \frac{1}{z}$ and

$$\int_1^\infty |\bar{c}(z)|^p \nu(dz) = \int_1^\infty \frac{\bar{\sigma}^p}{z^{p+1-\rho}} dz < \infty, \quad \forall p \geq 1.$$

Let $I_k = \{z \in [1, \infty) : k \leq z < k + 1\}$, $k \in \mathbb{N}$ and $m_k = \nu(I_k)$. Then for sufficiently large z , we have

$$\min\{|\partial_z \tilde{c}(s, z, x)|^2, |\tilde{c}(s, z, x)|^2\} \geq \underline{\sigma}^2 \times \frac{1}{z^4} \geq e^{-z^{\alpha_2}},$$

with some $0 < \alpha_2 < 1$. We also have

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \frac{1}{2} dz \geq 1_{I_k}(z) \frac{1}{2} \frac{dz}{z^{1-\alpha_0}},$$

with some $\alpha_0 \in (\alpha_2, 1)$. Moreover, since for any $p \geq 1$, $p + 1 - \alpha_0 > 1$, we have

$$\int_1^\infty \frac{|\tilde{c}(z)|^p}{z^{1-\alpha_0}} dz = \bar{\sigma} \int_1^\infty \frac{1}{z^{p+1-\alpha_0}} dz < \infty.$$

So **Hypothesis 2.3** and **Hypothesis 2.4** (a) are satisfied. Finally,

$$\left| \frac{\partial_x \tilde{c}(s, z, x)}{1 + \partial_x \tilde{c}(s, z, x)} \right| \leq \frac{\bar{\sigma} \times \frac{1}{z}}{1 + a \times \frac{1}{z}} \leq \max\left\{\frac{1}{1+a}, 1\right\} \times \bar{\sigma} \times \frac{1}{z},$$

so **Hypothesis 2.2** is satisfied as well. Then we can apply **Theorem 2.2(a)** for the equation (27) and (28). Since

$$\eta_3(\varepsilon) = \int_{(0, \varepsilon]} \bar{\sigma}^3 \times z^3 \mu(dz) = \frac{\bar{\sigma}^3}{3-\rho} \varepsilon^{3-\rho},$$

we obtain the estimates from **Theorem 2.3**. □

3 Abstract integration by parts framework

In order to obtain the main theorem (**Theorem 2.2**), we will apply some techniques of Malliavin calculus. So firstly, we give the abstract integration by parts framework introduced in [6]. This is a variant of the integration by parts framework given in [10].

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a subset $\mathcal{S} \subset \bigcap_{p=1}^\infty L^p(\Omega; \mathbb{R})$ such that for every $\phi \in C_p^\infty(\mathbb{R}^d)$ and every $F \in \mathcal{S}^d$, we have $\phi(F) \in \mathcal{S}$ (with C_p^∞ the space of smooth functions which, together with all the derivatives, have polynomial growth). A typical example of \mathcal{S} is the space of simple functionals, as in the standard Malliavin calculus. Another example is the space of "Malliavin smooth functionals".

Given a separable Hilbert space \mathcal{H} , we assume that we have a derivative operator $D : \mathcal{S} \rightarrow \bigcap_{p=1}^\infty L^p(\Omega; \mathcal{H})$ which is a linear application which satisfies

a)

$$D_h F := \langle DF, h \rangle_{\mathcal{H}} \in \mathcal{S}, \text{ for any } h \in \mathcal{H}, \quad (29)$$

b) Chain Rule: For every $\phi \in C_p^\infty(\mathbb{R}^d)$ and $F = (F_1, \dots, F_d) \in \mathcal{S}^d$, we have

$$D\phi(F) = \sum_{i=1}^d \partial_i \phi(F) DF_i, \quad (30)$$

Since $D_h F \in \mathcal{S}$, we may define by iteration the derivative operator of higher order $D^q : \mathcal{S} \rightarrow \bigcap_{p=1}^\infty L^p(\Omega; \mathcal{H}^{\otimes q})$ which verifies $\langle D^q F, \otimes_{i=1}^q h_i \rangle_{\mathcal{H}^{\otimes q}} = D_{h_q} D_{h_{q-1}} \dots D_{h_1} F$. We also denote $D_{h_1, \dots, h_q}^q F := D_{h_q} D_{h_{q-1}} \dots D_{h_1} F$. Then, $D_{h_1, \dots, h_q}^q F = D_{h_q} D_{h_1, \dots, h_{q-1}}^{q-1} F$ ($q \geq 2$).

For $F = (F_1, \dots, F_{\tilde{d}}) \in \mathcal{S}^{\tilde{d}}$, we define $\sigma_F = (\sigma_F^{i,j})_{i,j=1,\dots,\tilde{d}}$ to be the Malliavin covariance matrix with $\sigma_F^{i,j} = \langle DF_i, DF_j \rangle_{\mathcal{H}}$ and we denote

$$\Sigma_p(F) = \mathbb{E}(1/\det \sigma_F)^p. \quad (31)$$

For $\tilde{d} = 1$, which is the case that we discuss in this paper, $\det \sigma_F = \sigma_F = \langle DF, DF \rangle_{\mathcal{H}}$. We say that F is non-degenerated if $\Sigma_p(F) < \infty, \forall p \geq 1$.

We also assume that we have an Ornstein-Uhlenbeck operator $L : \mathcal{S} \rightarrow \mathcal{S}$ which is a linear operator satisfying the following duality formula:

Duality: For every $F, G \in \mathcal{S}$,

$$\mathbb{E}\langle DF, DG \rangle_{\mathcal{H}} = \mathbb{E}(FLG) = \mathbb{E}(GLF). \quad (32)$$

As an immediate consequence of the duality formula, we know that $L : \mathcal{S} \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is closable.

Definition 3.1. If $D^q : \mathcal{S} \subset L^2(\Omega) \rightarrow L^2(\Omega; \mathcal{H}^{\otimes q}), \forall q \geq 1$, are closable, then the triplet (\mathcal{S}, D, L) will be called an IbP (Integration by Parts) framework.

Now, we introduce the Sobolev norms. For any $l \geq 1, F \in \mathcal{S}$,

$$|F|_{1,l} = \sum_{q=1}^l |D^q F|_{\mathcal{H}^{\otimes q}}, \quad |F|_l = |F| + |F|_{1,l}. \quad (33)$$

We remark that $|F|_0 = |F|$ and $|F|_{1,l} = 0$ for $l = 0$. For $F = (F_1, \dots, F_d) \in \mathcal{S}^d$, we set

$$|F|_{1,l} = \sum_{i=1}^d |F_i|_{1,l}, \quad |F|_l = \sum_{i=1}^d |F_i|_l.$$

Moreover, we associate the following norms. For any $l, p \geq 1$,

$$\begin{aligned} \|F\|_{l,p} &= (\mathbb{E}|F|_l^p)^{1/p}, \quad \|F\|_p = (\mathbb{E}|F|^p)^{1/p}, \\ \|F\|_{L,l,p} &= \|F\|_{l,p} + \|LF\|_{l-2,p}. \end{aligned} \quad (34)$$

We denote by $\mathcal{D}_{l,p}$ the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{L,l,p}$:

$$\mathcal{D}_{l,p} = \overline{\mathcal{S}}^{\|\cdot\|_{L,l,p}}, \quad (35)$$

and

$$\mathcal{D}_{\infty} = \bigcap_{l=1}^{\infty} \bigcap_{p=1}^{\infty} \mathcal{D}_{l,p}, \quad \mathcal{H}_l = \mathcal{D}_{l,2}.$$

For an IbP framework (\mathcal{S}, D, L) , we now extend the operators from \mathcal{S} to \mathcal{D}_{∞} . For $F \in \mathcal{D}_{\infty}, p \geq 2$, there exists a sequence $F_n \in \mathcal{S}$ such that $\|F - F_n\|_p \rightarrow 0, \|F_m - F_n\|_{q,p} \rightarrow 0$ and $\|LF_m - LF_n\|_{q-2,p} \rightarrow 0$. Since D^q and L are closable, we can define

$$D^q F = \lim_{n \rightarrow \infty} D^q F_n \quad \text{in } L^p(\Omega; \mathcal{H}^{\otimes q}), \quad LF = \lim_{n \rightarrow \infty} LF_n \quad \text{in } L^p(\Omega). \quad (36)$$

We still associate the same norms introduced above for $F \in \mathcal{D}_{\infty}$.

Lemma 3.1. *The triplet $(\mathcal{D}_{\infty}, D, L)$ is an IbP framework.*

Proof. Here we just show that D verifies (29): For $F \in \mathcal{D}_{\infty}$ and $h \in \mathcal{H}$, we have $\langle DF, h \rangle_{\mathcal{H}} \in \mathcal{D}_{\infty}$.

In fact, for any $k \geq 1, p \geq 2$, any $F \in \mathcal{D}_{k+1,p}$, there is a sequence $F_n \in \mathcal{S}$ such that $\|F_n - F\|_{k+1,p} \rightarrow 0$. Then for any $u_1, \dots, u_k \in L^p(\Omega; \mathcal{H}), h \in \mathcal{H}$, any $n, m \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E} \langle D^k (\langle DF_m, h \rangle_{\mathcal{H}} - \langle DF_n, h \rangle_{\mathcal{H}}), u_1 \otimes \dots \otimes u_k \rangle_{\mathcal{H}^{\otimes k}}^{\frac{p}{2}} = \mathbb{E} |D_{u_k} D_{u_{k-1}} \dots D_{u_1} \langle D(F_m - F_n), h \rangle_{\mathcal{H}}|^{\frac{p}{2}} \\ & = \mathbb{E} |D_{u_k} D_{u_{k-1}} \dots D_{u_1} D_h (F_m - F_n)|^{\frac{p}{2}} = \mathbb{E} |D^{k+1} (F_m - F_n), h \otimes u_1 \otimes \dots \otimes u_k|_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} \\ & \leq \mathbb{E} |D^{k+1} (F_m - F_n)|_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} |h \otimes u_1 \otimes \dots \otimes u_k|_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} \rightarrow 0, \end{aligned}$$

which yields that $\mathbb{E} |D^k (\langle DF_m, h \rangle_{\mathcal{H}} - \langle DF_n, h \rangle_{\mathcal{H}})|_{\mathcal{H}^{\otimes k}}^p \rightarrow 0$. Therefore, $\langle DF, h \rangle_{\mathcal{H}} \in \mathcal{D}_{k,p}$ and (29) is verified. \square

The following lemma is useful in order to control the Sobolev norms $\|F\|_{L,l,q}$.

Lemma 3.2. *We fix $p \geq 2, l \geq 2$. Let $F \in L^1(\Omega)$ and let $F_n \in \mathcal{S}, n \in \mathbb{N}$ such that*

$$\begin{aligned} & i) \quad \mathbb{E} |F_n - F| \rightarrow 0, \\ & ii) \quad \sup_n \|F_n\|_{L,l,p} \leq K_{l,p} < \infty. \end{aligned}$$

Then for every $1 \leq \bar{p} < p$, we have $F \in \mathcal{D}_{l,\bar{p}}$ and $\|F\|_{L,l,\bar{p}} \leq K_{l,\bar{p}}$.

Proof. The Hilbert space $\mathcal{H}_l = \mathcal{D}_{l,2}$ equipped with the scalar product

$$\begin{aligned} \langle U, V \rangle_{L,l,2} & := \sum_{q=1}^l \mathbb{E} \langle D^q U, D^q V \rangle_{\mathcal{H}^{\otimes q}} + \mathbb{E} |UV| \\ & + \sum_{q=1}^{l-2} \mathbb{E} \langle D^q LU, D^q LV \rangle_{\mathcal{H}^{\otimes q}} + \mathbb{E} |LU \times LV| \end{aligned}$$

is the space of the functionals which are l -times differentiable in L^2 sense. By *ii)*, for $p \geq 2$, $\|F_n\|_{L,l,2} \leq \|F_n\|_{L,l,p} \leq K_{l,p}$. Then, applying Banach Alaoglu's theorem, there exists a functional $G \in \mathcal{H}_l$ and a subsequence (we still denote it by n), such that $F_n \rightarrow G$ weakly in the Hilbert space \mathcal{H}_l . This means that for every $Q \in \mathcal{H}_l$, $\langle F_n, Q \rangle_{L,l,2} \rightarrow \langle G, Q \rangle_{L,l,2}$. Therefore, by Mazur's theorem, we can construct some convex combination

$$G_n = \sum_{i=n}^{m_n} \lambda_i^n \times F_i \in \mathcal{S}$$

with $\lambda_i^n \geq 0, i = n, \dots, m_n$ and $\sum_{i=n}^{m_n} \lambda_i^n = 1$, such that

$$\|G_n - G\|_{L,l,2} \rightarrow 0.$$

In particular we have

$$\mathbb{E} |G_n - G| \leq \|G_n - G\|_{L,l,2} \rightarrow 0.$$

Also, we notice that by *i)*,

$$\mathbb{E} |G_n - F| \leq \sum_{i=n}^{m_n} \lambda_i^n \times \mathbb{E} |F_i - F| \rightarrow 0.$$

So we conclude that $F = G \in \mathcal{H}_l$. Thus, we have

$$\mathbb{E} (|G_n - F|_l^2) + \mathbb{E} (|LG_n - LF|_{l-2}^2) \leq \|G_n - F\|_{L,l,2}^2 \rightarrow 0.$$

By passing to a subsequence, we have $|G_n - F|_l + |LG_n - LF|_{l-2} \rightarrow 0$ almost surely. Now, for every $\bar{p} \in [1, p)$, we denote $Y_n := |G_n|_l^{\bar{p}} + |LG_n|_{l-2}^{\bar{p}}$ and $Y := |F|_l^{\bar{p}} + |LF|_{l-2}^{\bar{p}}$. Then, $Y_n \rightarrow Y$ almost surely, and

for any $\tilde{q} \in [\bar{p}, p]$,

$$\begin{aligned} \mathbb{E}|G_n|_{\tilde{l}}^{\tilde{q}} + \mathbb{E}|LG_n|_{\tilde{l}-2}^{\tilde{q}} &\leq \|G_n\|_{L, \tilde{l}, \tilde{q}}^{\tilde{q}} = \left\| \sum_{i=n}^{m_n} \lambda_i^n \times F_i \right\|_{L, \tilde{l}, \tilde{q}}^{\tilde{q}} \leq \left(\sum_{i=n}^{m_n} \lambda_i^n \times \|F_i\|_{L, \tilde{l}, \tilde{q}} \right)^{\tilde{q}} \\ &\leq \left(\sup_i \|F_i\|_{L, \tilde{l}, \tilde{q}} \times \sum_{i=n}^{m_n} \lambda_i^n \right)^{\tilde{q}} = \sup_i \|F_i\|_{L, \tilde{l}, \tilde{q}}^{\tilde{q}} \leq K_{\tilde{l}, \tilde{q}}^{\tilde{q}}. \end{aligned}$$

So $(Y_n)_{n \in \mathbb{N}}$ is uniformly integrable, and we have

$$\|F\|_{L, \tilde{l}, \bar{p}}^{\bar{p}} = \mathbb{E}(|F|_{\tilde{l}}^{\bar{p}}) + \mathbb{E}(|LF|_{\tilde{l}-2}^{\bar{p}}) = \mathbb{E}(Y) = \lim_{n \rightarrow \infty} \mathbb{E}(Y_n) \leq K_{\tilde{l}, \bar{p}}^{\bar{p}},$$

□

3.1 Main consequences: Convergence in total variation distance

We will use the abstract framework in [6] for the IbP framework $(\mathcal{D}_\infty, D, L)$, with D and L defined in (36). Using Malliavin type arguments, [6] proves the following results. The first result, concerning the density, is classical:

Lemma 3.3. *Let $F \in \mathcal{D}_\infty$. If $\Sigma_{6p+4}(F) < \infty$, then the law of random variable F has a density $p_F(x)$ which is p -times differentiable.*

In the following, we define the distances between random variables $F, G : \Omega \rightarrow \mathbb{R}$:

$$d_k(F, G) = \sup\{|\mathbb{E}(f(F)) - \mathbb{E}(f(G))| : \sum_{0 \leq i \leq k} \|f^{(i)}\|_\infty \leq 1\}$$

For $k = 1$, this is the Fortet Mourier distance (which is a variant of the Wasserstein distance), while for $k = 0$, this is the total variation distance and we denote it by d_{TV} . Now we present the second result concerning the total variation distance:

Lemma 3.4. *We fix some index l , some $r \in \mathbb{N}$ and some $\delta > 0$. We define $p_1 = 2(r(\frac{1}{\delta} - 1) + 2)$, $p_2 = \max\{6l + 4, 2(\frac{r+l}{\delta} - r + 2)\}$, $q_1 \geq r(\frac{1}{\delta} - 1) + 4$, $q_2 \geq \frac{r+l}{\delta} - r + 4$. Let $F, G \in \mathcal{D}_\infty$. Then one may find $C \in \mathbb{R}_+$, $p \in \mathbb{N}$ (depending on r and δ) such that*

$$i) \quad d_{TV}(F, G) \leq C(1 + \Sigma_{p_1}(F) + \Sigma_{p_1}(G) + \|F\|_{L, q_1, p} + \|G\|_{L, q_1, p}) \times d_r(F, G)^{1-\delta}, \quad (37)$$

and

$$ii) \quad \|p_F - p_G\|_{l, \infty} \leq C(1 + \Sigma_{p_2}(F) + \Sigma_{p_2}(G) + \|F\|_{L, q_2, p} + \|G\|_{L, q_2, p}) \times d_r(F, G)^{1-\delta}, \quad (38)$$

where $p_F(x)$ and $p_G(x)$ denote the density functions of F and G respectively.

Comment The significance of this lemma is the following. Suppose that one has already obtained an estimate of a "smooth" distance d_r between two random vectors F and G (in our case $r = 3$ in (17)). But we would like to control the total variation distance between them. In order to do this, one employs some integration by parts techniques which are developed in [6] and conclude the following. We need to assume that both F and G are "smooth" in the sense that $\|F\|_{L, q, p} + \|G\|_{L, q, p} < \infty$ for sufficiently large q, p . Moreover, we need some non degeneracy condition: both F and G are non-degenerated, that is $\Sigma_p(F) + \Sigma_p(G) < \infty$, with p large enough. Then (37) asserts that one may control d_{TV} by d_r , and the control is quasi optimal: we loose just a power $\delta > 0$ which we may take as small as we want. And (38) says that we may also control the distance between the derivatives of density functions by d_r .

Then we can get the following corollary.

Corollary 3.4.1. *We fix some index l , some $r \in \mathbb{N}$ and some $\delta > 0$. We define p_1, p_2, q_1, q_2 as in Lemma 3.4. Let $F_M \in \mathcal{D}_\infty$, $M \in \mathbb{N}$ such that for every $p \geq 1$,*

$$\sup_M (\|F_M\|_{L, q_1, p} + \Sigma_{p_1}(F_M)) \leq Q_{q_1, p, p_1} < \infty,$$

with Q_{q_1, p, p_1} a constant not dependent on M . Consider moreover some random variable F such that $d_r(F, F_M) \rightarrow 0$. Then there exists a constant $C > 0$ such that

$$i) \quad d_{TV}(F, F_M) \leq C d_r(F, F_M)^{1-\delta}.$$

Moreover, if $\sup_M (\|F_M\|_{L, q_2, p} + \Sigma_{p_2}(F_M)) \leq Q_{q_2, p, p_2} < \infty$, the law of F is absolutely continuous with smooth density p_F and one has

$$ii) \quad \|p_F - p_{F_M}\|_{L, \infty} \leq C d_r(F, F_M)^{1-\delta}.$$

Proof. By **Lemma 3.4**, for every $M < M'$, one has

$$d_{TV}(F_M, F_{M'}) \leq C d_r(F_M, F_{M'})^{1-\delta} \leq C [d_r(F_M, F)^{1-\delta} + d_r(F, F_{M'})^{1-\delta}]. \quad (39)$$

So $(F_M)_{M \in \mathbb{N}}$ is a Cauchy sequence in d_{TV} . It follows that it has a limit G . But since $d_r(F_M, F) \rightarrow 0$, it follows that $F = G$. Passing to the limit $M' \rightarrow \infty$ in (39), we get

$$d_{TV}(F_M, F) \leq C d_r(F_M, F)^{1-\delta}.$$

The proof of *ii*) is analogous. □

4 Malliavin calculus and stochastic differential equations with jumps

In this section, we present the integration by parts framework which will be used when we deal with the jump equations.

4.1 The splitting method

We consider a Poisson point measure $N_\nu(ds, dz)$ with compensator $\widehat{N}_\nu(ds, dz) = \nu(dz)ds$ on the state space $[1, \infty)$. We will make use of the noise $z \in [1, \infty)$ in order to apply the results from the previous section. We denote $I_k = [k, k+1)$ and $m_k = \nu(I_k)$, and we suppose that for every k , there exists $\varepsilon_k > 0$, such that

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k \times dz. \quad (40)$$

Remark. Under **Hypothesis 2.4 (a)**, the splitting condition (40) is satisfied with $\varepsilon_k = \frac{\varepsilon_*}{(k+1)^{1-\alpha}}$, for any $\alpha \leq \alpha_1$. If instead we assume **Hypothesis 2.4 (b)**, (40) is also satisfied, with $\varepsilon_k = \frac{\varepsilon_*}{k+1}$.

When (40) is satisfied, we are able to use the "**splitting method**" as follows. To begin we define the functions

$$a(y) = 1 - \frac{1}{1 - (4y - 1)^2} \quad \text{for } y \in [\frac{1}{4}, \frac{1}{2}] \quad (41)$$

$$\psi(y) = 1_{\{|y| \leq \frac{1}{4}\}} + 1_{\{\frac{1}{4} < |y| \leq \frac{1}{2}\}} e^{a(|y|)}. \quad (42)$$

We notice that $\psi \in C_c^\infty(\mathbb{R})$ and that its support is included in $[-\frac{1}{2}, \frac{1}{2}]$. We also notice that for every $q, p \in \mathbb{N}$ the function $y \mapsto |a^{(q)}(y)|^p \psi(y)$ is continuous and has support included in $[-\frac{1}{2}, \frac{1}{2}]$, so it is bounded: one may find $C_{q,p}$ such that

$$|a^{(q)}(y)|^p \psi(y) \leq C_{q,p} \quad \forall y \in \mathbb{R}. \quad (43)$$

We denote

$$\psi_k(y) = \psi(y - (k + \frac{1}{2})), \quad \theta_k(y) := \partial_y \ln \psi_k(y). \quad (44)$$

By (43) (which is uniform with respect to y), we have

$$\sup_k |(\ln \psi_k)^{(q)}(y)|^p \psi_k(y) \leq C_{q,p} \quad \forall y \in \mathbb{R}. \quad (45)$$

We denote

$$m(\psi) = \int_{-1/2}^{1/2} \psi(y) dy. \quad (46)$$

We consider a sequence of independent random variables Z^k such that

$$Z^k \sim 1_{I_k}(z) \frac{1}{m_k} \nu(dz).$$

This is the sequence of random variables which are involved in the representation of the measure $N_\nu(ds, dz)$ as long as $z \in [1, \infty)$ is concerned. We notice that, according to our hypothesis (40),

$$\mathbb{P}(Z^k \in dz) = 1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k \times dz.$$

Then we construct some independent random variables V^k, U^k, ξ^k with laws

$$\begin{aligned} \mathbb{P}(V^k \in dz) &= \frac{1}{m(\psi)} \psi(z - (k + \frac{1}{2})) dz \\ \mathbb{P}(U^k \in dz) &= \frac{1}{1 - \varepsilon_k m(\psi)} (\mathbb{P}(Z^k \in dz) - \varepsilon_k \psi(z - (k + \frac{1}{2})) dz) \\ \mathbb{P}(\xi^k = 1) &= \varepsilon_k m(\psi), \quad \mathbb{P}(\xi^k = 0) = 1 - \varepsilon_k m(\psi). \end{aligned} \quad (47)$$

We choose $\varepsilon_k < 1/m(\psi)$ so that $1 - \varepsilon_k m(\psi) > 0$. Using (40), one may check that $\mathbb{P}(U^k \in dz)$ is a positive measure and has mass one. So it is a probability measure. And finally one can easily check the identity of laws:

$$Z^k \sim \xi^k V^k + (1 - \xi^k) U^k. \quad (48)$$

In the following, we will work directly with $Z^k = \xi^k V^k + (1 - \xi^k) U^k$. This is possible because all the results that we discuss here concern the law of the random variables, and the law remains unchanged.

The Poisson point measure N_ν can be written as the following sum:

$$N_\nu(ds, dz) = \sum_{k=1}^{\infty} 1_{I_k}(z) N_{\nu_k}(ds, dz) = \sum_{k=1}^{\infty} N_{\nu_k}(ds, dz),$$

where $\nu_k(dz) = 1_{I_k}(z) \nu(dz)$ and N_{ν_k} is a Poisson point measure with intensity $\nu_k(dz) ds$.

The Poisson point measure N_{ν_k} can be represented by means of compound Poisson processes as follows. For each $k \in \mathbb{N}$, we denote by $T_i^k, i \in \mathbb{N}$ the jump times of a Poisson process $(J_t^k)_{t \in [0, T]}$ of parameter m_k , and we consider a sequence of independent random variables $Z_i^k \sim 1_{I_k}(z) \frac{\nu(dz)}{m_k}$, which are independent of

J^k as well. Then, for any $t > 0$ and $A \in \mathcal{B}([k, k+1))$, $N_{\nu_k}([0, t] \times A) = \sum_{i=1}^{J_t^k} 1_A(Z_i^k)$. And for each $k, i \in \mathbb{N}$,

we will split Z_i^k as $Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k) U_i^k$.

Remark. The law of Z_i^k could be very irregular and it is not possible to make integration by parts based on it. So we construct the V_i^k , which has all the good regularity properties in order to make Malliavin calculus. This is the idea of the splitting method. The splitting method presented here is analogous to the one in [10]. There, Bichteler, Gravereau and Jacod deal with 2 kinds of independent Poisson point measures. One is very regular, and smooth enough to make Malliavin calculus on it (in our paper, V_i^k play the same role). The other one can be arbitrary, and it may be very irregular (in our paper, it corresponds to U_i^k). But the difference is that instead of splitting the Poisson point measure, we split the random variables, and so this method can also be applied in a large class of different problems. For example, Bally, Caramellino and Poly use the splitting method to show the convergence in total variation distance in the central limit theorem in [5]. Other possible approaches to the Malliavin calculus for jump processes are given in the papers [24], [25], [35], [37] and the book [19] for example.

4.2 Malliavin calculus for Poisson point measures and space-time Brownian motions

In this section we present the IbP framework on a space where we have the Poisson point measure N_ν presented in the previous section and moreover we have a space-time Brownian motion $W_\nu(ds, dz)$ with covariance measure $\nu(dz)ds$, which is independent of N_ν . We recall that in Section 2.2 we have introduced the random variables $W_\nu(\varphi), N_\nu(\varphi)$ and the filtrations $(\mathcal{F}_t^W)_{t \in [0, T]}$, $(\mathcal{F}_t^N)_{t \in [0, T]}$, and we denote $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$. We present now the Malliavin calculus. We recall the random variables T_i^k , and $Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k)U_i^k$ introduced in the previous section and we take $\mathcal{G} = \sigma(U_i^k, \xi_i^k, T_i^k : k, i \in \mathbb{N})$ to be the σ -algebra associated to the noise from U_i^k, ξ_i^k, T_i^k , $k, i \in \mathbb{N}$. These are the noises which will not be involved in our calculus. We denote by $C_{\mathcal{G}, p}$ the space of the functions $f : \Omega \times \mathbb{R}^{m' \times m} \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that f is \mathcal{F}_T -measurable, and for each ω , the function $(v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n) \mapsto f(\omega, v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$ belongs to $C_p^\infty(\mathbb{R}^{m' \times m} \times \mathbb{R}^n)$, and for each $(v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$, the function $\omega \mapsto f(\omega, v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$ is \mathcal{G} -measurable. Then we define the space of simple functionals

$$\mathcal{S} = \{F = f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) : f \in C_{\mathcal{G}, p}, \varphi_1, \dots, \varphi_n \in L^2([1, \infty) \times [0, T], \nu \times \text{Leb}), m', m, n \in \mathbb{N}\}.$$

On the space \mathcal{S} we define the derivative operators

$$D_{(k_0, i_0)}^Z F = 1_{\{k_0 \leq m\}} 1_{\{i_0 \leq m'\}} \xi_{i_0}^{k_0} \frac{\partial f}{\partial v_{i_0}^{k_0}}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n), \quad k_0, i_0 \in \mathbb{N}$$

$$D_{(s, z)}^W F = \sum_{r=1}^n \frac{\partial f}{\partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \varphi_r(s, z), \quad (s, z) \in [0, T] \times [1, \infty).$$

We regard $D^Z F$ as an element of the Hilbert space l_2 (the space of the sequences $h = (h_i^k)_{k, i \in \mathbb{N}}$ with $\|h\|_{l_2}^2 = \sum_{k=1}^\infty \sum_{i=1}^\infty |h_i^k|^2 < \infty$) and $D^W F$ as an element of the Hilbert space $L^2([1, \infty) \times [0, T], \nu \times \text{Leb})$. Then

$$DF := (D^Z F, D^W F) \in l_2 \otimes L^2([1, \infty) \times [0, T], \nu \times \text{Leb}).$$

We also denote $D^{Z, W} F = DF$ and $\mathcal{H} = l_2 \otimes L^2([1, \infty) \times [0, T], \nu \times \text{Leb})$. And we have

$$\langle DF, DG \rangle_{\mathcal{H}} = \sum_{k=1}^\infty \sum_{i=1}^\infty D_{(k, i)}^Z F \times D_{(k, i)}^Z G + \int_{[0, T] \times [1, \infty)} D_{(s, z)}^W F \times D_{(s, z)}^W G \nu(dz)ds.$$

Moreover, we define the derivatives of order $q \in \mathbb{N}$ recursively:

$$D_{(k_1, i_1) \dots (k_q, i_q), (s_1, z_1) \dots (s_q, z_q)}^{Z, W, q} F := D_{(k_q, i_q), (s_q, z_q)}^{Z, W} D_{(k_{q-1}, i_{q-1}), (s_{q-1}, z_{q-1})}^{Z, W} \dots D_{(k_1, i_1), (s_1, z_1)}^{Z, W} F,$$

and we denote $D^q F = D^{Z, W, q} F$. We also denote $D^{Z, q}$ (respectively $D^{W, q}$) as the derivative D^Z (respectively D^W) of order q .

We recall the function θ_k defined in (44) and we define the Ornstein-Uhlenbeck operators L^Z, L^W and $L = L^Z + L^W$ (which verify the duality relation), with

$$L^Z F = - \sum_{k=1}^m \sum_{i=1}^{m'} (D_{(k, i)}^Z D_{(k, i)}^Z F + \xi_i^k D_{(k, i)}^Z F \times \theta_k(V_i^k)),$$

$$L^W F = \sum_{r=1}^n \frac{\partial f}{\partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) W_\nu(\varphi_r) - \sum_{l, r=1}^n \frac{\partial^2 f}{\partial w_l \partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \langle \varphi_l, \varphi_r \rangle_{L^2([1, \infty) \times [0, T], \nu \times \text{Leb})}.$$

One can check that the triplet (\mathcal{S}, D, L) is consistent with the IbP framework given in Section 3. The proof is left to Appendix 5.3.

In the following, we will close the operator D^q and L , so we will use the IbP framework $(\mathcal{D}_\infty, D, L)$ associated to (\mathcal{S}, D, L) in **Lemma 3.1**.

4.3 Malliavin calculus applied to stochastic differential equations with jumps

Now we will use the IbP framework presented in Section 4.2 for the equation (10).

Let $\theta : (0, 1] \rightarrow [1, \infty)$ be a function such that $\theta(z) = \frac{1}{z}$. By a change of variables, instead of dealing with equation (10), it is equivalent to consider the following equation.

$$\begin{aligned} \widehat{X}_t^M &= x + \int_0^t \int_{[1, M)} \widetilde{c}(s, z, \widehat{X}_{s-}^M) N_\nu(ds, dz) \\ &\quad + \int_0^t b_M(s, \widehat{X}_s^M) ds + \int_0^t \int_{\{z \geq M\}} \widetilde{c}(s, z, \widehat{X}_s^M) W_\nu(ds, dz), \end{aligned} \quad (49)$$

where $M = \frac{1}{\varepsilon}$, $\nu(dz) = \mu \circ \theta^{-1}(dz)$, $\widetilde{c}(s, z, x) = c(s, \frac{1}{z}, x)$,

$$b_M(s, x) = \int_{\{z \geq M\}} \widetilde{c}(s, z, x) \nu(dz), \quad (50)$$

and W_ν is the space-time Brownian motion with covariance measure $\nu(dz)ds$. One can check that \widehat{X}_t^M has the same law as X_t^ε .

Here we give two lemmas, concerning the Malliavin-Sobolev norms and the Malliavin covariance. We recall that ε_* and α are introduced in **Hypothesis 2.4** (b), and q^* is introduced in **Hypothesis 2.1**.

Lemma 4.1. *Assuming **Hypothesis 2.1** with $q^* \geq 2$ and **Hypothesis 2.4** (either 2.4(a) or 2.4(b)), we have $\widehat{X}_t^M \in \mathcal{D}_\infty$, and for all $p \geq 1, 2 \leq l \leq q^*$, there exists a constant $C_{l,p}(T)$ depending on l, p, x and T , such that $\sup_M \|\widehat{X}_t^M\|_{L^{l,p}} \leq C_{l,p}(T)$.*

Lemma 4.2. *Assume that **Hypothesis 2.1** with $q^* \geq 1$ and **Hypothesis 2.2, 2.3** hold true.*

a) *If we also assume **Hypothesis 2.4** (a), then for every $p \geq 1, t \in [0, T]$, we have*

$$\sup_M \mathbb{E}(1/\sigma_{\widehat{X}_t^M})^p \leq C_p, \quad (51)$$

with C_p a constant only depending on p and T .

b) *If we assume **Hypothesis 2.4** (b), then for every $p \geq 1, t \in [0, T]$ such that $t > \frac{4p\alpha}{\varepsilon_*}$, we have $\sup_M \mathbb{E}(1/\sigma_{\widehat{X}_t^M})^p \leq C_p$.*

The proofs of these lemmas are rather technical and are postponed for the Appendix (Section 5.1 and 5.2).

4.4 Proof of the main result (Theorem 2.2)

Proof. (a) By **Lemma 4.1** and **Lemma 4.2 a)**, we know that for any $\delta > 0$, for any $p, p_1 \geq 1, 2 \leq q \leq q^*$, with $q^* \geq \frac{3}{\delta} + 1$, there exists a constant $C_{q,p,p_1}(T)$ such that for any $M \geq 1, t \in [0, T]$, we have

$$\Sigma_{p_1}(\widehat{X}_t^M) + \|\widehat{X}_t^M\|_{L^{q,p}} \leq C_{q,p,p_1}(T).$$

By **Lemma 2.1**, we know that

$$d_3(\widehat{X}_t, \widehat{X}_t^M) = d_3(X_t, X_t^\varepsilon) \leq C\eta_3(\varepsilon).$$

Then applying **Corollary 3.4.1 i)** for $r = 3$, we have

$$d_{TV}(X_t, X_t^\varepsilon) = d_{TV}(\widehat{X}_t, \widehat{X}_t^M) \leq C d_3(\widehat{X}_t, \widehat{X}_t^M)^{1-\delta} \leq C_\delta \eta_3(\varepsilon)^{1-\delta}.$$

So we obtain (23). The proof of (24) is obtained by **Corollary 3.4.1 ii)**, since $q^* \geq \frac{3+l}{\delta} + 1$.

(b) The proof is almost the same. If $t > \frac{8\alpha(\frac{3}{\delta}-1)}{\varepsilon_*}$, then by **Lemma 4.2 b)**, $\Sigma_{p_1}(\widehat{X}_t^M) < \infty$ for $p_1 = 2(\frac{3}{\delta} - 1)$. So **Corollary 3.4.1 i)** still holds, and we can obtain (25). For

$$t > \max\left\{\frac{8\alpha}{\varepsilon_*}\left(\frac{3+l}{\delta} - 1\right), \frac{8\alpha(3l+2)}{\varepsilon_*}\right\},$$

by **Lemma 4.2 b)**, $\Sigma_{p_2}(\widehat{X}_t^M) < \infty$ for $p_2 = \max\{2(\frac{3+l}{\delta} - 1), 6l+4\}$. So **Corollary 3.4.1 ii)** still holds, and we obtain (26). \square

5 Appendix

5.1 Proof of Lemma 4.1

In the following, we will only work with the measure ν supported on $[1, \infty)$ and with the processes $(\widehat{X}_t)_{t \in [0, T]}$ and $(\widehat{X}_t^M)_{t \in [0, T]}$. So in order to simplify the notation, from now on we denote $\widehat{X}_t = X_t$ and $\widehat{X}_t^M = X_t^M$. We remark that $M = \frac{1}{\varepsilon}$ is generally not an integer, but for simplicity, we assume in the following that M is an integer.

Here is the idea of the proof. Since X_t^M is not a simple functional, we construct first the Euler scheme $(X_t^{n, M})_{t \in [0, T]}$ in subsection 5.1.1 and check that $X_t^{n, M} \rightarrow X_t^M$ in L^1 . We will prove that $\mathbb{E}|X_t^{n, M}|_l^p$ and $\mathbb{E}|LX_t^{n, M}|_l^p$ are bounded (uniformly in n, M) in subsection 5.1.3. Then based on **Lemma 3.2**, we obtain that $X_t^M \in \mathcal{D}_\infty$ and the norms $\|X_t^M\|_{L, l, p}$ are bounded (uniformly in M).

5.1.1 Construction of the Euler scheme

We take a time-partition $\mathcal{P}_t^n = \{r_j = \frac{jt}{n}, j = 0, \dots, n\}$ and a space-partition $\tilde{\mathcal{P}}_M^n = \{z_j = M + \frac{j}{n}, j = 0, 1, \dots\}$. We denote $\tau_n(r) = r_j$ when $r \in [r_j, r_{j+1})$, and denote $\gamma_n(z) = z_j$ when $z \in [z_j, z_{j+1})$. Let

$$\begin{aligned} X_t^{n, M} &= x + \int_0^t \int_{[1, M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n, M}) N_\nu(dr, dz) \\ &\quad + \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n, M}) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n, M}) W_\nu(dr, dz). \end{aligned} \quad (52)$$

Then we can obtain the following lemma immediately.

Lemma 5.1. *Assume that the **Hypothesis 2.1** holds true with $q^* \geq 1$. Then for any $p \geq 1, M \geq 1$, we have $\mathbb{E}|X_t^{n, M} - X_t^M|^p \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. We first notice that since $\bar{c}(z)$ (in **Hypothesis 2.1**) is decreasing,

$$\sup_{n \in \mathbb{N}} \int_1^\infty |\bar{c}(\gamma_n(z))|^2 \nu(dz) = |\bar{c}(1)|^2 \nu[1, 2] + \int_2^\infty |\bar{c}(\gamma_1(z))|^2 \nu(dz) \leq |\bar{c}(1)|^2 \nu[1, 2] + \int_1^\infty |\bar{c}(z)|^2 \nu(dz) \leq C < \infty. \quad (53)$$

Then (53) implies that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \int_0^T \int_{[1, \infty)} |\tilde{c}(s, z, x) - \tilde{c}(\tau_n(s), \gamma_n(z), x)|^2 \nu(dz) ds = 0, \quad (54)$$

and

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} \int_0^T \int_{[1, \infty)} |\tilde{c}(\tau_n(s), \gamma_n(z), x)|^2 \nu(dz) ds \leq C. \quad (55)$$

In the following proof, $C_p(T)$ will be a constant depending on p and T which may be changed from line to line. For $p \geq 2$, we write $\mathbb{E}|X_t^{n,M} - X_t^M|^p \leq C_p(T)(E_1 + E_2 + E_3)$, where

$$E_1 = \mathbb{E} \left| \int_0^t \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M) N_\nu(dr, dz) \right|^p,$$

$$E_2 = \mathbb{E} \left| \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) - b_M(r, X_r^M) dr \right|^p,$$

$$E_3 = \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M) W_\nu(dr, dz) \right|^p.$$

Then, compensating N_ν and using Burkholder's inequality (see for example the Theorem 2.11 in [26]),

$$\begin{aligned} E_1 &\leq C_p(T) \left[\mathbb{E} \int_0^t \left(\int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \right. \\ &\quad + \mathbb{E} \int_0^t \int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M)|^p \nu(dz) dr \\ &\quad + \mathbb{E} \int_0^t \left| \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M) \nu(dz) \right|^p dr \Big] \\ &\leq C_p(T) [R_n^1 + ((\bar{c}_2)^{\frac{p}{2}} + \bar{c}_p + (\bar{c}_1)^p) \int_0^t \mathbb{E}|X_{\tau_n(r)}^{n,M} - X_r^M|^p dr], \end{aligned}$$

with

$$\begin{aligned} R_n^1 &= \mathbb{E} \int_0^t \left(\int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^M) - \tilde{c}(r, z, X_{r-}^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\quad + \mathbb{E} \int_0^t \int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^M) - \tilde{c}(r, z, X_{r-}^M)|^p \nu(dz) dr \\ &\quad + \mathbb{E} \int_0^t \left| \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^M) - \tilde{c}(r, z, X_{r-}^M) \nu(dz) \right|^p dr. \end{aligned}$$

Since $|\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^M) - \tilde{c}(r, z, X_{r-}^M)|^p \leq |2\bar{c}(z)|^p \in L^1(\Omega \times [1, \infty) \times [0, T], \mathbb{P} \times \nu \times Leb)$, we apply the Lebesgue's dominated convergence theorem and we obtain that $R_n^1 \rightarrow 0$. Next,

$$\begin{aligned} E_2 &\leq C_p(T) \mathbb{E} \int_0^t \left| \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M) \nu(dz) \right|^p dr \\ &\leq C_p(T) [R_n^2 + (\bar{c}_1)^p \int_0^t \mathbb{E}|X_{\tau_n(r)}^{n,M} - X_r^M|^p dr], \end{aligned}$$

with

$$R_n^2 = \mathbb{E} \int_0^t \left| \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), z, X_r^M) - \tilde{c}(r, z, X_r^M) \nu(dz) \right|^p dr \rightarrow 0.$$

Finally, using Burkholder's inequality,

$$\begin{aligned} E_3 &\leq C_p(T) \mathbb{E} \int_0^t \int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M)|^2 \nu(dz) \Big|^{\frac{p}{2}} dr \\ &\leq C_p(T) [R_n^3 + (\bar{c}_2)^{\frac{p}{2}} \int_0^t \mathbb{E}|X_{\tau_n(r)}^{n,M} - X_r^M|^p dr], \end{aligned}$$

where (by (54)),

$$R_n^3 = \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_r^M) - \tilde{c}(r, z, X_r^M)|^2 \nu(dz) \right|^{\frac{p}{2}} dr \rightarrow 0.$$

Therefore, $\mathbb{E}|X_t^{n,M} - X_t^M|^p \leq C_p(T)[R_n + \int_0^t \mathbb{E}|X_{\tau_n(r)}^{n,M} - X_r^M|^p dr]$, with $R_n = R_n^1 + R_n^2 + R_n^3 \rightarrow 0$ as $n \rightarrow \infty$. One can easily check that $\mathbb{E}|X_t^{n,M} - X_{\tau_n(t)}^{n,M}|^p \rightarrow 0$, so we have $\mathbb{E}|X_t^{n,M} - X_t^M|^p \leq C_p(T)[\widetilde{R}_n + \int_0^t \mathbb{E}|X_r^{n,M} - X_r^M|^p dr]$, with $\widetilde{R}_n \rightarrow 0$ as $n \rightarrow \infty$. We conclude by using Gronwall's lemma. \square

Remark. Some results on the convergence of the Euler scheme of a jump-diffusion can be found for example in [32], [34]. The special thing in our paper is that we deal with the space-time Brownian motion instead of the classical Brownian motion, and this is why we need to assume (54).

Now we represent the jump's part of $(X_t^{n,M})_{t \in [0, T]}$ by means of compound Poisson processes. We recall that for each $k \in \mathbb{N}$, we denote by $T_i^k, i \in \mathbb{N}$ the jump times of a Poisson process $(J_t^k)_{t \in [0, T]}$ of parameter m_k , and we consider a sequence of independent random variables $Z_i^k \sim 1_{I_k}(z) \frac{\nu(dz)}{m_k}$, which are independent of J^k as well. Then we write

$$\begin{aligned} X_t^{n,M} &= x + \int_0^t \sum_{k=1}^{M-1} \int_{\{z \in I_k\}} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz) \\ &\quad + \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz) \\ &= x + \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M}) \\ &\quad + \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz). \end{aligned} \quad (56)$$

So for every $t \in [0, T]$, $X_t^{n,M}$ is a simple functional.

5.1.2 Preliminary estimates

In order to estimate the Sobolev norms of the Euler scheme, we need the following preliminary lemmas.

Lemma 5.2. *We fix $M \geq 1$. Let $y : \Omega \times [0, T] \times [M, \infty) \rightarrow \mathbb{R}$ be a function which is piecewise constant with respect to both t and z . We assume that $y_t(z)$ is progressively measurable with respect to \mathcal{F}_t (defined in (11)), $y_t(z) \in \mathcal{S}$, and $\mathbb{E}(\int_0^t \int_{\{z \geq M\}} |y_r(z)|^2 \nu(dz) dr) < \infty$. We denote $I_t(y) = \int_0^t \int_{\{z \geq M\}} y_r(z) W_\nu(dr, dz)$. Then for any $l \geq 1, p \geq 2$, there exists a constant $C_{l,p}(T)$ such that*

$$a) \quad \mathbb{E}|I_t(y)|_l^p \leq C_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr,$$

$$b) \quad \mathbb{E}|LI_t(y)|_l^p \leq C_{l,p}(T) [\mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |Ly_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr].$$

Proof. Proof of a): Let $C_{l,p}(T)$ be a constant depending on l, p and T which may change from one line to another. For any $l \geq 1$, we take $l_W \geq 0$ and $l_Z \geq 0$ such that $0 < l_W + l_Z \leq l$.

It is easy to check that

$$D_{(k_1, i_1) \dots (k_{l_Z}, i_{l_Z})}^{Z, l_Z} I_t(y) = \int_0^t \int_{\{z \geq M\}} D_{(k_1, i_1) \dots (k_{l_Z}, i_{l_Z})}^{Z, l_Z} y_r(z) W_\nu(dr, dz).$$

And by recurrence, one can show that

$$D_{(s_1, z_1) \cdots (s_{l_W}, z_{l_W})}^{W, l_W} I_t(y) = \int_0^t \int_{\{z \geq M\}} D_{(s_1, z_1) \cdots (s_{l_W}, z_{l_W})}^{W, l_W} y_r(z) W_\nu(dr, dz) + \sum_{j=1}^{l_W} D_{(s_j, z_j)}^{W, l_W - 1} y_{s_j}(z_j) 1_{s_j \leq t},$$

with

$$\widehat{(s_j, z_j)}^{l_W - 1} := (s_1, z_1) \cdots (s_{j-1}, z_{j-1})(s_{j+1}, z_{j+1}) \cdots (s_{l_W}, z_{l_W}).$$

We denote

$$\bar{y}_r(z)(k_1, i_1, \dots, k_{l_Z}, i_{l_Z}) := D_{(k_1, i_1) \cdots (k_{l_Z}, i_{l_Z})}^{Z, l_Z} y_r(z), \quad \bar{y}_r^{l_Z}(z) := D^{Z, l_Z} y_r(z) \in l_2^{\otimes l_Z}.$$

Then $D^{Z, l_Z} I_t(y) = I_t(\bar{y}^{l_Z})$, and

$$\begin{aligned} D_{(s_1, z_1) \cdots (s_{l_W}, z_{l_W})}^{W, l_W} D_{(k_1, i_1) \cdots (k_{l_Z}, i_{l_Z})}^{Z, l_Z} I_t(y) &= \int_0^t \int_{\{z \geq M\}} D_{(s_1, z_1) \cdots (s_{l_W}, z_{l_W})}^{W, l_W} \bar{y}_r(z)(k_1, i_1, \dots, k_{l_Z}, i_{l_Z}) W_\nu(dr, dz) \\ &+ \sum_{j=1}^{l_W} D_{(s_j, z_j)}^{W, l_W - 1} \bar{y}_{s_j}(z_j)(k_1, i_1, \dots, k_{l_Z}, i_{l_Z}) 1_{s_j \leq t}. \end{aligned}$$

Let $H_{l_Z, l_W, T} = l_2^{\otimes l_Z} \otimes L^2([0, T] \times [M, \infty), Leb \times \nu)^{\otimes l_W}$. We have

$$\begin{aligned} |D^{W, l_W} D^{Z, l_Z} I_t(y)|_{H_{l_Z, l_W, T}}^2 &= \int_{[0, T]^{l_W}} \int_{[M, \infty)^{l_W}} |D_{(s_1, z_1) \cdots (s_{l_W}, z_{l_W})}^{W, l_W} I_t(\bar{y}^{l_Z})|_{l_2^{\otimes l_Z}}^2 \nu(dz_1) ds_1 \cdots \nu(dz_{l_W}) ds_{l_W} \\ &\leq 2 \int_0^t \int_{\{z \geq M\}} |D^{W, l_W} \bar{y}_r^{l_Z}(z) W_\nu(dr, dz)|_{H_{l_Z, l_W, T}}^2 + l_W 2^{l_W} \int_0^t \int_{\{z \geq M\}} |D^{W, l_W - 1} \bar{y}_r^{l_Z}(z)|_{H_{l_Z, l_W - 1, T}}^2 \nu(dz) dr. \end{aligned}$$

Using Burkholder's inequality for Hilbert-space-valued martingales (see [30] for example), we have

$$\begin{aligned} \mathbb{E} |D^{W, l_W} D^{Z, l_Z} I_t(y)|_{H_{l_Z, l_W, T}}^p &\leq C_{l, p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{W, l_W} D^{Z, l_Z} y_r(z)|_{H_{l_Z, l_W, T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &+ \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{W, l_W - 1} D^{Z, l_Z} y_r(z)|_{H_{l_Z, l_W - 1, T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr. \end{aligned} \quad (57)$$

We recall that for $F \in \mathcal{D}_\infty$, we have $|D^{W, l_W} D^{Z, l_Z} F|_{H_{l_Z, l_W, T}} \leq |F|_{l_Z + l_W}$ (see the definition in (33)). Then (57) gives

$$\mathbb{E} |I_t(y)|_{1, l}^p \leq C_{l, p}(T) \sum_{l_Z + l_W \leq l} \mathbb{E} |D^{W, l_W} D^{Z, l_Z} I_t(y)|_{H_{l_Z, l_W, T}}^p \leq C_{l, p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr. \quad (58)$$

Finally, using Burkholder's inequality, we have

$$\mathbb{E} |I_t(y)|^p \leq C_{l, p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|^2 \nu(dz) \right)^{\frac{p}{2}} dr. \quad (59)$$

So a) is proved.

Proof of b): We first show that

$$L I_t(y) = I_t(Ly) + I_t(y). \quad (60)$$

We denote

$$I_k^t(f_k) = k! \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \int_{[M, +\infty)^k} f_k(s_1, \dots, s_k, z_1, \dots, z_k) W_\nu(ds_k, dz_k) \cdots W_\nu(ds_1, dz_1)$$

the multiple stochastic integral for a deterministic function f_k , which is square integrable with respect to $(\nu(dz)ds)^{\otimes k}$ and is symmetric with respect to the time variation (s_1, \dots, s_k) for each fixed (z_1, \dots, z_k) . Notice that $L^Z I_k^t(f_k) = 0$ and $L^W I_k^t(f_k) = kI_k^t(f_k)$. So, $LI_k^t(f_k) = kI_k^t(f_k)$. Then by the duality relation (32),

$$\mathbb{E}(I_k^t(f_k)L(I_t(y))) = \mathbb{E}(I_t(y) \times LI_k^t(f_k)) = k\mathbb{E}(I_t(y) \times I_k^t(f_k)). \quad (61)$$

On the other hand, using the isometry property and the duality relation,

$$\begin{aligned} \mathbb{E}(I_k^t(f_k) \times I_t(Ly)) &= k\mathbb{E} \int_0^t \int_{\{z \geq M\}} I_{k-1}^r(f_k(r, z, \cdot)) Ly_r(z) \nu(dz) dr \\ &= k \int_0^t \int_{\{z \geq M\}} \mathbb{E}[y_r(z) \times LI_{k-1}^r(f_k(r, z, \cdot))] \nu(dz) dr = k(k-1) \mathbb{E} \int_0^t \int_{\{z \geq M\}} y_r(z) I_{k-1}^r(f_k(r, z, \cdot)) \nu(dz) dr \\ &= k(k-1) \mathbb{E}(I_t(y) \times \int_0^t \int_{\{z \geq M\}} I_{k-1}^r(f_k(r, z, \cdot)) W_\nu(dr, dz)) \\ &= (k-1) \mathbb{E}(I_t(y) \times I_k^t(f_k)). \end{aligned}$$

Combining this with (61), we get

$$\mathbb{E}(I_k^t(f_k)(I_t(y) + I_t(Ly))) = k\mathbb{E}(I_k^t(f_k)I_t(y)) = \mathbb{E}[I_k^t(f_k) \times LI_t(y)]. \quad (62)$$

Since every element in $L^2(W)$ (defined by (12)) can be represented as the direct sum of multiple stochastic integrals, we have for any $F \in L^2(W)$,

$$\mathbb{E}[FLI_t(y)] = \mathbb{E}[F(I_t(Ly) + I_t(y))]. \quad (63)$$

For $G \in L^2(N)$, one has $L^W G = 0$ and $L^Z G \in L^2(N)$. Then by using duality and (13),

$$\mathbb{E}[GLI_t(y)] = \mathbb{E}[I_t(y)LG] = \mathbb{E}[I_t(y)L^Z G] = 0,$$

and by (13),

$$\mathbb{E}[G(I_t(Ly) + I_t(y))] = 0.$$

So,

$$\mathbb{E}[GLI_t(y)] = \mathbb{E}[G(I_t(Ly) + I_t(y))]. \quad (64)$$

Combining (63) and (64), for any $\tilde{G} \in L^2(W) \otimes L^2(N)$, we have $\mathbb{E}[\tilde{G}LI_t(y)] = \mathbb{E}[\tilde{G}(I_t(Ly) + I_t(y))]$, which proves (60).

Then, by **Lemma 5.2 a)**,

$$\begin{aligned} \mathbb{E}|LI_t(y)|_l^p &\leq 2^{p-1} (\mathbb{E} \int_0^t \int_{\{z \geq M\}} |Ly_r(z) W_\nu(dr, dz)|_l^p + \mathbb{E} \int_0^t \int_{\{z \geq M\}} |y_r(z) W_\nu(dr, dz)|_l^p) \\ &\leq C_{l,p}(T) \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |Ly_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr. \end{aligned}$$

□

We will also need the following lemma from [7] (Lemma 8 and Lemma 10), which is a consequence of the chain rule for D^q and L .

Lemma 5.3. *Let $F \in S^d$. For every $l \in \mathbb{N}$, if $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a $C^l(\mathbb{R}^d)$ function (l -times differentiable function), then*

$$a) \quad |\phi(F)|_{1,l} \leq |\nabla \phi(F)| |F|_{1,l} + C_l \sup_{2 \leq |\beta| \leq l} |\partial^\beta \phi(F)| |F|_{1,l-1}^l.$$

If $\phi \in C^{l+2}(\mathbb{R}^d)$, then

$$b) \quad |L\phi(F)|_l \leq |\nabla\phi(F)||LF|_l + C_l \sup_{2 \leq |\beta| \leq l+2} |\partial^\beta \phi(F)|(1 + |F|_{l+1}^{l+2})(1 + |LF|_{l-1}).$$

For $l = 0$, we have

$$c) \quad |L\phi(F)| \leq |\nabla\phi(F)||LF| + \sup_{|\beta|=2} |\partial^\beta \phi(F)||F|_{1,1}^2.$$

We finish this section with a first estimate concerning the operator L .

Lemma 5.4. *Under the **Hypothesis 2.4** (either 2.4(a) or 2.4(b)), for every $p \geq 2, \tilde{p} \geq 1, l \geq 0$, there exists a constant $C_{l,p,\tilde{p}}(T)$ such that*

$$\sup_{M \in \mathbb{N}} \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} \bar{c}(Z_i^k) |LZ_i^k|_{\tilde{p}}^p \right) \leq C_{l,p,\tilde{p}}(T). \quad (65)$$

Proof. We notice that (with ψ_k given in (44)), $LZ_i^k = \xi_i^k (\ln \psi_k)'(V_i^k)$ and $D^{W,l} LZ_i^k = 0$. Moreover,

$$D_{(r_1, m_1) \dots (r_l, m_l)}^{Z,l} LZ_i^k = \prod_{j=1}^l (\delta_{r_j k} \delta_{m_j i}) \xi_i^k (\ln \psi_k)^{(l+1)}(V_i^k),$$

with $\delta_{r,k}$ the Kronecker delta, so that

$$|LZ_i^k|_l = \xi_i^k \sum_{0 \leq \tilde{l} \leq l} |(\ln \psi_k)^{(\tilde{l}+1)}(V_i^k)|. \quad (66)$$

It follows that

$$\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} \bar{c}(Z_i^k) |LZ_i^k|_{\tilde{p}}^p \right) \leq C_{l,p,\tilde{p}} \sum_{0 \leq \tilde{l} \leq l} \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} \bar{c}(Z_i^k) \xi_i^k |(\ln \psi_k)^{(\tilde{l}+1)}(V_i^k)|_{\tilde{p}}^p \right).$$

Since $\bar{c}(Z_i^k) \xi_i^k = \bar{c}(V_i^k) \xi_i^k$, we may replace Z_i^k by V_i^k in the right hand side of the above estimate. This gives

$$C_{l,p,\tilde{p}} \sum_{0 \leq \tilde{l} \leq l} \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} \bar{c}(V_i^k) \xi_i^k |(\ln \psi_k)^{(\tilde{l}+1)}(V_i^k)|_{\tilde{p}}^p \right) = C_{l,p,\tilde{p}} \sum_{0 \leq \tilde{l} \leq l} \mathbb{E} \left| \int_0^t \int_{[1,M]} \int_{\{0,1\}} \bar{c}(v) \xi |(\ln \bar{\psi})^{(\tilde{l}+1)}(v)|_{\tilde{p}} \Lambda(ds, d\xi, dv) \right|^p,$$

where $\bar{\psi}(v) := \sum_{k=1}^{\infty} 1_{I_k}(v) \psi(v - (k + \frac{1}{2}))$ and Λ is a Poisson point measure on $\{0, 1\} \times [1, \infty)$ with compensator

$$\widehat{\Lambda}(ds, d\xi, dv) = \sum_{k=1}^{\infty} \left[\frac{\psi(v - (k + \frac{1}{2}))}{m(\psi)} 1_{I_k}(v) dv \times b(v, d\xi) \right] ds,$$

with $b(v, d\xi)$ the Bernoulli probability measure on $\{0, 1\}$ with parameter $\varepsilon_k m(\psi)$, if $v \in I_k$. Then by compensating Λ and using Burkholder's inequality (the same proof as for (5)),

$$\begin{aligned} & C_{l,p,\tilde{p}} \sum_{0 \leq \tilde{l} \leq l} \mathbb{E} \left| \int_0^t \int_{[1,M]} \int_{\{0,1\}} \bar{c}(v) \xi |(\ln \bar{\psi})^{(\tilde{l}+1)}(v)|_{\tilde{p}} \Lambda(ds, d\xi, dv) \right|^p \\ & \leq C_{l,p,\tilde{p}}(T) \sum_{0 \leq \tilde{l} \leq l} \left[\left(\int_0^t \int_{[1,M] \times \{0,1\}} |\bar{c}(v)|^2 \xi |(\ln \bar{\psi})^{(\tilde{l}+1)}(v)|_{\tilde{p}}^2 \widehat{\Lambda}(ds, d\xi, dv) \right)^{\frac{p}{2}} \right. \\ & \quad \left. + \int_0^t \int_{[1,M] \times \{0,1\}} |\bar{c}(v)|^p \xi |(\ln \bar{\psi})^{(\tilde{l}+1)}(v)|_{\tilde{p}}^{p\tilde{p}} \widehat{\Lambda}(ds, d\xi, dv) + \left| \int_0^t \int_{[1,M] \times \{0,1\}} \bar{c}(v) \xi |(\ln \bar{\psi})^{(\tilde{l}+1)}(v)|_{\tilde{p}} \widehat{\Lambda}(ds, d\xi, dv) \right|^p \right]. \quad (67) \end{aligned}$$

We notice that by (45),

$$\begin{aligned} & \int_0^t \int_{[1,M) \times \{0,1\}} |\bar{c}(v)|^p \xi |(\ln \bar{\psi})^{(\bar{l}+1)}(v)|^{p\bar{p}} \widehat{\Lambda}(ds, d\xi, dv) = t \sum_{k=1}^{M-1} \int_{I_k} \varepsilon_k m(\psi) |\bar{c}(v)|^p |(\ln \psi_k)^{(\bar{l}+1)}(v)|^{p\bar{p}} \frac{\psi_k(v)}{m(\psi)} dv \\ & \leq C_{\bar{l}, p, \bar{p}}(T) \sum_{k=1}^{M-1} \varepsilon_k \int_{I_k} |\bar{c}(v)|^p dv. \end{aligned}$$

Similar upper bounds hold for the two other terms in the right hand side of (67), so (67) is upper bounded by

$$C_{l, p, \bar{p}}(T) \left[\left(\sum_{k=1}^{M-1} \varepsilon_k \int_{I_k} |\bar{c}(v)|^2 dv \right)^{\frac{p}{2}} + \sum_{k=1}^{M-1} \varepsilon_k \int_{I_k} |\bar{c}(v)|^p dv + \left(\sum_{k=1}^{M-1} \varepsilon_k \int_{I_k} |\bar{c}(v)| dv \right)^p \right]. \quad (68)$$

If we assume the **Hypothesis 2.4** (a), then we have $\varepsilon_k = \varepsilon_*/(k+1)^{1-\alpha_0}$, with α_0 given in (7). So the above term is less than

$$C_{l, p, \bar{p}}(T) \left[\left(\int_1^\infty \frac{|\bar{c}(v)|^2}{v^{1-\alpha_0}} dv \right)^{\frac{p}{2}} + \int_1^\infty \frac{|\bar{c}(v)|^p}{v^{1-\alpha_0}} dv + \left(\int_1^\infty \frac{|\bar{c}(v)|}{v^{1-\alpha_0}} dv \right)^p \right],$$

which is upper bounded by a constant $C_{l, p, \bar{p}}(T)$ thanks to (7). On the other hand, if we assume the **Hypothesis 2.4** (b), then $\varepsilon_k = \varepsilon_*/(k+1)$. So (68) is upper bounded by a constant $C_{l, p, \bar{p}}(T)$ thanks to (9). □

5.1.3 Estimations of $\|X_t^{n, M}\|_{L, l, p}$

In this section, our aim is to prove the following lemma.

Lemma 5.5. *Under the **Hypothesis 2.1** with $q^* \geq 2$ and **Hypothesis 2.4** (either 2.4(a) or 2.4(b)), for all $p \geq 2, 0 \leq l \leq q^*$, there exists a constant $C_{l, p}(T)$ depending on l, p, x and T , such that*

$$a) \quad \sup_n \sup_M \mathbb{E} |X_t^{n, M}|_l^p \leq C_{l, p}(T), \quad (69)$$

and for $0 \leq l \leq q^* - 2$,

$$b) \quad \sup_n \sup_M \mathbb{E} |LX_t^{n, M}|_l^p \leq C_{l, p}(T). \quad (70)$$

Proof. In the following proof, $C_{l, p}(T)$ will be a constant which depends on l, p, x and T , and which may change from a line to another. $q^* \geq 2$ is fixed throughout the proof.

a) We prove (69) for $0 \leq l \leq q^*$ by recurrence on l .

Step 1 For $l = 0$, using Burkholder's inequality, **Hypothesis 2.1** and (55),

$$\begin{aligned} \mathbb{E} |X_t^{n, M}|^p & \leq C_{0, p}(T) [x^p + \mathbb{E} \left| \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n, M}) dr \right|^p + \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n, M}) W_\nu(dr, dz) \right|^p \\ & + \mathbb{E} \left| \int_0^t \int_{[1, M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n, M}) N_\nu(dr, dz) \right|^p] \\ & \leq C_{0, p}(T) [1 + \mathbb{E} \int_0^t \left| \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)}^{n, M}) \nu(dz) \right|^p dr \\ & + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n, M})|^2 \nu(dz) \right)^{\frac{p}{2}} dr + \mathbb{E} \int_0^t \left(\int_{[1, M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n, M})|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ & + \mathbb{E} \int_0^t \int_{[1, M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n, M})|^p \nu(dz) dr + \mathbb{E} \int_0^t \left| \int_{[1, M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n, M}) \nu(dz) \right|^p dr] \\ & \leq C_{0, p}(T). \end{aligned}$$

Step 2 Now we assume that (69) holds for $l - 1$, with $l \geq 1$ and for every $p \geq 2$, and we prove that it holds for l and for every $p \geq 2$. We write $\mathbb{E}|X_t^{n,M}|_l^p \leq C_{l,p}(T)(A_1 + A_2 + A_3)$, with

$$\begin{aligned} A_1 &= \mathbb{E} \left| \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr \right|_l^p, \\ A_2 &= \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz) \right|_l^p, \\ A_3 &= \mathbb{E} \left| \int_0^t \int_{[1, M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz) \right|_l^p. \end{aligned}$$

We notice that by **Hypothesis 2.1**, $\|b_M\|_{l,\infty} \leq \bar{c}_1$. Then using **Lemma 5.3 a)** and the recurrence hypothesis, we get

$$\begin{aligned} A_1 &\leq C_{l,p}(T) \mathbb{E} \int_0^t |b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|_l^p dr \\ &\leq C_{l,p}(T) [(\bar{c}_1)^p + \mathbb{E} \int_0^t |\partial_x b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |X_{\tau_n(r)}^{n,M}|_{1,l}^p dr \\ &\quad + \mathbb{E} \int_0^t \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{lp} dr] \\ &\leq C_{l,p}(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr]. \end{aligned} \tag{71}$$

Next, we estimate A_2 . By **Hypothesis 2.1**, for every n , $\|\tilde{c}(\tau_n(r), \gamma_n(z), \cdot)\|_{l,\infty} \leq |\bar{c}(\gamma_n(z))|$. Then using **Lemma 5.2 a)**, **Lemma 5.3 a)**, (53) and the recurrence hypothesis, we get

$$\begin{aligned} A_2 &\leq C_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\leq C_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\quad + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{2l} \nu(dz) \right)^{\frac{p}{2}} dr \\ &\quad + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\leq C_{l,p}(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr]. \end{aligned} \tag{72}$$

Finally we estimate A_3 . We notice that $D_{(r,m)}^Z Z_i^k = \xi_i^k \delta_{rk} \delta_{mi}$, $D_{(s,z)}^W Z_i^k = 0$, and for $l \geq 2$, $D_{(r_1, m_1) \dots (r_l, m_l), (s_1, z_1) \dots (s_l, z_l)}^{Z,W,l} Z_i^k = 0$. So we have $|Z_i^k|_{1,l}^p = |\xi_i^k|^p \leq 1$. By **Lemma 5.3 a)** for $d = 2$, **Hypothesis 2.1**, for any $k, i \in \mathbb{N}$,

$$\begin{aligned} &|\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \leq |\bar{c}(Z_i^k)| \\ &+ (|\partial_z \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|Z_i^k|_{1,l} + |X_{\tau_n(T_i^k)-}^{n,M}|_{1,l}) \\ &+ C_{l,p}(T) \sup_{2 \leq |\beta| \leq l} (|\partial_z^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|Z_i^k|_{1,l-1}^l + |X_{\tau_n(T_i^k)-}^{n,M}|_{1,l-1}^l) \\ &\leq C_{l,p}(T) \bar{c}(Z_i^k) (1 + |X_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l-1}^l). \end{aligned}$$

It follows that

$$\begin{aligned}
A_3 &\leq \mathbb{E} \left(\int_0^t \int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M})|_l N_\nu(dr, dz) \right)^p \\
&\leq C_{l,p}(T) \left(\int_0^t \int_{[1,M)} \tilde{c}(z) (1 + |X_{\tau_n(r)-}^{n,M}|_l + |X_{\tau_n(r)-}^{n,M}|_{l-1}^l) N_\nu(dr, dz) \right)^p \\
&\leq C_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr \right], \tag{73}
\end{aligned}$$

where the last inequality is obtained by using (5) and recurrence hypothesis. Then combining (71),(72) and (73),

$$\mathbb{E} |X_t^{n,M}|_l^p \leq C_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr \right]. \tag{74}$$

So $\mathbb{E} |X_{\tau_n(t)}^{n,M}|_l^p \leq C_{l,p}(T) \left[1 + \int_0^{\tau_n(t)} \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr \right]$. By Gronwall's lemma, we have

$$\mathbb{E} |X_{\tau_n(t)}^{n,M}|_l^p \leq C_{l,p}(T) e^{TC_{l,p}(T)}.$$

Substituting into (74), we conclude that

$$\sup_{n,M} \mathbb{E} |X_t^{n,M}|_l^p \leq C_{l,p}(T). \tag{75}$$

As a summary of the recurrence argument, we remark that the uniform bound in n, M of the operator D for $l = 0$ is due to the **Hypothesis 2.1**, and it propagates to larger l thanks to **Lemma 5.3 a)**.

b) Now we prove (70) for $0 \leq l \leq q^* - 2$, by recurrence on l .

Step 1 One has to check that (70) holds for $l = 0$. The proof is analogous to that in the following **Step 2**, but simpler. It is done by using **Lemma 5.3 c)**, (60), Burkholder's inequality, **Hypothesis 2.1, 2.4**, (53), (69) and Gronwall's lemma. So we skip it.

Step 2 Now we assume that (70) holds for $l - 1$, with $l \geq 1$ and for any $p \geq 2$ and we prove that it holds for l and for any $p \geq 2$. We write $\mathbb{E} |LX_t^{n,M}|_l^p \leq C_{l,p}(T) (B_1 + B_2 + B_3)$, with

$$\begin{aligned}
B_1 &= \mathbb{E} |L \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr|_l^p, \\
B_2 &= \mathbb{E} |L \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz)|_l^p, \\
B_3 &= \mathbb{E} |L \int_0^t \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz)|_l^p.
\end{aligned}$$

Using **Lemma 5.3 b)**, **Hypothesis 2.1**, the recurrence hypothesis and (69), we get

$$\begin{aligned}
B_1 &\leq C_{l,p}(T) \mathbb{E} \int_0^t |L b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|_l^p dr \\
&\leq C_{l,p}(T) [\mathbb{E} \int_0^t |\partial_x b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |LX_{\tau_n(r)}^{n,M}|_l^p dr \\
&\quad + \mathbb{E} \int_0^t \sup_{2 \leq |\beta| \leq l+2} |\partial_x^\beta b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p (1 + |X_{\tau_n(r)}^{n,M}|_{l+1}^{(l+2)p}) (1 + |LX_{\tau_n(r)}^{n,M}|_{l-1}^p) dr] \\
&\leq C_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)}^{n,M}|_l^p dr \right]. \tag{76}
\end{aligned}$$

Then by **Lemma 5.2 b)**, we get

$$\begin{aligned}
B_2 &\leq C_{l,p}(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |L\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^{\frac{p}{2}} dr] \\
&:= C_{l,p}(T) [B_{2,1} + B_{2,2}].
\end{aligned}$$

As a consequence of **Lemma 5.3 b)**, we have

$$\begin{aligned}
B_{2,1} &\leq C_{l,p}(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |LX_{\tau_n(r)}^{n,M}|_l^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l+2} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 (1 + |X_{\tau_n(r)}^{n,M}|_{l+1}^{2(l+2)}) (1 + |LX_{\tau_n(r)}^{n,M}|_{l-1}^2) \nu(dz))^{\frac{p}{2}} dr].
\end{aligned}$$

And using **Lemma 5.3 a)**,

$$\begin{aligned}
B_{2,2} &\leq C_{l,p}(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l}^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{2l} \nu(dz))^{\frac{p}{2}} dr].
\end{aligned}$$

Then by **Hypothesis 2.1**, (53), (69) and the recurrence hypothesis,

$$B_2 \leq C_{l,p}(T) [1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)}^{n,M}|^p dr]. \quad (77)$$

Now we estimate B_3 . By **Lemma 5.3 b)** for $d = 2$, **Hypothesis 2.1**, for any $k, i \in \mathbb{N}$,

$$\begin{aligned}
&|L\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \leq (|\partial_z \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l) \\
&+ C_{l,p}(T) \sup_{2 \leq |\beta| \leq l+2} (|\partial_z^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) \\
&\times (1 + |Z_i^k|_{l+1}^{l+2} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2}) (1 + |LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1}) \\
&\leq C_{l,p}(T) \bar{c}(Z_i^k) (1 + |LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2}) \times (|LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1}).
\end{aligned}$$

Then

$$\begin{aligned}
B_3 &\leq \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} |L\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \right)^p \\
&\leq C_{l,p}(T) \mathbb{E} \left[\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (1 + |LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2}) \right. \\
&\quad \left. + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times (|LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1}) \right]^p \\
&\leq C_{l,p}(T) (B_{3,1} + B_{3,2} + B_{3,3}),
\end{aligned}$$

where

$$B_{3,1} = \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LX_{\tau_n(T_i^k)-}^{n,M}|_l \right)^p,$$

$$\begin{aligned}
B_{3,2} &= \mathbb{E} \left| \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (|LZ_i^k|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LZ_i^k|_{l-1})^p \right|, \\
B_{3,3} &= \mathbb{E} \left| \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (1 + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1})^p \right|.
\end{aligned}$$

By (5),

$$\begin{aligned}
B_{3,1} &= \mathbb{E} \left| \int_0^t \int_{[1,M]} \bar{c}(z) |LX_{\tau_n(r)-}^{n,M}|_l N_\nu(dr, dz) \right|^p \\
&\leq C_{l,p}(T) \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr.
\end{aligned} \tag{78}$$

Using Schwartz's inequality, (5) and (69), we have

$$\begin{aligned}
&\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LZ_i^k|_{l-1} \right)^p \\
&\leq \left[\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{2(l+2)} \right)^p \right]^{\frac{1}{2}} \times \left[\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LZ_i^k|_{l-1}^2 \right)^p \right]^{\frac{1}{2}} \\
&= \left[\mathbb{E} \left| \int_0^t \int_{[1,M]} \bar{c}(z) |X_{\tau_n(r)-}^{n,M}|_{l+1}^{2(l+2)} N_\nu(dr, dz) \right|^p \right]^{\frac{1}{2}} \times \left[\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LZ_i^k|_{l-1}^2 \right)^p \right]^{\frac{1}{2}} \\
&\leq C_{l,p}(T) \left[\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LZ_i^k|_{l-1}^2 \right)^p \right]^{\frac{1}{2}}.
\end{aligned}$$

Then applying **Lemma 5.4**, we get

$$B_{3,2} \leq C_{l,p}(T). \tag{79}$$

By (5), (69) and recurrence hypothesis, we have

$$\begin{aligned}
B_{3,3} &= \mathbb{E} \left| \int_0^t \int_{[1,M]} \bar{c}(z) (1 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} \times |LX_{\tau_n(r)-}^{n,M}|_{l-1}) N_\nu(dr, dz) \right|^p \\
&\leq C_{l,p}(T).
\end{aligned} \tag{80}$$

So by (78),(79) and (80),

$$B_3 \leq C_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr \right]. \tag{81}$$

Then combining (76),(77) and (81),

$$\mathbb{E} |LX_t^{n,M}|_l^p \leq C_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr \right], \tag{82}$$

Using Gronwall's lemma as for (75), we conclude that

$$\sup_{n,M} \mathbb{E} |LX_t^{n,M}|_l^p \leq C_{l,p}(T). \tag{83}$$

As a summary of the recurrence argument, we remark that the uniform bound in n, M of the operator L for $l = 0$ is due to the **Hypothesis 2.1,2.4** and **Lemma 5.3 c)**, and it propagates to larger l thanks to **Lemma 5.3 b)**. \square

Proof of Lemma 4.1.

By **Lemma 5.1** and **Lemma 5.5**, as a consequence of **Lemma 3.2**, we have $X_t^M \in \mathcal{D}_{l,p}$ and $\sup_M \|X_t^M\|_{L,l,p} \leq C_{l,p}(T)$. \square

5.2 Proof of Lemma 4.2

In the following, we turn to the non-degeneracy of X_t^M . We consider the approximate equation (49)

$$X_t^M = x + \int_0^t \int_{[1,M]} \tilde{c}(r, z, X_{r-}^M) N_\nu(dr, dz) + \int_0^t b_M(r, X_r^M) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(r, z, X_r^M) W_\nu(dr, dz).$$

We can calculate the Malliavin derivatives of the Euler scheme and then by passing to the limit, we have

$$\begin{aligned} D_{(k,i)}^Z X_t^M &= 1_{\{k \leq M-1\}} 1_{\{i \leq J_t^k\}} \xi_i^k \partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k-}^M) + \int_{T_i^k}^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) D_{(k,i)}^Z X_{r-}^M N_\nu(dr, dz) \\ &+ \int_{T_i^k}^t \partial_x b_M(r, X_r^M) D_{(k,i)}^Z X_s^M dr + \int_{T_i^k}^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) D_{(k,i)}^Z X_s^M W_\nu(dr, dz). \end{aligned} \quad (84)$$

$$\begin{aligned} D_{(s,z_0)}^W X_t^M &= \int_s^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) D_{(s,z_0)}^W X_{r-}^M N_\nu(dr, dz) + \int_s^t \partial_x b_M(r, X_r^M) D_{(s,z_0)}^W X_r^M dr \\ &+ 1_{\{s \leq t\}} 1_{\{z_0 \geq M\}} \tilde{c}(s, z_0, X_s^M) + \int_s^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) D_{(s,z_0)}^W X_r^M W_\nu(dr, dz). \end{aligned} \quad (85)$$

We obtain now some explicit expressions for the Malliavin derivatives. We consider the tangent flow $(Y_t^M)_{t \in [0, T]}$ which is the solution of the linear equation

$$Y_t^M = 1 + \int_0^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) Y_{r-}^M N_\nu(dr, dz) + \int_0^t \partial_x b_M(r, X_r^M) Y_r^M dr + \int_0^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) Y_r^M W_\nu(dr, dz).$$

And using Itô's formula, $\bar{Y}_t^M = 1/Y_t^M$ verifies the equation

$$\begin{aligned} \bar{Y}_t^M &= 1 - \int_0^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) (1 + \partial_x \tilde{c}(r, z, X_{r-}^M))^{-1} \bar{Y}_{r-}^M N_\nu(dr, dz) - \int_0^t \partial_x b_M(r, X_r^M) \bar{Y}_r^M dr \\ &- \int_0^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) \bar{Y}_r^M W_\nu(dr, dz) + \frac{1}{2} \int_0^t \int_{\{z \geq M\}} |\partial_x \tilde{c}(r, z, X_r^M)|^2 \bar{Y}_r^M \nu(dz) dr. \end{aligned}$$

Applying **Hypothesis 2.1** with $q^* \geq 1$ and **Hypothesis 2.2**, with K_p a constant only depending on p , one also has (the proof is standard)

$$\mathbb{E}(\sup_{s \leq t} (|Y_s^M|^p + |\bar{Y}_s^M|^p)) \leq K_p < \infty. \quad (86)$$

Remark. Due to (4), we have

$$\max \left\{ \int_{[1,M]} |\bar{c}(z)|^p \nu(dz), \int_{[M,\infty)} |\bar{c}(z)|^p \nu(dz) \right\} \leq \int_{[1,\infty)} |\bar{c}(z)|^p \nu(dz) = \bar{c}_p,$$

so the constant in (86) is uniform with respect to M .

Then using the uniqueness of solution to the equation (84) and (85), one obtains

$$\begin{aligned} D_{(k,i)}^Z X_t^M &= 1_{\{k \leq M-1\}} 1_{\{i \leq J_t^k\}} \xi_i^k Y_t^M \bar{Y}_{T_i^k-}^M \partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k-}^M), \\ D_{(s,z_0)}^W X_t^M &= 1_{\{s \leq t\}} 1_{\{z_0 \geq M\}} Y_t^M \bar{Y}_s^M \tilde{c}(s, z_0, X_s^M). \end{aligned} \quad (87)$$

And the Malliavin covariance of X_t^M is

$$\sigma_{X_t^M} = \langle DX_t^M, DX_t^M \rangle_{\mathcal{H}} = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} |D_{(k,i)}^Z X_t^M|^2 + \int_0^T \int_{\{z \geq M\}} |D_{(s,z)}^W X_t^M|^2 \nu(dz) ds. \quad (88)$$

In the following, we denote $\lambda_t^M = \sigma_{X_t^M}$. So the aim is to prove that for every $p \geq 1$,

$$\mathbb{E}(|\lambda_t^M|^{-p}) \leq C_p. \quad (89)$$

We proceed in 5 steps.

Step 1 We notice that by (87) and (88)

$$\lambda_t^M = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k |Y_t^M|^2 |\bar{Y}_{T_i^k}^M|^2 |\partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k}^M)|^2 + |Y_t^M|^2 \int_0^t |\bar{Y}_s^M|^2 \int_{\{z \geq M\}} |\tilde{c}(s, z, X_s^M)|^2 \nu(dz) ds.$$

We recall the ellipticity hypothesis (**Hypothesis 2.3**): There exists a function $\underline{c}(z)$ such that

$$|\partial_z \tilde{c}(s, z, x)|^2 \geq \underline{c}(z) \quad \text{and} \quad |\tilde{c}(s, z, x)|^2 \geq \underline{c}(z).$$

In particular

$$\int_{\{z \geq M\}} |\tilde{c}(s, z, x)|^2 \nu(dz) \geq \int_{\{z \geq M\}} \underline{c}(z) \nu(dz),$$

so that

$$\lambda_t^M \geq Q_t^{-2} \times \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k) + t \int_{\{z \geq M\}} \underline{c}(z) \nu(dz) \right) \quad \text{with} \quad Q_t = \inf_{s \leq t} |Y_s^M \bar{Y}_t^M|.$$

We denote

$$\rho_t^M = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k), \quad \bar{\rho}_t^M = \sum_{k=M}^{\infty} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k), \quad \alpha^M = \int_{\{z \geq M\}} \underline{c}(z) \nu(dz).$$

By (86), $(\mathbb{E} \sup_{s \leq t} |Y_s^M \bar{Y}_t^M|^{4p})^{1/2} \leq C < \infty$, so that

$$\mathbb{E}(|\lambda_t^M|^{-p}) \leq C(\mathbb{E}(|\rho_t^M + t\alpha^M|^{-2p}))^{\frac{1}{2}}. \quad (90)$$

Step 2 Let $\Gamma(p) = \int_0^\infty s^{p-1} e^{-s} ds$. By a change of variables, we have the numerical equality

$$\frac{1}{(\rho_t^M + t\alpha^M)^p} = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} e^{-s(\rho_t^M + t\alpha^M)} ds$$

which, by taking expectation, gives

$$\mathbb{E}\left(\frac{1}{(\rho_t^M + t\alpha^M)^p}\right) = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) ds. \quad (91)$$

Step 3 (splitting). In order to compute $\mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)})$ we have to interpret ρ_t^M in terms of Poisson measures. We recall that we suppose the "splitting hypothesis" (40):

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k dz,$$

with $I_k = [k, k+1)$, $m_k = \nu(I_k)$. We also have the function ψ and $m(\psi) = \int_{\mathbb{R}} \psi(t) dt$. And we use the basic decomposition

$$Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k) U_i^k$$

where V_i^k, U_i^k, ξ_i^k are some independent random variables with laws given in (47).

For every k we consider a Poisson point measure $N_k(ds, d\xi, dv, du)$ with $\xi \in \{0, 1\}, v, u \in [1, \infty), s \in [0, T]$ with compensator

$$\begin{aligned}\widehat{N}_k(ds, d\xi, dv, du) &= \widehat{M}_k(d\xi, dv, du) \times ds \\ \text{with } \widehat{M}_k(d\xi, dv, du) &= b_k(d\xi) \times 1_{I_k}(v) \frac{1}{m(\psi)} \psi(v - (k + \frac{1}{2})) dv \\ &\quad \times \frac{1}{1 - \varepsilon_k m(\psi)} 1_{I_k}(u) (\mathbb{P}(Z_1^k \in du) - \varepsilon_k \psi(u - (k + \frac{1}{2}))) du.\end{aligned}$$

Here $b_k(d\xi)$ is the Bernoulli law of parameter $\varepsilon_k m(\psi)$. The intervals $I_k, k \in \mathbb{N}$ are disjoint so the Poisson point measures $N_k, k = 1, \dots, M-1$ are independent. Then

$$\sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k) = \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(\xi_i^k V_i^k + (1 - \xi_i^k) U_i^k) = \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} \xi \underline{c}(\xi v + (1 - \xi)u) N_k(ds, d\xi, dv, du).$$

In order to get compact notation, we put together all the measures $N_k, k \leq M-1$. Since they are independent we get a new Poisson point measure that we denote by Θ . And we have

$$\rho_t^M = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k) = \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} \xi \underline{c}(\xi v + (1 - \xi)v) \Theta(ds, d\xi, dv, du).$$

Step 4 Using Itô's formula,

$$\begin{aligned}\mathbb{E}(e^{-s\rho_t^M}) &= 1 + \mathbb{E} \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} (e^{-s(\rho_r^M + \xi \underline{c}(\xi v + (1 - \xi)v))} - e^{-s\rho_r^M}) \widehat{\Theta}(dr, d\xi, dv, du) \\ &= 1 - \int_0^t \mathbb{E}(e^{-s\rho_r^M}) dr \int_{\{0,1\}} \int_{[1,\infty)^2} (1 - e^{-s\xi \underline{c}(\xi v + (1 - \xi)v)}) \sum_{k=1}^{M-1} \widehat{M}_k(d\xi, dv, du).\end{aligned}$$

Solving the above equation we obtain

$$\mathbb{E}(e^{-s\rho_t^M}) = \exp(-t \sum_{k=1}^{M-1} \int_{\{0,1\}} \int_{[1,\infty)^2} (1 - e^{-s\xi \underline{c}(\xi v + (1 - \xi)v)}) \widehat{M}_k(d\xi, dv, du)).$$

We compute

$$\int_{\{0,1\} \times [1,\infty)^2} (1 - e^{-s\xi \underline{c}(\xi v + (1 - \xi)v)}) \widehat{M}_k(d\xi, dv, du) = \varepsilon_k m(\psi) \int_k^{k+1} (1 - e^{-s\underline{c}(v)}) \frac{1}{m(\psi)} \psi(v - (k + \frac{1}{2})) dv.$$

Since $\psi \geq 0$ and $\psi(z) = 1$ if $|z| \leq \frac{1}{4}$ it follows that the above term is larger than

$$\varepsilon_k \int_{k+\frac{1}{4}}^{k+\frac{3}{4}} (1 - e^{-s\underline{c}(v)}) dv.$$

Finally this gives

$$\begin{aligned}\mathbb{E}(e^{-s\rho_t^M}) &\leq \exp(-t \sum_{k=1}^{M-1} \varepsilon_k \int_{k+\frac{1}{4}}^{k+\frac{3}{4}} (1 - e^{-s\underline{c}(v)}) dv) \\ &= \exp(-t \int_1^M (1 - e^{-s\underline{c}(v)}) m(dv)),\end{aligned}$$

with

$$m(dv) = \sum_{k=1}^{\infty} \varepsilon_k 1_{(k+\frac{1}{4}, k+\frac{3}{4})}(v) dv. \tag{92}$$

In the same way, we get

$$\mathbb{E}(e^{-s\bar{\rho}_t^M}) \leq \exp(-t \int_M (1 - e^{-s\underline{c}(v)})m(dv)).$$

Notice that $t\alpha^M \geq \mathbb{E}(\bar{\rho}_t^M)$. Then using Jensen's inequality for the convex function $f(x) = e^{-sx}$, $s, x > 0$, we have

$$e^{-st\alpha^M} \leq e^{-s\mathbb{E}\bar{\rho}_t^M} \leq \mathbb{E}(e^{-s\bar{\rho}_t^M}) \leq \exp(-t \int_M (1 - e^{-s\underline{c}(v)})m(dv)).$$

So for every $M \in \mathbb{N}$, we get

$$\begin{aligned} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) &= e^{-st\alpha^M} \times \mathbb{E}(e^{-s\rho_t^M}) \\ &\leq \exp(-t \int_M (1 - e^{-s\underline{c}(v)})m(dv)) \times \exp(-t \int_1^M (1 - e^{-s\underline{c}(v)})m(dv)) \\ &= \exp(-t \int_1^\infty (1 - e^{-s\underline{c}(v)})m(dv)), \end{aligned} \quad (93)$$

and the last term does not depend on M .

Now we will use the Lemma 14 from [7], which states the following.

Lemma 5.6. *We consider an abstract measurable space E , a σ -finite measure η on this space and a non-negative measurable function $f : E \rightarrow \mathbb{R}_+$ such that $\int_E f d\eta < \infty$. For $t > 0$ and $p \geq 1$, we note*

$$\alpha_f(t) = \int_E (1 - e^{-tf(a)})\eta(da) \quad \text{and} \quad I_t^p(f) = \int_0^\infty s^{p-1} e^{-t\alpha_f(s)} ds.$$

We suppose that for some $t > 0$ and $p \geq 1$,

$$\underline{\lim}_{u \rightarrow \infty} \frac{1}{\ln u} \eta(f \geq \frac{1}{u}) > p/t, \quad (94)$$

then $I_t^p(f) < \infty$.

We will use the above lemma for $\eta = m$ and $f = \underline{c}$. So if we have

$$\underline{\lim}_{u \rightarrow \infty} \frac{1}{\ln u} m(\underline{c} \geq \frac{1}{u}) = \infty, \quad (95)$$

then for every $p \geq 1, t > 0, M \geq 1$, (91),(93) and Lemma 5.6 give

$$\begin{aligned} \mathbb{E}(\frac{1}{\rho_t^M + t\alpha^M})^{2p} &= \frac{1}{\Gamma(2p)} \int_0^\infty s^{2p-1} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) ds \\ &\leq \frac{1}{\Gamma(2p)} \int_0^\infty s^{2p-1} \exp(-t \int_1^\infty (1 - e^{-s\underline{c}(v)})m(dv)) ds < \infty. \end{aligned} \quad (96)$$

Finally using (90), we conclude that if (95) holds, then

$$\sup_M \mathbb{E}(\lambda_t^M)^{-p} < \infty. \quad (97)$$

Step 5 Now the only problem left is to compute $m(\underline{c} \geq \frac{1}{u})$. It seems difficult to discuss this in a completely abstract framework. So we suppose **Hypothesis 2.4 (a)**: There exists a constant $\varepsilon_* > 0$ and there are some $\alpha_1 > \alpha_2 > 0$ such that for every $k \in \mathbb{N}$,

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k dz \quad \text{with} \quad \varepsilon_k = \frac{\varepsilon_*}{(k+1)^{1-\alpha}}, \quad \text{for any } \alpha \in (\alpha_2, \alpha_1], \quad \text{and} \quad \underline{c}(z) \geq e^{-z^{\alpha_2}},$$

Then $\{z : \underline{c}(z) \geq \frac{1}{u}\} \supseteq \{z : (\ln u)^{1/\alpha_2} \geq z\}$. In particular, for $k \leq \lfloor (\ln u)^{1/\alpha_2} \rfloor - 1 := k(u)$, one has $I_k \subseteq \{z : \underline{c}(z) \geq \frac{1}{u}\}$. Then for u large enough, we compute

$$\begin{aligned} m(\underline{c} \geq \frac{1}{u}) &\geq \sum_{k=1}^{k(u)} m(I_k) \geq \frac{1}{2} \sum_{k=1}^{k(u)} \varepsilon_k \geq \frac{1}{2} \varepsilon_* \sum_{k=1}^{k(u)} \frac{1}{(k+1)^{1-\alpha}} \geq \frac{1}{2} \varepsilon_* \int_2^{(\ln u)^{1/\alpha_2}} \frac{1}{z^{1-\alpha}} dz \\ &= \frac{\varepsilon_*}{2\alpha} ((\ln u)^{\alpha/\alpha_2} - 2^\alpha). \end{aligned}$$

Since $\alpha > \alpha_2$, (95) is verified and we obtain (97).

Now we consider **Hypothesis 2.4** (b): We suppose that there exists a constant $\varepsilon_* > 0$ and there are some $\alpha > 0$ such that for every $k \in \mathbb{N}$,

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \varepsilon_k dz \quad \text{with} \quad \varepsilon_k = \frac{\varepsilon_*}{k+1}, \quad \text{and} \quad \underline{c}(z) \geq \frac{1}{z^\alpha}.$$

Now $\{z : \underline{c}(z) \geq \frac{1}{u}\} \supseteq \{z : z \leq u^{1/\alpha}\}$. Then for u large enough,

$$m(\underline{c} \geq \frac{1}{u}) \geq \frac{1}{2} \varepsilon_* \sum_{k=1}^{\lfloor u^{1/\alpha} \rfloor - 1} \frac{1}{k+1} \geq \frac{1}{2} \varepsilon_* \int_2^{u^{1/\alpha}} \frac{dz}{z} = \frac{1}{2} \varepsilon_* (\frac{1}{\alpha} \ln u - \ln 2).$$

And consequently

$$\liminf_{u \rightarrow \infty} \frac{1}{\ln u} m(\underline{c} \geq \frac{1}{u}) \geq \frac{\varepsilon_*}{2\alpha}.$$

Using **Lemma 5.6**, this gives: if

$$\frac{2p}{t} < \frac{\varepsilon_*}{2\alpha} \quad \Leftrightarrow \quad t > \frac{4p\alpha}{\varepsilon_*}$$

then

$$\sup_M \mathbb{E} \left(\frac{1}{\rho_t^M + t\alpha^M} \right)^{2p} < \infty,$$

and we have $\sup_M \mathbb{E}(\lambda_t^M)^{-p} < \infty$. □

5.3 Some proofs concerning Section 4.2

We will prove that the triplet (\mathcal{S}, D, L) defined in Section 4.2 is an IbP framework. Here, we only show that D^q is closable and L verifies the duality formula (32). To do so, we introduce the divergence operator δ . We denote the space of simple processes by

$$\mathcal{P} = \{u = ((\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, \sum_{r=1}^n u_r \varphi_r) : \bar{u}_i^k, u_r \in \mathcal{S}, \varphi_r \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times \text{Leb}), m', m, n \in \mathbb{N}\}.$$

For $u = ((\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, \sum_{r=1}^n u_r \varphi_r) \in \mathcal{P}$, we denote $u^Z = (\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}$ and $u^W = \sum_{r=1}^n u_r \varphi_r$, so that $u = (u^Z, u^W)$.

We notice that \mathcal{P} is dense in $L^2(\Omega; \mathcal{H})$, with $\mathcal{H} = l_2 \otimes L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times \text{Leb})$.

Then we define the divergence operator $\delta : \mathcal{P} \rightarrow \mathcal{S}$ by

$$\begin{aligned} \delta(u) &= \delta^Z(u^Z) + \delta^W(u^W) \\ \text{with} \quad \delta^Z(u^Z) &= - \sum_{k=1}^m \sum_{i=1}^{m'} (D_{(k,i)}^Z \bar{u}_i^k + \xi_i^k \bar{u}_i^k \times \theta_k(V_i^k)) \\ \delta^W(u^W) &= \sum_{r=1}^n u_r W_\nu(\varphi_r) - \sum_{r=1}^n \langle D^W u_r, \varphi_r \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times \text{Leb})}. \end{aligned}$$

We will show that δ satisfies the following duality formula: For every $F \in \mathcal{S}$, $u \in \mathcal{P}$,

$$\mathbb{E}\langle DF, u \rangle_{\mathcal{H}} = \mathbb{E}F\delta(u). \quad (98)$$

In fact, if we denote $\hat{V}_i^k(x)$ the sequence $(V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}$ after replacing V_i^k by x , then for any $m', m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}\langle D^Z F, u^Z \rangle_{l_2} &= \mathbb{E} \sum_{k=1}^m \sum_{i=1}^{m'} D_{(k,i)}^Z F \times \bar{u}_i^k \\ &= \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \xi_i^k \partial_{v_i^k} f(\omega, (V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \bar{u}_i^k(\omega, (V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \\ &= \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \int_{\mathbb{R}} \xi_i^k \partial_{v_i^k} f(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \times \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \frac{\psi_k(x)}{m(\psi)} dx \\ &= - \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \int_{\mathbb{R}} \xi_i^k f(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \times [\partial_{v_i^k} \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \\ &\quad + \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \frac{\partial_x \psi_k(x)}{\psi_k(x)}] \frac{\psi_k(x)}{m(\psi)} dx \\ &= - \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} F [D_{(k,i)}^Z \bar{u}_i^k + \xi_i^k \bar{u}_i^k \partial_x (\ln \psi_k(V_i^k))] = \mathbb{E}(F \delta^Z(u^Z)). \end{aligned}$$

On the other hand, since $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$ is a separable Hilbert space, we can assume without loss of generality that, in the definition of simple functionals, $(\varphi_1, \dots, \varphi_m, \dots)$ is the orthogonal basis of the space $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$.

Then with $p_r = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \varphi_r^2(s, z) \nu(dz) ds$, for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}\langle D^W F, u^W \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)} &= \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_+} D_{(s,z)}^W F \times \sum_{r=1}^n u_r \varphi_r(s, z) \nu(dz) ds \\ &= \mathbb{E} \sum_{r=1}^n \partial_{w_r} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) p_r \\ &= \sum_{r=1}^n \mathbb{E} \int_{\mathbb{R}} \partial_{w_r} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\ &\quad \times u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \frac{1}{\sqrt{2\pi p_r}} e^{-\frac{y^2}{2p_r}} dy \times p_r \\ &= - \sum_{r=1}^n \mathbb{E} \int_{\mathbb{R}} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\ &\quad \times [\partial_{w_r} u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\ &\quad - \frac{y}{p_r} u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n))] \frac{1}{\sqrt{2\pi p_r}} e^{-\frac{y^2}{2p_r}} dy \times p_r \\ &= \mathbb{E} F \left(\sum_{r=1}^n u_r W_\nu(\varphi_r) - \sum_{r=1}^n \langle D^W u_r, \varphi_r \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)} \right) = \mathbb{E}(F \delta^W(u^W)). \end{aligned}$$

Then (98) is proved. Using this duality formula recursively, we can show the closability of D^q . If there exists $u \in L^2(\Omega; \mathcal{H}^{\otimes q})$ such that $F_n \rightarrow 0$ in $L^2(\Omega)$ and $D^q F_n \rightarrow u$ in $L^2(\Omega; \mathcal{H}^{\otimes q})$, then for any $h_1, \dots, h_q \in \mathcal{P}$, $\mathbb{E}\langle u, h_1 \otimes \dots \otimes h_q \rangle_{\mathcal{H}^{\otimes q}} = \lim_{n \rightarrow \infty} \mathbb{E}\langle D^q F_n, h_1 \otimes \dots \otimes h_q \rangle_{\mathcal{H}^{\otimes q}} = \lim_{n \rightarrow \infty} \mathbb{E} F_n \delta(h_1 \delta(h_2 \dots \delta(h_q))) = 0$. Since $\mathcal{P}^{\otimes q}$ is dense in $L^2(\Omega; \mathcal{H}^{\otimes q})$, we conclude that $u = 0$. This implies that D^q is closable.

We notice that from the definition of δ and L , we get immediately that $LF = \delta(DF)$, $\forall F \in \mathcal{S}$. And if we replace u by DG in (98) for $G \in \mathcal{S}$, we get the duality formula of L (32).

Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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