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Total variation distance between a jump-equation and its Gaussian approximation

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Abstract

We deal with the stochastic differential equations with jumps. We replace the "small jumps" by a Brownian motion to make an approximation. In this paper, we prove that the approximation process converges to the original process in total variation distance. Moreover, we also give an estimate of the distance between the densities of the two processes. These are done by some integration by parts techniques in Malliavin calculus.

Key words: Stochastic differential equations with jumps, Malliavin calculus, Total variation distance, Gaussian approximation

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1 Introduction

In this paper we consider the stochastic equation with jumps

$$X_t = x + \int_0^t \int_{(0,1]} c(s, z, X_{s-}) N_\mu(ds, dz)$$

where N_μ is a Poisson random measure on $(0, 1]$ with compensator $\mu(dz)ds$ and c is a coefficient which verify strong regularity hypothesis (see **Hypothesis 2.1~2.4** in section 2.1). The typical example that we have in mind is $\mu(dz) = \frac{dz}{z^{1+\alpha}} 1_{\{z \in (0,1]\}}$, so this is a truncated stable process - however we keep in a general framework where μ is a measure which has infinite mass around zero. Our aim is to replace the "small jumps" by a space time Brownian motion:

$$\begin{aligned} X_t^\varepsilon &= x + \int_0^t \int_{\{z > \varepsilon\}} c(s, z, X_{s-}^\varepsilon) N_\mu(ds, dz) \\ &+ \int_0^t b_\varepsilon(s, X_s^\varepsilon) ds + \int_0^t \int_{(0, \varepsilon]} c(s, z, X_s^\varepsilon) W_\mu(ds, dz) \end{aligned} \quad (1)$$

where $W_\mu(ds, dz)$ is a time space Brownian motion (in the sense of Walsh [28]) with covariance $\mu(dz)ds$. And the coefficient b_ε is defined by

$$b_\varepsilon(s, x) = \int_{(0, \varepsilon]} c(s, z, x) \mu(dz).$$

The interest of such approximations appears in various frameworks.

Our main motivation comes from numerical computations. If $\mu(E) < \infty$ then there are a finite number of jumps in any interval of time, so X_t may be represented by means of a compound Poisson process which may be explicitly simulated. But if $\mu(E) = \infty$ this is no more possible (except very particular situations - see Talay Protter [27] for example) and then we have to truncate the "small jumps". This procedure is rather rough and gives a large error. So, in order to improve it, one may replace the "small jumps" from $(0, \varepsilon]$ by a stochastic integral with respect to $W_\mu(ds, dz)$. Note that the Poisson measure dN_μ is not compensated and this is why the drift corresponding to b_ε appears. This idea goes back to Asmussen Rosinski [3]. In the case of SDE 's driven by a Lévy process, Fournier [16] gives a precise estimate of the error and compare the approximation obtained just by truncation of the small jumps with the one obtained by adding a Gaussian noise as in (1). A enlightening discussion on the complexity of the two methods is also given. However, in that paper the strong error is considered, while in our paper we discuss the weak error.

A second motivation comes from modelization problems in chemistry and biology: we are concerned by reactions which are naturally modeled by means of jump processes; but there are two different regimes: one is very rapid but the jumps are small, and another one is much slower and the jumps are larger: see for example [1], [2], [4], [14], [22], [23]. In this case the regime corresponding to the rapid scale may be modeled by a stochastic integral with respect to a Gaussian process and the slow regime by a compound Poisson process. It may also be reasonable to consider an intermediary regime and this would be modeled by a drift term.

A third motivation is given by statistical problems: one aims to decide, in view of some observations, if the small variations come from small jumps or from a gaussian component - see for example [12], [15] and the references there. In this framework it is important to estimate the error in total variation sense. The authors explain that, if the error in total variation between the law of X_t and of X_t^ε goes to zero, then there is no way to construct a test which decide if the noise comes from small jumps or from Brownian motion. So, asymptotically, the two models contain the same information.

The basic idea of the proof is rather simple. If L (respectively L^ε) represents the infinitesimal operator of X (respectively of X^ε) then a development in Taylor series of order two gives

$$\|(L - L^\varepsilon)f\|_\infty \leq C \|f\|_{3, \infty} \int_{(0, \varepsilon]} |\bar{c}(z)|^3 \mu(dz)$$

where $\bar{c}(z) := \sup_{s \leq T} \sup_{x \in \mathbb{R}} |c(s, z, x)|$, and $\|f\|_{3,\infty} := \sum_{|\beta| \leq 3} \|\partial^\beta f\|_\infty$. Then a Trotter Cato type argument yields

$$\sup_{s \leq T} \|(P_t - P_t^\varepsilon)f\|_\infty \leq C \|f\|_{3,\infty} \int_{(0,\varepsilon]} |\bar{c}(z)|^3 \mu(dz)$$

where P_t (respectively P_t^ε) represents the semi group of X (respectively of X^ε).

The drawback of the above estimate is that it involves $\|f\|_{3,\infty}$ so it concerns just smooth test functions. The main contribution of our paper is to replace $\|f\|_{3,\infty}$ by $\|f\|_\infty$ so to prove convergence in total variation distance. This is done under non degeneracy and regularity properties for the coefficient c . Moreover, under these hypotheses, we prove that $\mathbb{P}(X_t(x) \in dy) = p_t(x, y)dy$ and $\mathbb{P}(X_t^\varepsilon(x) \in dy) = p_t^\varepsilon(x, y)dy$ with smooth densities $y \rightarrow p_t(x, y)$ and $y \rightarrow p_t^\varepsilon(x, y)$. And, for every k and every $\delta > 0$, we obtain

$$\|\partial_y^k p_t - \partial_y^k p_t^\varepsilon\|_\infty \leq C_{k,\delta} \left(\int_{(0,\varepsilon]} |\bar{c}(z)|^3 \mu(dz) \right)^{1-\delta}.$$

So we prove convergence in distribution norms. This is done by using a technology based on integration by parts (an abstract Malliavin calculus) developed in [7].

2 Main results

2.1 The basic equation and the hypotheses

We fix $T > 0$. We deal with the uni-dimensional jump equation

$$X_t = x + \int_0^t \int_{(0,1]} c(s, z, X_{s-}) N_\mu(ds, dz), \quad (2)$$

where N_μ is a Poisson point measure with intensity

$$\widehat{N}_\mu(ds, dz) = \mu(dz)ds$$

and μ is a σ -finite measure on $(0, 1]$, $t \in [0, T]$.

For some technical reasons which will be discussed in section 4, we introduce the following change of variables. Let $\theta : (0, 1] \rightarrow [1, \infty)$ be a function such that $\theta(z) = \frac{1}{z}$. We may define a new measure $\nu(dz) := \mu \circ \theta^{-1}(dz)$, then ν is a σ -finite measure on $[1, \infty)$. And we take a Poisson point measure $N_\nu(ds, dz)$ with intensity $\widehat{N}_\nu(ds, dz) = \nu(dz)ds$. Then by a change of variables, X_t has the same law as \widehat{X}_t , with \widehat{X}_t given by

$$\widehat{X}_t = x + \int_0^t \int_{[1,\infty)} \tilde{c}(s, z, \widehat{X}_{s-}) N_\nu(ds, dz), \quad (3)$$

where $\tilde{c}(s, z, x) := c(s, \frac{1}{z}, x)$.

Since all the results in this paper only concern the law of random variables, instead of dealing with equation (2), it is equivalent to consider the equation (3). So to simplify the notations, we let $\widehat{X}_t = X_t$. Since in the following we will work with the equation (3), we give our hypotheses in terms of \tilde{c} and ν (instead of c and μ).

Hypothesis 2.1 We assume that $s \mapsto \tilde{c}(s, z, x)$ is continuous, and there exists a non-negative function $\bar{c} : [1, \infty) \rightarrow \mathbb{R}_+$ and a constant q^* such that for every index β_1, β_2 , with $\beta_1 \leq q^*$ and $\beta_2 \leq q^*$, we have

$$\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} (|\tilde{c}(s, z, x)| + |\partial_z^{\beta_2} \partial_x^{\beta_1} \tilde{c}(s, z, x)|) \leq \bar{c}(z), \quad \forall z \in [1, \infty),$$

with

$$\int_{[1,\infty)} |\bar{c}(z)|^p \nu(dz) := \bar{c}_p < \infty, \quad \forall p \geq 1.$$

Hypothesis 2.2 We assume that there exists a non-negative function $\hat{c} : [1, \infty) \rightarrow \mathbb{R}_+$ such that $\int_{[1, \infty)} |\hat{c}(z)|^p \nu(dz) := \bar{c}_p < \infty, \quad \forall p \geq 1$, and

$$|\partial_x \tilde{c}(s, z, x)(1 + \partial_x \tilde{c}(s, z, x))^{-1}| \leq \hat{c}(z), \quad \forall s \in [0, T], x \in \mathbb{R}, z \in [1, \infty).$$

To avoid the complexity of notations, we shall assume $\hat{c}(z) = \bar{c}(z)$.

Hypothesis 2.3 We give the ellipticity hypothesis. There exists a non-negative function $\underline{c} : [1, \infty) \rightarrow \mathbb{R}_+$ such that for every $s \in [0, T], x \in \mathbb{R}, z \in [1, \infty)$,

$$|\partial_z \tilde{c}(s, z, x)|^2 \geq \underline{c}(z) \quad \text{and} \quad |\tilde{c}(s, z, x)|^2 \geq \underline{c}(z).$$

Hypothesis 2.4

Here we give a supplementary hypothesis concerning the measure ν .

Let $I_k = \{z \in [1, \infty) : k \leq z < k+1\}, k \in \mathbb{N}$ and $m_k = \nu(I_k)$.

a) **(strong sector condition)** We assume that there exists a constant $\varepsilon_* > 0$, such that for some $0 < \alpha_2 < \alpha_1 \leq 1$, we have

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \frac{\varepsilon_*}{z^{1-\alpha_1}} dz \quad \text{and} \quad \underline{c}(z) \geq e^{-z^{\alpha_2}}. \quad (4)$$

We will also work with the following alternative hypothesis:

b) We assume that there exists a constant $\varepsilon_* > 0$, such that for some $\alpha > 0$, we have

$$1_{I_k}(z) \frac{\nu(dz)}{m_k} \geq 1_{I_k}(z) \frac{\varepsilon_*}{z} dz \quad \text{and} \quad \underline{c}(z) \geq \frac{1}{z^\alpha}. \quad (5)$$

Remark In Chapter 4 and Appendix, when we apply some Malliavin techniques, we need hypothesis 2.1 to estimate the Malliavin-Sobolev norms, and hypotheses 2.2, 2.3 and 2.4 are needed in order to prove the non-degeneracy of the Malliavin covariance matrix.

The existence and uniqueness of the solution to the equation (3) are given by Kunita[20][21], so we omit here.

2.2 Approximation

In the equation (2), we may have an infinity of "small" jumps but we are not obliged to consider the compensated measure, because we assume that $\int_{(0,1]} |\bar{c}(\frac{1}{z})| \mu(dz) < \infty$. Our idea is to replace the "small" jumps by a drift + a Brownian motion.

We introduce the approximate equation. For $\varepsilon > 0$,

$$\begin{aligned} X_t^\varepsilon &= x + \int_0^t \int_{\{z > \varepsilon\}} c(s, z, X_{s-}^\varepsilon) N_\mu(ds, dz) \\ &\quad + \int_0^t b_\varepsilon(s, X_s^\varepsilon) ds + \int_0^t \int_{(0, \varepsilon]} c(s, z, X_s^\varepsilon) W_\mu(dz, ds) \end{aligned} \quad (6)$$

where

$$b_\varepsilon(s, x) = \int_{(0, \varepsilon]} c(s, z, x) \mu(dz)$$

and W_μ is a space- time Brownian motion with covariance measure $\mu(dz)ds$, which is independent of N_μ .

Let us discuss a little bit about this equation. We notice that we keep the "big" jumps with $z > \varepsilon$ but we eliminate the "small" jumps with $z \leq \varepsilon$. We replace the "small" jumps by the drift with coefficient b_ε and by the stochastic integral with coefficient c . Here this stochastic integral is driven by the so called space- time Brownian motion W_μ , as introduced by Walsh in [28].

We now precise the filtration that we consider. Let

$$\begin{aligned} \mathcal{F}_t^W &= \sigma(W_s(\varphi) : s \leq t, \varphi \text{ deterministic functions}), \\ \mathcal{F}_t^N &= \sigma(N_s(\varphi) : s \leq t, \varphi \text{ deterministic functions}), \\ \mathcal{F}_t &= \mathcal{F}_t^W \vee \mathcal{F}_t^N, \end{aligned} \quad (7)$$

where

$$W_t(\varphi) = \int_0^t \int_{(0,\varepsilon]} \varphi(s, z) W_\mu(ds, dz), \quad N_t(\varphi) = \int_0^t \int_{\{z>\varepsilon\}} \varphi(s, z) N_\mu(ds, dz).$$

So, X_t^ε is \mathcal{F}_t -measurable and X_t is \mathcal{F}_t^N -measurable.

We denote

$$L^2(W) = \{F \in \mathcal{F}_T^W : \mathbb{E}|F|^2 < \infty\}, \quad L^2(N) = \{G \in \mathcal{F}_T^N : \mathbb{E}|G|^2 < \infty\}. \quad (8)$$

Remark

Let $\Phi(s, z, \omega)$ be a process which is progressively measurable with respect to \mathcal{F}_s , which verifies that $(s, z) \mapsto \Phi(s, z, \omega)$ is continuous in $L^2(\Omega)$, and

$$\mathbb{E} \int_0^t \int_{(0,\varepsilon]} |\Phi(s, z, \omega)|^2 \mu(dz) ds < \infty.$$

Then for every $G \in L^2(N)$, we have

$$\mathbb{E}(G \times \int_0^t \int_{(0,\varepsilon]} \Phi(s, z, \omega) W_\mu(ds, dz)) = 0. \quad (9)$$

Now we write the infinitesimal operator of X_s and X_s^ε , respectively: For $\phi \in C_b^3(\mathbb{R})$ (the space of functions with continuous and bounded derivatives up to order 3),

$$L_s \phi(x) = \int_{(0,1]} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz),$$

$$\begin{aligned} L_s^\varepsilon \phi(x) &= \int_{\{z>\varepsilon\}} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz) \\ &\quad + \phi'(x) b_\varepsilon(s, x) + \frac{1}{2} \phi''(x) a_\varepsilon(s, x), \end{aligned} \quad (10)$$

where

$$a_\varepsilon(s, x) = \int_{(0,\varepsilon]} |c(s, z, x)|^2 \mu(dz).$$

And using Taylor's formula of order 2, we have

$$\begin{aligned} L_s \phi(x) &= \int_{\{z>\varepsilon\}} (\phi(x + c(s, z, x)) - \phi(x)) \mu(dz) \\ &\quad + \phi'(x) b_\varepsilon(s, x) + \frac{1}{2} \phi''(x) a_\varepsilon(s, x) + R_s(x), \end{aligned}$$

where

$$|R_s(x)| \leq \frac{1}{6} \|\phi\|_{3,\infty} \int_{(0,\varepsilon]} |c(s, z, x)|^3 \mu(dz),$$

with $\|\phi\|_{l,\infty} := \sum_{|\beta| \leq l} \|\partial^\beta \phi\|_\infty$, $l \geq 1$.

So we get

$$\|(L_s - L_s^\varepsilon)\phi\|_\infty = \|R_s\|_\infty \leq \frac{1}{6} \|\phi\|_{3,\infty} \eta_{\mathcal{B}}(\varepsilon), \quad (11)$$

with

$$\eta_p(\varepsilon) = \int_{(0,\varepsilon]} \left| \bar{c}\left(\frac{1}{z}\right) \right|^p \mu(dz) = \int_{[\frac{1}{\varepsilon},\infty)} |\bar{c}(z)|^p \nu(dz), \quad p \geq 1. \quad (12)$$

Then, we can give an estimate of the distance between the semigroups. Sometimes we write $X_t(x)$ instead of X_t and $X_t^\varepsilon(x)$ instead of X_t^ε to specify the dependence of x . We denote $P_t\phi(x) := \mathbb{E}\phi(X_t(x))$ and $P_t^\varepsilon\phi(x) := \mathbb{E}\phi(X_t^\varepsilon(x))$.

Lemma 2.1. *There exists a constant C depending on T such that for $\phi \in C_b^3(\mathbb{R})$ and $0 \leq t \leq T$, we have*

$$\|P_t\phi - P_t^\varepsilon\phi\|_\infty \leq C \|\phi\|_{3,\infty} \eta_3(\varepsilon). \quad (13)$$

Proof. Step 1 Trotter Cato method: We notice that

$$\partial_t P_t\phi(x) = L_t P_t\phi(x) \quad \text{and} \quad \partial_t P_t^\varepsilon\phi(x) = L_t^\varepsilon P_t^\varepsilon\phi(x).$$

Then using Leibniz-Newton's formula,

$$P_t\phi(x) - P_t^\varepsilon\phi(x) = \int_0^t \partial_s (P_{t-s}^\varepsilon P_s)\phi(x) ds = \int_0^t (P_{t-s}^\varepsilon (L_s - L_s^\varepsilon) P_s)\phi(x) ds$$

so we have

$$\begin{aligned} \|P_t\phi - P_t^\varepsilon\phi\|_\infty &\leq \int_0^t \|P_{t-s}^\varepsilon (L_s - L_s^\varepsilon) P_s\phi\|_\infty ds \\ &\leq \int_0^t \|(L_s - L_s^\varepsilon) P_s\phi\|_\infty ds \\ &\leq \frac{1}{6} \times \eta_3(\varepsilon) \int_0^t \|P_s\phi\|_{3,\infty} ds. \end{aligned} \quad (14)$$

The first and the second inequalities above are trivial and the third inequality is a consequence of (11).

Step 2 (propagation of regularity) In [21], Kunita has shown in Theorem 3.4.1 and Theorem 3.4.2 the regularity of the flow associated with the jump-diffusion. So in our case, we have

$$\begin{aligned} \|P_t\phi\|_{3,\infty} &\leq \mathbb{E}|\phi(X_t(x))| + \mathbb{E}|\partial_x\phi(X_t(x))| + \mathbb{E}|\partial_x^2\phi(X_t(x))| + \mathbb{E}|\partial_x^3\phi(X_t(x))| \\ &\leq \|\phi\|_{3,\infty} \mathbb{E}[1 + 3|\partial_x X_t(x)| + 3|\partial_x^2 X_t(x)| + |\partial_x^3 X_t(x)|] \leq \tilde{C} \|\phi\|_{3,\infty} \end{aligned} \quad (15)$$

Substituting (15) into (14), we obtain the result (13). \square

Remark

One may also consider an approximation equation just by cutting the small jumps:

$$\widetilde{X}_t^\varepsilon = x + \int_0^t \int_{\{z>\varepsilon\}} c(s, z, \widetilde{X}_{s-}^\varepsilon) N_\mu(ds, dz)$$

Then, if $\widetilde{L}_s^\varepsilon$ is the infinitesimal operator of $\widetilde{X}_s^\varepsilon$, we have

$$\|(L_s - \widetilde{L}_s^\varepsilon)\phi\|_\infty \leq \|\phi\|_{1,\infty} \eta_1(\varepsilon)$$

so that the same reasoning as above gives

$$\|P_t\phi - \widetilde{P}_t^\varepsilon\phi\|_\infty \leq C \|\phi\|_{1,\infty} \times \eta_1(\varepsilon) \rightarrow 0 \quad (16)$$

with $\widetilde{P}_t^\varepsilon\phi(x) := \mathbb{E}(\phi(\widetilde{X}_t^\varepsilon(x)))$.

So the gain in (13) is that we have $\eta_3(\varepsilon)$ instead of $\eta_1(\varepsilon)$ in (16), which means that we have a faster speed of convergence, but we also have $\|\phi\|_{3,\infty}$ instead of $\|\phi\|_{1,\infty}$, which means that the test function needs to be more regular. But if we can replace $\|\phi\|_{3,\infty}$ by $\|\phi\|_\infty$, then (13) is undoubtedly better than (16), and X_t^ε is a better approximation scheme than $\widetilde{X}_t^\varepsilon$.

2.3 The main theorem

Now we give the main results of this paper.

Theorem 2.2. *Assume that **Hypothesis 2.2, 2.3** hold true.*

(a) *We also assume **Hypothesis 2.4** (a). We fix $\delta > 0$ and we assume the **Hypothesis 2.1** with $q^* \geq \lfloor \frac{3}{\delta} \rfloor$. Then for every $t \in [0, T]$, X_t and X_t^ε have smooth densities. And there exists a constant C depending on δ, T such that*

$$\|P_t^\varepsilon \phi - P_t \phi\|_\infty \leq C \|\phi\|_\infty \eta_3(\varepsilon)^{1-\delta}, \quad (17)$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$.

We denote $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$ the density functions of X_t and X_t^ε respectively. We fix an index β and assume moreover that $q^ \geq \lfloor \frac{3+\beta}{\delta} \rfloor$. Then there exists a constant \bar{C} depending on δ, T and β such that*

$$\|\partial^\beta p_{X_t} - \partial^\beta p_{X_t^\varepsilon}\|_\infty \leq \bar{C} \eta_3(\varepsilon)^{1-\delta}. \quad (18)$$

(b) *We assume **Hypothesis 2.4** (b). We fix $\delta > 0$ and we assume the **Hypothesis 2.1** with $q^* \geq \lfloor \frac{3}{\delta} \rfloor$. Then for every $t \in [0, T]$ such that $t > \frac{8\alpha(\lfloor \frac{3}{\delta} \rfloor - 2)}{\varepsilon_*}$, there exists a constant C depending on δ, T such that*

$$\|P_t^\varepsilon \phi - P_t \phi\|_\infty \leq C \|\phi\|_\infty \eta_3(\varepsilon)^{1-\delta}, \quad (19)$$

with ε_*, α given in **Hypothesis 2.4** b).

For any index β , for $t > \frac{8\alpha(3\beta+2)}{\varepsilon_}$, X_t and X_t^M both have β -times differentiable densities $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$. Assume moreover that $q^* \geq \lfloor \frac{3+\beta}{\delta} \rfloor$, then for $t > \max\{\frac{8\alpha}{\varepsilon_*}(\lfloor \frac{3+\beta}{\delta} \rfloor - 2), \frac{8\alpha(3\beta+2)}{\varepsilon_*}\}$, there exists a constant \bar{C} depending on δ, T, β such that*

$$\|\partial^\beta p_{X_t} - \partial^\beta p_{X_t^\varepsilon}\|_\infty \leq \bar{C} \eta_3(\varepsilon)^{1-\delta}. \quad (20)$$

The proof of this theorem is left to section 4.4.

2.4 A typical example

We consider the following equation

$$X_t = x + \int_0^t \sigma(X_{s-}) dZ_s. \quad (21)$$

Here Z_t is a Lévy process of Lévy triplet $(0, 0, \mu)$:

$$Z_t = \sum_{s \leq t} \Delta_s(Z), \quad \Delta_s(Z) = Z_s - Z_{s-}$$

where for some $0 \leq b < 1$,

$$\mu(dz) = 1_{(0,1]}(z) \frac{dz}{z^{1+b}}.$$

We approximate (21) by

$$X_t^\varepsilon = x + \int_0^t \sigma(X_{s-}^\varepsilon) dZ_s^\varepsilon + b(\varepsilon) \int_0^t \sigma(X_s^\varepsilon) ds + c(\varepsilon) \int_0^t \sigma(X_s^\varepsilon) dB_s. \quad (22)$$

Here Z_t^ε is a Lévy process of Lévy triplet $(0, 0, 1_{\{z > \varepsilon\}} \mu(dz))$:

$$Z_t^\varepsilon = \sum_{s \leq t} \Delta_s(Z) 1_{\{\Delta_s(Z) > \varepsilon\}},$$

and B_t is a standard Brownian motion independent of $(Z_t^\varepsilon)_{t>0}$, and

$$b(\varepsilon) = \int_{(0,\varepsilon]} z\mu(dz), \quad c(\varepsilon) = \sqrt{\int_{(0,\varepsilon]} z^2\mu(dz)}.$$

Then we have

Theorem 2.3. *We assume that $\sigma(x) \in C_b^\infty(\mathbb{R})$ and there are universal constants $\bar{\sigma}, \underline{\sigma}$ such that $\underline{\sigma} \leq \sigma(x) \leq \bar{\sigma}$, $\underline{\sigma} \leq \sigma'(x) \leq \bar{\sigma}$, $\forall x \in \mathbb{R}$. Then for any $\delta > 0$, there is a constant $C > 0$ such that*

$$d_{TV}(X_t, X_t^\varepsilon) \leq C\varepsilon^{3-b-\delta}.$$

Moreover, X_t and X_t^ε both have smooth densities, for all $t \in [0, T]$. If we denote $p_{X_t}(x)$ and $p_{X_t^\varepsilon}(x)$ the density functions of X_t and X_t^ε respectively, then for any index β and any $\delta > 0$, there exists a constant $C > 0$ such that

$$\|\partial^\beta p_{X_t} - \partial^\beta p_{X_t^\varepsilon}\|_\infty \leq C\varepsilon^{3-b-\delta}.$$

Proof. We notice that

$$Z_t = \int_0^t \int_{(0,1]} z N_\mu(ds, dz),$$

where N_μ is a Poisson point measure with intensity $\mu(dz)ds$. Then one can check that (21) is the equation (2) with $c(s, z, x) = \sigma(x)z$, and (22) has the same law as

$$X_t^\varepsilon = x + \int_0^t \int_{\{z>\varepsilon\}} \sigma(X_{s-}^\varepsilon) z N_\mu(ds, dz) + b(\varepsilon) \int_0^t \sigma(X_s^\varepsilon) ds + \int_0^t \int_{(0,\varepsilon]} \sigma(X_s^\varepsilon) z W_\mu(ds, dz), \quad (23)$$

where W_μ is a space-time Brownian motion with covariance measure $\mu(dz)ds$, independent of N_μ .

Let $\theta : (0, 1] \rightarrow [1, \infty)$ be a function such that $\theta(z) = \frac{1}{z}$. By a change of variables,

$$\tilde{c}(s, z, x) = c(s, \frac{1}{z}, x) = \sigma(x) \times \frac{1}{z}, \quad \nu(dz) = \mu \circ \theta^{-1}(dz) = 1_{[1,\infty)}(z) \frac{dz}{z^{1-b}}.$$

Let $I_k = \{z \in [1, \infty) : k \leq z < k+1\}$, $k \in \mathbb{N}$ and $m_k = \nu(I_k)$. Then there exists a constant $\varepsilon_* = \frac{1}{2}$ such that for every k ,

$$1_{I_k} \frac{\nu(dz)}{m_k} \geq 1_{I_k} \varepsilon_* dz.$$

And

$$\min\{|\partial_z \tilde{c}(s, z, x)|^2, |\tilde{c}(s, z, x)|^2\} \geq \underline{\sigma}^2 \times \frac{1}{z^4} \geq e^{-z^a}$$

with some $0 < a < b$. So it satisfies **Hypothesis 2.3** and **Hypothesis 2.4** (a). And

$$\left| \frac{\partial_x \tilde{c}(s, z, x)}{1 + \partial_x \tilde{c}(s, z, x)} \right| \leq \frac{\bar{\sigma} \times \frac{1}{z}}{1 + \underline{\sigma} \times \frac{1}{z}} \leq \bar{\sigma} \times \frac{1}{z}.$$

So it satisfies **Hypothesis 2.2**. One can easily check that **Hypothesis 2.1** is also verified with

$$\bar{c}(z) = \bar{\sigma} \times \frac{1}{z},$$

so that we can apply **Theorem 2.2**(a) for the equation (21) and (23). Since

$$\eta_3(\varepsilon) = \int_{(0,\varepsilon]} \bar{\sigma}^3 \times z^3 \mu(dz) = \frac{\bar{\sigma}^3}{3-b} \varepsilon^{3-b},$$

we arrive at **Theorem 2.3**. □

3 Abstract integration by parts framework

In order to obtain the main theorem (**Theorem 2.2**), we will apply some techniques of Malliavin calculus. So firstly, we give the abstract integration by parts framework introduced in [7]. This is a variant of the integration by parts framework given in [11].

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a subset $\mathcal{S} \subset \bigcap_{p=1}^{\infty} L^p(\Omega; \mathbb{R})$ such that for every $\phi \in C_p^\infty(\mathbb{R}^d)$ and every $F \in \mathcal{S}^d$, we have $\phi(F) \in \mathcal{S}$ (with C_p^∞ the space of smooth functions which, together with all the derivatives, have polynomial growth). A typical example of \mathcal{S} is the space of simple functionals, as in the standard Malliavin calculus. Another example is the space of "Malliavin smooth functionals".

Given a separable Hilbert space \mathcal{H} , we assume that we have a derivative operator $D : \mathcal{S} \rightarrow \bigcap_{p=1}^{\infty} L^p(\Omega; \mathcal{H})$ which is a linear application which satisfies

a)

$$D_h F := \langle DF, h \rangle_{\mathcal{H}} \in \mathcal{S}, \text{ for any } h \in \mathcal{H}, \quad (24)$$

b) ChainRule: For every $\phi \in C_p^\infty(\mathbb{R}^d)$ and $F = (F_1, \dots, F_d) \in \mathcal{S}^d$, we have

$$D\phi(F) = \sum_{i=1}^d \partial_i \phi(F) DF_i, \quad (25)$$

Since $D_h F \in \mathcal{S}$, we may define by iteration the derivative operator of higher order $D^k : \mathcal{S} \rightarrow \bigcap_{p=1}^{\infty} L^p(\Omega; \mathcal{H}^{\otimes k})$ which verifies $\langle D^k F, \otimes_{i=1}^k h_i \rangle_{\mathcal{H}^{\otimes k}} = D_{h_k} D_{h_{k-1}} \dots D_{h_1} F$. We also denote $D_{h_1, \dots, h_k}^k F := \langle D^k F, \otimes_{i=1}^k h_i \rangle_{\mathcal{H}^{\otimes k}}$, for any $h_1, \dots, h_k \in \mathcal{H}$. Then, $D_{h_1, \dots, h_k}^k F = D_{h_k} D_{h_1, \dots, h_{k-1}}^{k-1} F$, ($k \geq 2$).

For $F = (F_1, \dots, F_d) \in \mathcal{S}^d$, we associate the Malliavin covariance matrix $\sigma_F = (\sigma_F^{i,j})_{i,j=1, \dots, d}$ with $\sigma_F^{i,j} = \langle DF_i, DF_j \rangle_{\mathcal{H}}$. And we denote

$$\Sigma_p(F) = \mathbb{E}(1/\det \sigma_F)^p. \quad (26)$$

We say that F is non-degenerated if $\Sigma_p(F) < \infty$, $\forall p \geq 1$.

We also assume that we have an Ornstein-Uhlenbeck operator $L : \mathcal{S} \rightarrow \mathcal{S}$ which is a linear operator satisfying the following duality formula:

Duality: For every $F, G \in \mathcal{S}$,

$$\mathbb{E}\langle DF, DG \rangle_{\mathcal{H}} = \mathbb{E}(FLG) = \mathbb{E}(GLF). \quad (27)$$

As an immediate consequence of the duality formula, we know that $L : \mathcal{S} \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is closable.

Definition 3.1. The triplet (\mathcal{S}, D, L) will be called an IbP(integration by parts) framework.

Now, we introduce the Sobolev norms. For any $l \geq 1$, $F \in \mathcal{S}$,

$$|F|_{1,l} = \sum_{q=1}^l |D^q F|_{\mathcal{H}^{\otimes q}}, \quad |F|_l = |F| + |F|_{1,l},$$

We remark that $|F|_0 = |F|$ and $|F|_{1,l} = 0$ for $l = 0$. For $F = (F_1, \dots, F_d) \in \mathcal{S}^d$, we set

$$|F|_{1,l} = \sum_{i=1}^d |F_i|_{1,l}, \quad |F|_l = \sum_{i=1}^d |F_i|_l,$$

Moreover, we associate the following norms. For any $l, p \geq 1$,

$$\begin{aligned}\|F\|_{l,p} &= (\mathbb{E}|F|_l^p)^{1/p}, \quad \|F\|_p = (\mathbb{E}|F|^p)^{1/p}, \\ \|F\|_{L,l,p} &= \|F\|_{l,p} + \|LF\|_{l-2,p}.\end{aligned}\tag{28}$$

We denote by $\mathcal{D}_{k,p}$ the closure of S with respect to the norm $\|\cdot\|_{L,k,p}$:

$$\mathcal{D}_{k,p} = \overline{S}^{\|\cdot\|_{L,k,p}},\tag{29}$$

and

$$\mathcal{D}_\infty = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \mathcal{D}_{k,p}, \quad \mathcal{H}_k = \mathcal{D}_{k,2}.$$

From now on, we will assume moreover that $D^k : S \subset L^2(\Omega) \rightarrow L^2(\Omega; \mathcal{H}^{\otimes k}), \forall k \geq 1$ are closable (if $F_n \rightarrow 0$ in $L^2(\Omega)$ and $D^k F_n \rightarrow G$ in $L^2(\Omega; \mathcal{H}^{\otimes k})$, then $G = 0$). Then we extend the operators from S to \mathcal{D}_∞ . For $F \in \mathcal{D}_\infty, p \geq 2$, there exists a sequence $F_n \in S$ such that $\|F - F_n\|_p \rightarrow 0, \|F_m - F_n\|_{k,p} \rightarrow 0$ and $\|LF_m - LF_n\|_{k-2,p} \rightarrow 0$. Since D^k and L are closable, we can define

$$D^k F = \lim_{n \rightarrow \infty} D^k F_n \quad \text{in } L^p(\Omega; \mathcal{H}^{\otimes k}), \quad LF = \lim_{n \rightarrow \infty} LF_n \quad \text{in } L^p(\Omega).\tag{30}$$

And we still associate the same norms introduced above for $F \in \mathcal{D}_\infty$.

Lemma 3.1. *The triplet $(\mathcal{D}_\infty, D, L)$ is an IbP framework.*

Proof. Here we just show that D verifies (24): For $F \in \mathcal{D}_\infty$ and $h \in \mathcal{H}$, we have $\langle DF, h \rangle_{\mathcal{H}} \in \mathcal{D}_\infty$.

In fact, for any $k \geq 1, p \geq 2$, any $F \in \mathcal{D}_{k+1,p}$, there is a sequence $F_n \in S$ such that $\|F_n - F\|_{k+1,p} \rightarrow 0$. Then for any $u_1, \dots, u_k \in L^p(\Omega; \mathcal{H}), h \in \mathcal{H}$, any $n, m \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\langle D^k(\langle DF_m, h \rangle_{\mathcal{H}} - \langle DF_n, h \rangle_{\mathcal{H}}), u_1 \otimes \dots \otimes u_k \rangle_{\mathcal{H}^{\otimes k}}^{\frac{p}{2}} &= \mathbb{E}|D_{u_k} D_{u_{k-1}} \dots D_{u_1} \langle D(F_m - F_n), h \rangle_{\mathcal{H}}|_{\mathcal{H}^{\otimes k}}^{\frac{p}{2}} \\ &= \mathbb{E}|D_{u_k} D_{u_{k-1}} \dots D_{u_1} D_h(F_m - F_n)|_{\mathcal{H}^{\otimes k}}^{\frac{p}{2}} = \mathbb{E}|D^{k+1}(F_m - F_n), h \otimes u_1 \otimes \dots \otimes u_k \rangle_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} \\ &\leq \mathbb{E}|D^{k+1}(F_m - F_n)|_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} |h \otimes u_1 \otimes \dots \otimes u_k|_{\mathcal{H}^{\otimes(k+1)}}^{\frac{p}{2}} \rightarrow 0,\end{aligned}$$

which yields that $\mathbb{E}|D^k(\langle DF_m, h \rangle_{\mathcal{H}} - \langle DF_n, h \rangle_{\mathcal{H}})|_{\mathcal{H}^{\otimes k}}^p \rightarrow 0$. Therefore, $\langle DF, h \rangle_{\mathcal{H}} \in \mathcal{D}_{k,p}$ and (24) is verified. \square

3.1 Main consequences

We will use the abstract framework in [7] for the IbP framework $(\mathcal{D}_\infty, D, L)$, with D and L defined in (30). Using Malliavin type arguments, one proves in [7] the following results. The first result concerning the density is classical.

Lemma 3.2. *We fix $q \in \mathbb{N}$. Let $F = (F^1, \dots, F^d) \in \mathcal{D}_\infty^d$. We assume that $\Sigma_{6q+4}(F) < \infty$. Then, the law of random variable F has a density $p_F(x)$ which is q -times differentiable.*

Now we present the second result. We define the following distances between random variables $F, G : \Omega \rightarrow \mathbb{R}^d$:

$$d_k(F, G) = \sup\{|E(f(F)) - E(f(G))| : \sum_{|\beta| \leq k} \|\partial^\beta f\|_\infty \leq 1\}$$

For $k = 1$, this is the Fortet Mourier distance (which appears as a variant of the Vasserstein distance), while for $k = 0$, this is the total variation distance and we denote it by d_{TV} .

We are now able to give the second result concerning the total variation distance:

Lemma 3.3. We fix some multi-index $\beta = (\beta_1, \dots, \beta_l)$, some $r \in \mathbb{N}$ and some $\delta > 0$. We define $p_1 = 2(\lfloor r(\frac{1}{\delta} - 1) \rfloor + 1)$, $p_2 = \max\{6l + 4, 2(\lfloor \frac{r+1}{\delta} - r \rfloor + 1)\}$, $k_1 = \lfloor r(\frac{1}{\delta} - 1) \rfloor + 3$, $k_2 = \lfloor \frac{r+1}{\delta} - r \rfloor + 3$. Let $F, G \in \mathcal{D}_\infty^d$. Then one may find $C \in \mathbb{R}_+$, $q \in \mathbb{N}$ (depending on r and δ) such that

$$i) \quad d_{TV}(F, G) \leq C(1 + \Sigma_{p_1}(F) + \Sigma_{p_1}(G) + \|F\|_{L, k_1, q} + \|G\|_{L, k_1, q}) \times d_r(F, G)^{1-\delta}, \quad (31)$$

and

$$ii) \quad \|\partial^\beta p_F - \partial^\beta p_G\|_\infty \leq C(1 + \Sigma_{p_2}(F) + \Sigma_{p_2}(G) + \|F\|_{L, k_2, q} + \|G\|_{L, k_2, q}) \times d_r(F, G)^{1-\delta}, \quad (32)$$

where $p_F(x)$ and $p_G(x)$ denote the density functions of F and G respectively.

Comment The significance of this lemma is the following: we are able to estimate a "smooth" distance d_r between two random vectors F and G . But we would like to control the total variation distance between them. In order to do this, one employs some integration by parts techniques which are developed in [7] and conclude the following: we need that both F and G are "smooth" in the sense that $\|F\|_{L, k, p} + \|G\|_{L, k, p} < \infty$ for sufficiently large k, p . Moreover, we need some non degeneracy condition: both F and G are "non degenerated", that is $\Sigma_p(F) + \Sigma_p(G) < \infty$, with p large enough. Then (31) asserts that one may control d_0 by d_r , and the control is quasi optimal: we loose just a power $\delta > 0$ which we may take as small as we want. And (32) says that we may also control the distance between the derivatives of density functions by d_r .

Then we can get the following corollary.

Corollary 3.3.1. Let $F_n \in \mathcal{D}_\infty^d$, $n \in \mathbb{N}$ such that for sufficiently large l and p

$$Q_{l,p}(F_n) := \|F_n\|_{L, l, p} + \Sigma_p(F_n) \leq Q_{l,p} < \infty, \quad \forall n \in \mathbb{N}.$$

Consider moreover some random variable F such that $d_r(F, F_n) \rightarrow 0$, with some $r \in \mathbb{N}$. Then for any $\delta > 0$, there exists a constant $C > 0$ such that

$$i) \quad d_{TV}(F, F_n) \leq C d_r(F, F_n)^{1-\delta}$$

Moreover, the law of F is absolutely continuous with smooth density p_F and one has

$$ii) \quad \|p_F - p_{F_n}\|_{l, \infty} \leq C d_r(F, F_n)^{1-\delta}.$$

Proof. Using **Lemma 3.3** i) we get, for every $n \leq m$

$$d_{TV}(F_m, F_n) \leq C d_r(F_m, F_n)^{1-\delta}$$

and since $d_r(F, F_m) \rightarrow 0$, it follows that $d_r(F_m, F_n) \rightarrow 0$ as $n, m \rightarrow \infty$. Using i), we conclude that the sequence $F_n, n \in \mathbb{N}$ is also Cauchy in the sense of total variation distance, so one may find G such that $d_{TV}(G, F_n) \rightarrow 0$. In particular this is true with respect to d_r so that $d_r(G, F_m) \rightarrow 0$. It follows that $G = F$. Then,

$$d_{TV}(F, F_n) = d_{TV}(G, F_n) = \lim_{m \rightarrow \infty} d_{TV}(F_m, F_n) \leq C \lim_{m \rightarrow \infty} d_r(F_m, F_n)^{1-\delta} = C d_r(F, F_n)^{1-\delta}$$

and the first assertion is proved. In order to prove the second assertion we see that by **Lemma 3.3** ii) we get

$$\|p_{F_m} - p_{F_n}\|_{l, \infty} \leq C d_r(F_m, F_n)^{1-\delta} \rightarrow 0.$$

So $p_{F_n}, n \in \mathbb{N}$ is Cauchy in $\|\cdot\|_{l, \infty}$ and consequently has a limit p . Since we have proved that F_n converges to F in total variation, for every measurable and bounded function φ we have

$$\mathbb{E}(\varphi(F)) = \lim_{n \rightarrow \infty} \mathbb{E}(\varphi(F_n)) = \lim_{n \rightarrow \infty} \int \varphi(x) p_{F_n}(x) dx = \int \varphi(x) p(x) dx.$$

So p represents the density of the law of F , and we get

$$\|p_F - p_{F_n}\|_{l, \infty} = \lim_{m \rightarrow \infty} \|p_{F_m} - p_{F_n}\|_{l, \infty} \leq C \lim_{m \rightarrow \infty} d_r(F_m, F_n)^{1-\delta} = C d_r(F, F_n)^{1-\delta}.$$

□

When we apply **Lemma 3.3**, we need to control the Sobolev norms $\|F\|_{L,k,q}$ and $\|G\|_{L,k,q}$. To do so, we will use the following auxiliary lemma.

Lemma 3.4. *We fix $p \geq 2, l \geq 2$. Let $F \in L^1(\Omega)$ and let $F_n \in \mathcal{S}$ such that*

$$\begin{aligned} i) \quad & \mathbb{E}|F_n - F| \rightarrow 0, \\ ii) \quad & \sup_n \|F_n\|_{L,l,p} \leq K_{l,p} < \infty, \end{aligned}$$

where $K_{l,p}$ is a constant which depend on l, p .

Then for every $1 \leq q < p$, we have $F \in \mathcal{D}_{l,q}$ and $\|F\|_{L,l,q} \leq K_{l,q}$.

Proof. We recall that the Hilbert space $\mathcal{H}_l = \mathcal{D}_{l,2}$ equipped with the scalar product

$$\begin{aligned} \langle U, V \rangle_l &:= \sum_{k=1}^l \mathbb{E} \langle D^k U, D^k V \rangle_{\mathcal{H}^{\otimes k}} + \mathbb{E}|UV| \\ &+ \sum_{k=1}^{l-2} \mathbb{E} \langle D^k LU, D^k LV \rangle_{\mathcal{H}^{\otimes k}} + \mathbb{E}|LU \times LV| \end{aligned}$$

is the space of the functionals which are l times differentiable in L^2 sense. Notice that $\|F_n\|_{L,l,2} \leq \|F_n\|_{L,l,p} \leq K_{l,p}$. Then, applying Banach Alaoglu's theorem, there exists a functional $G \in \mathcal{H}_l$ and a subsequence (we still denote by n), such that $F_n \rightarrow G$ weakly in the Hilbert space \mathcal{H}_l . This means that for every $Q \in \mathcal{H}_l$, $\langle F_n, Q \rangle_l \rightarrow \langle G, Q \rangle_l$. Therefore, by Mazur's theorem, we can construct some convex combination

$$G_n = \sum_{i=n}^{m_n} \lambda_i^n \times F_i \in \mathcal{S}$$

with $\lambda_i^n \geq 0, i = n, \dots, m_n$ and $\sum_{i=n}^{m_n} \lambda_i^n = 1$, such that

$$\|G_n - G\|_{L,l,2} \rightarrow 0.$$

In particular we have

$$\mathbb{E}|G_n - G| \leq \|G_n - G\|_{L,l,2} \rightarrow 0.$$

Also, we notice that by i),

$$\mathbb{E}|G_n - F| \leq \sum_{i=n}^{m_n} \lambda_i^n \times \mathbb{E}|F_i - F| \rightarrow 0.$$

So we conclude that

$$F = G \in \mathcal{H}_l.$$

Thus, we have

$$\mathbb{E}(|G_n - F|_l^2) + \mathbb{E}(|LG_n - LF|_{l-2}^2) \leq \|G_n - F\|_{L,l,2}^2 \rightarrow 0.$$

By passing to a subsequence, we have $|G_n - F|_l + |LG_n - LF|_{l-2} \rightarrow 0$ almost surely. Now, for every $q \in [1, p)$, we denote $Y_n := |G_n|_l^q + |LG_n|_{l-2}^q$ and $Y := |F|_l^q + |LF|_{l-2}^q$. Then, $Y_n \rightarrow Y$ almost surely, and for any $q' \in [q, p]$,

$$\begin{aligned} \mathbb{E}|G_n|_l^{q'} + \mathbb{E}|LG_n|_{l-2}^{q'} &\leq \|G_n\|_{L,l,q'}^{q'} = \left\| \sum_{i=n}^{m_n} \lambda_i^n \times F_i \right\|_{L,l,q'}^{q'} \leq \left(\sum_{i=n}^{m_n} \lambda_i^n \times \|F_i\|_{L,l,q'} \right)^{q'} \\ &\leq \left(\sup_i \|F_i\|_{L,l,q'} \times \sum_{i=n}^{m_n} \lambda_i^n \right)^{q'} = \sup_i \|F_i\|_{L,l,q'}^{q'} \leq K_{l,q'}^{q'}. \end{aligned}$$

So Y_n is uniformly integrable, and we have

$$\|F\|_{L,l,q}^q = \mathbb{E}(|F|_l^q) + \mathbb{E}(|LF|_{l-2}^q) = \mathbb{E}(Y) = \lim_{n \rightarrow \infty} \mathbb{E}(Y_n) \leq K_{l,q}^q,$$

□

4 Malliavin calculus and stochastic differential equations with jumps

In this section, we present the integration by parts framework which will be used when we deal with the jump equations.

4.1 Splitting's method

We consider a Poisson point measure $N_\nu(ds, dz)$ with compensator $\widehat{N}_\nu(ds, dz) = \nu(dz)ds$ on the state space $[1, \infty)$. We will make use of the noise $z \in [1, \infty)$ in order to apply the results from the previous section. We denote $I_k = [k, k+1)$ and $m_k = \nu(I_k)$. We suppose that for every k , there exists $\varepsilon_k > 0$ such that $1_{I_k}(z)\nu(dz) \geq 1_{I_k}(z)\varepsilon_k \times m_k \times dz$ in the sense that, for every $k \leq a < b < k+1$,

$$\nu(a, b) \geq \varepsilon_k \times m_k \times (b - a). \quad (33)$$

Once this hypothesis is verified we are able to use the "**splitting method**" as follows. To begin we define the functions

$$a(t) = 1 - \frac{1}{1 - (4t - 1)^2} \quad \text{for } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \quad (34)$$

$$\psi(t) = 1_{\{|t| \leq \frac{1}{4}\}} + 1_{\{\frac{1}{4} < |t| \leq \frac{1}{2}\}} e^{a(|t|)}. \quad (35)$$

We notice that $\psi \in C_c^\infty(\mathbb{R})$ and has the support included in $[-\frac{1}{2}, \frac{1}{2}]$. We also notice that for every $k, p \in \mathbb{N}$ the function $t \mapsto |a^{(k)}(t)|^p \psi(t)$ is continuous and has the support included in $[-\frac{1}{2}, \frac{1}{2}]$. Consequently it is bounded: one may find $C_{k,p}$ such that

$$|a^{(k)}(t)|^p \psi(t) \leq C_{k,p} \quad \forall t \in \mathbb{R}. \quad (36)$$

We denote

$$m(\psi) = \int_{-1/2}^{1/2} \psi(t) dt. \quad (37)$$

We go on and we construct our random variables. We consider a sequence of independent random variables Z^k such that

$$Z^k \sim 1_{I_k}(z) \frac{1}{m_k} \nu(dz).$$

This is the sequence of random variables which are involved in the representation of the measure $N_\nu(ds, dz)$ as long as $z \in [1, \infty)$ is concerned. We notice that, according to our hypothesis

$$1_{I_k}(z) \frac{1}{m_k} \nu(dz) \geq 1_{I_k}(z) \varepsilon_k dz.$$

Then we construct some independent random variables V^k, U^k, ξ^k with laws

$$\begin{aligned} \mathbb{P}(V^k \in dt) &= \frac{1}{m(\psi)} \psi\left(t - \left(k + \frac{1}{2}\right)\right) dt \\ \mathbb{P}(U^k \in dt) &= \frac{1}{1 - \varepsilon_k m(\psi)} \left(\mathbb{P}(Z^k \in dt) - \varepsilon_k \psi\left(t - \left(k + \frac{1}{2}\right)\right) dt \right) \\ \mathbb{P}(\xi^k = 1) &= \varepsilon_k m(\psi), \quad \mathbb{P}(\xi^k = 0) = 1 - \varepsilon_k m(\psi). \end{aligned}$$

Some commentaries: we choose $\varepsilon_k < 1/m(\psi)$ such that $1 - \varepsilon_k m(\psi) > 0$. Moreover, by (33), for every measurable function $f \geq 0$

$$\mathbb{E}(f(Z^k)) \geq \varepsilon_k \int_{I_k} f(z) dz \geq \varepsilon_k \int_{I_k} f(z) \psi\left(z - \left(k + \frac{1}{2}\right)\right) dz$$

so that $\mathbb{P}(U^k \in dt)$ is a positive measure (this is the reason of being of the hypothesis (33)). Moreover one may check that this measure has total mass equal to one, so it is a probability measure (this is necessary in order to construct a random variable with this law). And finally one can check the identity of laws:

$$Z^k \sim \xi^k V^k + (1 - \xi^k)U^k \quad (38)$$

In fact, for every measurable function f ,

$$\begin{aligned} \mathbb{E} f(\xi^k V^k + (1 - \xi^k)U^k) &= \varepsilon_k m(\psi) \mathbb{E} f(V^k) + (1 - \varepsilon_k m(\psi)) \mathbb{E} f(U^k) \\ &= \varepsilon_k m(\psi) \times \frac{1}{m(\psi)} \int_{\mathbb{R}} f(t) \psi(t - (k + \frac{1}{2})) dt + (1 - \varepsilon_k m(\psi)) \frac{1}{1 - \varepsilon_k m(\psi)} [\mathbb{E} f(Z^k) - \varepsilon_k \int_{\mathbb{R}} f(t) \psi(t - (k + \frac{1}{2})) dt] \\ &= \mathbb{E} f(Z^k). \end{aligned}$$

In the following, we will work directly with the equality: $Z^k = \xi^k V^k + (1 - \xi^k)U^k$. This is possible because all the results that we discuss here concern the law of the random variables, and the law remains unchanged.

The Poisson point measure N_ν can be written as the following sum:

$$N_\nu(ds, dz) = \sum_{k=1}^{\infty} 1_{I_k}(z) N_{\nu_k}(ds, dz) = \sum_{k=1}^{\infty} N_{\nu_k}(ds, dz),$$

where $\nu_k(dz) = 1_{I_k}(z) \nu(dz)$ and N_{ν_k} is a Poisson point measure with intensity $\nu_k(dz) ds$.

The Poisson point measure N_{ν_k} can be constituted by compound Poisson processes. We take J_t^k a Poisson process of intensity m_k , with $m_k = \nu(I_k)$ as before. We denote by T_i^k the jump times of J_t^k , $k, i \in \mathbb{N}$ and we consider a sequence of independent random variables $Z_i^k \sim 1_{I_k}(z) \frac{\nu(dz)}{m_k}$ (Z_i^k are independent copies of Z^k). Then for any $t > 0$ and $A \in \mathcal{B}([k, k + 1))$,

$$N_{\nu_k}([0, t] \times A) = \sum_{i=1}^{J_t^k} 1_A(Z_i^k).$$

So now for each $k, i \in \mathbb{N}$, we will split Z_i^k as

$$Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k)U_i^k.$$

4.2 Malliavin calculus for Poisson point measures and space-time Brownian motions

In this section we present the IbP framework on a space where we have the Poisson point measure N_ν presented in the previous section and moreover we have a space-time Brownian motion $W_\nu(dt, dz)$ with covariance measure $\nu(dz) ds$, which is independent of N_ν . The random measure W_ν is defined as follows. We consider a Gaussian family $W_\nu(\varphi)$, $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$ which is a family of centered Gaussian random variables with covariance

$$\mathbb{E}(W_\nu(\varphi) W_\nu(\psi)) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \varphi(s, z) \psi(s, z) \nu(dz) ds.$$

We denote by $W_\nu(ds, dz)$ the random measure such that

$$W_\nu(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \varphi(s, z) W_\nu(ds, dz).$$

We recall that the Malliavin calculus with respect to W_ν has already been introduced in [10] and we will follow the same approach as there.

We begin to introduce the space of simple functionals \mathcal{S} . We denote by \mathcal{F}^N respectively by \mathcal{F}^W the σ fields associated to N_ν respectively to W_ν , and $\mathcal{F} = \mathcal{F}^N \vee \mathcal{F}^W$. We recall the random variables V_i^k , $k, i \in \mathbb{N}$ introduced in the previous section "Splitting method": $Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k) U_i^k$. We take $\mathcal{G} = \sigma(U_i^k, \xi_i^k, T_i^k : k, i \in \mathbb{N})$ to be the σ -algebra associated to the noise from U_i^k, ξ_i^k, T_i^k , $k, i \in \mathbb{N}$. We denote by $C_{\mathcal{G}, p}$ the space of the functions $f : \Omega \times \mathbb{R}^{m' \times m} \times \mathbb{R}^n$ such that f is an \mathcal{F} -measurable function, and for each ω , the function $(v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n) \mapsto f(\omega, v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$ belongs to $C_p^\infty(\mathbb{R}^{m' \times m} \times \mathbb{R}^n)$, and for each $(v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$, the function $\omega \mapsto f(\omega, v_1^1, \dots, v_{m'}^m, w_1, \dots, w_n)$ is \mathcal{G} -measurable. And then we define the space of simple functionals by

$$\mathcal{S} = \{F = f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) : f \in C_{\mathcal{G}, p}, \varphi_1, \dots, \varphi_n \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb), m', m, n \in \mathbb{N}\},$$

with Leb the Lebesgue measure. On the space \mathcal{S} we define the derivative operators by

$$D_{(k_0, i_0)}^Z F = 1_{\{k_0 \leq m\}} 1_{\{i_0 \leq m'\}} \xi_{i_0}^{k_0} \frac{\partial f}{\partial v_{i_0}^{k_0}}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n), \quad k_0, i_0 \in \mathbb{N}$$

$$D_{(s, z)}^W F = \sum_{r=1}^n \frac{\partial f}{\partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \varphi_r(s, z), \quad (s, z) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

We look to $D^Z F$ as an element of the Hilbert space l_2 (the space of the sequences $h = (h_i^k)_{k, i \in \mathbb{N}}$ with $|h|_{l_2}^2 = \sum_{k=1}^\infty \sum_{i=1}^\infty |h_i^k|^2 < \infty$) and to $D^W F$ as an element of the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$. Then

$$DF := (D^Z F, D^W F) \in l_2 \otimes L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb).$$

We also denote $D^{Z, W} F = DF$ and $\mathcal{H} = l_2 \otimes L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$. And we have

$$\langle DF, DG \rangle_{\mathcal{H}} = \sum_{k=1}^\infty \sum_{i=1}^\infty D_{(k, i)}^Z F \times D_{(k, i)}^Z G + \int_{\mathbb{R}_+ \times \mathbb{R}_+} D_{(s, z)}^W F \times D_{(s, z)}^W G \nu(dz) ds.$$

Moreover, we can define the derivatives of order $q \in \mathbb{N}$ recursively:

$$D_{(k_1, i_1) \dots (k_q, i_q), (s_1, z_1) \dots (s_q, z_q)}^{Z, W, q} F := D_{(k_q, i_q), (s_q, z_q)}^{Z, W} D_{(k_{q-1}, i_{q-1}), (s_{q-1}, z_{q-1})}^{Z, W} \dots D_{(k_1, i_1), (s_1, z_1)}^{Z, W} F,$$

and we denote $D^q F = D^{Z, W, q} F$.

Let us now consider the "logarithmic derivatives". We recall the function ψ defined in (35) and we denote

$$\psi_k(s) = \psi(s - (k + \frac{1}{2})), \quad \theta_k(s) := \partial_s \ln \psi_k(s). \quad (39)$$

By (36) (which is uniformed with respect to t), we have

$$\sup_k \left| (\ln \psi_k)^{(r)}(s) \right|^p \psi_k(s) \leq C_{r, p} \quad \forall s \in \mathbb{R}. \quad (40)$$

And we define the Ornstein-Uhlenbeck operators L^Z , L^W and $L = L^Z + L^W$ (which verify the duality relation), with

$$L^Z F = - \sum_{k=1}^m \sum_{i=1}^{m'} (D_{(k, i)}^Z D_{(k, i)}^Z F + \xi_i^k D_{(k, i)}^Z F \times \theta_k(V_i^k)),$$

$$L^W F = \sum_{r=1}^n \frac{\partial f}{\partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) W_\nu(\varphi_r)$$

$$- \sum_{l, r=1}^n \frac{\partial^2 f}{\partial w_l \partial w_r}(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \langle \varphi_l, \varphi_r \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)}.$$

One can easily check that the triplet (\mathcal{S}, D, L) is consistent with the IbP framework given in section 3. Here, we only show that D^q is closable and L verifies the duality formula (27). To do so, we introduce the divergence operator δ . We denote the space of simple processes by

$$\mathcal{P} = \{u = ((\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, \sum_{r=1}^n u_r \varphi_r) : \bar{u}_i^k, u_r \in \mathcal{S}, \varphi_r \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb), m', m, n \in \mathbb{N}\}. \quad (41)$$

For $u = ((\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, \sum_{r=1}^n u_r \varphi_r) \in \mathcal{P}$, we denote $u^Z = (\bar{u}_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}$ and $u^W = \sum_{r=1}^n u_r \varphi_r$, so that $u = (u^Z, u^W)$.

We notice that \mathcal{P} is dense in $L^2(\Omega; \mathcal{H})$, with $\mathcal{H} = l_2 \otimes L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$.

Then we define the divergence operator $\delta : \mathcal{P} \rightarrow \mathcal{S}$ by

$$\begin{aligned} \delta(u) &= \delta^Z(u^Z) + \delta^W(u^W) \\ \text{with } \delta^Z(u^Z) &= - \sum_{k=1}^m \sum_{i=1}^{m'} (D_{(k,i)}^Z \bar{u}_i^k + \xi_i^k \bar{u}_i^k \times \theta_k(V_i^k)) \\ \delta^W(u^W) &= \sum_{r=1}^n u_r W_\nu(\varphi_r) - \sum_{r=1}^n \langle D^W u_r, \varphi_r \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)}. \end{aligned}$$

We will show that δ satisfies the following duality formula: For every $F \in \mathcal{S}$, $u \in \mathcal{P}$,

$$\mathbb{E}\langle DF, u \rangle_{\mathcal{H}} = \mathbb{E}F\delta(u). \quad (42)$$

In fact, if we denote $\hat{V}_i^k(x)$ the sequence $(V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}$ after replacing V_i^k by x , then for any $m', m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}\langle D^Z F, u^Z \rangle_{l_2} &= \mathbb{E} \sum_{k=1}^m \sum_{i=1}^{m'} D_{(k,i)}^Z F \times \bar{u}_i^k \\ &= \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \xi_i^k \partial_{v_i^k} f(\omega, (V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \bar{u}_i^k(\omega, (V_{i_0}^{k_0})_{\substack{1 \leq i_0 \leq m' \\ 1 \leq k_0 \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) \\ &= \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \int_{\mathbb{R}} \xi_i^k \partial_{v_i^k} f(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \times \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \frac{\psi_k(x)}{m(\psi)} dx \\ &= - \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} \int_{\mathbb{R}} \xi_i^k f(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \times [\partial_{v_i^k} \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \\ &\quad + \bar{u}_i^k(\omega, \hat{V}_i^k(x), (W_\nu(\varphi_j))_{j=1}^n) \frac{\partial_x \psi_k(x)}{\psi_k(x)}] \frac{\psi_k(x)}{m(\psi)} dx \\ &= - \sum_{k=1}^m \sum_{i=1}^{m'} \mathbb{E} F [D_{(k,i)}^Z \bar{u}_i^k + \xi_i^k \bar{u}_i^k \partial_x (\ln \psi_k(V_i^k))] = \mathbb{E}(F \delta^Z(u^Z)). \end{aligned}$$

On the other hand, since $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$ is a separable Hilbert space, we can assume without loss of generality that, in the definition of simple functionals, $(\varphi_1, \dots, \varphi_m, \dots)$ is the orthogonal base of the space $L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)$.

Then with $p_r = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \varphi_r^2(s, z) \nu(dz) ds$, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E} \langle D^W F, u^W \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)} &= \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_+} D_{(s,z)}^W F \times \sum_{r=1}^n u_r \varphi_r(s, z) \nu(dz) ds \\
&= \mathbb{E} \sum_{r=1}^n \partial_{w_r} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, (W_\nu(\varphi_j))_{j=1}^n) p_r \\
&= \sum_{r=1}^n \mathbb{E} \int_{\mathbb{R}} \partial_{w_r} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\
&\quad \times u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \frac{1}{\sqrt{2\pi p_r}} e^{-\frac{y^2}{2p_r}} dy \times p_r \\
&= - \sum_{r=1}^n \mathbb{E} \int_{\mathbb{R}} f(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\
&\quad \times [\partial_{w_r} u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n)) \\
&\quad - \frac{y}{p_r} u_r(\omega, (V_i^k)_{\substack{1 \leq i \leq m' \\ 1 \leq k \leq m}}, W_\nu(\varphi_1), \dots, W_\nu(\varphi_{r-1}), y, W_\nu(\varphi_{r+1}), \dots, W_\nu(\varphi_n))] \frac{1}{\sqrt{2\pi p_r}} e^{-\frac{y^2}{2p_r}} dy \times p_r \\
&= \mathbb{E} F \left(\sum_{r=1}^n u_r W_\nu(\varphi_r) - \sum_{r=1}^n \langle D^W u_r, \varphi_r \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, \nu \times Leb)} \right) = \mathbb{E} (F \delta^W(u^W)).
\end{aligned}$$

Then (42) is proved. Using this duality formula recursively, we can show the closability of D^q . If there exists $u \in L^2(\Omega; \mathcal{H}^{\otimes q})$ such that $F_n \rightarrow 0$ in $L^2(\Omega)$ and $D^q F_n \rightarrow u$ in $L^2(\Omega; \mathcal{H}^{\otimes q})$, then for any $h_1, \dots, h_q \in \mathcal{P}$, $\mathbb{E} \langle u, h_1 \otimes \dots \otimes h_q \rangle_{\mathcal{H}^{\otimes q}} = \lim_{n \rightarrow \infty} \mathbb{E} \langle D^q F_n, h_1 \otimes \dots \otimes h_q \rangle_{\mathcal{H}^{\otimes q}} = \lim_{n \rightarrow \infty} \mathbb{E} F_n \delta(h_1 \delta(h_2 \dots \delta(h_q))) = 0$. Since $\mathcal{P}^{\otimes q}$ is dense in $L^2(\Omega; \mathcal{H}^{\otimes q})$, we conclude that $u = 0$. This implies that D^q is closable.

We notice that from the definition of δ and L , we get immediately that $LF = \delta(DF)$, $\forall F \in \mathcal{S}$. And if we replace u by DG in (42) for $G \in \mathcal{S}$, we get the duality formula of L (27).

In the following, we will use the IbP framework $(\mathcal{D}_\infty, D, L)$ associated to (\mathcal{S}, D, L) in **Lemma 3.1**.

4.3 Malliavin calculus applied to stochastic differential equations with jumps

Now we apply the IbP framework presented in section 4.2 to the equation (6).

Let $\theta : (0, 1] \rightarrow [1, \infty)$ be a function such that $\theta(z) = \frac{1}{z}$. By a change of variables, instead of dealing with equation (6), it is equivalent to consider the following equation.

$$\begin{aligned}
X_t^M &= x + \int_0^t \int_{[1, M)} \tilde{c}(s, z, X_{s-}^M) N_\nu(ds, dz) \\
&\quad + \int_0^t b_M(s, X_s^M) ds + \int_0^t \int_{\{z \geq M\}} \tilde{c}(s, z, X_s^M) W_\nu(ds, dz),
\end{aligned} \tag{43}$$

where $M = \frac{1}{\varepsilon}$, $\nu(dz) = \mu \circ \theta^{-1}(dz)$, $\tilde{c}(s, z, x) = c(s, \frac{1}{z}, x)$,

$$b_M(s, x) = \int_{\{z \geq M\}} \tilde{c}(s, z, x) \nu(dz), \tag{44}$$

and W_ν is the space-time Brownian motion with covariance measure $\nu(dz) ds$.

It's easy to check that X_t^M has the same law as X_t^ε .

Then we are able to represent the jump's part of X_t^M by compound Poisson processes. We recall that $I_k = [k, k+1)$ and we take J_t^k a Poisson process of intensity m_k , with $m_k = \nu(I_k)$ as in section 4.1. We denote by T_i^k the jump times of J_t^k , $k, i \in \mathbb{N}$ and we consider a sequence of independent random variables

$Z_i^k \sim 1_{I_k}(z) \frac{\nu(dz)}{m_k}$. Then we write

$$\begin{aligned}
X_t^M &= x + \int_0^t \sum_{k=1}^{M-1} \int_{\{z \in I_k\}} \tilde{c}(s, z, X_{s-}^M) N_\nu(ds, dz) \\
&\quad + \int_0^t b_M(s, X_s^M) ds + \int_0^t \int_{\{z \geq M\}} \tilde{c}(s, z, X_s^M) W_\nu(ds, dz) \\
&= x + \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \tilde{c}(T_i^k, Z_i^k, X_{T_i^k-}^M) \\
&\quad + \int_0^t b_M(s, X_s^M) ds + \int_0^t \int_{\{z \geq M\}} \tilde{c}(s, z, X_s^M) W_\nu(ds, dz). \tag{45}
\end{aligned}$$

Here we give two lemmas, concerning the Sobolev norms and the covariance matrices. We recall ε_* , α introduced in **Hypothesis 2.4**, and q^* introduced in **Hypothesis 2.1**.

Lemma 4.1. *Assuming **Hypothesis 2.1**, we have $X_t^M \in \mathcal{D}_\infty$, and for all $p \geq 1, 1 \leq l \leq q^*$, there exists a constant $C_{l,p}(T)$ depending on l, p, T , such that $\sup_M \|X_t^M\|_{L,l,p} \leq C_{l,p}(T)$.*

Lemma 4.2. *Assume that **Hypothesis 2.2, 2.3** hold true.*

(a) *If we also assume **Hypothesis 2.4** (a), then for every $p \geq 1, t \in [0, T]$, we have*

$$\sup_M \mathbb{E} \left| \frac{1}{\det \sigma_{X_t^M}} \right|^p \leq C_p, \tag{46}$$

with C_p a constant only depending on p .

(b) *If we assume **Hypothesis 2.4** (b), then for every $p \geq 1, t \in [0, T]$ such that $t > \frac{4p\alpha}{\varepsilon_*}$, we have*

$$\sup_M \mathbb{E} \left| \frac{1}{\det \sigma_{X_t^M}} \right|^p \leq C_p,$$

with C_p a constant only depending on p .

The proofs of these lemmas are rather technical and are proved in the Appendix (section 5.1 and 5.2).

4.4 Proof of Theorem 2.2

Now we give the proof of **Theorem 2.2**, by using the two lemmas established in section 4.3.

Proof. (a) By **Lemma 4.1** and **Lemma 4.2** a), we know that for any $\delta > 0$, for any $p, q \geq 1, 1 \leq k \leq q^*$, $t > 0$, with $q^* \geq \lfloor \frac{3}{\delta} \rfloor$, there exists a constant $C_{p,q,k}(T)$ such that for any $M \geq 1$, we have

$$\Sigma_p(X_t^M) + \|X_t^M\|_{L,k,q} \leq C_{p,q,k}(T).$$

Since X_t^M has the same law as X_t^ε with $\varepsilon = \frac{1}{M}$, by **Lemma 2.1**, we have

$$\mathbb{E} |\phi(X_t) - \phi(X_t^M)| \leq C \|\phi\|_{3,\infty} \tilde{\eta}_3(M),$$

where $\tilde{\eta}_p(M) := \int_{\{z \geq M\}} |\bar{c}(z)|^p \nu(dz)$, $p \in \mathbb{N}$. We remark that $\tilde{\eta}_p(M) = \eta_p(\varepsilon)$. This means that

$$d_3(X_t, X_t^M) \leq C \eta_3(\varepsilon).$$

Then applying **Corollary 3.3.1** i) for $r = 3$, we have

$$d_{TV}(X_t, X_t^M) \leq C d_3(X_t, X_t^M)^{1-\delta} \leq C_\delta \eta_3(\varepsilon)^{1-\delta}.$$

Because X_t^M has the same law as X_t^ε with $\varepsilon = \frac{1}{M}$, we arrive at (17). And (18) is obtained by applying **Corollary 3.3.1** ii).

(b) The proof is almost the same. If

$$t > \frac{8\alpha(\lfloor \frac{3}{\delta} \rfloor - 2)}{\varepsilon_*},$$

then by **Lemma 4.2** b), $\Sigma_p(X_t^M) < \infty$ for $1 \leq p \leq 2(\lfloor \frac{3}{\delta} \rfloor - 2)$. So **Corollary 3.3.1** i) still holds, and we can obtain (19). For

$$t > \max\left\{\frac{8\alpha}{\varepsilon_*}(\lfloor \frac{3+\beta}{\delta} \rfloor - 2), \frac{8\alpha(3\beta+2)}{\varepsilon_*}\right\},$$

by **Lemma 4.2** b), $\Sigma_p(X_t^M) < \infty$ for $1 \leq p \leq \max\{2(\lfloor \frac{3+\beta}{\delta} \rfloor - 2), 6\beta + 4\}$. So **Corollary 3.3.1** ii) still holds, and we can arrive at (20). \square

5 Appendix

5.1 Proof of Lemma 4.1

5.1.1 Construction of Euler scheme

We need to prove that $X_t^M \in \mathcal{D}_\infty$. Our approach is based on **Lemma 3.4**. Since the solution X_t^M of the equation (45) is not a simple functional, we construct the Euler scheme. We take time partition $\mathcal{P}_t^n = \{r_k = \frac{kt}{n}, k = 0, \dots, n\}$ and space partition $\tilde{\mathcal{P}}_M^n = \{z_k = M + \frac{k}{n}, k = 0, 1, \dots\}$. We denote $\tau_n(r) = r_k$ when $r \in [r_k, r_{k+1})$, and denote $\gamma_n(z) = z_k$ when $z \in [z_k, z_{k+1})$. Let

$$\begin{aligned} X_t^{n,M} &= x + \int_0^t \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz) \\ &\quad + \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz). \end{aligned} \quad (47)$$

Then we can obtain the following lemma immediately.

Lemma 5.1. *Assume that the **Hypothesis 2.1** holds true with $q^* \geq 1$, then for any $p \geq 1, M \geq 1$, we have $\mathbb{E}|X_t^{n,M} - X_t^M|^p \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. For $p \geq 2$, we write $\mathbb{E}|X_t^{n,M} - X_t^M|^p \leq C_p(E_1 + E_2 + E_3)$, where C_p is a constant only depending on p , and

$$E_1 = \mathbb{E} \left| \int_0^t \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_r^M) N_\nu(dr, dz) \right|^p,$$

$$E_2 = \mathbb{E} \left| \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) - b_M(r, X_r^M) dr \right|^p,$$

$$E_3 = \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M) W_\nu(dr, dz) \right|^p.$$

Then, compensating N_ν and using Burkholder's inequality,

$$\begin{aligned}
E_1 &\leq C_p(T) \mathbb{E} \int_0^t \left(\int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\
&+ \mathbb{E} \int_0^t \int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M)|^p \nu(dz) dr \\
&+ \mathbb{E} \int_0^t \left| \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) - \tilde{c}(r, z, X_{r-}^M) \nu(dz) \right|^p dr \\
&\leq C_p(T) [R_n^1 + ((\bar{c}_2)^{\frac{p}{2}} + \bar{c}_p + (\bar{c}_1)^p) \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M} - X_r^M|^p dr],
\end{aligned}$$

with $C_p(T)$ a constant depending on p and T , and

$$\begin{aligned}
R_n^1 &= \mathbb{E} \int_0^t \left(\int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{r-}^M) - \tilde{c}(r, z, X_{r-}^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\
&+ \mathbb{E} \int_0^t \int_{[1,M)} |\tilde{c}(\tau_n(r), z, X_{r-}^M) - \tilde{c}(r, z, X_{r-}^M)|^p \nu(dz) dr \\
&+ \mathbb{E} \int_0^t \left| \int_{[1,M)} \tilde{c}(\tau_n(r), z, X_{r-}^M) - \tilde{c}(r, z, X_{r-}^M) \nu(dz) \right|^p dr \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$.

And

$$\begin{aligned}
E_2 &\leq \bar{C}_p(T) \mathbb{E} \int_0^t \left| \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M) \nu(dz) \right|^p dr \\
&\leq \bar{C}_p(T) [R_n^2 + (\bar{c}_1)^p \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M} - X_r^M|^p dr],
\end{aligned}$$

with $\bar{C}_p(T)$ a constant depending on p, T , and

$$R_n^2 = \mathbb{E} \int_0^t \left| \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), z, X_r^M) - \tilde{c}(r, z, X_r^M) \nu(dz) \right|^p dr \rightarrow 0.$$

Using Burkholder's inequality,

$$\begin{aligned}
E_3 &\leq \hat{C}_p(T) \mathbb{E} \int_0^t \int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) - \tilde{c}(r, z, X_r^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\
&\leq \hat{C}_p(T) [R_n^3 + (\bar{c}_2)^{\frac{p}{2}} \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M} - X_r^M|^p dr],
\end{aligned}$$

with $\hat{C}_p(T)$ a constant depending on p, T , and

$$\begin{aligned}
R_n^3 &= \mathbb{E} \int_0^t \int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), z, X_r^M) - \tilde{c}(r, z, X_r^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \\
&+ \mathbb{E} \int_0^t \int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_r^M) - \tilde{c}(\tau_n(r), z, X_r^M)|^2 \nu(dz) \right)^{\frac{p}{2}} dr \rightarrow 0.
\end{aligned}$$

Therefore, $\mathbb{E} |X_t^{n,M} - X_t^M|^p \leq \widetilde{C}_p(T) [R_n + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M} - X_r^M|^p dr]$, with $\widetilde{C}_p(T)$ a constant depending on p and T , $R_n \rightarrow 0$ as $n \rightarrow \infty$. Since one can easily check that $\mathbb{E} |X_t^{n,M} - X_{\tau_n(t)}^{n,M}|^p \rightarrow 0$, we have $\mathbb{E} |X_t^{n,M} - X_t^M|^p \leq \widetilde{C}_p(T) [\widetilde{R}_n + \int_0^t \mathbb{E} |X_r^{n,M} - X_r^M|^p dr]$, with $\widetilde{R}_n \rightarrow 0$ as $n \rightarrow \infty$. Then we conclude by Gronwall's lemma. \square

5.1.2 Auxiliary lemmas for the operator D^q

Now we establish such a lemma.

Lemma 5.2. *We fix $M \geq 1$. Let $y : \Omega \times [0, T] \times [M, \infty) \rightarrow \mathbb{R}$. We assume that $y_t(z)$ is progressively measurable with respect to \mathcal{F}_t (defined in (7)), $y_t(z) \in \mathcal{S}$, and $\mathbb{E}(\int_0^t \int_{\{z \geq M\}} |y_r(z)|^2 \nu(dz) dr) < \infty$ and $y_t(z)$ is piecewise constant with respect to both t and z . Let $I_t(y) = \int_0^t \int_{\{z \geq M\}} y_r(z) W_\nu(dr, dz)$. Then for any $l \geq 1, p \geq 2$, there exists a constant $C_{l,p}(T)$ such that*

$$\mathbb{E}|I_t(y)|_l^p \leq C_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr.$$

Proof. It's easy to check that $\forall l \geq 1$,

$$D_{(k_1, i_1) \dots (k_l, i_l)}^{Z, l} I_t(y) = \int_0^t \int_{\{z \geq M\}} D_{(k_1, i_1) \dots (k_l, i_l)}^{Z, l} y_r(z) W_\nu(dr, dz).$$

And by recurrence, one can show that for any $\bar{l} \geq 1$,

$$D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} I_t(y) = \int_0^t \int_{\{z \geq M\}} D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} y_r(z) W_\nu(dr, dz) + \sum_{j=1}^{\bar{l}} D_{(s_j, z_j)}^{W, \bar{l}-1} y_{s_j}(z_j) \mathbf{1}_{s_j \leq t},$$

with

$$\widehat{(s_j, z_j)}^{\bar{l}-1} := (s_1, z_1) \cdots (s_{j-1}, z_{j-1}) (s_{j+1}, z_{j+1}) \cdots (s_{\bar{l}}, z_{\bar{l}}).$$

We denote

$$\bar{y}_r(z)(k_1, i_1, \dots, k_l, i_l) := D_{(k_1, i_1) \dots (k_l, i_l)}^{Z, l} y_r(z), \quad \bar{y}_r^l(z) := D^{Z, l} y_r(z).$$

Then $D^{Z, l} I_t(y) = I_t(\bar{y}^l)$, and

$$\begin{aligned} & D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} D_{(k_1, i_1) \dots (k_l, i_l)}^{Z, l} I_t(y) \\ &= \int_0^t \int_{\{z \geq M\}} D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} \bar{y}_r(z)(k_1, i_1, \dots, k_l, i_l) W_\nu(dr, dz) + \sum_{j=1}^{\bar{l}} D_{(s_j, z_j)}^{W, \bar{l}-1} \bar{y}_{s_j}(z_j)(k_1, i_1, \dots, k_l, i_l) \mathbf{1}_{s_j \leq t}. \end{aligned}$$

Let

$$H_{l, \bar{l}, T} = l^{\otimes l} \otimes L^2([0, T] \times [M, \infty), \text{Leb} \times \nu)^{\otimes \bar{l}}.$$

So for any $\bar{l}, l \in \mathbb{N}$, we have

$$\begin{aligned} & |D^{W, \bar{l}} D^{Z, l} I_t(y)|_{H_{l, \bar{l}, T}}^2 = \int_{[0, T]^{\bar{l}}} \int_{[M, \infty)^{\bar{l}}} |D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} I_t(\bar{y}^l)|_{l^{\otimes l}}^2 \nu(dz_1) \cdots \nu(dz_{\bar{l}}) ds_{\bar{l}} \\ & \leq 2 \left| \int_0^t \int_{\{z \geq M\}} D_{(s_1, z_1) \dots (s_{\bar{l}}, z_{\bar{l}})}^{W, \bar{l}} \bar{y}_r^l(z) W_\nu(dr, dz) \right|_{H_{l, \bar{l}, T}}^2 + \bar{l}^{\bar{l}} \int_0^t \int_{\{z \geq M\}} |D^{W, \bar{l}-1} \bar{y}_r^l(z)|_{H_{l, \bar{l}-1, T}}^2 \nu(dz) dr. \end{aligned}$$

Using Burkholder's inequality for Hilbert-valued martingales (see [24] for example), we have

$$\begin{aligned} \mathbb{E} |D^{W, \bar{l}} D^{Z, l} I_t(y)|_{H_{l, \bar{l}, T}}^p & \leq \bar{C}_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{W, \bar{l}} D^{Z, l} y_r(z)|_{H_{l, \bar{l}, T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ & \quad + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{W, \bar{l}-1} D^{Z, l} y_r(z)|_{H_{l, \bar{l}-1, T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr, \end{aligned} \quad (48)$$

where $\bar{C}_{l,p}(T)$ is a constant depending on l, p, T .

We notice that for some random variable F ,

$$D^{Z,W,l}F = \sum_{j=0}^l \binom{l}{j} D^{W,l-j} D^{Z,j}F,$$

with $\binom{l}{j}$ the binomial coefficients. And we denote

$$|D^{Z,W,l}F|_{H_{l,T}}^2 = \sum_{j=0}^l \binom{l}{j} |D^{W,l-j} D^{Z,j}F|_{H_{j,l-j,T}}^2,$$

so that

$$|F|_l^2 \leq C_l \sum_{i=1}^l |D^{Z,W,i}F|_{H_{i,T}}^2.$$

Then by (48), for any $l \geq 1$,

$$\begin{aligned} \mathbb{E}|D^{Z,W,l}I_t(y)|_{H_{l,T}}^p &\leq C_{l,p} \sum_{j=0}^l \mathbb{E}|D^{W,l-j} D^{Z,j}I_t(y)|_{H_{j,l-j,T}}^p \\ &\leq \hat{C}_{l,p}(T) \mathbb{E} \int_0^t \sum_{j=0}^l \left(\int_{\{z \geq M\}} |D^{W,l-j} D^{Z,j}y_r(z)|_{H_{j,l-j,T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\quad + \mathbb{E} \int_0^t \sum_{j=0}^l \left(\int_{\{z \geq M\}} |D^{W,l-j-1} D^{Z,j}y_r(z)|_{H_{j,l-j-1,T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\leq \tilde{C}_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{Z,W,l}y_r(z)|_{H_{l,T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |D^{Z,W,l-1}y_r(z)|_{H_{l-1,T}}^2 \nu(dz) \right)^{\frac{p}{2}} dr, \end{aligned}$$

where $C_{l,p}$ is a constant depending on l, p and $\hat{C}_{l,p}(T)$ and $\tilde{C}_{l,p}(T)$ are constants depending on l, p, T . Summing all the derivatives up to order l , we get

$$\begin{aligned} \mathbb{E}|I_t(y)|_{1,l}^p &\leq \bar{C}_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_{1,l}^2 \nu(dz) \right)^{\frac{p}{2}} dr + \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_{l-1}^2 \nu(dz) \right)^{\frac{p}{2}} dr \\ &\leq 2\bar{C}_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr. \end{aligned} \quad (49)$$

Finally, we notice that by applying Burkholder's inequality, we have

$$\mathbb{E}|I_t(y)|^p \leq \bar{C}_{l,p}(T) \mathbb{E} \int_0^t \left(\int_{\{z \geq M\}} |y_r(z)|^2 \nu(dz) \right)^{\frac{p}{2}} dr, \quad (50)$$

with $\bar{C}_{l,p}(T), \bar{C}_{l,p}(T)$ constants depending on l, p, T .

So summing up (49) and (50), we get the final result. \square

We will also need the following lemma from [8], which is a consequence of the chain rule of D^q .

Lemma 5.3. *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ a C^∞ function and $F \in \mathcal{S}^d$, then $\forall l \in \mathbb{N}$,*

$$|\phi(F)|_{1,l} \leq |\nabla \phi(F)| |F|_{1,l} + C_l \sup_{2 \leq |\beta| \leq l} |\partial^\beta \phi(F)| |F|_{1,l-1}.$$

5.1.3 Estimates of $\mathbb{E}|X_t^{n,M}|_l^p$

Now we prove that $\mathbb{E}|X_t^{n,M}|_l^p$ is bounded, uniformly with respect to n and M (see (54)).

Proof. Without loss of generality, we assume $p \geq 2$. We will prove that for $l \leq q^*$, $\sup_{n,M} \mathbb{E}|X_t^{n,M}|_l^p \leq C_{l,p}(T)$ by recurrence on l . Assume that for $l-1$, there exists a constant $C_{l-1,p}(T)$ such that $\sup_{n,M} \mathbb{E}|X_t^{n,M}|_{l-1}^p \leq C_{l-1,p}(T)$. Then for l , we write $\mathbb{E}|X_t^{n,M}|_l^p \leq C_p(A_1 + A_2 + A_3)$, with C_p a constant depending on p and

$$\begin{aligned} A_1 &= \mathbb{E} \left| \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr \right|_l^p, \\ A_2 &= \mathbb{E} \left| \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz) \right|_l^p, \\ A_3 &= \mathbb{E} \left| \int_0^t \int_{[1,M]} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz) \right|_l^p. \end{aligned}$$

We notice that by **Hypothesis 2.1**,

$$\|b_M\|_{l,\infty} \leq \bar{c}_1.$$

Then by **Lemma 5.3**, we get

$$\begin{aligned} A_1 &\leq T^{p-1} \mathbb{E} \int_0^t |b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|_l^p dr \\ &\leq C_{l,p}^1(T) [(\bar{c}_1)^p + \mathbb{E} \int_0^t |\partial_x b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |X_{\tau_n(r)}^{n,M}|_{1,l}^p dr \\ &\quad + \mathbb{E} \int_0^t \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{lp} dr] \\ &:= C_{l,p}^1(T) [A_{1,1} + A_{1,2}], \quad \text{respectively.} \end{aligned}$$

The recurrence hypothesis gives,

$$A_{1,2} \leq (\bar{c}_1)^p \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{lp} dr \leq T(\bar{c}_1)^p C_{l-1,lp}(T).$$

So we have

$$A_1 \leq \bar{C}_{l,p}^1(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr], \quad (51)$$

where $C_{l,p}^1(T), \bar{C}_{l,p}^1(T)$ are constants depending on l, p, T .

Then by **Lemma 5.2, Lemma 5.3**, we get

$$\begin{aligned} A_2 &\leq C_{l,p}^2(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^\frac{p}{2} dr \\ &\leq \hat{C}_{l,p}^2(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l}^2 \nu(dz))^\frac{p}{2} dr \\ &\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{2l} \nu(dz))^\frac{p}{2} dr \\ &\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^\frac{p}{2} dr \\ &:= \hat{C}_{l,p}^2(T) [A_{2,1} + A_{2,2} + A_{2,3}], \quad \text{respectively.} \end{aligned}$$

We notice that by **Hypothesis 2.1** and the recurrence hypothesis,

$$\begin{aligned} A_{2,2} + A_{2,3} &\leq (\bar{c}_2)^{\frac{p}{2}} \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{lp} dr + T(\bar{c}_2)^{\frac{p}{2}} \\ &\leq T(\bar{c}_2)^{\frac{p}{2}} (C_{l-1,lp}(T) + 1). \end{aligned}$$

So, we have

$$A_2 \leq \bar{C}_{l,p}^2(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr], \quad (52)$$

where $C_{l,p}^2(T), \hat{C}_{l,p}^2(T), \bar{C}_{l,p}^2(T)$ are constants depending on l, p, T .

We notice that

$$D_{(r,m),(s,z)}^{Z,W} Z_i^k = \xi_i^k \delta_{rk} \delta_{mi}$$

and for $l \geq 2$,

$$D_{(r_1,m_1)\dots(r_l,m_l),(s_1,z_1)\dots(s_l,z_l)}^{Z,W,l} Z_i^k = 0,$$

where here δ_{ij} is the Kronecker delta. So we have $|Z_i^k|_{1,l}^p \leq |\xi_i^k|^p \leq 1$. By **Lemma 5.3, Hypothesis 2.1**, for any $k, i \in \mathbb{N}$,

$$\begin{aligned} &|\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \leq |\bar{c}(Z_i^k)| \\ &+ (|\partial_z \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|Z_i^k|_{1,l} + |X_{\tau_n(T_i^k)-}^{n,M}|_{1,l}) \\ &+ C_l^3 \sup_{2 \leq |\beta| \leq l} (|\partial_z^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|Z_i^k|_{1,l-1}^l + |X_{\tau_n(T_i^k)-}^{n,M}|_{1,l-1}^l) \\ &\leq \bar{C}_l^3 \bar{c}(Z_i^k) (1 + |X_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l-1}^l), \end{aligned}$$

where C_l^3, \bar{C}_l^3 are constants depending on l . Then by compensating N_ν , Burkholder's inequality, **Hypothesis 2.1**, and recurrence hypothesis,

$$\begin{aligned} A_3 &\leq \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} |\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \right)^p \leq C_{l,p}^3 \mathbb{E} \left[\sum_{k=1}^{M-1} \sum_{i=1}^{J_i^k} \bar{c}(Z_i^k) (1 + |X_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l-1}^l) \right]^p \\ &= C_{l,p}^3 \mathbb{E} \left| \int_0^t \int_{[1,M]} \bar{c}(z) (1 + |X_{\tau_n(r)-}^{n,M}|_l + |X_{\tau_n(r)-}^{n,M}|_{l-1}^l) N_\nu(dr, dz) \right|^p \\ &\leq \hat{C}_{l,p}^3(T) \mathbb{E} \int_0^t \left(\int_{[1,M]} |\bar{c}(z)|^2 (1 + |X_{\tau_n(r)-}^{n,M}|_l^2 + |X_{\tau_n(r)-}^{n,M}|_{l-1}^{2l}) \nu(dz) \right)^{\frac{p}{2}} dr \\ &+ \mathbb{E} \int_0^t \int_{[1,M]} |\bar{c}(z)|^p (1 + |X_{\tau_n(r)-}^{n,M}|_l^p + |X_{\tau_n(r)-}^{n,M}|_{l-1}^{lp}) \nu(dz) dr \\ &+ \mathbb{E} \int_0^t \left(\int_{[1,M]} |\bar{c}(z)| (1 + |X_{\tau_n(r)-}^{n,M}|_l + |X_{\tau_n(r)-}^{n,M}|_{l-1}^l) \nu(dz) \right)^p dr \\ &\leq \bar{C}_{l,p}^3(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr], \quad (53) \end{aligned}$$

where $C_{l,p}^3$ is a constant depending on l, p and $C_{l,p}^3(T), \hat{C}_{l,p}^3(T), \bar{C}_{l,p}^3(T)$ are constants depending on l, p, T . Then combining (51)(52)(53),

$$\mathbb{E} |X_t^{n,M}|_l^p \leq \tilde{C}_{l,p}(T) [1 + \int_0^t \mathbb{E} |X_{\tau_n(r)}^{n,M}|_l^p dr],$$

with $\tilde{C}_{l,p}(T)$ a constant depending on l, p, T . By Gronwall's lemma, we have

$$\mathbb{E} |X_{\tau_n(t)}^{n,M}|_l^p \leq \tilde{C}_{l,p}(T) e^{T \tilde{C}_{l,p}(T)}.$$

So we conclude that

$$\mathbb{E}|X_t^{n,M}|_l^p \leq C_{l,p}(T). \quad (54)$$

□

5.1.4 Auxiliary lemmas for the operator L

Now we turn to the estimate of L . We first give the following lemma concerning the operator $L = L^Z + L^W$.

Lemma 5.4. *We fix $M \geq 1$. Let H be a Hilbert space and $y : \Omega \times [0, T] \times [M, \infty) \rightarrow H$. We assume that $y_t(z)$ is progressively measurable with respect to \mathcal{F}_t , $y_t(z) \in \mathcal{S}$, and $\mathbb{E}(\int_0^t \int_{\{z \geq M\}} |y_r(z)|^2 \nu(dz) dr) < \infty$ and $y_t(z)$ is piecewise constant with respect to both t and z . Let $I_t(y) = \int_0^t \int_{\{z \geq M\}} y_r(z) W_\nu(dr, dz)$. Then $\forall l \geq 1, p \geq 2, \exists C_{l,p}(T)$ such that*

$$\mathbb{E}|LI_t(y)|_l^p \leq C_{l,p}(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |Ly_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr].$$

Proof. We first show that

$$LI_t(y) = I_t(Ly) + I_t(y). \quad (55)$$

In fact, we denote

$$I_k^t(f_k) = k! \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \int_{[M, +\infty)^k} f_k(s_1, \dots, s_k, z_1, \dots, z_k) W_\nu(ds_k, dz_k) \cdots W_\nu(ds_1, dz_1)$$

the multiple stochastic integral for a deterministic function f_k , which is square integrable with respect to $(\nu(dz)ds)^{\otimes k}$ and which satisfies:

$$\begin{aligned} & \text{For any permutation } \pi : \{1, \dots, k\} \rightarrow \{1, \dots, k\}, \\ & f_k(s_1, \dots, s_k, z_1, \dots, z_k) = f_k(s_{\pi(1)}, \dots, s_{\pi(k)}, z_1, \dots, z_k). \end{aligned}$$

Then by duality,

$$\mathbb{E}(I_k^t(f_k)L(I_t(y))) = \mathbb{E}(I_t(y) \times LI_k^t(f_k)).$$

Notice that $L^Z I_k^t(f_k) = 0$ and $L^W I_k^t(f_k) = kI_k^t(f_k)$. So, $LI_k^t(f_k) = kI_k^t(f_k)$ and we obtain

$$\mathbb{E}(I_k^t(f_k)L(I_t(y))) = k\mathbb{E}(I_t(y) \times I_k^t(f_k)). \quad (56)$$

On the other hand, by isometric property and duality,

$$\begin{aligned} \mathbb{E}(I_k^t(f_k) \times I_t(Ly)) &= k\mathbb{E} \int_0^t \int_{\{z \geq M\}} I_{k-1}^r(f_k(r, z, \cdot)) Ly_r(z) \nu(dz) dr \\ &= k \int_0^t \int_{\{z \geq M\}} \mathbb{E}[y_r(z) \times LI_{k-1}^r(f_k(r, z, \cdot))] \nu(dz) dr = k(k-1) \mathbb{E} \int_0^t \int_{\{z \geq M\}} y_r(z) I_{k-1}^r(f_k(r, z, \cdot)) \nu(dz) dr \\ &= k(k-1) \mathbb{E}(I_t(y) \times \int_0^t \int_{\{z \geq M\}} I_{k-1}^r(f_k(r, z, \cdot)) W_\nu(dr, dz)) \\ &= (k-1) \mathbb{E}(I_t(y) \times I_k^t(f_k)). \end{aligned}$$

Then,

$$\mathbb{E}(I_k^t(f_k)(I_t(y) + I_t(Ly))) = k\mathbb{E}(I_k^t(f_k)I_t(y)). \quad (57)$$

Combining (56) and (57), we get

$$\mathbb{E}[I_k^t(f_k) \times LI_t(y)] = \mathbb{E}[I_k^t(f_k)(I_t(Ly) + I_t(y))].$$

Since every element in $L^2(W)$ (defined by (8)) can be represented as the direct sum of multiple stochastic integrals, we have for any $F \in L^2(W)$,

$$\mathbb{E}[FLI_t(y)] = \mathbb{E}[F(I_t(Ly) + I_t(y))]. \quad (58)$$

For $G \in L^2(N)$, by duality and (9),

$$\mathbb{E}[GLI_t(y)] = \mathbb{E}[I_t(y)LG] = \mathbb{E}[I_t(y)L^Z G] = 0,$$

and

$$\mathbb{E}[G(I_t(Ly) + I_t(y))] = 0.$$

So,

$$\mathbb{E}[GLI_t(y)] = \mathbb{E}[G(I_t(Ly) + I_t(y))]. \quad (59)$$

Combining (58)(59), for any $\tilde{G} \in L^2(W) \otimes L^2(N)$, we have

$$\mathbb{E}[\tilde{G}LI_t(y)] = \mathbb{E}[\tilde{G}(I_t(Ly) + I_t(y))].$$

Therefore, we obtain (55).

Then, by **Lemma 5.2**,

$$\begin{aligned} \mathbb{E}|LI_t(y)|_l^p &\leq 2^{p-1}(\mathbb{E}|\int_0^t \int_{\{z \geq M\}} Ly_r(z)W_\nu(dr, dz)|_l^p + \mathbb{E}|\int_0^t \int_{\{z \geq M\}} y_r(z)W_\nu(dr, dz)|_l^p) \\ &\leq C_{l,p}(T)[\mathbb{E}\int_0^t (\int_{\{z \geq M\}} |Ly_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr + \mathbb{E}\int_0^t (\int_{\{z \geq M\}} |y_r(z)|_l^2 \nu(dz))^{\frac{p}{2}} dr]. \end{aligned}$$

□

We will also need the following lemma from [8], which is a consequence of the ‘‘chain rule’’ of L .

Lemma 5.5. *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ a C^∞ function and $F \in \mathcal{S}^d$, then $\forall l \in \mathbb{N}$,*

$$|L\phi(F)|_l \leq |\nabla\phi(F)||LF|_l + C_l \sup_{2 \leq |\beta| \leq l+2} |\partial^\beta \phi(F)|(1 + |F|_{l+1}^{l+2})(1 + |LF|_{l-1}).$$

For $l = 0$, we have

$$|L\phi(F)| \leq |\nabla\phi(F)||LF| + \sup_{|\beta|=2} |\partial^\beta \phi(F)||F|_{1,l}^2.$$

5.1.5 Estimates of $\mathbb{E}|LX_t^{n,M}|_l^p$

Now we prove that $\mathbb{E}|LX_t^{n,M}|_l^p$ is bounded, uniformly with respect to n and M (see (69)).

Proof. Without loss of generality, we assume $p \geq 2$. We will prove that for $l \leq q^* - 2$, $\sup_{n,M} \mathbb{E}|LX_t^{n,M}|_l^p \leq C_{l,p}(T)$ by recurrence on l . Assume that for $l-1$, there exists a constant $C_{l-1,p}(T)$ such that $\sup_{n,M} \mathbb{E}|LX_t^{n,M}|_{l-1}^p \leq C_{l-1,p}(T)$. Then for l , we write $\mathbb{E}|LX_t^{n,M}|_l^p \leq C_p(B_1 + B_2 + B_3)$, with C_p a constant depending on p and

$$B_1 = \mathbb{E}|L \int_0^t b_M(\tau_n(r), X_{\tau_n(r)}^{n,M}) dr|_l^p,$$

$$\begin{aligned}
B_2 &= \mathbb{E} |L \int_0^t \int_{\{z \geq M\}} \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M}) W_\nu(dr, dz)|_l^p, \\
B_3 &= \mathbb{E} |L \int_0^t \int_{[1, M)} \tilde{c}(\tau_n(r), z, X_{\tau_n(r)-}^{n,M}) N_\nu(dr, dz)|_l^p.
\end{aligned}$$

By **Lemma 5.5**, we get

$$\begin{aligned}
B_1 &\leq T^{p-1} \mathbb{E} \int_0^t |L b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|_l^p dr \\
&\leq C_{l,p}^1(T) [\mathbb{E} \int_0^t |\partial_x b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p |L X_{\tau_n(r)}^{n,M}|_l^p dr \\
&\quad + \mathbb{E} \int_0^t \sup_{2 \leq |\beta| \leq l+2} |\partial_x^\beta b_M(\tau_n(r), X_{\tau_n(r)}^{n,M})|^p (1 + |X_{\tau_n(r)}^{n,M}|_{l+1}^{(l+2)p}) (1 + |L X_{\tau_n(r)}^{n,M}|_{l-1}^p) dr] \\
&:= C_{l,p}^1(T) [B_{1,1} + B_{1,2}], \quad \text{respectively.}
\end{aligned}$$

We notice that by **Hypothesis 2.1**, (54) and the recurrence hypothesis,

$$B_{1,2} \leq \tilde{C}_{l,p}^1(T) [1 + \int_0^t (\mathbb{E} |L X_{\tau_n(r)}^{n,M}|_{l-1}^{2p})^{\frac{1}{2}} dr] \leq \tilde{C}_{l,p}^1(T) [1 + T \sqrt{C_{l-1,2p}(T)}].$$

So we have

$$B_1 \leq \bar{C}_{l,p}^1(T) [1 + \int_0^t \mathbb{E} |L X_{\tau_n(r)}^{n,M}|_l^p dr], \quad (60)$$

where $C_{l,p}^1(T)$, $\tilde{C}_{l,p}^1(T)$, $\bar{C}_{l,p}^1(T)$ are constants depending on l, p, T .

Then by **Lemma 5.4**, we get

$$\begin{aligned}
B_2 &\leq C_{l,p}^2(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |L \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|_l^2 \nu(dz))^{\frac{p}{2}} dr] \\
&:= C_{l,p}^2(T) [B_{2,1} + B_{2,2}], \quad \text{respectively.}
\end{aligned}$$

We notice that by **Lemma 5.5**,

$$\begin{aligned}
B_{2,1} &\leq \widehat{C}_{l,p}^2(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |L X_{\tau_n(r)}^{n,M}|_l^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l+2} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 (1 + |X_{\tau_n(r)}^{n,M}|_{l+1}^{2(l+2)}) (1 + |L X_{\tau_n(r)}^{n,M}|_{l-1}^2) \nu(dz))^{\frac{p}{2}} dr].
\end{aligned}$$

And by **Lemma 5.3**,

$$\begin{aligned}
B_{2,2} &\leq \hat{C}_{l,p}^2(T) [\mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} |\partial_x \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l}^2 \nu(dz))^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t (\int_{\{z \geq M\}} \sup_{2 \leq |\beta| \leq l} |\partial_x^\beta \tilde{c}(\tau_n(r), \gamma_n(z), X_{\tau_n(r)}^{n,M})|^2 |X_{\tau_n(r)}^{n,M}|_{1,l-1}^{2l} \nu(dz))^{\frac{p}{2}} dr].
\end{aligned}$$

We notice that by **Hypothesis 2.1**, (54) and the recurrence hypothesis,

$$\begin{aligned}
B_2 &\leq \bar{C}_{l,p}^2(T) [1 + \int_0^t \mathbb{E} |L X_{\tau_n(r)}^{n,M}|_l^p dr + \int_0^t (\mathbb{E} |L X_{\tau_n(r)}^{n,M}|_{l-1}^{2p})^{\frac{1}{2}} dr] \\
&\leq \tilde{C}_{l,p}^2(T) [1 + \int_0^t \mathbb{E} |L X_{\tau_n(r)}^{n,M}|_l^p dr], \quad (61)
\end{aligned}$$

where $C_{l,p}^2(T), \widehat{C}_{l,p}^2(T), \hat{C}_{l,p}^2(T), \bar{C}_{l,p}^2(T), \tilde{C}_{l,p}^2(T)$ are constants depending on l, p, T .

We notice that (with ψ_k given in (39)),

$$LZ_i^k = (\xi_i^k)^2 (\ln \psi_k)' (V_i^k)$$

and

$$D_{(r_1, m_1) \dots (r_l, m_l), (s_1, z_1) \dots (s_l, z_l)}^{Z, W, l} LZ_i^k = \prod_{n=1}^l (\delta_{r_n k} \delta_{m_n i}) (\xi_i^k)^{l+2} (\ln \psi_k)^{(l+1)} (V_i^k).$$

So using (40), we get

$$\mathbb{E} \sup_{k, i \in \mathbb{N}} |LZ_i^k|_l^p \leq \bar{C}_{l,p}, \quad (62)$$

with $\bar{C}_{l,p}$ a constant depending on l, p .

By **Lemma 5.5, Hypothesis 2.1**, for any $k, i \in \mathbb{N}$,

$$\begin{aligned} & |L\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \leq (|\partial_z \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) (|LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l) \\ & + C_l^3 \sup_{2 \leq |\beta| \leq l+2} (|\partial_z^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})| + |\partial_x^\beta \tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|) \\ & \times (1 + |Z_i^k|_{l+1}^{l+2} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2}) (1 + |LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1}) \\ & \leq \bar{C}_l^3 \bar{c}(Z_i^k) (1 + |LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} + |LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1} \\ & + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times (|LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1})), \end{aligned}$$

where C_l^3, \bar{C}_l^3 are constants depending on l .

Then

$$\begin{aligned} B_3 & \leq \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} |L\tilde{c}(\tau_n(T_i^k), Z_i^k, X_{\tau_n(T_i^k)-}^{n,M})|_l \right)^p \\ & \leq C_{l,p}^3 \mathbb{E} \left[\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (1 + |LZ_i^k|_l + |LX_{\tau_n(T_i^k)-}^{n,M}|_l + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} + |LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1} \right. \\ & \left. + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times (|LZ_i^k|_{l-1} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1})) \right]^p \\ & \leq \tilde{C}_{l,p}^3 (B_{3,1} + B_{3,2} + B_{3,3}), \end{aligned}$$

where $C_{l,p}^3, \tilde{C}_{l,p}^3$ are constants depending on l, p and

$$\begin{aligned} B_{3,1} & = \mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LX_{\tau_n(T_i^k)-}^{n,M}|_l \right)^p, \\ B_{3,2} & = \mathbb{E} \left[\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (|LZ_i^k|_l + |LZ_i^k|_{l-1} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LZ_i^k|_{l-1}) \right]^p, \\ B_{3,3} & = \mathbb{E} \left[\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) (1 + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} + |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1} + |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LX_{\tau_n(T_i^k)-}^{n,M}|_{l-1}) \right]^p. \end{aligned}$$

By compensating N_ν , Burkholder's inequality and **Hypothesis 2.1**,

$$\begin{aligned}
B_{3,1} &= \mathbb{E} \left| \int_0^t \int_{[1,M)} \bar{c}(z) |LX_{\tau_n(r)-}^{n,M}|_l N_\nu(dr, dz) \right|^p \\
&\leq C_{l,p}^3(T) \left[\mathbb{E} \int_0^t \left(\int_{[1,M)} |\bar{c}(z)|^2 |LX_{\tau_n(r)-}^{n,M}|_l^2 \nu(dz) \right)^{\frac{p}{2}} dr + \mathbb{E} \int_0^t \int_{[1,M)} |\bar{c}(z)|^p |LX_{\tau_n(r)-}^{n,M}|_l^p \nu(dz) dr \right] \\
&\quad + \mathbb{E} \int_0^t \left(\int_{[1,M)} |\bar{c}(z)| |LX_{\tau_n(r)-}^{n,M}|_l \nu(dz) \right)^p dr \\
&\leq C_{l,p}^3(T) \left((\bar{c}_2)^{\frac{p}{2}} + \bar{c}_p + (\bar{c}_1)^p \right) \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr, \tag{63}
\end{aligned}$$

with \bar{c}_p given in **Hypothesis 2.1**, and $C_{l,p}^3(T)$ a constant depending on l, p, T .

Using (62) and (54), we have

$$\begin{aligned}
\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |LZ_i^k|_l \right)^p &\leq \mathbb{E} \left(\sup_{k,i \in \mathbb{N}} |LZ_i^k|_l \right)^p \times \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k)^p \\
&\leq \left(\mathbb{E} \sup_{k,i \in \mathbb{N}} |LZ_i^k|_l^{2p} \right)^{\frac{1}{2}} \left(\mathbb{E} \left| \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) \right|^{2p} \right)^{\frac{1}{2}} \leq \sqrt{\bar{C}_{l,2p}} \left(\mathbb{E} \int_0^t \int_{[1,M)} \bar{c}(z) N_\nu(dr, dz) \right)^{\frac{1}{2}} \leq \hat{C}_{l,p}^3(T) \tag{64}
\end{aligned}$$

and

$$\mathbb{E} \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \bar{c}(Z_i^k) |X_{\tau_n(T_i^k)-}^{n,M}|_{l+1}^{l+2} \times |LZ_i^k|_{l-1} \right)^p \leq \sqrt{\bar{C}_{l-1,2p}} \left(\mathbb{E} \int_0^t \int_{[1,M)} \bar{c}(z) |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} N_\nu(dr, dz) \right)^{\frac{1}{2}} \leq \hat{C}_{l,p}^3(T), \tag{65}$$

where $\bar{C}_{l,2p}$ is given in (62), and $\hat{C}_{l,p}^3(T)$ is a constant depending on l, p, T .

So by (64) and (65),

$$B_{3,2} \leq \hat{C}_{l,p}^3(T), \tag{66}$$

with $\hat{C}_{l,p}^3(T)$ a constant depending on l, p, T .

Then by compensating N_ν , Burkholder's inequality, **Hypothesis 2.1**, (54) and recurrence hypothesis, we have

$$\begin{aligned}
B_{3,3} &= \mathbb{E} \left| \int_0^t \int_{[1,M)} \bar{c}(z) (1 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} + |LX_{\tau_n(r)-}^{n,M}|_{l-1} + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} \times |LX_{\tau_n(r)-}^{n,M}|_{l-1}) N_\nu(dr, dz) \right|^p \\
&\leq \tilde{C}_{l,p}^3(T) \left[\mathbb{E} \int_0^t \int_{[1,M)} |\bar{c}(z)|^2 (1 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{2(l+2)} + |LX_{\tau_n(r)-}^{n,M}|_{l-1}^2 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{2(l+2)} \times |LX_{\tau_n(r)-}^{n,M}|_{l-1}^2) \nu(dz) \right]^{\frac{p}{2}} dr \\
&\quad + \mathbb{E} \int_0^t \int_{[1,M)} |\bar{c}(z)|^p (1 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{(l+2)p} + |LX_{\tau_n(r)-}^{n,M}|_{l-1}^p + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{(l+2)p} \times |LX_{\tau_n(r)-}^{n,M}|_{l-1}^p) \nu(dz) dr \\
&\quad + \mathbb{E} \int_0^t \int_{[1,M)} |\bar{c}(z)| (1 + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} + |LX_{\tau_n(r)-}^{n,M}|_{l-1} + |X_{\tau_n(r)-}^{n,M}|_{l+1}^{l+2} \times |LX_{\tau_n(r)-}^{n,M}|_{l-1}) \nu(dz) \right]^p dr \leq \bar{C}_{l,p}^3(T), \tag{67}
\end{aligned}$$

where $\tilde{C}_{l,p}^3(T), \bar{C}_{l,p}^3(T)$ are constants depending on l, p, T . So by (63)(66)(67),

$$B_3 \leq \bar{C}_{l,p}^3(T) \left[1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr \right], \tag{68}$$

with $\bar{C}_{l,p}^3(T)$ a constant depending on l, p, T . Then combining (60)(61)(68),

$$\mathbb{E} |LX_t^{n,M}|_l^p \leq \bar{C}_{l,p}(T) \left[1 + \int_0^t \mathbb{E} |LX_{\tau_n(r)-}^{n,M}|_l^p dr \right],$$

with $\tilde{C}_{l,p}(T)$ a constant depending on l, p, T . By Gronwall's lemma, we have

$$\mathbb{E}|LX_{T_n(t)}^{n,M}|^p \leq \tilde{C}_{l,p}(T)e^{T\tilde{C}_{l,p}(T)}.$$

So we conclude that

$$\mathbb{E}|LX_t^{n,M}|^p \leq C_{l,p}(T). \quad (69)$$

□

Finally, by **Lemma 5.1** and (54)(69), as a consequence of **Lemma 3.4**, we have $X_t^M \in \mathcal{D}_{l,p}$ and $\sup_M \|X_t^M\|_{L,l,p} \leq C_{l,p}(T)$. So we finish the proof of **Lemma 4.1**.

5.2 Proof of Lemma 4.2

In the following, we turn to the non-degeneracy of X_t^M . We consider the approximating equation (43)

$$X_t^M = x + \int_0^t \int_{[1,M]} \tilde{c}(r, z, X_{r-}^M) N_\nu(dr, dz) + \int_0^t b_M(r, X_r^M) dr + \int_0^t \int_{\{z \geq M\}} \tilde{c}(r, z, X_r^M) W_\nu(dr, dz).$$

We can calculate the Malliavin derivatives of the Euler scheme and then by passing to the limit, we have

$$\begin{aligned} D_{(k,i)}^Z X_t^M &= 1_{\{k \leq M-1\}} \xi_i^k \partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k-}^M) + \int_{T_i^k}^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) D_{(k,i)}^Z X_{r-}^M N_\nu(dr, dz) \\ &+ \int_{T_i^k}^t \partial_x b_M(r, X_r^M) D_{(k,i)}^Z X_s^M dr + \int_{T_i^k}^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) D_{(k,i)}^Z X_s^M W_\nu(dr, dz). \end{aligned} \quad (70)$$

$$\begin{aligned} D_{(s,z_0)}^W X_t^M &= \int_s^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) D_{(s,z_0)}^W X_{r-}^M N_\nu(dr, dz) + \int_s^t \partial_x b_M(r, X_r^M) D_{(s,z_0)}^W X_r^M dr \\ &+ 1_{\{s \leq t\}} 1_{\{z_0 \geq M\}} \tilde{c}(s, z_0, X_s^M) + \int_s^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) D_{(s,z_0)}^W X_r^M W_\nu(dr, dz). \end{aligned} \quad (71)$$

Now we employ the "variance of constant method". We consider the tangent flow

$$Y_t^M = 1 + \int_0^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) Y_{r-}^M N_\nu(dr, dz) + \int_0^t \partial_x b_M(r, X_r^M) Y_r^M dr + \int_0^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) Y_r^M W_\nu(dr, dz).$$

And by Itô's formula, the inverse of Y is given by

$$\begin{aligned} \hat{Y}_t^M &= 1 - \int_0^t \int_{[1,M]} \partial_x \tilde{c}(r, z, X_{r-}^M) (1 + \partial_x \tilde{c}(r, z, X_{r-}^M))^{-1} \hat{Y}_{r-}^M N_\nu(dr, dz) - \int_0^t \partial_x b_M(r, X_r^M) \hat{Y}_r^M dr \\ &- \int_0^t \int_{\{z \geq M\}} \partial_x \tilde{c}(r, z, X_r^M) \hat{Y}_r^M W_\nu(dr, dz) + \frac{1}{2} \int_0^t \int_{\{z \geq M\}} |\partial_x \tilde{c}(r, z, X_r^M)|^2 \hat{Y}_r^M \nu(dz) dr. \end{aligned}$$

Applying **Hypothesis 2.2**, one also has

$$\mathbb{E}(\sup_{s \leq t} (|Y_s^M|^p + |\hat{Y}_s^M|^p)) < \infty. \quad (72)$$

Then using the uniqueness of solution to the equation (70) and (71), one obtains

$$D_{(k,i)}^Z X_t^M = 1_{\{k \leq M-1\}} \xi_i^k Y_t^M \hat{Y}_{T_i^k-}^M \partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k-}^M), \quad D_{(s,z_0)}^W X_t^M = 1_{\{s \leq t\}} 1_{\{z_0 \geq M\}} Y_t^M \hat{Y}_s^M \tilde{c}(s, z_0, X_s^M). \quad (73)$$

And the Malliavin covariance matrix of X_t^M is

$$\sigma_{X_t^M} = \langle DX_t^M, DX_t^M \rangle_{\mathcal{H}} = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} |D_{(k,i)}^Z X_t^M|^2 + \int_0^T \int_{\{z \geq M\}} |D_{(s,z)}^W X_t^M|^2 \nu(dz) ds. \quad (74)$$

In the following, we denote $\lambda_t^M = \det \sigma_{X_t^M}$. So the aim is to prove that for every $p \geq 1$,

$$\mathbb{E}(|\lambda_t^M|^{-p}) \leq C_p. \quad (75)$$

We proceed in 5 steps.

Step 1 We notice that by (73) and (74)

$$\lambda_t^M = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k Y_t^2 \widehat{Y}_{T_i^k}^2 |\partial_z \tilde{c}(T_i^k, Z_i^k, X_{T_i^k}^M)|^2 + Y_t^2 \int_0^t \widehat{Y}_s^2 \int_{\{z \geq M\}} |\tilde{c}(s, z, X_s^M)|^2 \nu(dz) ds.$$

We recall the ellipticity hypothesis (**Hypothesis 2.3**): there exists a function $\underline{c}(z)$ such that

$$|\partial_z \tilde{c}(s, z, x)|^2 \geq \underline{c}(z) \quad \text{and} \quad |\tilde{c}(s, z, x)|^2 \geq \underline{c}(z).$$

In particular

$$\int_{\{z \geq M\}} |\tilde{c}(s, z, x)|^2 \nu(dz) \geq \int_{\{z \geq M\}} \underline{c}(z) \nu(dz)$$

so that

$$\lambda_t^M \geq Q_t^{-2} \times \left(\sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k) + t \int_{\{z \geq M\}} \underline{c}(z) \nu(dz) \right) \quad \text{with} \quad Q_t = \inf_{s \leq t} Y_s^M \widehat{Y}_t^M$$

We denote

$$\rho_t^M = \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k), \quad \bar{\rho}_t^M = \sum_{k=M}^{\infty} \sum_{i=1}^{J_t^k} \xi_i^k \underline{c}(Z_i^k), \quad \alpha^M = \int_{\{z \geq M\}} \underline{c}(z) \nu(dz)$$

By (72), with $C = \mathbb{E}(\sup_{s \leq t} |Y_s^M \widehat{Y}_t^M|^{4p})^{1/2}$,

$$\mathbb{E}(|\lambda_t^M|^{-p}) \leq C(\mathbb{E}(|\rho_t^M + t\alpha^M|^{-2p}))^{\frac{1}{2}}. \quad (76)$$

Step 2 We denote by $\Gamma(p)$ the "Gamma function". Explicitly, for $p \geq 1$, $\Gamma(p) = \int_0^\infty s^{p-1} e^{-s} ds$. By a change of variables, we have the numerical equality

$$\frac{1}{(\rho_t^M + t\alpha^M)^p} = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} e^{-s(\rho_t^M + t\alpha^M)} ds$$

which, by taking expectation, gives

$$\mathbb{E}\left(\frac{1}{(\rho_t^M + t\alpha^M)^p}\right) = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) ds. \quad (77)$$

Step 3 (splitting). In order to compute $\mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)})$ we have to interpret ρ_t^M in terms of Poisson measures. We recall that we suppose the "splitting hypothesis" (33):

$$1_{I_k} \nu(dz) \geq \varepsilon_k m_k \times 1_{I_k} dz,$$

with $I_k = [k, k + 1)$, $m_k = \nu(I_k)$. We also have the function ψ and $m(\psi) = \int_{\mathbb{R}} \psi(t) dt$. Then we have the basic decomposition

$$Z_i^k = \xi_i^k V_i^k + (1 - \xi_i^k) U_i^k$$

where V_i^k, U_i^k, ξ_i^k are some independent random variables with laws

$$\begin{aligned} \mathbb{P}(V_i^k \in dt) &= \frac{1}{m(\psi)} \psi(t - (k + \frac{1}{2})) dt \\ \mathbb{P}(U_i^k \in dt) &= \frac{1}{1 - \varepsilon_k m(\psi)} (\mathbb{P}(Z_i^k \in dt) - \varepsilon_k \psi(t - (k + \frac{1}{2})) dt) \\ \mathbb{P}(\xi_i^k = 1) &= \varepsilon_k m(\psi), \quad \mathbb{P}(\xi_i^k = 0) = 1 - \varepsilon_k m(\psi). \end{aligned}$$

Then for every k we consider a Poisson point measure $N_k(d\xi, dv, du)$ with $\xi \in \{0, 1\}, v, u \in [1, \infty)$ with compensator

$$\begin{aligned} \widehat{N}_k(d\xi, dv, du) &= b_k(d\xi) \times \frac{1}{m(\psi)} \psi(v - (k + \frac{1}{2})) dv \\ &\times \frac{1}{1 - \varepsilon_k m(\psi)} (\mathbb{P}(Z_1^k \in du) - \varepsilon_k \psi(u - (k + \frac{1}{2}))) du. \end{aligned}$$

Here $b_k(d\xi)$ is the Bernoulli law of parameter $\varepsilon_k m(\psi)$. Recall that ψ has support $[-\frac{1}{2}, \frac{1}{2}]$ so that $t \mapsto \psi(t - (k + \frac{1}{2}))$ is supported on I_k (which are disjoint sets). So the Poisson point measures N_k are independent. Then

$$\sum_{i=1}^{J_t^k} \xi_i^k \mathcal{L}(Z_i^k) = \sum_{i=1}^{J_t^k} \xi_i^k \mathcal{L}(\xi_i^k V_i^k + (1 - \xi_i^k) U_i^k) = \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} \xi \mathcal{L}(\xi v + (1 - \xi)u) N_k(ds, d\xi, dv, du).$$

In order to get compact notation, we put together all the measures $N_k, k \leq M - 1$. Since they are independent we get a new Poisson point measure that we denote by Λ . And we have

$$\begin{aligned} \rho_t^M &= \sum_{k=1}^{M-1} \sum_{i=1}^{J_t^k} \xi_i^k \mathcal{L}(Z_i^k) = \sum_{k=1}^{M-1} \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} \xi \mathcal{L}(\xi v + (1 - \xi)u) N_k(ds, d\xi, dv, du) \\ &= \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} \xi \mathcal{L}(\xi v + (1 - \xi)v) \Lambda(ds, d\xi, dv, du). \end{aligned}$$

Step 4 Using Itô's formula ,

$$\begin{aligned} \mathbb{E}(e^{-s\rho_t^M}) &= 1 + \mathbb{E} \int_0^t \int_{\{0,1\}} \int_{[1,\infty)^2} (e^{-s(\rho_r^M + \xi \mathcal{L}(\xi v + (1 - \xi)v))} - e^{-s\rho_r^M}) \widehat{\Lambda}(dr, d\xi, dv, du) \\ &= 1 - \int_0^t \mathbb{E}(e^{-s\rho_r^M}) dr \int_{\{0,1\}} \int_{[1,\infty)^2} (1 - e^{-s\xi \mathcal{L}(\xi v + (1 - \xi)v)}) \Theta(d\xi, dv, du) \end{aligned}$$

with

$$\Theta(d\xi, dv, du) = \sum_{k=1}^{M-1} \widehat{N}_k(d\xi, dv, du).$$

Solving the above equation we obtain

$$\begin{aligned} \mathbb{E}(e^{-s\rho_t^M}) &= \exp(-t \int_{\{0,1\}} \int_{[1,\infty)^2} (1 - e^{-s\xi \mathcal{L}(\xi v + (1 - \xi)u)}) \Theta(d\xi, dv, du)) \\ &= \exp(-t \sum_{k=1}^{M-1} \int_{\{0,1\}} \int_{[1,\infty)^2} (1 - e^{-s\xi \mathcal{L}(\xi v + (1 - \xi)u)}) \widehat{N}_k(d\xi, dv, du)). \end{aligned}$$

And we compute

$$\begin{aligned} & \int_{\{0,1\} \times [1,\infty)^2} (1 - e^{-s\xi \underline{c}(\xi v + (1-\xi)u)}) \widehat{N}_k(d\xi, dv, du) \\ &= \varepsilon_k m(\psi) \int_k^{k+1} (1 - e^{-s\underline{c}(v)}) \frac{1}{m(\psi)} \psi(v - (k + \frac{1}{2})) dv. \end{aligned}$$

Since $\psi \geq 0$ and $\psi(t) = 1$ if $|t| \leq \frac{1}{4}$ it follows that the above term is larger than

$$\varepsilon_k \int_{k+\frac{1}{4}}^{k+\frac{3}{4}} (1 - e^{-s\underline{c}(v)}) dv.$$

Finally this gives

$$\begin{aligned} \mathbb{E}(e^{-s\rho_t^M}) &\leq \exp(-t \sum_{k=1}^{M-1} \varepsilon_k \int_{k+\frac{1}{4}}^{k+\frac{3}{4}} (1 - e^{-s\underline{c}(v)}) dv) \\ &= \exp(-t \int_1^M (1 - e^{-s\underline{c}(v)}) m(dv)), \end{aligned}$$

with

$$m(dv) = \sum_{k=1}^{\infty} \varepsilon_k \mathbf{1}_{(k+\frac{1}{4}, k+\frac{3}{4})}(v) dv. \quad (78)$$

By the same way, we get

$$\mathbb{E}(e^{-s\bar{\rho}_t^M}) \leq \exp(-t \int_M^{\infty} (1 - e^{-s\underline{c}(v)}) m(dv)).$$

Then using Jensen's inequality for the convex function $f(x) = e^{-sx}$, $s, x > 0$, we have

$$e^{-st\alpha^M} = \exp(-s\mathbb{E} \sum_{k=M}^{\infty} \sum_{i=1}^{J_t^k} \underline{c}(Z_i^k)) \leq e^{-s\mathbb{E}\bar{\rho}_t^M} \leq \mathbb{E}(e^{-s\bar{\rho}_t^M}) \leq \exp(-t \int_M^{\infty} (1 - e^{-s\underline{c}(v)}) m(dv)).$$

So we get

$$\begin{aligned} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) &= e^{-st\alpha^M} \times \mathbb{E}(e^{-s\rho_t^M}) \\ &\leq \exp(-t \int_M^{\infty} (1 - e^{-s\underline{c}(v)}) m(dv)) \times \exp(-t \int_1^M (1 - e^{-s\underline{c}(v)}) m(dv)) \\ &= \exp(-t \int_1^{\infty} (1 - e^{-s\underline{c}(v)}) m(dv)). \end{aligned} \quad (79)$$

Now we will use the lemma 14 from [8], which states the followings.

Lemma 5.6. *We consider an abstract measurable space E , a σ -finite measure ν_* on this space and a non-negative measurable function $f : E \rightarrow \mathbb{R}_+$ such that $\int_E f d\nu_* < \infty$. For $t > 0$ and $p \geq 1$, we note*

$$\alpha_f(t) = \int_E (1 - e^{-tf(a)}) \nu_*(da) \quad \text{and} \quad I_t^p(f) = \int_0^{\infty} s^{p-1} e^{-t\alpha_f(s)} ds.$$

We suppose that for some $t > 0$ and $p \geq 1$,

$$\underline{\lim}_{u \rightarrow \infty} \frac{1}{\ln u} \nu_*(f \geq \frac{1}{u}) > p/t, \quad (80)$$

then $I_t^p(f) < +\infty$.

So if we have

$$\lim_{u \rightarrow \infty} \frac{1}{\ln u} m(\underline{c} \geq \frac{1}{u}) = \infty, \quad (81)$$

then for every $p \geq 1, t > 0$, (77)(79) and **Lemma 5.6** give

$$\begin{aligned} \mathbb{E}\left(\frac{1}{\rho_t^M + t\alpha^M}\right)^{2p} &= \frac{1}{\Gamma(2p)} \int_0^\infty s^{2p-1} \mathbb{E}(e^{-s(\rho_t^M + t\alpha^M)}) ds \\ &\leq \frac{1}{\Gamma(2p)} \int_0^\infty s^{2p-1} \exp(-t \int_1^\infty (1 - e^{-s\underline{c}(v)}) m(dv)) < \infty. \end{aligned} \quad (82)$$

Finally by (76),

$$\mathbb{E}(\lambda_t^M)^{-p} < \infty. \quad (83)$$

Step5 Now the only problem left is to compute $m(\underline{c} \geq \frac{1}{u})$. It seems difficult to discuss this in a completely abstract framework. So we suppose the **Hypothesis 2.4** (a): There exists a constant $\varepsilon_* > 0$, for some $0 < \alpha_2 < \alpha_1 \leq 1$,

$$1_{I_k} \frac{\nu(dz)}{m_k} \geq 1_{I_k} \frac{\varepsilon_*}{z^{1-\alpha_1}} dz \quad \text{and} \quad \underline{c}(z) \geq e^{-z^{\alpha_2}}.$$

Then $\{z : \underline{c}(z) \geq \frac{1}{u}\} \supseteq \{z : (\ln u)^{1/\alpha_2} \geq z\}$. In particular, for $k \leq (\ln u)^{1/\alpha_2} - 1$, one has $I_k \subseteq \{z : \underline{c}(z) \geq \frac{1}{u}\}$.

We can take

$$\varepsilon_k = \frac{\varepsilon_*}{(k+1)^{1-\alpha_1}}$$

Then for u large enough such that $\lfloor (\ln u)^{1/\alpha_2} \rfloor \geq 2$, we compute

$$\begin{aligned} m(\underline{c} \geq \frac{1}{u}) &\geq \frac{1}{2} \sum_{k=1}^{\lfloor (\ln u)^{1/\alpha_2} \rfloor - 1} \varepsilon_k \geq \frac{\varepsilon_*}{2} \sum_{k=1}^{\lfloor (\ln u)^{1/\alpha_2} \rfloor - 1} \frac{1}{(k+1)^{1-\alpha_1}} \geq \frac{\varepsilon_*}{2} \int_2^{(\ln u)^{1/\alpha_2}} \frac{1}{z^{1-\alpha_1}} dz \\ &= \frac{\varepsilon_*}{2\alpha_1} ((\ln u)^{\alpha_1/\alpha_2} - 2^{\alpha_1}). \end{aligned}$$

Since $\alpha_1 > \alpha_2$, (81) is verified and we obtain (83).

Now we consider **Hypothesis 2.4** (b): We suppose that there exists a constant $\varepsilon_* > 0$, for some $\alpha > 0$,

$$1_{I_k} \frac{\nu(dz)}{m_k} \geq 1_{I_k} \frac{\varepsilon_*}{z} dz \quad \text{and} \quad \underline{c}(z) \geq \frac{1}{z^\alpha}.$$

Now $\{z : \underline{c}(z) \geq \frac{1}{u}\} \supseteq \{z : z \leq u^{1/\alpha}\}$ and $\varepsilon_k = \frac{\varepsilon_*}{k+1}$. Then for u large enough such that $\lfloor u^{1/\alpha} \rfloor \geq 2$,

$$m(\underline{c} \geq \frac{1}{u}) \geq \frac{\varepsilon_*}{2} \sum_{k=1}^{\lfloor u^{1/\alpha} \rfloor - 1} \frac{1}{k+1} \geq \frac{\varepsilon_*}{2} \int_2^{u^{1/\alpha}} \frac{dz}{z} = \frac{\varepsilon_*}{2} (\frac{1}{\alpha} \ln u - \ln 2).$$

And consequently

$$\lim_{u \rightarrow \infty} \frac{1}{\ln u} m(\underline{c} \geq \frac{1}{u}) \geq \frac{\varepsilon_*}{2\alpha}.$$

Using **Lemma 5.6**, this gives: if

$$\frac{2p}{t} < \frac{\varepsilon_*}{2\alpha} \quad \Leftrightarrow \quad t > \frac{4p\alpha}{\varepsilon_*}$$

then

$$\mathbb{E}\left(\frac{1}{\rho_t^M + t\alpha^M}\right)^{2p} < \infty,$$

and we have $\mathbb{E}(\lambda_t^M)^{-p} < \infty$.

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