



HAL
open science

Output-feedback repetitive control for minimum-phase nonlinear systems with arbitrarily relative degree

Daniele Astolfi

► **To cite this version:**

Daniele Astolfi. Output-feedback repetitive control for minimum-phase nonlinear systems with arbitrarily relative degree. Third IFAC Conference on Modelling, Identification and Control of Nonlinear Systems MICNON 2021, Sep 2021, Tokyo (virtual), Japan. hal-03351596

HAL Id: hal-03351596

<https://hal.science/hal-03351596>

Submitted on 22 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Output-feedback repetitive control for minimum-phase nonlinear systems with arbitrarily relative degree[★]

Daniele Astolfi^{*}

^{*} *Univ Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP
UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne,
France. (daniele.astolfi@univ-lyon1.fr)*

Abstract: A new design of output-feedback repetitive control scheme for nonlinear minimum-phase systems with arbitrary relative degree and globally Lipschitz nonlinearities is proposed. This work extends the recent results in Astolfi et al. [2021]. The delay of the repetitive control scheme is represented by a transport equation. A high-gain observer and a forwarding-based feedback law are employed to steer the desired output to zero in presence of periodic signals (references and/or perturbations) and model uncertainties.

Keywords: Repetitive Control, nonlinear Systems, PDEs, high-gain observers, forwarding.

1. INTRODUCTION

Repetitive control (RC) schemes are based on the main idea that a delay can be used to generate any periodic signal. RC schemes are therefore very popular in regulation problems of periodic signals, i.e. reference tracking and/or disturbance rejection, and used in many control applications in both continuous-time, Hara et al. [1988], and discrete-time domain, Tomizuka et al. [1989]. See, more recently, Mattavelli and Marafao [2004], Kurniawan et al. [2014], Blanken et al. [2019] and references therein. Although this vast success, from the theoretical point of view, few works addressed the problem of using RC-schemes for continuous-time nonlinear systems. In view of the difficulties of dealing with infinite-dimensional systems (i.e., the delay), a common approach is to focus on some finite-dimensional RC approximation based on the use of low-pass filters (see, e.g., Weiss and Häfele [1999]) or on its equivalent harmonic representation (see, e.g., Astolfi et al. [2019], Ghosh and Paden [2000]). Recently, a new stability analysis approach for an exact RC-scheme (i.e. infinite-dimensional) has been proposed in the context of nonlinear systems which are strictly dissipative, under the assumption that the relative degree between the control input and the regulated output is zero, see Califano and Macchelli [2019].

This work addresses the problem of designing an output-feedback RC-scheme for minimum-phase nonlinear systems having a relative degree larger than one and with globally Lipschitz nonlinearities. It relies on the recent developments proposed in Astolfi et al. [2021] where the same class of systems is considered but with a partial-state feedback design. A new design is proposed, allowing to extend the class of systems to which RC-approaches apply. With respect to Astolfi et al. [2021], it is employed a different transport-equation design which is able to achieve

an input-to-state stability property with respect to the estimation error. As a consequence, the state-feedback law, based on forwarding technique, can be turned into output-feedback by means of classical high-gain observers, see, e.g., Atassi and Khalil [1999].

Notation. \mathbb{R} denotes the space of real numbers and \mathbb{C} the space of complex numbers, $\mathbb{R}_+ := [0, \infty)$ and \mathbb{N} denotes the space of positive integers, i.e. $\mathbb{N} := \{1, 2, \dots, \infty\}$. For a function $w : (t, x) \in \mathbb{R}_+ \times [0, 1] \mapsto w(t, x) \in \mathbb{R}$, the notation w_t (respectively, w_x) denotes the partial derivative of w w.r.t the variable t (respectively, w.r.t. the variable x). When a function w depends only on the variable of the time t (resp. space x), its derivative is simply denoted by \dot{w} (resp. w'); $L^2(0, 1)$ denotes the Hilbert space of real-valued square-integrable functions over the interval $(0, 1)$ and $H^1(0, 1) \subset L^2(0, 1)$ is the Hilbert space of real-valued absolutely continuous functions over $[0, 1]$ with square-integrable derivative; $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ denotes their respectively induced norms (see, also, Astolfi et al. [2021]). A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of *class- \mathcal{K}_∞* if α is continuous, increasing, $\alpha(0) = 0$ and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$.

2. MAIN MOTIVATION

Consider a system of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{d}(t)) + \mathbf{g}(\mathbf{x}, \mathbf{d}(t))\mathbf{u} \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{d}(t)) \end{aligned} \quad (1)$$

with system state $\mathbf{x} \in \mathbb{R}^{n_x}$, control input $\mathbf{u} \in \mathbb{R}$ and output $\mathbf{y} \in \mathbb{R}$. The problem of periodic output regulation consists in designing a regulator for system (1) so that the output \mathbf{y} satisfies

$$\lim_{t \rightarrow \infty} e(t) := \lim_{t \rightarrow \infty} (\mathbf{y}(t) - \mathbf{r}(t)) = 0$$

for some T -periodic reference $\mathbf{r} \in \mathbb{R}$ and in presence of possible T -periodic disturbances $\mathbf{d} \in \mathbb{R}^{n_d}$. In Repetitive-Control design, the solution to such a problem is typically pursued by including a regulator of the form¹

^{*} This research was partially supported by the French Grant ANR ODISSE (ANR-19-CE48-0004-01).

¹ Here $s \in \mathbb{C}$ represents the Laplace variable.

$$R(s) := \frac{\exp(-Ts)}{1 - \exp(-Ts)} \quad (2)$$

inside the control-feedback loop. The main purpose of (2), based on the so called *internal-model principle* (Pauonon et al. [2008]), is to generate any desired periodic steady-state input, see for instance Hara et al. [1988]. However, the analysis and the design of a RC-based stabilizer-feedback for systems (1) is not straightforward since the Laplace transformation cannot be used in such a nonlinear framework, and different tools need to be used.

An alternatively approach, pursued in Astolfi et al. [2021], Califano et al. [2018], Califano and Macchelli [2019], consists in rewriting the scheme (2) by using an equivalent transport equation representation. In particular, the main result in Astolfi et al. [2021] is that of developing a new RC-feedback control law for systems (1) having a well-defined relative degree and that can be rewritten, after a change of coordinates, into the form

$$\begin{cases} \dot{z} = f(t, z, \xi_1), \\ \dot{\xi}_i = \xi_{i+1}, & i = 1, \dots, r-1 \\ \dot{\xi}_r = q(t, z, \xi_1, \dots, \xi_r) + u, \\ e = \xi_1, \end{cases} \quad (3)$$

where $(z, \xi) \in \mathbb{R}^n \times \mathbb{R}^r$ is the system state, with the $z \in \mathbb{R}^n$ -dynamics being the so-called zero-dynamics and $\xi := (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$ representing the derivatives of the output $e \in \mathbb{R}$ that we aim at regulating to zero, and $u \in \mathbb{R}$ the control input. Note that with respect to the representation (1), we have

$$\xi_1 = \mathbf{h}(\mathbf{x}, \mathbf{d}(t)) - \mathbf{r}(t),$$

and the explicit dependence on \mathbf{d}, \mathbf{r} in (3) is compactly substituted by the dependence on time t . Finally, the zero-dynamics in (3) is supposed to possess some good input-to-state properties as specified later. The RC-design for system (3) proposed in Astolfi et al. [2021] has then the form

$$\begin{cases} \eta_t(t, x) = -\frac{1}{T}\eta_x(t, x) & \forall (t, x) \in \mathbb{R}_+ \times [0, 1], \\ \eta(t, 0) = \eta(t, 1) + \theta(t) & \forall t \in \mathbb{R}_+, \\ u(t) = \gamma(\theta(t), \eta(t, x)) & \forall (t, x) \in \mathbb{R}_+ \times [0, 1], \end{cases} \quad (4)$$

where $(t, x) \in \mathbb{R}_+ \times [0, 1]$ and $\eta \in L^2(0, 1)$, the signal θ is a linear combination of the state variables ξ of the form

$$\theta := \xi_r + \sum_{i=1}^{r-1} g^{r-i} a_i \xi_i,$$

and the feedback law $\gamma : \mathbb{R} \times L^2(0, 1) \rightarrow \mathbb{R}$ is designed according to the so-called *forwarding* approach. The drawback of such an approach is that the knowledge of θ , and therefore of the state components (ξ_1, \dots, ξ_r) , is required.

The objective of this work is to extend such an approach to the context of output feedback, in which the sole use of $e = \xi_1$ is employed for the RC-feedback law (4). To this end, recall that a common routine in output-feedback control is to replace the state information ξ by an estimate $\hat{\xi}$ provided by an observer. In the context of minimum-phase systems, such an observer is typically chosen as a high-gain observer, see, e.g., Atassi and Khalil [1999]. However, for this to work, an input-to-state stability (ISS) with respect to estimation error is needed, see Andrieu

and Praly [2009]. As a consequence, when considering the feedback (4), such an approach cannot be followed since simple computations can show that the ISS property with respect to estimation error acting at the boundary conditions $\eta(t, 0)$ may not be verified, see, for instance Tanwani et al. [2016]. The main result of this work is therefore that of proposing a variation of the scheme (4) by developing a pure output feedback RC-scheme for systems of the form (3), with the signal θ acting in a distributed way and not at the boundary of the η -dynamics.

3. MAIN RESULT

Following Astolfi et al. [2021], consider a system of the form (3) and suppose that the following assumptions hold.

Assumption 1. The function q in (3) is globally Lipschitz, C^2 in its arguments and periodic with respect to the first argument.

Assumption 2. The zero-dynamics $\dot{z} = f(t, z, 0)$ of system (3) admits a unique C^2 T -periodic bounded solution $\bar{z}(t)$ which is globally uniformly stable and input-to-state stable.

In particular, there exists a positive definite function $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ and class \mathcal{K}_∞ functions $\underline{\alpha}, \bar{\alpha}$ and real numbers $\alpha, \rho > 0$ satisfying

$$\underline{\alpha}(|z - \bar{z}(t)|) \leq V(t, z - \bar{z}(t)) \leq \bar{\alpha}(|z - \bar{z}(t)|) \quad (5)$$

for all $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n$ and

$$\langle \nabla V(t, z - \bar{z}(t)), f(t, z, \xi_1) - f(t, \bar{z}(t), 0) \rangle \leq -\alpha|z - \bar{z}(t)|^2 + \rho|\xi_1|^2 \quad (6)$$

for all $(t, z, \xi_1) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$.

Under previous assumptions, an output-feedback repetitive-control based regulator can be designing by combining the following elements.

- A high-gain observer of the form:

$$\begin{cases} \dot{\hat{\xi}}_i = \hat{\xi}_i + \ell^i b_i (e - \hat{\xi}_1), & i = 1 \dots, r-1 \\ \dot{\hat{\xi}}_r = u_s + \ell^r b_r (e - \hat{\xi}_1), \end{cases} \quad (7a)$$

where $\ell \geq 1$ is the so-called high-gain parameter, the parameters $b_i > 0$, $i = 1, \dots, r$ have to be properly chosen and the term u_s is an extra input to be defined. Note that following Wang et al. [2015], the high-gain observer (7a) could be also substituted by a low-power high-gain observer Astolfi and Marconi [2015].

- The core of the RC scheme, also denoted as *internal-model unit* in output regulation theory, see, e.g., Astolfi et al. [2019], embedding a delay represented as a transport equation:

$$\begin{cases} \eta_t(t, x) = -\frac{1}{T}\eta_x(t, x) + \exp(\beta x)\hat{\theta} \\ \eta(t, 0) = \eta(t, 1) \\ \eta(0, x) = \eta_0(x) \end{cases} \quad (7b)$$

defined on $(t, x) \in \mathbb{R}_+ \times [0, 1]$, with β being a parameter to be chosen and $\hat{\theta}$ a signal to be defined.

- The forwarding-based (see, e.g., Marx et al. [2020, 2021], Astolfi et al. [2021]) stabilizing feedback control law:

$$\begin{cases} u = u_s + u_{im} \\ u_s = -\kappa\hat{\theta} \\ u_{im} = \mu \int_0^1 (\eta(t, x) - M(x)\hat{\theta})M(x)dx, \end{cases} \quad (7c)$$

where $\mu > 0$ is a positive parameters, $\hat{\theta}$ is defined as

$$\hat{\theta} := \hat{\xi}_r + \sum_{i=1}^{r-1} g^{r-i} a_i \hat{\xi}_i, \quad (7d)$$

with $g > 0$, $a_i > 0$, $i = 1, \dots, r-1$, parameters to be defined, and the function $M : [0, 1] \rightarrow \mathbb{R}$ is defined as the solution of the following two-boundary value problem

$$\begin{cases} \frac{1}{T} M'(x) = \kappa M(x) + \exp(\beta x), \\ M(0) = M(1) \end{cases} \quad (8)$$

which can be explicitly computed as

$$M(x) = \begin{cases} \exp(kTx) \bar{M} + T \frac{\exp(\beta x) - \exp(\kappa T x)}{\beta - \kappa T}, & \beta \neq \kappa T, \\ \exp(kTx) \bar{M} + xT \exp(\kappa T x), & \beta = \kappa T, \end{cases}$$

$$\bar{M} = \begin{cases} \frac{\exp(\beta) + \exp(\kappa T)}{(\kappa T - \beta)(1 - \exp(\kappa T))}, & \beta \neq \kappa T, \\ \frac{T \exp(\kappa T)}{1 - \exp(\kappa T)}, & \beta = \kappa T. \end{cases} \quad (9)$$

The following main result can be stated.

Theorem 1. Suppose Assumptions 1 and 2 hold. Let the parameters a_i , b_i of the regulator (7) be such that the corresponding polynomial

$$p_a(\lambda) := \lambda^{r-1} + a_{r-1} \lambda^{r-2} + \dots + a_2 \lambda + a_1$$

$$p_b(\lambda) := \lambda^r + b_r \lambda^{r-1} + \dots + b_2 \lambda + b_1$$

are Hurwitz and let $\mu > 0$ and $\beta \neq 0$ be fixed. Then, there exists $g^ \geq 1$ and, for any $g > g^*$, there exists $\kappa^* \geq 1$ such that, for any $\kappa \geq \kappa^*$ there exists $\ell^* \geq 1$ (the values of g^* , κ^* , ℓ^* depends only on the Lipschitz constants of f, q , the parameters α, ρ defined in Assumptions 1, 2, and the choice of a_1, b_i) such that the following statements hold true for any $\ell \geq \ell^*$.*

- (1) *For any initial condition $(z_0, \xi_0, \hat{\xi}_0, \eta_0) \in \mathbb{R}^n \times \mathbb{R}^r \times L^2(0, 1)$, the closed-loop system (3), (7), (8), admits a unique solution $(z, \xi, \hat{\xi}, \eta) \in C^0([0, \infty); \mathbb{R}^{n+2r} \times L^2(0, 1))$, which is bounded forward in time, namely*

$$|z(t)| + |\xi(t)| + |\hat{\xi}(t)| + \|\eta(t, \cdot)\|_{L^2} \leq \delta, \quad \forall t \geq 0$$

for some $\delta > 0$.

- (2) *Any solution of the closed-loop system the closed-loop system (3), (7), (8), starting from $\mathbb{R}^n \times \mathbb{R}^r \times L^2(0, 1)$ satisfies $\lim_{t \rightarrow \infty} e(t) = 0$.*

The result of Theorem 1 mainly states that the proposed design (7) guarantees asymptotic regulation to zero of the error e , namely periodic reference tracking and disturbance rejection is achieved for systems (1) that can be rewritten, via a change of coordinates, in the form (3), with a pure output-feedback design which uses only the regulated output e . Furthermore, the proposed design is robust with respect to model uncertainties. Indeed, the control law (7) is parametrized by some parameters. and their values depend only on the Lipschitz constants of the functions f, q and the ISS-properties α, ρ of the zero-dynamics (see Assumption 2). In other words, the exact knowledge of the function f, q characterizing system (3), and so of $\mathbf{f}, \mathbf{g}, \mathbf{h}$ of system (1), is not needed for the design of the proposed RC-scheme.

As one can see comparing equations (7b) with (4), the main difference with respect to the design proposed in Astolfi et al. [2021] is that the driving term of the transport equation is acting in a distributed fashion and not at the boundary. This, in turns, is able to guarantee some ISS properties with respect to the estimated $\hat{\theta}$. In practice, once the high-gain observer (7a) is converged, one recovers a state-feedback equation

$$\eta_t(t, x) = -\frac{1}{T} \eta_x(t, x) + \exp(\beta x) \theta$$

which in turns guarantee the signal θ (and therefore ξ_1) to converge asymptotically to zero. Such a modification is however not trivial to analyze as one has to prove that the internal-model property of (7b) is still verified, i.e. the system

$$\begin{cases} \eta_t(t, x) = -\frac{1}{T} \eta_x(t, x), & \eta(t, 0) = \eta(t, 1), \\ u_{im}^{ss} = \mu \int_0^1 \eta(t, x) M(x) dx, \end{cases} \quad (10)$$

is such that the output u_{im}^{ss} can generate any arbitrarily periodic signal.

Finally, note that for unitary systems of the form

$$\begin{cases} \dot{z} = f(t, z, e) \\ \dot{e} = q(t, z, e) + u \end{cases}$$

the proposed control law (7) reads

$$\begin{cases} \eta_t(t, x) = -\frac{1}{T} \eta_x(t, x) + \exp(\beta x) e, \\ \eta(t, 0) = \eta(t, 1), \quad \eta(0, x) = \eta_0(x), \\ u = -\kappa e + \mu \int_0^1 M(x) (\eta(t, x) - M(x) e) dx, \end{cases}$$

with M chosen as in (8).

4. PROOF OF THEOREM 1

The proof is organized as follows. First, it is shown that system (7b) possesses the so-called ‘‘internal-model property’’, i.e. system (10) can generate any periodic signal. Then, a suitable change of coordinates for the closed-loop dynamics is proposed. Existence of solution is briefly discussed and the stability and convergence analysis is finally addressed. Fundamental properties developed in the works Astolfi et al. [2021], Marx et al. [2020] and Marx et al. [2021] will be used throughout the proof.

4.1 Internal Model Property

Consider system (7b), (7c). When needed, it will be used the following equivalent operator-form (see, e.g. Tucsnak and Weiss [2009]) given by

$$\begin{cases} \dot{\eta} = \mathcal{S}\eta + \mathcal{G}\hat{\theta} \\ u = -\kappa\hat{\theta} + \mu\mathcal{M}^*(\eta - \mathcal{M}\hat{\theta}) \end{cases} \quad (11)$$

where $\mathcal{S} = -\frac{1}{T} \partial_x$, $\mathcal{G} = \exp(\beta x)$ and with \mathcal{M}^* denoting the adjoint of M so that $\mathcal{M}^* \eta = \int_0^1 \eta(t, x) M(x) dx$. The representation (11) can be obtained following for instance [Marx et al., 2021, Example 2,4].

The following results, concerning some observability and stability properties of the operators in (11), are stated.

Lemma 1. For any $\beta \neq 0$, the pair $(\mathcal{S}^, \mathcal{G}^*)$ defined in (11) is approximately observable in infinite time².*

² See, e.g., [Tucsnak and Weiss, 2009, Definition 6.1.1] for a precise definition of approximate observability.

Proof. In order to prove the desired result it suffices to check that the adjoint system, namely

$$\begin{cases} T\eta_t(t, x) + \eta_x(t, x) = 0 \\ \eta(t, 0) = \eta(t, 1) \\ y(t) = \int_0^1 \exp(\beta x) \eta(t, x) dx \end{cases}$$

is approximately observable from the output y . To this end, suppose that y is constantly equal to zero. Then, by computing an integration by parts, this implies

$$\begin{aligned} y(t) &= \int_0^1 \exp(\beta x) \eta(t, x) dx \\ &= \frac{1}{\beta} [\exp(\beta x) \eta(t, x)]_0^1 - \frac{1}{\beta} \int_0^1 \exp(\beta x) \eta_x(t, x) dx \\ &= \frac{1}{\beta} (\exp(\beta) - 1) \eta(t, 0) + \frac{T}{\beta} y(t). \end{aligned}$$

If $\beta = T$, it yields

$$0 = y(t) = \frac{1}{\beta} (\exp(\beta) - 1) \eta(t, 0) + y(t)$$

for all $t \geq 0$. Otherwise, for $\beta \neq T$,

$$0 = y(t) = \int_0^1 \exp(\beta x) \eta(t, x) dx = \frac{\exp(\beta) - 1}{\beta - T} \eta(t, 0)$$

for all $t \geq 0$. In both cases, one obtain $\eta(t, 0) = 0$ for all $t \geq 0$. Therefore, following the proof in [Marx et al., 2020, Theorem 2], the latter implies that $\eta(t, x) = 0$ for all $(t, x) \in \mathbb{R}_+ \times [0, 1]$ concluding the desired result. More details are given also in [Marx et al., 2021, Example 5]. An alternative proof consists in following [Russell, 1978, Section 4] and using an eigenfunction decomposition of the operators \mathcal{S}, \mathcal{G} . \square

Lemma 2. For any $\beta \neq 0$, the pair $(\mathcal{S}, \mathcal{M}^*)$ defined in (11) is approximately observable in infinite time.

Proof. The definition of M given in (8) is solution of the Sylvester equation

$$-\kappa \mathcal{M} = \mathcal{S} \mathcal{M} + \mathcal{G}. \quad (12)$$

Indeed, the spectrum of $-\kappa$ and \mathcal{S} is disjoint and the solution is unique. In particular, it is given by (9). Recall that by Lemma 1, the pair $(\mathcal{S}^*, \mathcal{G}^*)$ is approximately observable, the operator \mathcal{S} is skew-adjoint, and the matrix

$$\begin{pmatrix} -\kappa + \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

is full rank for any λ in the spectrum of \mathcal{S} . Hence one can apply Proposition 1 of Marx et al. [2021] to conclude the proof. \square

Lemma 3. For any $\beta \neq 0$, $\mu > 0$, the operator

$$\mathcal{A}_{cl} = \begin{pmatrix} -\kappa - \mu \mathcal{M}^* \mathcal{M} & \mathcal{M}^* \\ \mathcal{G} & \mathcal{S} \end{pmatrix}$$

generates a strongly stable C_0 semigroup.

Proof. The proof is a direct application of Theorems 1, 2 and Proposition 1 of Marx et al. [2021] and it is therefore just sketched. As noted in the previous lemma, \mathcal{M} given in (8) is solution of the Sylvester equation (12). Indeed, the spectrum of $-\kappa$ and \mathcal{S} is disjoint and the solution is unique, as given by (9). Hence, the system

$$\begin{cases} \dot{\hat{\theta}} = -\kappa \hat{\theta} + \mu \mathcal{M}^* (\eta - \mathcal{M} \hat{\theta}) \\ \dot{\eta} = \mathcal{S} \eta + \mathcal{G} \hat{\theta} \end{cases}$$

is transformed, by using the change of coordinates

$$\eta \mapsto \tilde{\eta} := \eta - \mathcal{M} \hat{\theta}$$

into a system in triangular form

$$\dot{\chi} = \tilde{\mathcal{A}}_{cl} \chi, \quad \tilde{\mathcal{A}}_{cl} := \begin{pmatrix} -\kappa & \mu \mathcal{M}^* \\ 0 & (\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*) \end{pmatrix},$$

where $\chi := (\hat{\theta}, \tilde{\eta})$. Now, by Lemma 1, the pair $(\mathcal{S}^*, \mathcal{G}^*)$ is approximately observable in infinite time and therefore, in view of Proposition 1 of Marx et al. [2021], also the pair $(\mathcal{S}, \mathcal{M}^*)$ is approximately observable. As a consequence, since the operator \mathcal{S} is skew-adjoint, the operator $(\mathcal{S} - \mu \mathcal{M} \mathcal{M}^*)$ is dissipative. Furthermore using LaSalle Invariance Principle for infinite-dimensional systems (see, e.g., Theorem 3.1 in Slemrod [1989]) and the lower triangular structure of $\tilde{\mathcal{A}}_{cl}$, one can conclude that the origin of χ is globally asymptotically stable, i.e., the operator \mathcal{A}_{cl} generates a strongly stable semigroup, concluding the proof. Technical details follow from Marx et al. [2021]. \square

Finally, similarly to [Astolfi et al., 2021, Lemma 1], the following result concerning the internal-model property of the feedback regulator (7) can be stated.

Lemma 4. For any C^2 T -periodic function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ and any $\mu, \beta \neq 0$, there exists $\bar{\eta}_0 \in L^2(0, 1)$, such that the solution to the system

$$\begin{aligned} \bar{\eta}_t(t, x) &= -\frac{1}{T} \bar{\eta}_x(t, x), & \bar{\eta}(t, 0) &= \bar{\eta}(t, 1), & \bar{\eta}(0, x) &= \bar{\eta}_0(x), \\ y(t) &= \mu \int_0^1 \bar{\eta}(t, x) M(x) dx \end{aligned} \quad (13)$$

defined on $(t, x) \in \mathbb{R}_+ \times [0, 1]$, with M defined by (8), satisfies $y(t) \equiv \psi(t)$ for all $t \in \mathbb{R}_+$.

Proof. Since the function ψ is periodic and C^2 , the signal ψ can thought as generated by an autonomous infinite-dimensional system of the form

$$\begin{aligned} w_t(t, x) - \frac{1}{T} w_x(t, x) &= 0, & w(t, 1) &= w(t, 0), \\ w(0, x) &= w_0(x), & \psi(t) &= w(t, 1), \end{aligned}$$

with $(t, x) \in \mathbb{R}_+ \times [0, 1]$ and suited initial conditions $w_0(x) \in L^2(0, 1)$. Equivalently, it is represented with the operator form

$$\dot{w} = \mathcal{S} w, \quad \psi = \mathcal{E} w.$$

Now, by using the operator-form introduced above, consider the following system

$$\begin{cases} \dot{w} = \mathcal{S} w \\ \dot{\hat{\theta}} = -\kappa \hat{\theta} - \mathcal{E} w + \mathcal{M}^* (\bar{\eta} - \mathcal{M} \hat{\theta}) \\ \dot{\bar{\eta}} = \mathcal{S} \bar{\eta} + \mathcal{G} \hat{\theta} \end{cases}$$

Let study now the existence of a solution Π_{θ}, Π_{η} to the following infinite-dimensional Sylvester equation

$$\begin{aligned} \Pi_{\theta} \mathcal{S} &= -(\kappa + \mathcal{M}^* \mathcal{M}) \Pi_{\theta} + \mathcal{M}^* \Pi_{\eta} - \mathcal{E} \\ \Pi_{\eta} \mathcal{S} &= \mathcal{S} \Pi_{\eta} + \mathcal{G} \Pi_{\theta}. \end{aligned} \quad (14)$$

Since the operator \mathcal{A}_{cl} defined in Lemma 3 generates a strongly stable semigroup, the Sylvester equation (14) is well defined, see Lemma 1 in Paunonen et al. [2008] and the existence of a solution is ensured. Furthermore, it is possible to verify that the conditions of Theorem 6 in Paunonen et al. [2008] are verified. Therefore, by Theorem 5 in Paunonen et al. [2008], one can conclude from the second equation of (14) that $\Pi_{\theta} = 0$, and thus from the first equation of (14) that $\mathcal{M}^* \Pi_{\eta} - \mathcal{E} = 0$. Selecting $\bar{\eta}(0) = \Pi_{\eta} w(0)$ and coming back in the explicit form (13) yields the desired result since $y = \mathcal{M}^* \Pi_{\eta} w = \mathcal{E} w = \psi$. \square

4.2 Change of coordinates

Given \bar{z} from Assumption 2, let $\psi(t) := -q(t, \bar{z}(t), 0, \dots, 0)$ and let $\mu > 0$ be fixed. In view of Assumptions 1 and 2, ψ is C^2 and T -periodic. Then, let $\bar{\eta}_0$ be the initial conditions given by Lemma 4 so that

$$\mu \int_0^1 \bar{\eta}(t, x) M(x) dx = \psi(t) = -q(t, \bar{z}(t), 0). \quad (15)$$

Now, consider the following changes of coordinates for the closed-loop system dynamics (3)-(7)

$$\begin{aligned} z &\mapsto \zeta := z - \bar{z}(t) \\ \xi_i &\mapsto x_i := g^{r-i} \xi_i, & i = 1, \dots, r-1, \\ \xi_r &\mapsto \theta := \xi_r + \sum_{i=1}^{r-1} \ell^{r-i} a_i \xi_i, \\ \hat{\xi}_i &\mapsto \varepsilon_i := \ell^{(r-i)} (\hat{\xi}_i - \xi_i), & i = 1, \dots, r, \\ \eta &\mapsto \phi := \eta(\cdot, x) - \bar{\eta}(\cdot, x) - M(x) \hat{\theta} \end{aligned}$$

and denote $x := (x_1, \dots, x_{r-1}) \in \mathbb{R}^{r-1}$, $\varepsilon := (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{R}^r$. First, note that $\hat{\theta}$ reads

$$\hat{\theta} = \theta + \varepsilon_r + \sum_{i=1}^{r-1} a_i \left(\frac{g}{\ell}\right)^{r-i} \varepsilon_i = \theta + \Psi \varepsilon$$

with

$$\Psi := \left(a_1 \left(\frac{g}{\ell}\right)^{r-1}, \dots, a_{r-1} \left(\frac{g}{\ell}\right), 1 \right). \quad (16)$$

The input u is then transformed into

$$u = -\kappa \theta + \mu \int_0^1 \phi(t, x) M(x) dx - (\kappa + \mu m) \Psi \varepsilon + \psi(t)$$

with $m := \int_0^1 M(x)^2 dx$. in light of (15). As a consequence, the closed-loop system dynamics reads

$$\begin{cases} \dot{\zeta} = F(t, \zeta, g^{1-r} C_{r-1} x) \\ \dot{x} = g(A_{r-1} - B_{r-1} K_a) x + g B_{r-1} \theta \\ \dot{\theta} = \Delta(t, \zeta, x, \theta) - (\kappa + \mu m) \Psi \varepsilon - \kappa \theta + y \\ \dot{\varepsilon} = \ell(A_r - K_b C_r) \varepsilon - B_r [\Delta(t, \zeta, x, \theta) + y] \\ y = \mu \int_0^1 \phi(t, x) M(x) dx \end{cases} \quad (17a)$$

where the triplet (A_i, B_i, C_i) is in prime form, i.e.,

$$A_i := \begin{pmatrix} 0_{i-1,1} & I_{i-1} \\ 0 & 0 \dots 0 \end{pmatrix}, \quad B_i := \begin{pmatrix} 0_{i-1,1} & 1 \end{pmatrix}^\top, \quad C_i := \begin{pmatrix} 1 & 0_{1 \times i-1} \end{pmatrix},$$

$K_a = (a_1, \dots, a_{r-1})$, $K_b = (b_1, \dots, b_r)$, the function F, Δ are defined as

$$F(t, \zeta, g^{1-r} C_{r-1} x) := f(t, \zeta + \bar{z}(t), g^{1-r} C_{r-1} x) - f(t, \bar{z}(t), 0),$$

$$\Delta(t, \zeta, x, \theta) :=$$

$$\begin{aligned} & q(t, \zeta + \bar{z}(t), g^{1-r} x_1, \dots, g^{-1} x_{r-1}, \theta - K_a x) \\ & - q(t, \bar{z}(t), 0) + g \sum_{i=1}^{r-2} a_i x_{i+1} + g a_{r-1} (\theta - \sum_{i=1}^{r-1} a_i x_i), \end{aligned}$$

and the ϕ -dynamics is given by

$$\begin{cases} \dot{\phi}_t(t, x) = -\frac{1}{T} \phi_x(t, x) - M(x) y \\ \quad - M(x) (\Delta(t, \zeta, x, \theta) - (\kappa + \mu m) \Psi \varepsilon) \\ \phi(t, 0) = \phi(t, 1) \\ \phi(0, x) = \phi_0(x) \end{cases} \quad (17b)$$

defined on $(t, x) \in \mathbb{R}_+ \times [0, 1]$, with

$$\phi_0(x) = \eta_0(x) - \bar{\eta}_0(x) - M(x) \hat{\theta}(0) \in L^2(0, 1).$$

Recall that the function q is globally Lipschitz by Assumption 1. Furthermore, $\Delta(t, 0, 0, 0) = 0$. As a consequence, by using the definition of the matrix Ψ in (16), there exists a $L > 0$, independent of g, ℓ, κ such that the following inequalities hold

$$\begin{aligned} |\Psi_a| &\leq L, & |(\kappa + \mu m) \Psi \varepsilon| &\leq L \kappa |\varepsilon|, \\ |\Delta(t, \zeta, x, \theta)| &\leq L (|\zeta| + |g|x| + |g|\theta|), \end{aligned} \quad (18)$$

for all $(\zeta, x, \theta, \varepsilon) \in \mathbb{R}^{n+2r}$ and for any $g \geq 1$ and $\ell \geq \max\{g, 1\}$. Finally, system (17) can be also compactly written as

$$\dot{p} = \mathcal{F}(t, p) \quad (19)$$

with $p = (\zeta, x, \theta, \varepsilon, \phi) \in \mathbb{R}^{n+2r} \times L^2(0, 1)$, and associated norm $|p| := |\zeta| + |x| + |\theta| + |\varepsilon| + |\phi|_{L^2(0,1)}$.

4.3 Existence of solutions

The following lemma concerning the existence of solutions to (17) is stated. Its proof is similar to Lemma 2 in Astolfi et al. [2021]. Details are omitted for space reasons.

Lemma 5. (Well-posedness of (19)). *For any $t_0 > 0$ and any initial conditions $p_0 \in \mathbb{R}^{n+2r} \times L^2(0, 1)$, respectively $p_0 \in \mathbb{R}^{n+2r} \times H^1(0, 1)$, with the compatibility condition $\phi_0(0) = \phi_0(1)$, there exists a sufficiently small constant $\tau > 0$ such that system (19) admits a unique solution satisfying $p \in C^0([t_0, t_0 + \tau]; \mathbb{R}^{n+2r} \times L^2(0, 1))$, respectively $p \in C^1([t_0, t_0 + \tau]; \mathbb{R}^{n+2r} \times L^2(0, 1)) \cap C^0([t_0, t_0 + \tau]; \mathbb{R}^{n+2r} \times H^1(0, 1))$.*

4.4 Stability of the origin of the closed-loop system

Once the existence of solutions has been established thanks to Lemma 5, it is possible to prove that the origin of (19) is globally asymptotically stable. To this end, consider the Lyapunov functional

$$W(t, p) = g^2 U_1(t, \zeta, x) + U_2(\theta, \varepsilon) + U_3(\phi)$$

$$U_1(t, \zeta, x) = V(t, \zeta) + x^\top P_a x,$$

$$U_2(\theta, \varepsilon) = \theta^\top P_b \varepsilon, \quad U_3(\phi) = 2\mu \int_0^1 \phi(t, x)^2 dx,$$

where the matrices P_a, P_b are defined as solution of the following Lyapunov matrix equations

$$\begin{aligned} P_a (A_{r-1} - B_{r-1} K_a) + (A_{r-1} - B_{r-1} K_a)^\top P_a &= -2I, \\ P_b (A_r - K_b C_r) + (A_r - K_b C_r)^\top P_b &= -2I, \end{aligned}$$

since the matrices $(A_{r-1} - B_{r-1} K_a)$ and $(A_r - K_b C_r)$ are Hurwitz due to the choice of the parameters a_i, b_i in the statement of the theorem. In view of Assumption 2, the function W satisfies, for all $(t, p) \in \mathbb{R}_+ \times \mathbb{R}^{n+2r} \times L^2(0, 1)$,

$$\underline{\alpha}_W(|p|) \leq W(t, p) \leq \bar{\alpha}_W(|p|), \quad (20)$$

for some class \mathcal{K}_∞ functions $\underline{\alpha}_W, \bar{\alpha}_W$. Then, compute the time derivative of U_1 . Using Assumption 2, it yields

$$\begin{aligned} \dot{U}_1 &\leq -\alpha |\zeta|^2 + \rho g^{1-r} |C_{r-1} x|^2 - 2g |x|^2 + 2g x^\top P_a B_{r-1} \theta \\ &\leq -\alpha |\zeta|^2 - (g - \rho) |x|^2 + g |P_a|^2 \theta^2 \end{aligned}$$

for all $g \geq 1$, where in the second step it is used the Young's inequality and the fact that $g^{1-r} \leq 1$ for all $g \geq 1$. Then, consider the time derivatives of U_2 . By using the Lipschitz inequalities in (18), it gives

$$\begin{aligned} \dot{U}_2 &= -2\kappa \theta^2 + 2\theta y + 2\theta \Delta(t, \zeta, x, \theta) - 2\theta (\kappa + \mu m) \Psi \varepsilon \\ &\quad - 2\ell |\varepsilon| - 2\varepsilon^\top B_r (\Delta(t, \zeta, x, \theta) + y) \\ &\leq - (2\kappa - c_1(g)) \theta^2 - (2\ell - c_2(g, \kappa)) |\varepsilon| \\ &\quad + 2y^2 + 2|\zeta|^2 + 2|x|^2 \end{aligned}$$

with $c_1(g) = 2 + 2L^2 + L^2g^2 + 2Lg$, and $c_2(g, \kappa) = 1 + \kappa^2 + L^2 + 2L^2g^2$. Finally, following similar computations in [Marx et al., 2021, Appendix], and using the property $\int_0^1 \phi_x(t, x)\phi(t, x)dx = 0$, due to the boundary conditions $\phi(t, 1) = \phi(t, 0)$, and recalling the definition of y in (17), the time-derivative of U_3 gives

$$\begin{aligned} \dot{U}_3 &= -4y^2 - 2y(\Delta(t, \zeta, x, \theta) - (\kappa + \mu m)\Psi\varepsilon) \\ &\leq -3y^2 + 2L^2(|\zeta|^2 + g^2|x|^2 + g^2|\theta|^2 + \kappa^2|\varepsilon|^2). \end{aligned}$$

Finally, by combining all previous inequalities together, the time derivative of W gives

$$\begin{aligned} \dot{W} &\leq -(g^2\alpha - 2L^2 - 2)|\zeta|^2 - (g^3 - \rho g^2 - 2L^2g^2 - 2)|x|^2 \\ &\quad - (2\kappa - c_1(g) - g^3|P_a|^2 - 2L^2g^2)\theta^2 \\ &\quad - (2\ell - c_2(g, \kappa) - 2L^2\kappa^2)|\varepsilon|^2 - y^2. \end{aligned}$$

Hence, since $c_1(g)$ depends only g and $c_2(\kappa)$ on κ , it is always possible to select g, κ and ℓ large enough so that there exists a positive number $\epsilon > 0$ such that

$$\begin{aligned} \dot{W} &\leq -\epsilon \left[|\zeta|^2 + |x|^2 + |\theta|^2 + |\varepsilon|^2 - \left(\int_0^2 \phi(t, x)M(x)dx \right)^2 \right] \\ &\leq 0. \end{aligned}$$

Asymptotic stability of $p = 0$ in the $\mathbb{R}^{n+2r} \times L^2(0, 1)$ -topology can be therefore proved following [Marx et al., 2021, Theorem 1] and in particular by using the following arguments: completeness of solutions, applying Barbalat's lemma, precompactness of solutions and LaSalle's invariance principle arguments for infinite-dimensional systems (see [Slemrod, 1989, Theorem 3.1]). The proof concludes by noting that the origin of (17) implies $x = 0$ and therefore $e = x_1 = 0$. \square

5. CONCLUSIONS

A new output-feedback RC-scheme for minimum-phase nonlinear systems with arbitrarily relative degree is proposed. This work is cast into a global context (i.e. with globally Lipschitz nonlinearities) but similar results could be proved into a semi-global framework by relaxing these assumptions. Future works envision also to enlarge the class of nonlinear systems by considering systems in the original coordinates with sector-bounded or monotonic nonlinearities.

ACKNOWLEDGEMENT

The author thanks Swann Marx and Vincent Andrieu for their precious help in preparing this work.

REFERENCES

- V. Andrieu and L. Praly. A unifying point of view on output feedback designs for global asymptotic stabilization. *Automatica*, 45(8):1789–1798, 2009.
- D. Astolfi and L. Marconi. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11):3059–3064, 2015.
- D. Astolfi, L. Praly, and L. Marconi. Francis-wonham nonlinear viewpoint in output regulation of minimum phase systems. *IFAC-PapersOnLine*, 52(16):532–537, 2019.
- D. Astolfi, S. Marx, and N. van de Wouw. Repetitive control design based on forwarding for nonlinear minimum-phase systems. *Automatica*, 129:109671, 2021.
- A.N. Atassi and H.K. Khalil. A separation principle for the stabilization of a class of nonlinear systems. *IEEE Trans. on Automatic Control*, 44(9):1672–1687, 1999.
- L. Blanken, S. Koekebakker, and T. Oomen. Multivariable repetitive control: Decentralized designs with application to continuous media flow printing. *IEEE/ASME Transactions on Mechatronics*, 2019.
- F. Califano and A. Macchelli. A stability analysis based on dissipativity of linear and nonlinear repetitive control. *IFAC-PapersOnLine*, 52(2):40–45, 2019.
- F. Califano, M. Bin, A. Macchelli, and C. Melchiorri. Stability analysis of nonlinear repetitive control schemes. *IEEE control systems letters*, 2(4):773–778, 2018.
- J. Ghosh and B. Paden. Nonlinear repetitive control. *IEEE Transactions on Automatic Control*, 45(5):949–954, 2000.
- S. Hara, Y. Yamamoto, T. Omata, and M. Nakano. Repetitive control system: A new type servo system for periodic exogenous signals. *IEEE Transactions on automatic control*, 33(7):659–668, 1988.
- E. Kurniawan, Z. Cao, O. Mahendra, and R. Wardoyo. A survey on robust repetitive control and applications. In *2014 IEEE International Conference on Control System, Computing and Engineering*, pages 524–529, 2014.
- S. Marx, L. Brivadis, and D. Astolfi. Forwarding design for stabilization of acoupled transport equation/ode with a cone-bounded input nonlinearity. In *59th IEEE Conference on Decision and Control*, 2020.
- S. Marx, L. Brivadis, and D. Astolfi. Forwarding techniques for the global stabilization of dissipative infinite-dimensional systems coupled with an ode. *Mathematics of Control, Signals and Systems*, 2021.
- P. Mattavelli and F.P. Marafao. Repetitive-based control for selective harmonic compensation in active power filters. *IEEE Transactions on Industrial Electronics*, 51(5):1018–1024, 2004.
- L. Paunonen, S. Pohjolainen, and T. Hämäläinen. On infinite-dimensional sylvester equation and the internal model principle. *Proceedings of the Mathematical Theory of Networks and Systems*, Blacksburg, VA, 2008.
- D.L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *Siam Review*, 20(4):639–739, 1978.
- M. Slemrod. Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Math. of Contr., Sign. and Syst.*, 2(3):847–857, 1989.
- A. Tanwani, C. Prieur, and S. Tarbouriech. Input-to-state stabilization in h_1 -norm for boundary controlled linear hyperbolic pdes with application to quantized control. In *IEEE 55th Conference on Decision and Control*, pages 3112–3117, 2016.
- M. Tomizuka, T.C. Tsao, and K.K. Chew. Analysis and synthesis of discrete-time repetitive controllers. *Jour. of Dyn. Systems, Measur., and Control*, 111:353–358, 1989.
- M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Springer, 2009.
- L. Wang, D. Astolfi, H. Su, L. Marconi, and A. Isidori. Output stabilization for a class of nonlinear systems via high-gain observer with limited gain power. *IFAC-PapersOnLine*, 48(11):730–735, 2015.
- G. Weiss and M. Häfele. Repetitive control of mimo systems using h^∞ design. *Automatica*, 35(7):1185–1199, 1999.