Sufficient metric conditions for synchronization of leader-connected homogeneous nonlinear multi-agent systems

Mattia Giaccagli, Vincent Andrieu, Daniele Astolfi, Giacomo Casadei

Sufficient metric conditions for synchronization of leader-connected homogeneous nonlinear multi-agent systems

Mattia Giaccagli* **Vincent Andrieu*** **Daniele Astolfi*** **Giacomo Casadei****

* *Univ Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France.*
** *Laboratoire Ampere Dpt. EEA of the École Centrale de Lyon, Université de Lyon, 69134 Ecully, France.*

Abstract: We address the exponential synchronization problem of a leader-connected network of identical nonlinear systems. We suppose that each system can be made contractive by a static feedback law obtained via sufficient metric-based conditions. A distributed diffusive coupling feedback is then designed in order to solve the problem in two frameworks: the case of input-affine systems allowed to exchange all the state information to its neighbour; and the case of nonlinear systems linear in the input allowed to exchange only a possibly nonlinear output.

Keywords: Synchronization, contraction, multi-agent systems, incremental stability.

1. INTRODUCTION

Due to the many applications in which agents interact in order to accomplish task and to achieve goals, the control community has devoted a huge attention to networks analysis and to the design of control architecture. For a network to achieve a certain task, we often require the agents to achieve an agreement and to this end, synchronization and consensus are the mathematical features to express such an agreement. For instance, this is the case for power networks Dörfler et al. [2013], heat networks Scholten et al. [2016] and robot swarms Olfati-Saber [2006].

First, researchers focused on the problem of networks of linear systems and fundamental results for linear systems can be found in Scardovi and Sepulchre [2008] for homogeneous systems and in Wieland et al. [2011] for heterogeneous networks. Nowadays, the focus of the community is placed on networks of nonlinear systems. Many approaches have been considered in the literature of homogeneous nonlinear networks: among the may results, it is worth recalling passivity conditions in Arcak [2007], dissipativity in Stan and Sepulchre [2007] and ISS in Casadei et al. [2019a,b]. One of the most popular approaches is based on high-gain type arguments (see Isidori et al. [2014], Panteley and Loría [2017]) inherited from high-gain observers theory. This technique has however well known limits which brought researchers to investigate alternative tools such as nonlinear integral control Pavlov et al. [2009, 2018] which incorporates both low and high gain arguments.

Naturally, a property associated to synchronization is incremental stability, namely the property that any two solutions of a system converge to each other (see, e.g., Lohmiller and Slotine [1998], Forni and Sepulchre [2013], Angeli [2002], Simpson-Porco and Bullo [2014], Andrieu et al. [2013] and the references therein). By enforcing this for every system in a network, we will have the systems *converge* to each other and thus achieve synchronization. More recently, authors have focused on incremental stability as a framework to solve the problem of synchronization and contraction theory as tool for the design of control laws, see, e.g., Andrieu et al. [2013], Andrieu and Tarbouriech [2019], Yin et al. [2021].

In this work we investigate metric-based conditions for exponential synchronization of identical multi-agent systems described by nonlinear dynamics in presence of a leader. With respect to notable articles in the literature such as Scardovi and Sepulchre [2008] (where the state synchronization problem via output exchange for linear systems is addressed with a dynamical regulator) and Pavlov et al. [2009] (where incremental passivity conditions are proposed for nonlinear dynamics linear in the inputs and with linear outputs), our contribution is twofold. First, we consider input affine dynamics and the agents are allowed to exchange the full-state with their neighborhoods: we generalize the results in Andrieu et al. [2018] in which the control was linear in the dynamics. Second, we consider the case in which agents are allowed to exchange only a nonlinear output with their neighborhoods while the control is linear in the state-dynamics: inspired by the observer construction given in Andrieu et al. [2020], we generalize the results in Pavlov et al. [2009] where the output was linear.

This paper is organized as follows. In Section 2, we recall some preliminaries of graph theory and contraction theory. Then, we provide the main results in Section 3 concerning the sufficient conditions to achieve synchronization via static state-feedback and static output feedback distributed control laws. An illustration is provided in Section 4. Conclusions and future perspectives are drawn in Section 5.

Notation. $\mathbb{R}$ denotes the set of real numbers and $|\cdot|$ the standard Euclidean norm. Given a C^1 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and a C^1 vector field $\zeta : \mathbb{R}^n \mapsto \mathbb{R}^n$, we define the Lie derivative of P along ζ as

$$\mathcal{L}_\zeta P(x) := \mathfrak{d}_\zeta P(x) + P(x)\frac{\partial \zeta}{\partial x}(x) + \frac{\partial \zeta}{\partial x}(x)^\top P(x)$$

where

$$\mathfrak{d}_\zeta P(x) := \lim_{h\to 0} \frac{P(x+h\zeta(x)) - P(x)}{h}$$

for all $x \in \mathbb{R}^n$.

2. PRELIMINARIES

2.1 Graph Theory

In a general framework, a communication graph is described by a triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ in which $\mathcal{V}$ is a set of N *nodes* $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is a set of *edges* e_{jk} that models the interconnection between nodes with the flow of information from node j to node k *weighted* by the (k, j)-th entry $a_{kj} \geq 0$ of the *adjacency matrix* $A \in \mathbb{R}^{N\times N}$. We denote by $L \in \mathbb{R}^{N\times N}$ the *Laplacian matrix* of the graph, defined as

$$\ell_{kj} = -a_{kj} \text{ for } k \neq j, \qquad \ell_{kj} = \sum_{i=1}^{N} a_{ki} \text{ for } k = j,$$

where $\ell_{j,k}$ is the (j, k)-th entry of L.

In this work we will consider a network of N homogeneous multi-agent systems, i.e. described by identical dynamics, which are connected according to a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$, fulfilling the following assumption.

Assumption 1. The graph $\mathcal{G}$ is leader connected, namely it contains at least one spanning tree with the leader as a root. Furthermore, its Laplacian L is diagonalizable.

As a consequence, according to Assumption 1, the Laplacian matrix L can be partitioned as

$$L = \begin{pmatrix} 0 & 0 \\ L_{21} & L_{22} \end{pmatrix} \tag{1}$$

where the matrix L_{22} is a positive definite matrix satisfying (see Godsil and Royle [2001])

$$L_{22} \geq \mu I \ , \tag{2}$$

for some $\mu > 0$ satisfying $\mu \leq \underline{\lambda}(L_{22})$, where $\underline{\lambda}(L_{22})$ denotes the smallest eigenvalue of L_{22}.

2.2 Metric conditions for contraction

Consider a system

$$\dot{x} = f(x) \tag{3}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a C^1 vector field. We indicate with $\mathcal{X}(x^\circ, t)$ the trajectory of the system with initial condition $x^\circ \in \mathbb{R}^n$ evaluated at time t.

Definition 1. We say that system (3) *is Incrementally Globally Exponentially Stable (δGES in short) if there exist positive real numbers k, c such that*

$$|\mathcal{X}(x_1, t) - \mathcal{X}(x_2, t)| \leq k|x_1 - x_2| \exp(-ct)$$

for all $t \geq 0$ in the time domain of existence of the solution and for all initial conditions $x_1, x_2 \in \mathbb{R}^n$.

A sufficient condition for (3) to be δGES, is the existence of a metric for with the associated distance is decreasing along trajectories, as stated in the following proposition. For a proof, see Theorem 1 in Giaccagli et al. [2020] and references therein.

Proposition 1. Consider system (3) *and assume there exists a symmetric and positive definite matrix function $P : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, and positive real numbers $\underline{p}, \bar{p}, q > 0$ satisfying the following*

$$\underline{p}I \leq P(x) \leq \bar{p}I$$
$$\mathcal{L}_f P(x) \leq -qP(x)$$

for all $x \in \mathbb{R}^n$. Then, system (3) *is δGES.*

To conclude the section, we recall the definition of Killing vector field.

Definition 2. Consider a C^1 functions $P : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, and $g : \mathbb{R}^n \to \mathbb{R}^n$. We say that g is a Killing vector field for P if $\mathcal{L}_g P(x) = 0$ for all $x \in \mathbb{R}^n$.

Note that, if both P and g are constant matrices, the Killing vector field conditions is trivially satisfied.

3. MAIN RESULTS

3.1 Synchronization via State-feedback

In this section we consider a network of N identical agents described by the following dynamics

$$\dot{x}_i = f(x_i) + g(x_i)u_i \qquad i = 1, \ldots, N, \tag{4}$$

in which $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}$ is the control input. We denote

$$\mathbf{x} := \mathrm{col}(x_1, \ldots, x_N) \in \mathbb{R}^{Nn}.$$

Our objective is to design a distributed control law stabilizing the dynamics (4) on the synchronization manifold $\mathcal{D}$ defined by

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^{Nn} \mid x_1 = x_2 = \cdots = x_N\}, \tag{5}$$

where the state variables of different systems agree with each other. For every $\mathbf{x}$ in $\mathbb{R}^{Nn}$, we denote the Euclidean distance to the set $\mathcal{D}$ by $|\mathbf{x}|_\mathcal{D}$. We define our synchronization problem as follows.

Problem 1. The control laws $u_i = \phi_i(\mathbf{x})$, $i = 1\ldots, N$ solve the uniform exponential synchronization *problem for* (4) *if the following conditions hold:*

(1) For all non-communicating pair (i, j) (i.e., $(i, j) \notin \mathcal{E}$),

$$\frac{\partial \phi_i}{\partial x_j}(\mathbf{x}) = \frac{\partial \phi_j}{\partial x_i}(\mathbf{x}) = 0 \ , \ \forall \mathbf{x} \in \mathbb{R}^{Nn}.$$

(2) For all $\mathbf{x} \in \mathcal{D}$, $\phi_i(\mathbf{x}) = 0$ (i.e., ϕ_i is zero on $\mathcal{D}$).

(3) The manifold $\mathcal{D}$ of the closed-loop system

$$\dot{x}_i = f(x_i) + g(x_i)\phi_i(\mathbf{x}), \quad i = 1, \ldots, N$$

is uniformly exponentially stable, i.e., there exist positive constants k and $c > 0$ such that for all $\mathbf{x}^\circ$ in $\mathbb{R}^{Nn}$ and for all t in the time domain of existence of solution

$$|\mathcal{X}(\mathbf{x}^\circ, t)|_\mathcal{D} \leq k \exp(-ct)\, |\mathbf{x}^\circ|_\mathcal{D}, \tag{6}$$

where $\mathcal{X}(\mathbf{x}^\circ, t)$ denotes the solution of (4) *initiated from $\mathbf{x}^\circ$.*

We can now state the first main result of this work.

Theorem 1. Suppose Assumption 1 holds and assume that there exists a C^1 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ taking symmetric positive definite values such that the following conditions hold.

- *The Control Matrix Function (CMF) condition holds:*
$$\begin{aligned} &\mathcal{L}_f P(x) - \rho P(x)g(x)g(x)^\top P(x) \leq -qP(x)\ , \\ &\underline{p}\,\mathrm{I} \leq P(x) \leq \overline{p}\,\mathrm{I}\ , \end{aligned} \tag{7}$$
for all $x \in \mathbb{R}^n$ and for some positive real numbers $\underline{p}, \overline{p}, q, \rho > 0$.
- *g is a Killing vector field for P, i.e.,*
$$\mathcal{L}_g P(x) = 0\,, \qquad \forall x \in \mathbb{R}^n\ . \tag{8}$$
- *There exists $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that*
$$\frac{\partial \alpha}{\partial x}(x)^\top = P(x)g(x)\,, \qquad \forall x \in \mathbb{R}^n\ . \tag{9}$$

Then, the distributed control law $u_i = \phi_i(\mathbf{x})$ with
$$\phi_i(\mathbf{x}) = -\kappa \sum_{j=1}^{N} \ell_{ij}\alpha(x_j)\,, \tag{10}$$
solves the global uniform exponential synchronization problem for (4), for any $\kappa \geq \frac{\rho}{\mu}$, with ρ satisfying (7) and μ satisfying (2).

Proof. Consider, without loss of generality, that x_1 is the leader node in the network,

i.e., its dynamics is given by $\dot{x}_1 = f(x_1)$. Define $n-1$ error coordinates $e := (e_2, \ldots, e_N)$ with $e_i := x_i - x_1$ and $z = x_1$. These errors display the following dynamics
$$\dot{e}_i \;=\; f(z+e_i) - f(z) - \kappa\, \left[\sum_{j=1}^{N} \ell_{ij} g(z+e_i)\alpha(z+e_j)\right].$$
Using the fact that
$$\sum_{j=1}^{N} \ell_{ij} g(z+e_i)\alpha(z) = g(z+e_i)\alpha(z)\sum_{j=1}^{N}\ell_{ij} = 0\,,$$
such a term can be added to the e_i dynamics to obtain
$$\begin{aligned} \dot{e}_i \;=&\; f(z+e_i) - f(z) \\ &-\kappa\sum_{j=1}^{N}\ell_{ij} g(z+e_i)\big[\alpha(z+e_j)-\alpha(z)\big] \end{aligned}$$
for all $i = 2, \ldots, N$. Note that in this new error coordinates, the manifold $\mathcal{D}$ defined in (5) corresponds simply to the origin of the e-dynamics, i.e.
$$\mathcal{D} = \{(z,e) \in \mathbb{R}^{Nn} : e = 0\}.$$
Consider the function $\Gamma(s,t)$ defined as
$$\Gamma(1,0) = e\,, \quad \Gamma(0,0) = 0\,, \quad \Gamma(s,0) = \gamma(s)\,,$$
where γ is any C^1 with its i-th component Γ_i defined as solution of the following ordinary differential equation
$$\begin{aligned} &\frac{\partial \Gamma_i}{\partial t}(s,t) = f(Z(t)+\Gamma_i(s,t)) - f(Z(t)) \\ &-\kappa\sum_{j=1}^{N}\ell_{ij} g(Z(t)+\Gamma_i(s,t))(\alpha(Z(t)+\Gamma_j(s,t))-\alpha(Z(t))). \end{aligned}$$
where $Z(t)$ is the solution at time t of $\dot{z} = f(z)$ initiated from z. For each path γ, we consider the function V_i, $i = 2, \ldots, N$, defined by
$$V_i(t) = \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s,t)^\top P(Z(z,t)+\Gamma_i(s,t))\frac{\partial \Gamma_i}{\partial s}(s,t)ds\ .$$
Note that we have for all (k,l) in $\{1,\ldots,n\}^2$
$$\begin{aligned} &\frac{d}{dt}(P(Z(t)+\Gamma_i(s,t))_{kl} = \\ &\qquad \frac{\partial P_{kl}}{\partial z}(Z(t)+\Gamma_i(s,t))\left[f(Z(t)+\frac{\partial \Gamma_i}{\partial t}(s,t)\right]. \end{aligned}$$
This implies for all vector ν in $\mathbb{R}^n$
$$\begin{aligned} \frac{d}{dt}\nu^\top P(Z(t)+\Gamma_i(s,t))v &= \nu^\top \mathfrak{d}_f P(Z(t)+\Gamma_i(t))\nu \\ &- \kappa(\alpha(Z(t)+\Gamma_j(s,t)) - \alpha(Z(t))) \\ &\times \sum_{j=1}^{N}\ell_{ij}\nu^\top \mathfrak{d}_g P(Z(t)+\Gamma_i(t))\nu \end{aligned}$$
By using the Killing vector field (8) and the integrability assumption (9), the time-derivative of V_i can be computed as follows
$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s,t)^\top \mathcal{L}_f P(Z(t)+\Gamma_i(s,t))\frac{\partial \Gamma_i}{\partial s}(s,t) \\ &- \kappa\frac{\partial \Gamma_i}{\partial s}(s,t)^\top \sum_{j=1}^{N}\ell_{ij} P(Z(t)+\Gamma_i(s,t))g(Z(t)+\Gamma_i(s,t)) \\ &\times g(Z(t)+\Gamma_j(s,t))^\top P(Z(t)+\Gamma_j(s,t))\frac{\partial \Gamma_j}{\partial s}(s,t)ds. \end{aligned}$$
Note that
$$\begin{aligned} &\sum_{i=2}^{N}\dot{V}_i(t) = -\kappa v(t)^\top L_{22} v(t) \\ &\int_0^1 \sum_{i=2}^{N}\frac{\partial \Gamma_i}{\partial s}(s,t)^\top \mathcal{L}_f P(Z(t)+\Gamma_i(s,t))\frac{\partial \Gamma_i}{\partial s}(s,t)ds, \end{aligned}$$
where v is defined as
$$v(t) = \begin{bmatrix} g(Z(t)+\Gamma_2(s,t))^\top P(Z(t)+\Gamma_2(s,t))\frac{\partial \Gamma_2}{\partial s}(s,t)) \\ \vdots \\ g(Z(t)+\Gamma_N(s,t))^\top P(Z(t)+\Gamma_N(s,t))\frac{\partial \Gamma_N}{\partial s}(s,t)) \end{bmatrix}.$$
Hence, by using (2), we get,
$$\begin{aligned} &\sum_{i=2}^{N}\dot{V}_i(t) \leq -\mu\kappa v(t)^\top v(t) \\ &\qquad\int_0^1 \sum_{i=2}^{N}\frac{\partial \Gamma_i}{\partial s}(s,t)^\top \mathcal{L}_f P(z+\Gamma_i(s,t)\frac{\partial \Gamma_i}{\partial s}(s,t)ds, \end{aligned}$$
which implies
$$\begin{aligned} \sum_{i=2}^{N}\dot{V}_i(t) &\leq \sum_{i=2}^{N}\int_0^1 \frac{\partial \Gamma_i}{\partial s}(s,t)^\top[\mathcal{L}_f P(z+\Gamma_i(s,t)) \\ &- \mu\kappa P(Z(t)+\Gamma_i(s,t))g(Z(t)+\Gamma_i(s,t))^\top \\ &g(Z(t)+\Gamma_i(s,t))P(Z(t)+\Gamma_i(s,t))]\frac{\partial \Gamma_i}{\partial s}(s,t)ds\ . \end{aligned}$$
Selecting any $\kappa \geq \frac{\rho}{\mu}$ and employing the CMF condition (7), we obtain
$$\sum_{i=2}^{N}\dot{V}_i(t) \leq -q\sum_{i=2}^{N}V_i(t)$$

and therefore

$$\sum_{i=2}^{N} V_i(t) \leq \exp(-qt) \sum_{i=2}^{N} V_i(0) \ .$$

Note that

$$\sum_{i=2}^{N} V_i(t) \geq \underline{p} \sum_{i=2}^{n} e_i(t)^2$$

which yields exponential convergence to $\mathcal{D}$. Furthermore, to obtain inequality (6), we follow [Andrieu et al., 2020, Proposition 4] and consider a sequence of minimizing paths $(\gamma_k)_{k\in\mathbb{N}}$ such that $\gamma_k(1) = e$, $\gamma_k(0) = 0$. This concludes the proof. □

Remark 1. In the case of a network of linear systems of the form

$$\dot{x}_i = Ax_i + Bu_i,$$

the conditions of Theorem 1 reduces to the following Riccati-like inequality $PA^\top + A^\top P - \rho PBB^\top P \leq -qP$, with the function $\alpha(x) := B^\top Px$, and the Killing vector condition (8) automatically satisfied. In other words, we find the conditions in [Li et al., 2009, Section II.C]. Unfortunately, unlike the linear case, the extension of the proposed conditions to the non-leader case are non-trivial due to the presence of the state-dependent metric P. When P is constant, state-synchronization without leader can be achieved by following for instance Zhang et al. [2014], Andrieu et al. [2018] or Andrieu and Tarbouriech [2019].

3.2 Synchronization via Output-Feedback

We suppose now that the dynamics of the agents are described by

$$\begin{aligned} \dot{x}_i &= f(x_i) + u_i \qquad i = 1, \dots, N, \\ y_i &= h(x_i), \end{aligned} \tag{11}$$

in which $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^n$ is the control input, and $y_i \in \mathbb{R}$ is the exchange output. We denote

$$\mathbf{x} := \mathrm{col}(x_1, \dots, x_N) \in \mathbb{R}^{Nn}, \quad \mathbf{y} := \mathrm{col}(y_1, \dots, y_N) \in \mathbb{R}^n.$$

Similarly to the previous section, our objective is to design a distributed control law stabilizing the synchronization manifold $\mathcal{D}$ defined in (5). However, we restrict in this section to the case of output synchronization laws, i.e. the agents cannot exchange with their neighbors the full state information x_i but only the output y_i. In particular, we define such a problem as follows.

Problem 2. The control laws $u_i = \phi_i(x_i, \mathbf{y})$, $i = 1 \dots, N$ solve the uniform exponential synchronization *problem for* (11) *if the following conditions hold:*

(1) For all non-communicating pair (i, j) (i.e., $(i, j) \notin \mathcal{E}$),

$$\frac{\partial \phi_i}{\partial y_j}(x_i, \mathbf{y}) = \frac{\partial \phi_j}{\partial y_i}(x_j, \mathbf{y}) = 0 \ , \ \forall x_i \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^N.$$

(2) For all $X \in \mathcal{D}$, $\phi(x_i, \mathbf{y}) = 0$ (i.e., ϕ is zero on $\mathcal{D}$).

(3) The manifold $\mathcal{D}$ of the closed-loop system

$$\dot{x}_i = f(x_i) + \phi_i(x_i, \mathbf{y}), \quad i = 1, \dots, N,$$

is uniformly exponentially stable, i.e., there exist positive constants k and $\lambda > 0$ such that for all $X^\circ \in \mathbb{R}^{Nn}$

$$|\mathcal{X}(\mathbf{x}^\circ, t)|_{\mathcal{D}} \leq k \exp(-ct) \, |\mathbf{x}^\circ|_{\mathcal{D}},$$

where $\mathcal{X}(\mathbf{x}^\circ, t)$ denotes the solution initiated from $\mathbf{x}^\circ$, holds for all t in the time domain of existence of solution.

For a leader-connected network of agents described by (11), we have the following set of sufficient conditions to solve the previous problem.

Theorem 2. Suppose Assumption 1 holds and suppose that there exists a C^1 function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ taking symmetric positive definite values such that the following conditions hold.

- *The Control Matrix Function (CMF) condition holds:*

$$\begin{aligned} &\mathcal{L}_f P(x) - \rho \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \leq -qP(x) \ , \\ &\underline{p}\,\mathrm{I} \leq P(x) \leq \overline{p}\,\mathrm{I} \ , \end{aligned} \tag{12}$$

  for all $x \in \mathbb{R}^n$ and for some positive constants $\underline{p}, \overline{p}, q, \rho$.

- *The vector field $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined as*

$$\alpha(x) = P^{-1}(x) \frac{\partial h}{\partial x}(x)^\top \tag{13}$$

  is a Killing vector for P, i.e.

$$\mathcal{L}_\alpha P(x) = 0 \qquad \forall x \in \mathbb{R}^n. \tag{14}$$

Then the distributed control law $u_i = \phi_i(x_i, \mathbf{y})$ with

$$\phi_i(x_i, \mathbf{y}) = -\kappa \sum_{j=1}^{N} \ell_{ij} \alpha(x_i) y_j \tag{15}$$

solves the global uniform exponential synchronization problem for system (11) *for any $\kappa \geq \frac{\rho}{\mu}$, with ρ satisfying* (12) *and μ satisfying* (2).

Proof. Consider, without loss of generality, that x_1 is the leader node in the network. We define $n - 1$ error coordinates $e := (e_2, \dots, e_N)$ with $e_i := x_i - x_1$ and $z := x_1$. The dynamics of these errors reads

$$\begin{aligned} \dot{e}_i &= f(e_i + z) - f(z) \\ &\quad - \kappa \sum_{j=1}^{N} \ell_{ij} \alpha(e_i + z)(h(e_j + z) - h(z)) \end{aligned}$$

for all $i = 2, \dots, N$. Consider now the function $\Gamma(s, t)$ defined as

$$\Gamma(1, 0) = e \ , \ \Gamma(0, 0) = 0 \ , \ \Gamma(s, 0) = \gamma(s)$$

where γ is any C^1 path with its i-th component Γ_i defined as solution of the following ordinary differential equation

$$\begin{aligned} &\frac{\partial \Gamma_i}{\partial t}(s, t) = f(Z(t) + \Gamma_i(s, t)) - f(Z(t)) \\ &- \kappa \sum_{j=1}^{N} \ell_{ij} \alpha(\Gamma_i(s, t) + Z(t))(h(\Gamma_j(s, t) + Z(t)) - h(Z(t))). \end{aligned}$$

Consider the function V_i, $i = 2, \dots, N$, defined by

$$V_i(t) = \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s, t)^\top P(Z(t) + \Gamma_i(s, t)) \frac{\partial \Gamma_i}{\partial s}(s, t) ds \ .$$

Employing the Killing vector field assumption (14) and the definition of α in (13), we compute the time-derivative of V_i as

$$\begin{aligned} \dot{V}_i(t) = \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s}(s, t) \Big[&\mathcal{L}_f P(Z(t) + \Gamma_i(s, t)) \\ &- \kappa \sum_{j=1}^{N} \ell_{ij} \frac{\partial h^\top}{\partial z}(Z(t)) \frac{\partial h}{\partial z}(Z(t)) \Big] \frac{\partial \Gamma_i^\top}{\partial s}(s, t) ds \end{aligned}$$

Hence

$$\sum_{i=2}^{N} \dot{V}_i(t) = \int_0^1 \sum_{i=2}^{N} \frac{\partial \Gamma_i^\top}{\partial s}(s,t)\Big[\mathcal{L}_f P(Z(t)+\Gamma_i(s,t)) - \kappa \sum_{j=1}^{N} \ell_{ij} \frac{\partial h^\top}{\partial z}(Z(t))\frac{\partial h}{\partial z}(Z(t))\Big]\frac{\partial \Gamma_i^\top}{\partial s}(s,t)ds$$

$$= \int_0^1 \sum_{i=2}^{N} \frac{\partial \Gamma_i^\top}{\partial s}(s,t)\mathcal{L}_f P(Z(t)+\Gamma_i(s,t))\frac{\partial \Gamma_i}{\partial s}(s,t) - \kappa\mu v(t)^\top v(t)\ ds$$

where we defined

$$v(t) = \begin{bmatrix} \frac{\partial h}{\partial z}(Z(t))\frac{\partial \Gamma_2}{\partial s}(s,t) \\ \vdots \\ \frac{\partial h}{\partial z}(Z(t))\frac{\partial \Gamma_N}{\partial s}(s,t) \end{bmatrix}.$$

Consequently, we obtain

$$\sum_{i=2}^{N} \dot{V}_i(t) \le -q \sum_{i=2}^{N} V_i(t)$$

for any $\kappa \ge \frac{\varrho}{\mu}$. Finally, since

$$\sum_{i=2}^{N} V_i(t) \ge \underline{p} \sum_{i=2}^{n} e_i(t)^2\ ,$$

it yields the result following the same arguments of the proof of Theorem 1. □

Remark 2. In the case of a network of linear systems of the form

$$\dot{x}_i = Ax_i + u_i, \qquad y_i = Cx_i$$

the conditions of Theorem 1 reduces to the following Riccati-like inequality $PA^\top + A^\top P - \rho C^\top C \le -qP$ with the function $\alpha(x) := P^{-1}C^\top$, and the Killing vector condition (8) automatically satisfied. In other words, we find the conditions in [Li et al., 2009, Section II.B].

Remark 3. The conditions of Theorem 2 are inspired by the conditions for observer design based on Riemannian distances, see, e.g., Sanfelice and Praly [2012], Andrieu et al. [2018]. Furthermore, for a constant metric P and systems dynamics (11) in the so-called prime form (or observability canonical form), we recover the conditions of Proposition 1 in Isidori et al. [2014], based on high-gain observer theory.

4. ILLUSTRATION

Consider the multiagent network system with 5 nodes described in Figure 1 and represented by the Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix},$$

with $\mu = 0.18$. Each agent dynamics is described by (11) with $x_i \in \mathbb{R}^2$, and the functions f, h defined as

$$f(z) = \begin{pmatrix} z_2 + \epsilon \sin(z_1) \\ -z_1 + \epsilon z_2 \cos(z_i) + \epsilon^2 \cos(z_1)\sin(z_1) \end{pmatrix}, \qquad (16)$$
$$h(z) = (z_2 + \epsilon \sin(z_1))$$

for any $z = (z_1, z_2) \in \mathbb{R}^2$ and for some $0 < \epsilon < 1$. It can be verified that the conditions (12) of Theorem 2 are satisfied with the metric P defined as

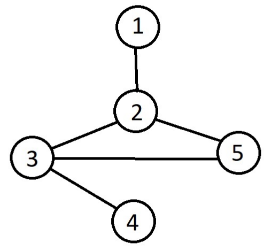

Fig. 1. Network structure

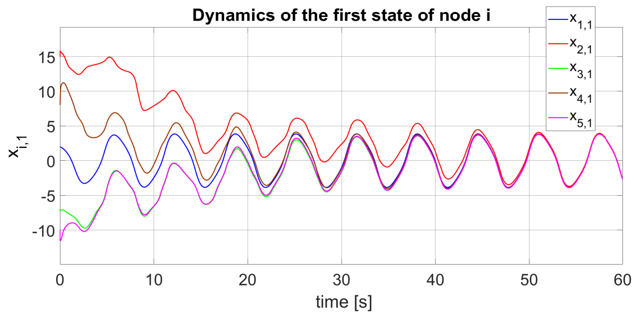

Fig. 2. Dynamics of the first state of node i

$$P(z) = \begin{pmatrix} 2[1-\epsilon\cos(z_1)+\epsilon^2\cos^2(z_1)] & 1-2\epsilon\cos(z_1) \\ 1-2\epsilon\cos(z_1) & 2 \end{pmatrix},$$

and constants $\underline{p} = 1, \bar{p} = 3, q = 3$, any $\varrho \ge 2$. Moreover, the Killing vector property (14) is verified with the function α defined as

$$\alpha(z) = \frac{1}{3}\begin{pmatrix} -1+4\epsilon\cos(z_1) \\ 4\epsilon^2\cos^2(z_1) - 3\epsilon\cos(z_1) + 2 \end{pmatrix}.$$

Therefore, all conditions of Theorem 2 are satisfied and the control law (15) is able to achieve state synchronization of the network with $\kappa \ge \frac{\varrho}{\mu} = 11.12$. The rate of convergence c in item (3) of Problem 2 is approximately computed as $c = q\mu\sqrt{\underline{p}/\bar{p}}$, corresponding to $c = 0.18\sqrt{3}$ in our example.

Figures 2, 3 show respectively the trajectories of the first and second state of each node $i = 1, \ldots, 5$ of the system (11), (16) where $\epsilon = \frac{1}{2}$ and with the control law (15) with $\kappa = 12$ and initial condition $\mathbf{x}(0) = [[(2,-1),(15,5),(-7,16),(8,14),(-10,-16)]]$. After the transient behaviour of the first seconds of the simulation, state-synchronization is achieved to a unique oscillating mode, corresponding to the first node of the graph.

5. CONCLUSION

We proposed a set of sufficient conditions to solve the problem of state synchronization for a leader-connected network of homogeneous agents described by a nonlinear dynamics into two scenarios: the case of input-affine nonlinear systems in which the agents exchange the full state to their neighbours, and the case of nonlinear systems dynamics linear in the inputs in which the agents are allowed to exchange only a certain nonlinear output. The conditions are based on a metric analysis and guarantees exponential synchronization of the network.

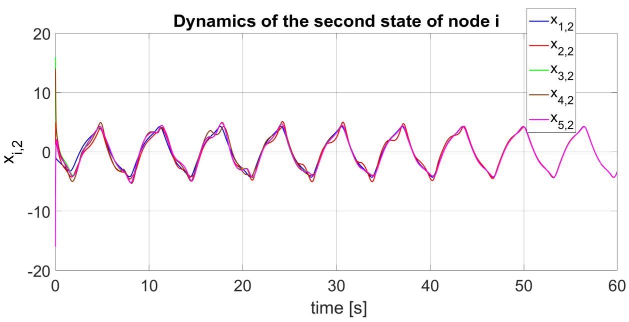

Fig. 3. Dynamics of the second state of node i

This paper is a preliminary work in order to extend of the results in Scardovi and Sepulchre [2008] and in Pavlov et al. [2009] to the case of input-affine nonlinear systems with nonlinear outputs. Future works will consider also the case of non-leader connected networks and potentially heterogeneous dynamics as in Isidori et al. [2014].

ACKNOWLEDGMENTS

The authors would like to thank Fulvio Forni for many fruitful discussions.

REFERENCES

V. Andrieu and S. Tarbouriech. LMI conditions for contraction and synchronization. *IFAC-PapersOnLine*, 52(16):616–621, 2019.

V. Andrieu, B. Jayawardhana, and L. Praly. On transverse exponential stability and its use in incremental stability, observer and synchronization. In *52nd IEEE Conference on Decision and Control*, pages 5915–5920, 2013.

V. Andrieu, B. Jayawardhana, and S. Tarbouriech. Some results on exponential synchronization of nonlinear systems. *IEEE Transactions on Automatic Control*, 63(4): 1213–1219, 2018.

V. Andrieu, B. Jayawardhana, and L. Praly. Characterizations of global transversal exponential stability. *IEEE Transactions on Automatic Control*, 2020.

D. Angeli. A lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8): 1380–1390, 2007.

G. Casadei, D. Astolfi, A. Alessandri, and L. Zaccarian. Synchronization in networks of identical nonlinear systems via dynamic dead zones. *IEEE Control Systems Letters*, 3(3):667–672, 2019a.

G. Casadei, D. Astolfi, A. Alessandri, and L. Zaccarian. Synchronization of interconnected linear systems via dynamic saturation redesign. *IFAC-PapersOnLine*, 52 (16):622–627, 2019b.

F. Dörfler, M. Chertkov, and F. Bullo. Synchronization in complex oscillator networks and smart grids. *Proceedings of the National Academy of Sciences*, 110(6): 2005–2010, 2013.

F. Forni and R. Sepulchre. A differential lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3):614–628, 2013.

M. Giaccagli, D. Astolfi, V. Andrieu, and L. Marconi. Sufficient conditions for output reference tracking for nonlinear systems: a contractive approach. In *59th IEEE Conference on Decision and Control*, pages 4580–4585, 2020.

C. Godsil and G. Royle. *Algebraic graph theory*. Springer, 2001.

A. Isidori, L. Marconi, and G. Casadei. Robust output synchronization of a network of heterogeneous nonlinear agents via nonlinear regulation theory. *IEEE Transactions on Automatic Control*, 59(10):2680–2691, 2014.

Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1):213–224, 2009.

W. Lohmiller and J.E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

R. Olfati-Saber. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3):401–420, 2006.

E. Panteley and A. Loría. Synchronization and dynamic consensus of heterogeneous networked systems. *IEEE Transactions on Automatic Control*, 62(8):3758–3773, 2017.

A. Pavlov, E. Steur, and N. van de Wouw. Controlled synchronization via nonlinear integral coupling. In *48h IEEE Conference on Decision and Control held jointly with 28th Chinese Control Conference*, pages 5263–5268, 2009.

A. Pavlov, A.V. Proskurnikov, E. Steur, and N. van de Wouw. Synchronization of networked oscillators under nonlinear integral coupling. *IFAC-PapersOnLine*, 51 (33):56–61, 2018.

R. Sanfelice and L. Praly. Convergence of nonlinear observers on $\mathbb{R}^n$ with a riemannian metric (part i). *IEEE Transactions on Automatic Control*, 57(7):1709–1722, 2012.

L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. In *47th IEEE Conference on Decision and Control*, pages 546–551, 2008.

T. Scholten, C. De Persis, and P. Tesi. Modeling and control of heat networks with storage: the single-producer multiple-consumer case. *IEEE Transactions on Control Systems Technology*, 25(2):414–428, 2016.

J.W. Simpson-Porco and F. Bullo. Contraction theory on riemannian manifolds. *Systems & Control Letters*, 65: 74–80, 2014.

G.B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256–270, 2007.

P. Wieland, R. Sepulchre, and F. Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068–1074, 2011.

H. Yin, B. Jayawardhana, and R. Reyes-Báez. Pinning synchronization of heterogeneous multi-agent nonlinear systems via contraction analysis. *IEEE Control Systems Letters*, 2021.

F. Zhang, H. Trentelman, and J. Scherpen. Fully distributed robust synchronization of networked lur'e systems with incremental nonlinearities. *Automatica*, 50 (10):2515–2526, 2014.