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Beyond Helly graphs: the diameter problem on absolute retracts ^{*}

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Abstract. A subgraph H of a graph G is called a retract of G if it is the image of some idempotent endomorphism of G . We say that H is an absolute retract of some graph class \mathcal{C} if it is a retract of any $G \in \mathcal{C}$ of which it is an isochromatic and isometric subgraph. In this paper, we study the complexity of computing the diameter within the absolute retracts of various hereditary graph classes. First, we show how to compute the diameter within absolute retracts of bipartite graphs in randomized $\tilde{\mathcal{O}}(m\sqrt{n})$ time. Even on the proper subclass of cube-free modular graphs it is, to our best knowledge, the first subquadratic-time algorithm for diameter computation. For the special case of chordal bipartite graphs, it can be improved to linear time, and the algorithm even computes all the eccentricities. Then, we generalize these results to the absolute retracts of k -chromatic graphs, for every $k \geq 3$. Finally, we study the diameter problem within the absolute retracts of planar graphs and split graphs.

Keywords: absolute retract · chordal bipartite graphs · split graphs · planar graphs · diameter computation.

1 Introduction

One of the most basic graph properties is the diameter of a graph (maximum number of edges on a shortest path). It is a rough estimate of the maximum delay in order to send a message in a communication network [32], but it also got used in the literature for various other purposes [2, 74]. The complexity of computing the diameter has received tremendous attention in the Graph Theory community [1, 14, 18, 20, 26, 25, 24, 29–31, 34, 43–45, 41, 47, 49, 65]. Indeed, while this can be done in $\mathcal{O}(nm)$ time for any n -vertex m -edge graph, via a simple reduction to breadth-first search, breaking this quadratic barrier (in the size $n + m$ of the input) happens to be a challenging task. In fact, under plausible complexity assumptions such as the Strong Exponential-Time Hypothesis (SETH), the optimal running time for computing the diameter is essentially in

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$\mathcal{O}(nm)$ — up to sub-polynomial factors [71]. This negative result holds even if we restrict ourselves to bipartite graphs or split graphs [1, 13]. However, on the positive side, several recent works have identified important graph classes for which we can achieve for the diameter problem $\mathcal{O}(m^{2-\epsilon})$ time, or even better $\mathcal{O}(mn^{1-\epsilon})$ time, for some $\epsilon > 0$. Next, we focus on a few such classes that are most relevant to our work. Specifically, we call $G = (V, E)$ a Helly graph if every family of pairwise intersecting balls of G (of arbitrary radius and center) have a nonempty common intersection. The Helly graphs are a broad generalization of many better-known graph classes in Structural Graph Theory, such as: trees, interval graphs, strongly chordal graphs and dually chordal graphs [4]. Furthermore, a celebrated theorem in Metric Graph Theory is that every graph is an isometric (distance-preserving) subgraph of some Helly graph [40, 57]. Other properties of Helly graphs were also thoroughly investigated in prior works [7, 8, 10, 23, 35–38, 63, 69, 70]. In particular, as far as we are concerned here, there is a randomized $\tilde{\mathcal{O}}(m\sqrt{n})$ -time algorithm in order to compute the diameter within n -vertex m -edge Helly graphs [43].

Recall that an endomorphism of a graph G is an edge-preserving mapping of G to itself. A retraction is an idempotent endomorphism. If H is the image of G by some retraction (in particular, H is a subgraph of G) then, we call H a retract of G . The notion of retract has applications in some discrete facility location problems [56], and it is useful in characterizing some important graph classes. For instance, the median graphs are exactly the retracts of hypercubes [3]. We here focus on the relation between retracts and Helly graphs, that is as follows (for other classes related to the Helly graphs and considered recently, see [42, 21, 39, 28, 15, 45]). For some class \mathcal{C} of reflexive graphs (*i.e.*, with a loop at every vertex), let us define the absolute retracts of \mathcal{C} as those H such that, whenever H is an isometric subgraph of some $G \in \mathcal{C}$, H is a retract of G . Absolute retracts find their root in Geometry, where they got studied for various metric spaces [60]. In the special case of the class of all reflexive graphs, the absolute retracts are exactly the Helly (reflexive) graphs [55]. Motivated by this characterization of Helly graphs, and the results obtained in [43] for the diameter problem on this graph class, we here consider the following notion of absolute retracts, for irreflexive graphs. – Unless stated otherwise, all graphs considered in this paper are irreflexive. – Namely, let us first recall that a subgraph H of a graph G is isochromatic if it has the same chromatic number as G . Then, given a class of (irreflexive) graphs \mathcal{C} , the absolute retracts of \mathcal{C} are those H such that, whenever H is an isometric and isochromatic subgraph of some $G \in \mathcal{C}$, H is a retract of G . We refer the reader to [5, 6, 9, 56, 54, 59, 61, 64, 68, 66, 67], where this notion got studied for various graph classes.

Our results. In this paper, we prove new structural and algorithmic properties of the absolute retracts of various hereditary graph classes, such as: bipartite graphs, k -chromatic graphs (for any $k \geq 3$), split graphs and planar graphs. Our focus is about the diameter problem on these graph classes but, on our way, we uncover several nice properties of the shortest-path distribution of their absolute retracts, that may be of independent interest.

First, in Sec. 2, we consider the absolute retracts of bipartite graphs and some important subclasses of the latter. Recall that the diameter of a bipartite graph can unlikely be computed in subquadratic time. We prove that the diameter of absolute bipartite retracts can be computed in $\tilde{O}(m\sqrt{n})$ time (Theorem 2). For that, we observe that in the square of such graph G , its two partite sets induce Helly graphs. This result complements the known relations between Helly graphs and absolute retracts of bipartite graphs [6]. Then, roughly, we show how to compute the diameter of G from the diameter of both Helly graphs (actually, from the knowledge of the peripheral vertices in these graphs, *i.e.*, their vertices with maximal eccentricity). Absolute bipartite retracts properly contain all cube-free modular graphs, and so, the cube-free median graphs and chordal bipartite graphs [5]. Therefore, as a byproduct of our Theorem 2, we get the first truly subquadratic-time algorithm for computing the diameter within the cube-free modular graphs. However, the structure of absolute bipartite retracts is far more complex than cube-free modular graphs: in fact, every bipartite graph is an isometric subgraph of some absolute bipartite retract [66].

Recently [39], we announced an $\mathcal{O}(m\sqrt{n})$ -time algorithm in order to compute all the eccentricities in a Helly graph. However, extending this result to the absolute retracts of bipartite graphs appears to be a more challenging task. We manage to do so for the subclass of chordal bipartite graphs, for which we achieve a linear-time algorithm in order to compute all the eccentricities. For that, we use the stronger result that in the square of such graph, its two partite sets induce strongly chordal graphs.

In Sec. 3, we generalize our above framework to the absolute retracts of k -chromatic graphs, for any $k \geq 3$. Notice that we are not aware of any prior work showing the usefulness of (efficiently computable) proper colorings for faster diameter computation. Our positive results in Sec. 2 and 3 rely on some Helly-type properties of the graph classes considered. We complement those with a hardness result in Sec. 4, that hints that the weaker property of being an absolute retract of some well-structured graph class is not sufficient on its own for faster diameter computation. Specifically, we prove that under SETH, there is no $\mathcal{O}(mn^{1-\epsilon})$ -time algorithm for the diameter problem, for any $\epsilon > 0$, on the class of absolute retracts of split graphs. This negative result follows from an elegant characterization of this subclass of split graphs in [59].

Finally, in Sec. 5, we consider the absolute planar retracts. While there now exist several truly subquadratic-time algorithms for the diameter problem on all planar graphs [20, 45, 49] the existence of a quasi linear-time algorithm for this problem has remained so far elusive, and it is sometimes conjectured that no such algorithm exists [20]. We give evidence that finding such algorithm for the absolute retracts of planar graphs is already a hard problem on its own. Specifically, we prove that every planar graph is an isometric subgraph of some absolute retract of planar graphs. This result mirrors the aforementioned property that every graph isometrically embeds in a Helly graph [40, 57].

Let us mention that all graph classes considered here are polynomial-time recognizable. For all that, we do *not* need to execute these recognition algorithms before we can compute the diameter of these graphs.

Notations. We mostly follow the graph terminology from [12, 33]. All graphs considered are finite, simple, unweighted and connected. For a graph $G = (V, E)$, let the (open) neighbourhood of a vertex v be defined as $N_G(v) = \{u \in V \mid uv \in E\}$ and its closed neighbourhood as $N_G[v] = N_G(v) \cup \{v\}$. Similarly, for a vertex-subset $S \subseteq V$, let $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$, and let $N_G[S] = N_G(S) \cup S$. The distance between two vertices $u, v \in V$ equals the minimum number of edges on a uv -path, and it is denoted $d_G(u, v)$. Let $I_G(u, v) = \{w \in V \mid d_G(u, v) = d_G(u, w) + d_G(w, v)\}$. The ball of center v and radius r is defined as $N_G^r[v] = \{u \in V \mid d_G(u, v) \leq r\}$. Furthermore, let the eccentricity of a vertex v be defined as $e_G(v) = \max_{u \in V} d_G(u, v)$. The diameter and the radius of a graph G are defined as $\text{diam}(G) = \max_{v \in V} e_G(v)$ and $\text{rad}(G) = \min_{v \in V} e_G(v)$, respectively. A vertex $v \in V$ is called central if $e_G(v) = \text{rad}(G)$, and peripheral if $e_G(v) = \text{diam}(G)$. We introduce additional terminology where it is needed throughout the paper.

2 Bipartite graphs

The study of the absolute retracts of bipartite graphs dates back from Hell [53], and since then many characterizations of this graph class were proposed [5]. This section is devoted to the diameter problem on this graph class. In Sec. 2.1, we propose a randomized $\tilde{O}(m\sqrt{n})$ -time algorithm for this problem. Then, we consider the chordal bipartite graphs in Sec. 2.2, that have been proved in [5] to be a subclass of the absolute retracts of bipartite graphs. For the chordal bipartite graphs, we present a deterministic linear-time algorithm in order to compute all the eccentricities. Before going further, let us introduce a few additional terminology. For a connected bipartite graph G , we denote its two partite sets by V_0 and V_1 . A half-ball is the intersection of a ball with one of the two partite sets of G . Finally, for $i \in \{0, 1\}$, let H_i be the graph with vertex-set V_i and an edge between every two vertices with a common neighbour in G .

2.1 Faster diameter computation

We start with the following characterization of the absolute bipartite retracts:

Theorem 1 ([5]). $G = (V, E)$ is an absolute retract of bipartite graphs if and only if the collection of half-balls of G satisfies the Helly property.

This above Theorem 1 leads us to the following simple observation about the internal structure of the absolute retracts of bipartite graphs:

Lemma 1. If $G = (V_0 \cup V_1, E)$ is an absolute retract of bipartite graphs then both H_0 and H_1 are Helly graphs.

Next, we prove that in order to compute $\text{diam}(G)$, with G an absolute retract of bipartite graphs, it is sufficient to compute the peripheral vertices of the Helly graphs H_0 and H_1 .

Lemma 2. *If $G = (V_0 \cup V_1, E)$ is an absolute bipartite retract such that $\text{diam}(H_0) \leq \text{diam}(H_1)$ then, $\text{diam}(G) \in \{2\text{diam}(H_1), 2\text{diam}(H_1)+1\}$. Moreover, if $\text{diam}(G) \geq 3$ then we have $\text{diam}(G) = 2\text{diam}(H_1) + 1$ if and only if:*

- $\text{diam}(H_1) = 1$;
- or $\text{diam}(H_0) = \text{diam}(H_1)$ and, for some $i \in \{0, 1\}$, there exists a peripheral vertex of H_i whose all neighbours in G are peripheral vertices of H_{1-i} .

The remaining of Sec. 2.1 is devoted to the computation of all the peripheral vertices in both Helly graphs H_0 and H_1 . While there exists a truly subquadratic-time algorithm for computing the diameter of a Helly graph [43], we observe that in general, we cannot compute H_0 and H_1 in truly subquadratic time from G . Next, we adapt [43, Theorem 2], for the Helly graphs, to our needs.

Lemma 3. *If $G = (V_0 \cup V_1, E)$ is an absolute bipartite retract then, for any k , we can compute in $\mathcal{O}(km)$ time the set of vertices of eccentricity at most k in H_0 (resp., in H_1).*

Proof (Sketch). By symmetry, we only need to prove the result for H_0 . Let $U = \{v \in V_0 \mid e_{H_0}(v) \leq k\}$ be the set to be computed. We consider the more general problem of computing, for any t , a partition $\mathcal{P}_t = (A_1^t, A_2^t, \dots, A_{p_t}^t)$ of V_0 , in an arbitrary number p_t of subsets, subject to the following constraints:

- For every $1 \leq i \leq p_t$, let $C_i^t := \bigcap_{v \in A_i^t} N_G^t[v]$. Let $B_i^t := C_i^t \cap V_0$ if t is even and let $B_i^t := C_i^t \cap V_1$ if t is odd (for short, $B_i^t = C_i^t \cap V_{t \pmod{2}}$). We impose the sets B_i^t to be nonempty and pairwise disjoint.

Indeed, under these two conditions above, we have $U \neq \emptyset$ if and only if, for any partition \mathcal{P}_{2k} as described above, $p_{2k} = 1$. Furthermore if it is the case then $U = B_1^{2k}$. To construct the desired partition, we proceed by induction over t . If $t = 0$ then, let $V_0 = \{v_1, v_2, \dots, v_{p_0}\}$. We just set $\mathcal{P}_0 = (\{v_0\}, \{v_1\}, \dots, \{v_{p_0}\})$ (each set is a singleton), and for every $1 \leq i \leq p_0$ let $B_i^0 = A_i^0 = \{v_i\}$. Else, we construct \mathcal{P}_t from \mathcal{P}_{t-1} . Specifically, for every $1 \leq i \leq p_{t-1}$, we let $W_i^t := N_G(B_i^{t-1})$. Then, starting from $j := 0$ and $\mathcal{F} := \mathcal{P}_{t-1}$, we proceed as follows until we have $\mathcal{F} = \emptyset$. We pick a vertex u s.t. $\#\{i \mid A_i^{t-1} \in \mathcal{F}, u \in W_i^t\}$ is maximized (the maximality of u ensures that all sets B_i^t will be pairwise disjoint). Then, we set $A_j^t := \bigcup\{A_i^{t-1} \mid A_i^{t-1} \in \mathcal{F}, u \in W_i^t\}$ and $B_j^t := \bigcap\{W_i^t \mid A_i^{t-1} \in \mathcal{F}, u \in W_i^t\}$. We add the new subset A_j^t to \mathcal{P}_t , we remove all the subsets $A_i^{t-1}, u \in W_i^t$ from \mathcal{F} , then we set $j := j + 1$. Overall, by using standard lists and pointer structures, each inductive step takes $\mathcal{O}(n + m)$ time.

The base case of our above induction is trivially correct. In order to prove correctness of our inductive step, we use Theorem 1 in order to prove that for each $1 \leq i \leq p_t$ we get $W_i^t = V_{t \pmod{2}} \cap \left(\bigcap_{v \in A_i^{t-1}} N_G^t[v] \right)$. Doing so, for each

subset A_j^t created at step t , we have $B_j^t = V_{t \pmod{2}} \cap \left(\bigcap_{v \in A_j^t} N_G^t[v] \right)$, as desired. Finally, observe that all the subsets B_j^t are nonempty since they at least contain the vertex $u \in V_{t \pmod{2}}$ that is selected in order to create A_j^t . \square

We use Lemma 3 when the diameters of H_0 and H_1 are in $\mathcal{O}(\sqrt{n})$. For larger values of diameters, we use a randomized procedure.

Lemma 4 (Theorem 3 in [43]). *For a Helly graph H s.t. $\text{diam}(H) > 3k = \omega(\log |V(H)|)$, one can compute with high probability its diameter and all the peripheral vertices in $\tilde{\mathcal{O}}(|E(H)| \cdot |V(H)|/k)$ time.*

It is important to note that, in the algorithmic procedure of Lemma 4, we just need to perform a BFS from randomly selected vertices. As any BFS in H_0 or H_1 can be simulated with a BFS in G , we can implement this procedure in order to compute $\text{diam}(H_i)$, for $i \in \{0, 1\}$, in $\tilde{\mathcal{O}}(mn/\text{diam}(H_i))$ time with high probability. Combined with Lemma 3, we get:

Theorem 2. *If $G = (V_0 \cup V_1, E)$ is an absolute retract of bipartite graphs then, with high probability, we can compute $\text{diam}(G)$ in $\tilde{\mathcal{O}}(m\sqrt{n})$ time.*

We suspect that Theorem 2 can be derandomized by using a recent technique from [39, Theorem 3]. This is left for future work.

2.2 Chordal bipartite graphs

We improve Theorem 2 for the special case of chordal bipartite graphs. Recall (amongst many characterizations) that a bipartite graph is chordal bipartite if and only if every induced cycle has length four [51]. It was proved in [5] that every chordal bipartite graph is an absolute retract of bipartite graphs.

Theorem 3. *If $G = (V, E)$ is chordal bipartite then we can compute all the eccentricities (and so, the diameter) in linear time.*

We subdivide our proof of Theorem 3 into four main steps.

The chordal structure of the partite sets. A graph is chordal if it has no induced cycle of length more than three. It is strongly chordal if it is chordal and it does not contain any n -sun ($n \geq 3$) as an induced subgraph [46]. We use the following characterization of the haf-sets of chordal bipartite graphs:

Lemma 5 ([62]). *If $G = (V_0 \cup V_1, E)$ is chordal bipartite, then H_0 and H_1 are strongly chordal graphs.*

Computation of a clique-tree. The same as in Sec. 2.1, in general we cannot compute H_0 and H_1 from G in subquadratic time. In order to overcome this issue, we use a more compact representation of the latter. Specifically, for a graph $H = (V, E)$, a clique-tree is a tree T whose nodes are the maximal cliques of H and such that, for every $v \in V$, the maximal cliques of H containing v induce a connected subtree T_v of T . It is well-known that H is chordal if and only if it has a clique-tree [19, 48, 73]. By using standard results on dual hypertrees [11, 72], we obtain that:

Lemma 6. *If $G = (V_0 \cup V_1, E)$ is chordal bipartite then, we can compute a clique-tree for H_0 and H_1 in linear time.*

Computation of all the eccentricities in the partite sets. Next, we propose a new algorithm in order to compute all the eccentricities of a strongly chordal graph H , being given a clique-tree. There already exist linear-time algorithms for computing all the eccentricities of a strongly chordal graphs, being given by its adjacency list [17, 39, 43]. However, in general these algorithms do not run in time linear in the size of a clique-tree. We often use in our proof the *clique-vertex incidence graph* of H , *i.e.*, the bipartite graph I_H whose partite sets are the vertices and the maximal cliques of H , and such that there is an edge between every vertex of H and every maximal clique of H containing it.

Let us first recall the following result about Helly graphs:

Lemma 7 ([35]). *If H is Helly then, for every vertex v we have $e_H(v) = d_H(v, C(H)) + \text{rad}(H)$, where $C(H)$ denotes the set of central vertices of H .*

Hence, by Lemma 7, we are left computing $C(H)$. It starts with computing one central vertex. Define, for every vertex v and vertex-subset C , $d_H(v, C) = \min_{c \in C} d_H(v, c)$. Following [27], we call a set C gated if, for every $v \notin C$, there exists a vertex $v^* \in N_H^{d_H(v, C)-1}[v] \cap (\bigcap \{N_H(c) \mid c \in C, d_H(v, c) = d_H(v, C)\})$ (such vertex v^* is called a gate of v).

Lemma 8 ([22]). *Every clique in a chordal graph is a gated set.*

Lemma 9 ([43]). *If T is a clique-tree of a chordal graph H then, for every clique C of H , for every $v \notin C$ we can compute $d_H(v, C)$ and a corresponding gate v^* in total $\mathcal{O}(w(T))$ time, where $w(T)$ denotes the sum of cardinalities of all the maximal cliques of H .*

For every $u, v \in V$ and $k \leq d_H(u, v)$, the set $L_H(u, k, v) = \{x \in I_H(u, v) \mid d_H(u, x) = k\}$ is called a slice. We also need the following result:

Lemma 10 ([22]). *Every slice in a chordal graph is a clique.*

Now, consider the procedure described in Algorithm 1.

Lemma 11 (special case of Theorem 5 in [43]). *Algorithm 1 outputs a central vertex of H .*

Algorithm 1 Computation of a central vertex.

Require: A strongly chordal graph H .

```

1:  $v \leftarrow$  an arbitrary vertex of  $H$ 
2:  $u \leftarrow$  a furthest vertex from  $v$ , i.e.,  $d_H(u, v) = e_H(v)$ 
3:  $w \leftarrow$  a furthest vertex from  $u$ , i.e.,  $d_H(u, w) = e_H(u)$ 
4: for all  $r \in \{\lceil e_H(u)/2 \rceil, \lceil (e_H(u) + 1)/2 \rceil, 1 + \lceil e_H(u)/2 \rceil\}$  do
5:   Set  $C := L(w, r, u)$  //  $C$  is a clique by Lemma 10
6:   for all  $v \notin C$  do
7:     Compute  $d_H(v, C)$  and a corresponding gate  $v^*$  // whose existence follows from
       Lemma 8
8:   Set  $S := \{v^* \mid d_H(v, C) = r\}$  // gates of vertices at max. distance from  $C$ 
9:   for all  $c \in C$  do
10:    if  $S \subseteq N_H(c)$  then
11:      return  $c$ 

```

By using dynamic programming on a clique-tree in order to compute, for each candidate vertex $c \in C$, its number of neighbours in S , we get:

Lemma 12. *If T is a clique-tree of a strongly chordal graph H then, we can implement Algorithm 1 in order to run in $\mathcal{O}(w(T))$ time, where $w(T)$ denotes the sum of cardinalities of all the maximal cliques of H .*

We need one more result about the center of strongly chordal graphs:

Lemma 13 ([35, 36]). *If H is strongly chordal then, its center $C(H)$ induces a strongly chordal graph of radius ≤ 1 .*

By Lemma 13, given a central vertex c of H , we can compute $C(H)$ by local search in the neighbourhood at distance two around c . For doing that efficiently, we also need the following nice characterization of strongly chordal graphs. Recall that the *clique-vertex incidence graph* of H is a bipartite graph whose partite sets are the vertices and the maximal cliques of H , respectively; there is an edge between every vertex and every maximal clique in which this vertex is contained.

Lemma 14 ([16, 46]). *H is strongly chordal if and only if its clique-vertex incidence graph I_H is chordal bipartite.*

By Lemma 14, we can apply the techniques of Sec. 2.1 to the clique-vertex incidence graph of any strongly chordal H . In particular, by combining Lemma 3 with the dynamic programming technique of Lemma 12, we obtain:

Proposition 1. *If T is a clique-tree of a strongly chordal graph $H = (V, E)$ then, we can compute its center $C(H)$ in $\mathcal{O}(w(T))$ time.*

Computation of all the eccentricities in G . Before proving Theorem 3, we need a final ingredient. Let us first generalize Lemma 2 as follows.

Lemma 15. *If $G = (V_0 \cup V_1, E)$ is an absolute retract of bipartite graphs then, the following holds for every $i \in \{0, 1\}$ and $v \in V_i$:*

- If $e_{H_i}(v) \leq \text{rad}(H_{1-i}) - 1$ then, $e_G(v) = 2e_{H_i}(v) + 1 = 2\text{rad}(H_{1-i}) - 1$.
- If $e_{H_i}(v) = \text{rad}(H_{1-i})$ then, $e_G(v) = 2\text{rad}(H_{1-i})$ if and only if $N_G(v) \subseteq C(H_{1-i})$ and, for every $u \in V_{1-i}$, we have $d_{H_{1-i}}(u, N_G(v)) \leq \text{rad}(H_{1-i}) - 1$ (otherwise, $e_G(v) = 2\text{rad}(H_{1-i}) + 1$).
- If $e_{H_i}(v) \geq \text{rad}(H_{1-i}) + 1$ then, $e_G(v) = 2e_{H_i}(v)$ if and only if we have $e_{H_{1-i}}(u) < e_{H_i}(v)$ for some $u \in N_G(v)$ (otherwise, $e_G(v) = 2e_{H_i}(v) + 1$).

Of the three cases in the above Lemma 15, the real algorithmic challenge is the case $e_{H_i}(v) = \text{rad}(H_{1-i})$, for some $i \in \{0, 1\}$. We solve this case by using similar techniques as for Proposition 1, which concludes the proof of Theorem 3.

3 k -chromatic graphs

Recall that a proper k -coloring of $G = (V, E)$ is any mapping $c : V \rightarrow \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E$. The chromatic number of G is the least k such that it has a proper k -coloring, and a k -chromatic graph is a graph whose chromatic number is equal to k . We study the diameter problem within the absolute retracts of k -chromatic graphs, for every $k \geq 3$.

Our approach requires such graphs to be equipped with a proper k -coloring. While this is a classic NP-hard problem for every $k \geq 3$ [58], it can be done in polynomial time for absolute retracts of k -chromatic graphs [9]. By using a standard greedy coloring approach, we first improve this result as follows:

Proposition 2. *There is a linear-time algorithm such that, for every $k \geq 3$, if the input G is an absolute retract of k -chromatic graphs, then it computes a proper k -coloring of G .*

In the remainder of the section, we always assume the input graph G to be given with a proper k -coloring. We sometimes use the fact that, for an absolute retract, such proper k -coloring is unique up to permuting the colour classes [68]. Now, let us recall the following characterization of absolute retracts:

Theorem 4 ([68]). *Let $k \geq 3$. The graph $G = (V, E)$ is an absolute retract of k -chromatic graphs if and only if for any proper k -coloring c , every peripheral vertex v is adjacent to all vertices u with $c(u) \neq c(v)$, or it is covered¹ and $G \setminus v$ is an absolute retract of k -chromatic graphs.*

A special case of Theorem 4 leads to a linear-time algorithm in order to decide whether an absolute k -chromatic retract has diameter at most two. For those graphs with diameter at least three, we propose a generalization of Lemma 2. Specifically, for each colour i , let $V_i := \{v \in V \mid c(v) = i\}$ be called a colour class. For every $v \in V_i$, $e_i(v) := \max\{d_G(u, v) \mid u \in V_i\}$. A vertex $v \in V_i$ is i -peripheral if it maximizes $e_i(v)$. Finally, let $d_i := \max\{e_i(v) \mid v \in V_i\}$.

¹ A vertex v is covered by another vertex w if $N_G(v) \subseteq N_G(w)$ (a covered vertex is called embeddable in [68]).

Lemma 16. *Let $G = (V, E)$ be an absolute retract of k -chromatic graphs for some $k \geq 3$, and let c be a corresponding proper k -coloring. Then, $\max_{1 \leq i \leq k} d_i \leq \text{diam}(G) \leq 1 + \max_{1 \leq i \leq k} d_i$. Moreover, if $\text{diam}(G) \geq 3$, then we have $\text{diam}(G) = 1 + \max_{1 \leq i \leq k} d_i$ if and only if:*

- either $\max_{1 \leq i \leq k} d_i = 2$;
- or, for some $i \neq j$ s.t. $d_i = d_j$ is maximized, there is some i -peripheral vertex whose all neighbours coloured j are j -peripheral.

We end up sketching the computation, for each colour i , of the value d_i and of the i -peripheral vertices. Our strategy is as follows. First, we prove that we can reduce our study to the case $k = 3$. This is done by using another, more algorithmic, characterization of absolute retracts [9].

Lemma 17. *Let $G = (V, E)$ be an absolute retract of k -chromatic graphs for some $k \geq 3$, and let c be a corresponding proper k -coloring. For every distinct colours i_1, i_2, i_3 , the subgraph $H := G[V_{i_1} \cup V_{i_2} \cup V_{i_3}]$ is isometric. Moreover, H is an absolute retract of 3-chromatic graphs.*

Next, we deal with the case when d_i is sufficiently small. For that, we extend the techniques of Lemma 3 to the absolute 3-chromatic retracts. Correctness of our approach follows from the following property of these graphs: if v_1, v_2, \dots, v_t are vertices coloured i then, for any $r \geq 2$ and any colour j , the balls $N_G^r[v_1], N_G^r[v_2], \dots, N_G^r[v_t]$ intersect in colour j if and only if they also intersect in colour i .

Lemma 18. *Let $G = (V, E)$ be an absolute retract of 3-chromatic graphs, and let c be a corresponding proper 3-coloring. For each colour i and $D \geq 2$, we can compute in $\mathcal{O}(Dm)$ time the set $U_i := \{v \in V_i \mid e_i(u) \leq D\}$.*

Finally, we address the case when d_i is large. A function is called unimodal if every local minimum is also a global minimum. It is known that the eccentricity function of a Helly graph is unimodal [35], and this property got used in [39] in order to compute all the eccentricities in this graph class in subquadratic time. We prove that a similar, but weaker property holds for each colour class of absolute retracts:

Lemma 19. *Let $G = (V, E)$ be an absolute retract of k -chromatic graphs for some $k \geq 3$, and let c be a corresponding proper k -coloring. For each colour i and any $u \in V_i$ s.t. $e_i(u) \geq (d_i + 5)/2 \geq 7$, there exists a $u' \in V_i$ s.t. $d_G(u, u') = 2$ and $e_i(u') = e_i(u) - 2$.*

We apply this almost-unimodality property to the computation of the d_i 's:

Lemma 20. *Let $G = (V, E)$ be an absolute retract of k -chromatic graphs for some $k \geq 3$, let c be a corresponding proper k -coloring, and let i be such that $d_i \geq 8D + 5 = \omega(\log n)$. Then, with high probability, we can compute in total $\tilde{\mathcal{O}}(mn/D)$ time the value d_i and the i -peripheral vertices.*

By combining Lemmas 17-20, we get:

Theorem 5. *If $G = (V, E)$ is an absolute k -chromatic retract, for some $k \geq 3$, then we can compute its diameter with high probability in $\tilde{\mathcal{O}}(m\sqrt{n})$ time.*

4 Split graphs

Recall that $G = (V, E)$ is a split graph if its vertex-set V can be partitioned into a clique K and a stable set S . Such partition, that may not be unique, can be computed in linear time [50]. In contrast to Sec. 2 and 3, we prove that:

Theorem 6. *For any $\epsilon > 0$, there exists a $c(\epsilon)$ s.t., under SETH, we cannot compute the diameter in $\mathcal{O}(n^{2-\epsilon})$ time on the absolute retracts of split graphs of order n and clique-number at most $c(\epsilon) \log n$.*

Proof (Sketch). The result holds for general split graphs [13]. Let $G = (K+S, E)$ be any split graph. In order to decide whether $\text{diam}(G) \leq 2$ or $\text{diam}(G) = 3$, we may remove first all vertices v s.t. $N_G(v) = K \setminus v$ (i.e., because $e_G(v) \leq 2$ and v is simplicial). By applying this above pruning rule until it can no more be done, we get a split graph G' with a unique partition $K' + S'$ [50]. All such graphs are absolute split retracts [59]. \square

5 Planar graphs

Our last (non-algorithmic) section is about the absolute retracts of planar graphs

Theorem 7 ([54]). *A planar graph G is an absolute retract of planar graphs if and only if it is maximal planar and, in an embedding of G in the plane, any triangle bounding a face of G belongs to a subgraph of G isomorphic to K_4 .*

To our best knowledge, there has been no relation uncovered between the absolute retracts of planar graphs and other important planar graph subclasses. We make a first step in this direction. Specifically, we prove the following two results.

Proposition 3. *Every planar 3-tree is an absolute retract of planar graphs.*

Theorem 8. *Every connected planar graph is an isometric subgraph of some absolute planar retract. In particular, there are absolute retracts of planar graphs with arbitrarily large treewidth.*

We stress that the proof of Theorem 8 is constructive, and that it leads to a polynomial-time algorithm in order to construct an absolute planar retract in which the input planar graph G isometrically embeds. In contrast to our result, the smallest Helly graph in which a graph G isometrically embeds may be exponential in its size [52].

The existence of an almost linear-time algorithm for computing the diameter of planar graphs is an important open problem. We see our Theorem 8 as evidence that answering to this problem for the absolute planar retracts would be already an important intermediate step toward a full resolution.

References

1. Abboud, A., Vassilevska Williams, V., Wang, J.R.: Approximation and fixed parameter subquadratic algorithms for radius and diameter in sparse graphs. pp. 377–391. SIAM (2016). <https://doi.org/10.1137/1.9781611974331.ch28>
2. Albert, R., Jeong, H., Barabási, A.L.: Diameter of the world-wide web. *nature* **401**(6749), 130–131 (1999)
3. Bandelt, H.J.: Retracts of hypercubes. *Journal of graph theory* **8**(4), 501–510 (1984)
4. Bandelt, H.J., Chepoi, V.: Metric graph theory and geometry: a survey. *Contemporary Mathematics* **453**, 49–86 (2008)
5. Bandelt, H.J., Dählmann, A., Schütte, H.: Absolute retracts of bipartite graphs. *Discrete Applied Mathematics* **16**(3), 191–215 (1987)
6. Bandelt, H.J., Farber, M., Hell, P.: Absolute reflexive retracts and absolute bipartite retracts. *Discrete Applied Mathematics* **44**(1-3), 9–20 (1993)
7. Bandelt, H.J., Pesch, E.: Dismantling absolute retracts of reflexive graphs. *European Journal of Combinatorics* **10**(3), 211–220 (1989)
8. Bandelt, H.J., Pesch, E.: A Radon theorem for Helly graphs. *Archiv der Mathematik* **52**(1), 95–98 (1989)
9. Bandelt, H.J., Pesch, E.: Efficient characterizations of n -chromatic absolute retracts. *Journal of Combinatorial Theory, Series B* **53**(1), 5–31 (1991)
10. Bandelt, H.J., Prisner, E.: Clique graphs and Helly graphs. *Journal of Combinatorial Theory, Series B* **51**(1), 34–45 (1991)
11. Beeri, C., Fagin, R., Maier, D., Yannakakis, M.: On the desirability of acyclic database schemes. *Journal of the ACM (JACM)* **30**(3), 479–513 (1983)
12. Bondy, J.A., Murty, U.S.R.: *Graph theory*, Graduate Texts in Mathematics, vol. 244. Springer-Verlag London (2008)
13. Borassi, M., Crescenzi, P., Habib, M.: Into the square: On the complexity of some quadratic-time solvable problems. *Electronic Notes in Theoretical Computer Science* **322**, 51–67 (2016)
14. Borassi, M., Crescenzi, P., Trevisan, L.: An axiomatic and an average-case analysis of algorithms and heuristics for metric properties of graphs. In: *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. pp. 920–939. SIAM (2017)
15. Bousquet, N., Thomassé, S.: VC-dimension and Erdős–Pósa property. *Discrete Mathematics* **338**(12), 2302–2317 (2015)
16. Brandstädt, A.: Classes of bipartite graphs related to chordal graphs. *Discrete Applied Mathematics* **32**(1), 51–60 (1991)
17. Brandstädt, A., Dragan, F., Chepoi, V., Voloshin, V.: Dually chordal graphs. *SIAM Journal on Discrete Mathematics* **11**(3), 437–455 (1998)
18. Bringmann, K., Husfeldt, T., Magnusson, M.: Multivariate Analysis of Orthogonal Range Searching and Graph Distances. *Algorithmica* pp. 1–24 (2020)
19. Buneman, P.: A characterisation of rigid circuit graphs. *Discrete Mathematics* **9**(3), 205–212 (1974)
20. Cabello, S.: Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs. *ACM Transactions on Algorithms (TALG)* **15**(2), 1–38 (2018)
21. Chalopin, J., Chepoi, V., Genevois, A., Hirai, H., Osajda, D.: Helly groups. *Tech. Rep. arXiv:2002.06895*, arXiv (2020)

22. Chang, G., Nemhauser, G.: The k -domination and k -stability problems on sun-free chordal graphs. *SIAM Journal on Algebraic Discrete Methods* **5**(3), 332–345 (1984)
23. Chastand, M., Laviolette, F., Polat, N.: On constructible graphs, infinite bridged graphs and weakly cop-win graphs. *Discrete Mathematics* **224**(1-3), 61–78 (2000)
24. Chepoi, V.: On distances in benzenoid systems. *Journal of chemical information and computer sciences* **36**(6), 1169–1172 (1996)
25. Chepoi, V., Dragan, F., Estellon, B., Habib, M., Vaxès, Y.: Diameters, centers, and approximating trees of δ -hyperbolic geodesic spaces and graphs. In: *Symposium on Computational Geometry (SocG)*. pp. 59–68. ACM (2008)
26. Chepoi, V., Dragan, F., Vaxès, Y.: Center and diameter problems in plane triangulations and quadrangulations. In: *Symposium on Discrete Algorithms (SODA'02)*. pp. 346–355 (2002)
27. Chepoi, V., Dragan, F.: A linear-time algorithm for finding a central vertex of a chordal graph. In: *ESA*. pp. 159–170. Springer (1994)
28. Chepoi, V., Estellon, B., Vaxès, Y.: Covering planar graphs with a fixed number of balls. *Discrete & Computational Geometry* **37**(2), 237–244 (2007)
29. Corneil, D., Dragan, F., Habib, M., Paul, C.: Diameter determination on restricted graph families. *Discrete Applied Mathematics* **113**(2-3), 143–166 (2001)
30. Coudert, D., Ducoffe, G., Popa, A.: Fully polynomial FPT algorithms for some classes of bounded clique-width graphs. *ACM Transactions on Algorithms (TALG)* **15**(3), 1–57 (2019)
31. Damaschke, P.: Computing giant graph diameters. In: *International Workshop on Combinatorial Algorithms (IWOCA)*. pp. 373–384. Springer (2016)
32. De Rumeur, J.: *Communications dans les réseaux de processeurs*. Masson Paris (1994)
33. Diestel, R.: *Graph Theory*. Graduate Texts in Mathematics, Springer (2010). <https://doi.org/10.1007/978-3-662-53622-3>, 4th edition
34. Dragan, F.: Almost diameter of a house-hole-free graph in linear time via LexBFS. *Discrete applied mathematics* **95**(1-3), 223–239 (1999)
35. Dragan, F.: *Centers of graphs and the Helly property*. Ph.D. thesis, Moldova State University (1989)
36. Dragan, F.: Domination in quadrangle-free Helly graphs. *Cybernetics and Systems Analysis* **29**(6), 822–829 (1993)
37. Dragan, F., Brandstädt, A.: r -dominating cliques in graphs with hypertree structure. *Discrete Mathematics* **162**(1-3), 93–108 (1996)
38. Dragan, F., Guarnera, H.: Obstructions to a small hyperbolicity in Helly graphs. *Discrete Mathematics* **342**(2), 326 – 338 (2019)
39. Dragan, F., Guarnera, H.: Helly-gap of a graph and vertex eccentricities. Tech. Rep. arXiv:2005.01921, arXiv (2020)
40. Dress, A.: Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces. *Advances in Mathematics* **53**(3), 321–402 (1984)
41. Ducoffe, G.: A New Application of Orthogonal Range Searching for Computing Giant Graph Diameters. In: *Symposium on Simplicity in Algorithms (SOSA)* (2019)
42. Ducoffe, G.: Distance problems within Helly graphs and k -Helly graphs. Tech. Rep. arXiv:2011.00001, arXiv preprint (2020)
43. Ducoffe, G., Dragan, F.: A story of diameter, radius and Helly property **77**(3), 435–453 (2021)
44. Ducoffe, G., Habib, M., Viennot, L.: Fast diameter computation within split graphs. In: *International Conference on Combinatorial Optimization and Applications*. pp. 155–167. Springer (2019)

45. Ducoffe, G., Habib, M., Viennot, L.: Diameter computation on H -minor free graphs and graphs of bounded (distance) VC-dimension. In: Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). pp. 1905–1922. SIAM (2020)
46. Farber, M.: Characterizations of strongly chordal graphs. *Discrete Mathematics* **43**(2-3), 173–189 (1983)
47. Farley, A., Proskurowski, A.: Computation of the center and diameter of outerplanar graphs. *Discrete Applied Mathematics* **2**(3), 185–191 (1980)
48. Gavril, F.: The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B* **16**(1), 47–56 (1974)
49. Gawrychowski, P., Kaplan, H., Mozes, S., Sharir, M., Weimann, O.: Voronoi diagrams on planar graphs, and computing the diameter in deterministic $\tilde{O}(n^{5/3})$ time. In: Symposium on Discrete Algorithms (SODA). pp. 495–514. SIAM (2018)
50. Golumbic, M.: *Algorithmic graph theory and perfect graphs*, vol. 57. Elsevier (2004)
51. Golumbic, M., Goss, C.: Perfect elimination and chordal bipartite graphs. *Journal of Graph Theory* **2**(2), 155–163 (1978)
52. Guarnera, H., Dragan, F., Leitert, A.: Injective hulls of various graph classes. Tech. Rep. arXiv:2007.14377, arXiv (2020)
53. Hell, P.: Rétractions de graphes. Ph.D. thesis, Thèse (Ph. D.: Mathématiques)–Université de Montréal. 1972. (1972)
54. Hell, P.: Absolute planar retracts and the four color conjecture. *Journal of Combinatorial Theory, Series B* **17**(1), 5–10 (1974)
55. Hell, P., Rival, I.: Absolute retracts and varieties of reflexive graphs. *Canadian journal of mathematics* **39**(3), 544–567 (1987)
56. Hell, P.: Absolute retracts in graphs. In: *Graphs and Combinatorics*, pp. 291–301. Springer (1974)
57. Isbell, J.: Six theorems about injective metric spaces. *Commentarii Mathematici Helvetici* **39**(1), 65–76 (1964)
58. Johnson, D.S., Garey, M.R.: *Computers and intractability: A guide to the theory of NP-completeness*. WH Freeman (1979)
59. Klavžar, S.: Absolute retracts of split graphs. *Discrete Mathematics* **134**(1-3), 75–84 (1994)
60. Klisowski, J.: A survey of various modifications of the notions of absolute retracts and absolute neighborhood retracts. In: *Colloquium Mathematicum*. vol. 46, pp. 23–35. Institute of Mathematics Polish Academy of Sciences (1982)
61. Kloks, T., Wang, Y.L.: On retracts, absolute retracts, and foldings in cographs. *Optimization Letters* **12**(3), 535–549 (2018)
62. Le, H.O., Le, V.: Hardness and structural results for half-squares of restricted tree convex bipartite graphs. *Algorithmica* **81**(11), 4258–4274 (2019)
63. Lin, M., Szwarcfiter, J.: Faster recognition of clique-Helly and hereditary clique-Helly graphs. *Information Processing Letters* **103**(1), 40–43 (2007)
64. Loten, C.: Absolute retracts and varieties generated by chordal graphs. *Discrete mathematics* **310**(10-11), 1507–1519 (2010)
65. Olariu, S.: A simple linear-time algorithm for computing the center of an interval graph. *International Journal of Computer Mathematics* **34**(3-4), 121–128 (1990)
66. Pesch, E.: Minimal extensions of graphs to absolute retracts. *Journal of graph theory* **11**(4), 585–598 (1987)
67. Pesch, E.: Products of absolute retracts. *Discrete mathematics* **69**(2), 179–188 (1988)

68. Pesch, E., Poguntke, W.: A characterization of absolute retracts of n -chromatic graphs. *Discrete mathematics* **57**(1-2), 99–104 (1985)
69. Polat, N.: Convexity and fixed-point properties in Helly graphs. *Discrete Mathematics* **229**(1-3), 197–211 (2001)
70. Polat, N.: On constructible graphs, locally Helly graphs, and convexity. *Journal of Graph Theory* **43**(4), 280–298 (2003)
71. Roditty, L., Vassilevska Williams, V.: Fast approximation algorithms for the diameter and radius of sparse graphs. In: *Proceedings of the forty-fifth annual ACM symposium on Theory of computing (STOC)*. pp. 515–524 (2013)
72. Tarjan, R., Yannakakis, M.: Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM Journal on computing* **13**(3), 566–579 (1984)
73. Walter, J.R.: Representations of rigid cycle graphs. Ph.D. thesis, Wayne State University, Department of Mathematics (1972)
74. Watts, D., Strogatz, S.: Collective dynamics of ‘small-world’ networks. *Nature* **393**(6684), 440–442 (1998)