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► **To cite this version:**

Nicolas Vanspranghe, Francesco Ferrante, Christophe Prieur. Stabilization of the Wave Equation by the Mean of a Saturating Dirichlet Feedback. MICNON 2021 - Third IFAC Conference on Modelling, Identification and Control of Nonlinear Systems, Sep 2021, Tokyo, Japan. 10.1016/j.ifacol.2021.10.394 . hal-03349953

HAL Id: hal-03349953

<https://hal.science/hal-03349953>

Submitted on 21 Sep 2021

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Stabilization of the Wave Equation by the Mean of a Saturating Dirichlet Feedback

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Abstract: In this paper, we consider the wave equation with Dirichlet boundary control subject to a nonlinearity, the kind of which includes (but is not restricted to) pointwise saturation mappings. The case where only a subset of the boundary is actuated is allowed. Initial data is taken in the optimal energy space associated with Dirichlet boundary control – which means that we deal with (very) weak solutions. Using nonlinear semigroup techniques, we prove that the associated closed-loop system is asymptotically stable. Some numerical simulations are given to illustrate the stability result.

Keywords: Infinite-dimensional systems, stability of nonlinear systems, control of partial differential equations, control and estimation of wave equations and systems of elasticity, semigroup and operator theory.

1. INTRODUCTION

The general context of this paper is the study of infinite-dimensional systems in the presence of nonlinearities in the feedback loop, and in particular, saturation mappings, which represent those most commonly met in applications. Even for finite-dimensional linear systems, saturating actuators can be a source of instability and give rise to unbounded trajectories – see (Tarbouriech et al., 2011, Example 1.1) for instance. In the case of infinite-dimensional systems, an early abstract treatment is given in Slemrod (1989), where the feedback is *bounded* with respect to the state space and the saturation concerns its norm. Here, we consider wave dynamics with actuation in the Dirichlet boundary conditions. The precise formulation of the control problem is given next.

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with smooth boundary Γ , as depicted in Figure 1 for instance. We are given a relatively open non-empty subset Γ_0 of Γ and we consider the following control system:

$$\partial_{tt}u - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1a)$$

$$u(\sigma, t) = -g(U(\sigma, t)) \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (1b)$$

$$u(\sigma, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (1c)$$

where $U(\sigma, t)$ is the control input, $\Gamma_1 \triangleq \Gamma \setminus \Gamma_0$, and g is a real scalar function satisfying the following assumptions:

- g is continuous and nondecreasing;
- $g(s) = 0$ if and only if $s = 0$;
- g is globally Lipschitz continuous.

Our prototype case is when g is a pointwise saturation mapping, which satisfies the assumptions stated above.

In the literature, the problem of boundary stabilization of the wave equation with saturating input (and more generally, nonlinear boundary dissipation) has been mainly considered in the Neumann boundary conditions. In Prieur

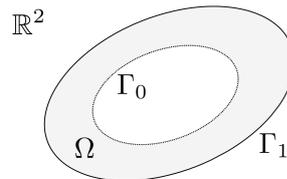


Fig. 1. An admissible configuration in \mathbb{R}^2 . Here, the two parts Γ_0 and Γ_1 of the boundary Γ are disconnected.

et al. (2016), asymptotic stability of a one-dimensional wave equation with a boundary velocity feedback subject to a cone-bounded nonlinearity (which includes saturation mappings) is studied. Non-uniform decay rates for two and three-dimensional waves are given in Xu and Xu (2019), for solutions with smooth initial data. When the nonlinearity has linear growth at infinity (which of course excludes the saturating case), uniform decay rate can be achieved, as in Lasiecka and Tataru (1993).

However, the Dirichlet case is less investigated when it comes to nonlinear feedback. It appears that the natural energy space for the wave equation with $L^2(\Gamma)$ -Dirichlet control is

$$\mathcal{H} \triangleq L^2(\Omega) \times H^{-1}(\Omega), \quad (2)$$

where $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$. We give some additional notation that is required to describe the control action under consideration here. First, the differential operator $-\Delta$ associated with homogeneous Dirichlet boundary condition defines a strictly positive self-adjoint unbounded operator on $L^2(\Omega)$, which we denote by A , with bounded inverse A^{-1} . Recall that $H^{-1}(\Omega)$ is a Hilbert space if endowed with the scalar product

$$(v_1, v_2)_{H^{-1}(\Omega)} \triangleq (A^{-1/2}v_1, A^{-1/2}v_2)_{L^2(\Omega)}, \quad (3)$$

where $A^{-1/2}$ is the square root of A^{-1} – see (Curtain and Zwart, 2012, Example 2.2.5). The norm induced by this

scalar product is equivalent to that of the dual space. Let us now define the following (continuous) energy functional on \mathcal{H} :

$$\mathcal{E}(u, v) \triangleq \frac{1}{2} \left\{ \|u\|_{L^2(\Omega)}^2 + \|v\|_{H^{-1}(\Omega)}^2 \right\}. \quad (4)$$

If a solution $[u, u']$ to (1) is smooth enough, one can derive the following energy identity:

$$\frac{d}{dt} \mathcal{E}(u(t), u'(t)) = \int_{\Gamma_0} g(U(\sigma, t)) \partial_\nu [A^{-1} u'(t)] d\sigma, \quad (5)$$

where ∂_ν denotes the outward normal derivative. Note that the right-hand side integral is on the actuated region Γ_0 of the boundary Γ . Equation (5) motivates the following choice of feedback:

$$U(t) = -\partial_\nu [A^{-1} u'(t)], \quad (6)$$

which then makes the energy nonincreasing. If $u'(t)$ is in $L^2(\Omega)$, then it follows from standard elliptic regularity theory that $A^{-1} u'(t)$ belongs to $H^2(\Omega)$ – see for instance (Brezis, 2010, Theorem 9.25); thus, $\partial_\nu [A^{-1} u'(t)]$ is well-defined in (say) $L^2(\Gamma)$. It is proved in Lasiecka and Triggiani (1992) that, in the linear case (i.e. $g = \text{id}$), the closed-loop system defined by (1) and (6) is exponentially stable with respect to the norm of \mathcal{H} when $\Gamma = \Gamma_0$ (the result still holds under additional assumptions when $\Gamma \neq \Gamma_0$). To the best of our knowledge, there has been no work extending stability analysis of the aforementioned system to the nonlinear case. In this paper, we shall prove that the initial-and-boundary value problem given by (1) and (6) is well-posed in the Hadamard sense on the energy space \mathcal{H} and that the zero equilibrium is globally asymptotically stable.

The remainder of the paper is organized as follows. In Section 2, the main results of the paper are stated, namely the well-posedness and asymptotic stability of the associated evolution semigroup; then, the functional formulation of the closed-loop system defined by (1)-(6) is precised. In Section 3, some useful properties of the generator of the semigroup are shown, allowing us to prove the main theorems. Finally, finite element simulations are given in Section 4 for illustrative purpose.

Notation. The norm of a given Banach space E is denoted by $\|\cdot\|_E$ and the duality bracket $\langle \phi, x \rangle_E$ is used to write $\phi(x)$ for any vector $x \in E$ and continuous linear form $\phi \in E'$. We denote by $W^{1,p}(0, T; E)$ the set of E -valued functions f defined on $[0, T]$ such that, for some $h \in L^p(0, T; H)$, $f(t) = f(0) + \int_0^t h(s) ds$; then, we say that $f' = h$ in the sense of E -valued distributions. If E is also a Hilbert space, then its scalar product is written $(\cdot, \cdot)_E$. We use dx to denote the Lebesgue measure on \mathbb{R}^d and $d\sigma$ to denote the induced surface measure on Γ , as in the formula $-\int_\Omega \Delta uv dx = \int_\Omega \nabla u \cdot \nabla v dx - \int_\Gamma \partial_\nu uv d\sigma$. Finally, we denote by $\mathcal{C}_c^\infty(\Omega)$ the space of compactly supported and infinitely differentiable (real-valued) functions on Ω .

2. PRELIMINARIES AND MAIN RESULTS

2.1 Statement of the main results

We associate with the closed-loop system a nonlinear unbounded operator \mathcal{A} on \mathcal{H} with domain $\mathcal{D}(\mathcal{A})$, so that (1)-(6) is recast into a first-order Cauchy problem on \mathcal{H} :

$$\dot{X}(t) + \mathcal{A}(X(t)) = 0, \quad (7a)$$

$$X(0) = X_0. \quad (7b)$$

Proper definition of the operator \mathcal{A} is given in the next subsection. We recall that *strong* solutions to (7) are defined as absolutely continuous \mathcal{H} -valued functions satisfying (7a) for a.e. t in the sense of strong differentiation in \mathcal{H} , with initial data in $\mathcal{D}(\mathcal{A})$. *Weak* solutions are defined as uniform limits of strong solutions. The well-posedness of the closed-loop dynamics is stated in the following theorem.

Theorem 1. *The nonlinear operator \mathcal{A} associated with the closed-loop system (1)-(6) is densely defined, maximal monotone and thus generates a strongly continuous semigroup $\{\mathcal{S}_t\}$ of (nonlinear) contractions on the energy space \mathcal{H} . Moreover, strong solutions $[u, u']$ satisfy the energy identity*

$$\frac{d}{dt} \mathcal{E}(u(t), u'(t)) = \int_{\Gamma_0} g(-\partial_\nu [A^{-1} u']) \partial_\nu [A^{-1} u'] d\sigma, \quad (8)$$

in the scalar distribution sense on $(0, +\infty)$.

Once the maximal monotone property in Theorem 1 is proved, that the evolution problem is well-posed in the Hadamard sense follows from Kato's theorem and standard nonlinear semigroup theory – see (Showalter, 2013, Chapter IV) for instance. Now, we claim that the closed-loop system is globally asymptotically stable around the zero equilibrium.

Theorem 2. *The evolution semigroup $\{\mathcal{S}_t\}$ associated with the closed-loop system (1)-(6) satisfies*

$$\lim_{t \rightarrow +\infty} \|\mathcal{S}_t[u_0, v_0]\|_{\mathcal{H}} = 0 \quad \text{for all } [u_0, v_0] \in \mathcal{H}. \quad (9)$$

Together with the contraction property of $\{\mathcal{S}_t\}$, (9) means that the zero equilibrium is globally asymptotically stable.

Proofs of Theorems 1 and 2 are given in Section 3. Let us give a sketch of the proof of Theorem 2, which relies on a LaSalle invariance argument and the Holmgren's uniqueness theorem in its global version. It suffices to prove that the ω -limit set of a given “smooth” initial condition $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$ is reduced to zero. With that in mind, we show that the ω -limit is comprised of trajectories $[w, w']$ obeying (1) with the dissipative term $-g(-\partial_\nu [A^{-1} w']) \partial_\nu [A^{-1} w']$ vanishing for a.e. (σ, t) in $\Gamma_0 \times (0, +\infty)$. This is seen using the energy identity (8). As a consequence, $A^{-1} w'$ is a (smoother) solution to the wave equation with homogeneous Dirichlet boundary condition and vanishing Cauchy data on $\Gamma_0 \times (0, +\infty)$. At this point, the global version of Holmgren theorem allows us to conclude. Of course, we need some compactness properties of the trajectories, which are obtained by investigating the properties of the generator \mathcal{A} , and in particular, those of the nonlinear resolvent operator $(\mathcal{A} + \lambda \text{id})^{-1}$. This is discussed in Section 3 as well.

2.2 Functional settings and definition of the generator

We continue by completing the brief definitions given in the introduction. We first define A as the isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ associated with the Laplacian operator and Dirichlet boundary conditions: for all $(u, w) \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\langle Au, w \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \triangleq \int_\Omega \nabla u \cdot \nabla w dx. \quad (10)$$

Identifying $L^2(\Omega)$ with its topological dual so that $L^2(\Omega)$ is viewed as a subset of $H^{-1}(\Omega)$, we can consider A as a strictly positive self-adjoint unbounded operator on $L^2(\Omega)$ with domain

$$\mathcal{D}(A) \triangleq \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}. \quad (11)$$

Since Ω is smooth, as mentioned earlier, elliptic regularity results give the explicit characterization

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad (12)$$

with the graph norm of $\mathcal{D}(A)$ being equivalent to the norm of $H^2(\Omega)$. This also proves that A^{-1} is a compact operator on $L^2(\Omega)$. As a consequence, using spectral decomposition, we can define the powers A^s of A for any real s , as well as Hilbertian structures for the associated domains $\mathcal{D}(A^s)$ – see (Temam, 2012, Section 2.1). In particular, we recover

$$\mathcal{D}(A^{1/2}) = H_0^1(\Omega), \quad \text{and } \mathcal{D}(A^{-1/2}) = H^{-1}(\Omega). \quad (13)$$

In the sequel, we shall also use that A can be extended, with the same symbol, as an isomorphism between $L^2(\Omega)$ and $\mathcal{D}(A^{-1}) \simeq \mathcal{D}(A)'$.

Furthermore, recall that the trace operator $\gamma : H^1(\Omega) \rightarrow H^{1/2}(\Gamma)$ admits a right inverse D satisfying

$$Df = u, \quad \text{where } \begin{cases} \Delta u = 0 & \text{in the sense of distributions,} \\ u|_\Gamma = f & \text{on } \Gamma. \end{cases} \quad (14)$$

We know from the theory of non-homogeneous elliptic boundary value problems that D can be extended as a continuous operator mapping $L^2(\Gamma)$ onto the interpolated Sobolev space $H^{1/2}(\Omega)$, which is continuously embedded into $L^2(\Omega)$. For the time being, it is enough that we see D as a continuous linear operator between $L^2(\Gamma)$ and $L^2(\Omega)$.

We denote by D^* the adjoint operator of D , which is uniquely defined as a continuous mapping between $L^2(\Omega)$ and $L^2(\Gamma)$ by

$$(D^*u, h)_{L^2(\Gamma)} = (u, Dh)_{L^2(\Omega)} \quad (15)$$

for all u in $L^2(\Omega)$ and h in $L^2(\Gamma)$. We have the identity

$$-D^*Au = \partial_\nu u \quad \text{for all } u \in \mathcal{D}(A), \quad (16)$$

which is shown using Green formula and the characterization of $\mathcal{D}(A)$ given by (12). Also, to take into account that only Γ_0 is subject to the control action, we define a projection operator P on $L^2(\Gamma)$ as follows:

$$[Ph](\sigma) \triangleq \mathbb{1}_{\Gamma_0}(\sigma)h(\sigma) \quad \text{for all } h \in L^2(\Gamma). \quad (17)$$

Using (16), we can write an operator-theoretic formulation of the control input:

$$U(t) = D^*u'(t) \quad (18)$$

As for the nonlinearity, we allow ourselves a little abuse of notation and denote by g the (nonlinear) Lipschitz mapping on $L^2(\Gamma)$ defined by

$$g(h)(\sigma) = g(h(\sigma)) \quad \text{for all } h \in L^2(\Gamma). \quad (19)$$

We can finally introduce the unbounded nonlinear operator \mathcal{A} associated with the closed-loop system. It is defined as follows:

$$\mathcal{D}(\mathcal{A}) \triangleq \left\{ [u, v] \in \mathcal{H} : \begin{cases} v \in L^2(\Omega) \\ A[u + DPg(D^*v)] \in H^{-1}(\Omega) \end{cases} \right\} \quad (20)$$

and, for all $[u, v] \in \mathcal{D}(\mathcal{A})$,

$$\mathcal{A}[u, v] \triangleq [-v, Au + ADPg(D^*v)]. \quad (21)$$

If u and v belong to $L^2(\Omega)$, then $A[u + DPg(D^*v)]$ is in $\mathcal{D}(A^{-1})$, which contains $H^{-1}(\Omega)$; therefore, the definition above makes sense. The isomorphism properties of A yield

$$[u, v] \in \mathcal{D}(\mathcal{A}) \iff \begin{cases} v \in L^2(\Omega), \\ u + DPg(D^*v) \in H_0^1(\Omega). \end{cases} \quad (22)$$

Note that, in general, it does not mean that $u \in H^1(\Omega)$.

3. PROPERTIES OF THE GENERATOR AND PROOFS

In this section, we discuss some properties of the nonlinear operator \mathcal{A} and we give the proofs of the main results.

3.1 Proof of well-posedness

The following lemma is adapted from (Showalter, 2013, Lemma 2.1 and Theorem 2.1). Its proof relies on the Brouwer fixed-point theorem in finite dimension followed by an application of the Galerkin method.

Lemma 1. *Let E be a separable reflexive Banach space and $f \in E'$. Assume $\Theta : E \rightarrow E'$ verifies the following assumptions:*

- (1) Θ is monotone i.e. $\langle \Theta(x_1) - \Theta(x_2), x_1 - x_2 \rangle_E \geq 0$ for all $x_1, x_2 \in E$;
- (2) Θ is bounded i.e. $S \subset E$ bounded implies $\Theta(S)$ bounded in E' ;
- (3) For all $x_1, x_2 \in E$, the scalar function $t \mapsto \langle \Theta(x_1 + tx_2), x_2 \rangle_E$ is continuous.

If for some $\rho > 0$, $\|x\|_E > \rho$ implies $\langle \Theta(x), x \rangle_E > \langle f, x \rangle_E$, then there exists $x \in E$ such that $\Theta(x) = f$.

It is used in the proof of Theorem 1, which we give now.

Proof of Theorem 1. The proof is split into four steps.

Step 1: Monotonicity of the operator. Let $[u_1, v_1]$ and $[u_2, v_2]$ in $\mathcal{D}(\mathcal{A})$. We write $u \triangleq u_1 - u_2$ and $v \triangleq v_1 - v_2$. Then,

$$\begin{aligned} (\mathcal{A}[u_1, v_1] - \mathcal{A}[u_2, v_2], [u_1, v_1] - [u_2, v_2])_{\mathcal{H}} &= -(v, u)_{L^2(\Omega)} \\ &\quad + (A^{1/2}[u + DPg(D^*v_1) - DPg(D^*v_2)], A^{-1/2}v)_{L^2(\Omega)}. \end{aligned} \quad (23)$$

Since $A^{-1/2}v \in \mathcal{D}(A^{1/2})$ and $A^{1/2}$ is self-adjoint, we obtain

$$\begin{aligned} (\mathcal{A}[u_1, v_1] - \mathcal{A}[u_2, v_2], [u_1, v_1] - [u_2, v_2])_{\mathcal{H}} &= (DPg(D^*v_1) - DPg(D^*v_2), v)_{L^2(\Omega)} \\ &= (Pg(D^*v_1) - Pg(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma)} \\ &= (g(D^*v_1) - g(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma_0)} \geq 0, \end{aligned} \quad (24)$$

where we use the nondecreasingness of the nonlinearity g . Monotonicity of the nonlinear operator \mathcal{A} is now proved.

Step 2: Surjectivity. Let $\lambda > 0$. Let us prove that $\text{Range}(\mathcal{A} + \lambda \text{id}) = \mathcal{H}$. We consider the equation

$$\mathcal{A}[u, v] + \lambda[u, v] = [f_1, f_2] \quad \text{in } \mathcal{H} \quad (25)$$

and we must find a solution $[u, v] \in \mathcal{D}(\mathcal{A})$ for any given $[f_1, f_2]$ in \mathcal{H} . Let assume for a moment that such $[u, v]$ exists. Then, v satisfies

$$A[\lambda^{-1}v + DPg(D^*v)] + \lambda v = f_2 - \lambda^{-1}Af_1 \quad \text{in } \mathcal{D}(A^{-1}). \quad (26)$$

Applying A^{-1} to (26) yields

$$\lambda^{-1}v + DPg(D^*v) + \lambda A^{-1}v = A^{-1}f_2 - \lambda^{-1}f_1 \quad \text{in } L^2(\Omega). \quad (27)$$

The left-hand side (resp. right-hand side) of (27) defines a linear form $\Theta(v) \in L^2(\Omega)'$ (resp. $\Psi \in L^2(\Omega)'$). Rewriting (27) as a variational problem, we seek $v \in L^2(\Omega)$ such that

$$\Theta(v) = \Psi \quad \text{in } L^2(\Omega)'. \quad (28)$$

Now, let us examine the mapping $\Theta : v \in L^2(\Omega) \mapsto \Theta(v) \in L^2(\Omega)'$ in order to apply Lemma 1. First, Θ is *monotone*. Let v_1 and v_2 in $L^2(\Omega)$; then,

$$\begin{aligned} & \langle \Theta(v_1) - \Theta(v_2), v_1 - v_2 \rangle_{L^2(\Omega)} \\ &= (g(D^*v_1) - g(D^*v_2), D^*[v_1 - v_2])_{L^2(\Gamma_0)} \\ &+ \lambda^{-1} \|v_1 - v_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{H^{-1}(\Omega)}^2 \geq 0. \end{aligned} \quad (29)$$

Also, Θ is *bounded*, since g is a Lipschitz continuous mapping on $L^2(\Gamma)$, A^{-1} is a continuous linear operator on $L^2(\Omega)$, D^* is continuous from $L^2(\Omega)$ into $L^2(\Gamma)$, and likewise, DP is continuous from $L^2(\Gamma)$ into $L^2(\Omega)$. Continuity of the function $t \mapsto \langle \Theta(v_1 + tv_2), v_2 \rangle_{L^2(\Omega)}$ follows from the same arguments. Finally, let $v \in L^2(\Omega)$; again, we have

$$\begin{aligned} & \langle \Theta(v), v \rangle_{L^2(\Omega)} - \langle \Psi, v \rangle_{L^2(\Omega)} \\ & \geq \lambda^{-1} \|v\|_{L^2(\Omega)}^2 - \|\Psi\|_{L^2(\Omega)'} \|v\|_{L^2(\Omega)} \rightarrow +\infty \end{aligned} \quad (30)$$

as $\|v\|_{L^2(\Omega)}$ goes to $+\infty$. As a consequence, the lemma provides the existence of $v \in L^2(\Omega)$ satisfying (28). Letting $u \triangleq \lambda^{-1}[f_1 + v]$, we infer from (26) that $[u, v]$ belongs to $\mathcal{D}(\mathcal{A})$ and that (25) is satisfied.

Step 3: Denseness of the domain. We now prove that $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$. Let $[u, v] \in \mathcal{H}$ and $\epsilon > 0$. First, $A^{-1}v \in H_0^1(\Omega)$, so we can pick $\phi \in \mathcal{C}_c^\infty(\Omega)$ such that $\|A^{-1}v - \phi\|_{H_0^1(\Omega)}^2 \leq \epsilon$. Then, by continuity of A from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$,

$$\|v - A\phi\|_{H^{-1}(\Omega)}^2 \leq C\epsilon. \quad (31)$$

where $C > 0$ does not depend on ϵ or the data. Moreover, pick $\psi \in \mathcal{C}_c^\infty(\Omega)$ such that $\|u - \psi\|_{L^2(\Omega)}^2 \leq \epsilon$. Since $\phi \in \mathcal{C}_c^\infty(\Omega)$, using (16) we see that

$$D^*A\phi = -\partial_\nu \phi = 0, \quad (32)$$

and $g(0) = 0$; also, $A\phi \in L^2(\Omega)$, thus $[\psi, A\phi] \in \mathcal{D}(\mathcal{A})$ and

$$\|[u, v] - [\psi, A\phi]\|_{\mathcal{H}}^2 \leq (1 + C)\epsilon. \quad (33)$$

Step 4: Conclusion. It is now proved that \mathcal{A} is a maximal monotone operator \mathcal{H} . The well-posedness of the evolution semigroup $\{\mathcal{S}_t\}$ follows from standard nonlinear semigroup theory – see (Showalter, 2013, Proposition 3.1 and Theorem 4.1A) for instance. It remains to prove the energy identity (8). Let $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$ and $[u(t), u'(t)] \triangleq \mathcal{S}_t[u_0, v_0]$. We have

$$\mathcal{E}(u(t), u'(t)) = \frac{1}{2} \|[u(t), u'(t)]\|_{\mathcal{H}}^2 \quad (34)$$

Hence, $t \mapsto \mathcal{E}(u(t), u'(t))$ is absolutely continuous, and for a.e. $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(u(t), u'(t)) &= -([u(t), u'(t)], \mathcal{A}[u(t), u'(t)])_{\mathcal{H}} \\ &= \int_{\Gamma_0} g(-\partial_\nu[A^{-1}u'(t)]) \partial_\nu[A^{-1}u'(t)] \, d\sigma \end{aligned} \quad (35)$$

where we use the calculations made in Step 1 and (16). \square

3.2 Proof of the stability result

As mentioned earlier, we need additional properties of the solutions in order to carry out the invariance argument

and prove Theorem 2. The next proposition shows that the nonlinear resolvent of the generator \mathcal{A} is well-behaved.

Proposition 1. *For any $\lambda > 0$, the (nonlinear) resolvent operator $(\mathcal{A} + \lambda \text{id})^{-1}$ is well-defined on \mathcal{H} and compact.*

Recalling that $\mathcal{A}(0) = 0$ and applying (Dafermos and Slemrod, 1973, Theorem 3), we obtain an important and useful consequence of Proposition 1: for any $[u_0, v_0]$ in \mathcal{H} ,

$$\{\mathcal{S}_t[u_0, v_0]\}_{t \geq 0} \text{ is a relatively compact subset of } \mathcal{H}. \quad (36)$$

The proof of the proposition is given next.

Proof of Proposition 1. Let $\lambda > 0$. That $\mathcal{A} + \lambda \text{id}$ is onto comes from Step 2 in the proof of Theorem 1. As for uniqueness, let $[f_1, f_2] \in \mathcal{H}$ and consider two solutions $[u_1, v_1]$ and $[u_2, v_2]$ to (25). Then, as in Step 2 of the proof of Theorem 1, we obtain

$$\lambda^{-1}[v_1 - v_2] + DP[g(D^*v_1) - g(D^*v_2)] + \lambda A^{-1}[v_1 - v_2] = 0. \quad (37)$$

Taking the scalar product in $L^2(\Omega)$ of (37) with $v_1 - v_2$ yields

$$\begin{aligned} & \lambda^{-1} \|v_1 - v_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{H^{-1}(\Omega)}^2 \\ &+ (g(D^*v_1) - g(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma_0)} = 0. \end{aligned} \quad (38)$$

Since g is nondecreasing, we infer from (38) that $v_1 = v_2$; thus, $[u_1, v_1] = [u_2, v_2]$ and $(\mathcal{A} + \lambda \text{id})^{-1}$ is well-defined.

Step 1. In what follows, we let $[u, v] \triangleq (\mathcal{A} + \lambda \text{id})^{-1}[f_1, f_2]$. We start with an estimate of the $L^2(\Omega)$ -norm of v . Again, v satisfies (27) and we obtain

$$\begin{aligned} & \lambda^{-1} \|v\|_{L^2(\Omega)}^2 + (g(D^*v), D^*v)_{L^2(\Gamma_0)} + \lambda \|v\|_{H^{-1}(\Omega)}^2 \\ &= (A^{-1}f_2 - \lambda^{-1}f_1, v)_{L^2(\Omega)} \\ &= (A^{-1/2}f_2, A^{-1/2}v)_{L^2(\Omega)} - \lambda^{-1}(f_1, v)_{L^2(\Omega)}, \end{aligned} \quad (39)$$

where we used that $A^{-1/2}$ is self-adjoint. Since the terms on the left-hand side of (39) are all nonnegative, we have

$$\begin{aligned} & \lambda^{-1} \|v\|_{L^2(\Omega)}^2 + \lambda \|v\|_{H^{-1}(\Omega)}^2 \\ & \leq \frac{1}{2\epsilon_1} \|f_2\|_{H^{-1}(\Omega)}^2 + \frac{\epsilon_1}{2} \|v\|_{H^{-1}(\Omega)}^2 \\ &+ \frac{1}{2\epsilon_2\lambda} \|f_1\|_{L^2(\Omega)}^2 + \frac{\epsilon_2}{2\lambda} \|v\|_{L^2(\Omega)}^2, \end{aligned} \quad (40)$$

where we used Cauchy-Schwarz and Young inequalities, and ϵ_1 and ϵ_2 are arbitrary positive parameter. If we chose, say, $\epsilon_1 = 2\lambda$ and $\epsilon_2 = 1$, we finally obtain the estimate

$$\|v\|_{L^2(\Omega)}^2 \leq (1/2) \|f_2\|_{H^{-1}(\Omega)}^2 + \|f_1\|_{L^2(\Omega)}^2. \quad (41)$$

Step 2. Next, we need a estimate of u in a stronger norm. We rely on the theory of elliptic regularity in the non-homogeneous case, and in particular on (Lions and Magenes, 1961, Théorème 10.1). Recall that u satisfies

$$A[u + DPg(D^*v)] + \lambda v = f_2 \quad \text{in } H^{-1}(\Omega). \quad (42)$$

Let ϕ be an arbitrary test-function in $\mathcal{C}_c^\infty(\Omega)$. Then, $\phi \in \mathcal{D}(A)$. We apply A^{-1} to (42) and take the scalar product in $L^2(\Omega)$ with $A\phi$ to obtain

$$- \int_{\Omega} u \Delta \phi \, dx + \lambda \int_{\Omega} v \phi \, dx = \langle f_2, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad (43)$$

where we used the fact that

$$(DPg(D^*v), A\phi)_{L^2(\Omega)} = (g(D^*v), D^*A\phi)_{L^2(\Gamma_0)} = 0 \quad (44)$$

since $D^*A\phi = -\partial_\nu \phi = 0$ as $\phi \in \mathcal{C}_c^\infty(\Omega)$. Recall that f_2 , as an element of $H^{-1}(\Omega)$, can be identified as a distribution. Then, since (43) holds for any $\phi \in \mathcal{C}_c^\infty(\Omega)$, we see that

$$-\Delta u + \lambda v = f_2 \quad \text{in the sense of distributions.} \quad (45)$$

We already know that $u \in L^2(\Omega)$, and we infer from (45) that the distribution $-\Delta u$ belongs to $H^{-1}(\Omega)$. As a consequence of elliptic regularity theory,

$$\text{The trace } u|_{\Gamma} \text{ is well-defined in } H^{-1/2}(\Gamma). \quad (46)$$

Denoting by γ the trace operator, we have

$$\gamma[u + Dg(D^*v)] = 0 \quad (47)$$

because, by definition of $\mathcal{D}(\mathcal{A})$, $u + Dg(D^*v) \in H_0^1(\Omega)$. Then, since $\gamma Dh = h$ for any $h \in L^2(\Gamma)$, we finally obtain

$$\begin{cases} -\Delta u = -\lambda v + f_2 & \text{in } H^{-1}(\Omega), \\ u|_{\Gamma} = -\mathbf{1}_{\Gamma_0} g(-\partial_{\nu}[A^{-1}v]) & \text{in } L^2(\Gamma). \end{cases} \quad (48)$$

Using the elliptic regularity theorem, we deduce from (48) that u enjoys the extra regularity $u \in H^{1/2}(\Omega)$, and that for some $C_1 > 0$ that does not depend on the data,

$$\|u\|_{H^{1/2}(\Omega)}^2 \leq C_1 \left\{ \|Pg(D^*v)\|_{L^2(\Gamma)}^2 + \|f_2 - \lambda v\|_{H^{-1}(\Omega)}^2 \right\}. \quad (49)$$

Recall that P and g are Lipschitz continuous on $L^2(\Gamma)$, and $g(0) = 0$. Also, D^* is linear continuous between $L^2(\Omega)$ and $L^2(\Gamma)$. As a result, there exists another constant $C_2 > 0$ such that

$$\begin{aligned} \|u\|_{H^{1/2}(\Omega)}^2 &\leq C_1 C_2 \|v\|_{L^2(\Omega)}^2 + 2C_1 \lambda^2 \|v\|_{H^{-1}(\Omega)}^2 \\ &\quad + 2C_1 \|f_2\|_{H^{-1}(\Omega)}^2. \end{aligned} \quad (50)$$

Step 3: Conclusion. Combining (41) and (50), we have shown that the image of a bounded set of \mathcal{H} by $(\mathcal{A} + \lambda \text{id})^{-1}$ is bounded in $H^{1/2}(\Omega) \times L^2(\Omega)$, which is compactly embedded into $L^2(\Omega) \times H^{-1}(\Omega)$. Therefore, the nonlinear operator $(\mathcal{A} + \lambda \text{id})^{-1}$ is compact. \square

Now that all the tools are at our disposal, we can finally prove the asymptotic stability theorem.

Proof of Theorem 2. In the monotone case, it is sufficient to prove that (9) holds for all initial data in the domain $\mathcal{D}(\mathcal{A})$. Indeed, let $X \in \mathcal{H}$ and $\tilde{X} \in \mathcal{D}(\mathcal{A})$ such that $\|X - \tilde{X}\|_{\mathcal{H}} \leq \epsilon$ for some $\epsilon > 0$. Then,

$$\|\mathcal{S}_t X\|_{\mathcal{H}} \leq \|\mathcal{S}_t \tilde{X}\|_{\mathcal{H}} + \|\mathcal{S}_t X - \mathcal{S}_t \tilde{X}\|_{\mathcal{H}} \quad \text{for all } t \geq 0. \quad (51)$$

Since we have a semigroup of contractions,

$$\|\mathcal{S}_t X - \mathcal{S}_t \tilde{X}\|_{\mathcal{H}} \leq \|X - \tilde{X}\|_{\mathcal{H}} \leq \epsilon \quad \text{for all } t \geq 0. \quad (52)$$

If (9) is satisfied for initial conditions in $\mathcal{D}(\mathcal{A})$, then, for sufficiently large t , we have $\|\mathcal{S}_t X\|_{\mathcal{H}} \leq 2\epsilon$.

That being said, let us pick $X \in \mathcal{D}(\mathcal{A})$. We use a LaSalle-type invariance argument. Recall that the ω -limit set of X (with respect to the evolution semigroup $\{\mathcal{S}_t\}$) is defined by

$$\omega(X) \triangleq \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \mathcal{S}_t X}, \quad (53)$$

where the closure is taken in \mathcal{H} . It can be characterized as follows: $W \in \mathcal{H}$ belongs to $\omega(X)$ if and only if there exists an increasing sequence $\{t_n\} \in \mathbb{R}^{\mathbb{N}}$ such that $t_n \rightarrow +\infty$ and

$$\mathcal{S}_{t_n} X \rightarrow W \quad \text{as } n \rightarrow +\infty. \quad (54)$$

Since the (semi)trajectory $\{\mathcal{S}_t X\}_{t \geq 0}$ is relatively compact, $\omega(X)$ is a nonempty (positively) invariant compact set, and $\text{dist}(\mathcal{S}_t X, \omega(X)) \rightarrow 0$ as $t \rightarrow +\infty$ – see (Haraux, 1991, Théorème 1.1.8). Furthermore, recall from Kato's theorem that

$$t \mapsto \|\mathcal{A}(\mathcal{S}_t X)\|_{\mathcal{H}} \text{ is bounded.} \quad (55)$$

As a consequence, using the characterization of the ω -limit given by (54) and applying (Crandall and Pazy, 1969, Lemma 2.3), we obtain that $\omega(X) \subset \mathcal{D}(\mathcal{A})$.

Now, it remains to prove that $\omega(X)$ is reduced to $\{0\}$. Since $t \mapsto \mathcal{E}(\mathcal{S}_t X)$ is bounded and nonincreasing, it must converge to some $\mathcal{E}_{\infty} \geq 0$ as t goes to $+\infty$. It follows from (54) and the continuity of \mathcal{E} that $\mathcal{E}(W) = \mathcal{E}_{\infty}$ for any $W \in \omega(X)$. Let $W \in \omega(X)$. Since $\mathcal{S}_t(\omega(X)) \subset \omega(X)$, we must have $\mathcal{E}(\mathcal{S}_t W) = \mathcal{E}_{\infty}$ for all $t \geq 0$. We write $[w(t), w'(t)] \triangleq \mathcal{S}_t W$. Using the energy identity (8), we obtain

$$\int_0^{\tau} \int_{\Gamma_0} g(-\partial_{\nu}[A^{-1}w']) \partial_{\nu}[A^{-1}w'] \, d\sigma \, dt = 0 \quad (56)$$

holding for all $\tau \geq 0$. Using the assumptions on the nonlinearity g , it comes from (56) that

$$\partial_{\nu}[A^{-1}w'](\sigma, t) = 0 \quad \text{for a.e. } (\sigma, t) \in \Gamma_0 \times (0, +\infty). \quad (57)$$

Now, let us have a look at $z \triangleq A^{-1}w'$. Since $[u, u']$ is a (strong) solution to (1)-(6), we have in particular $w' \in \mathcal{C}(\mathbb{R}^+, H^{-1}(\Omega)) \cap W^{1,\infty}(0, +\infty; \mathcal{D}(A^{-1}))$; and w satisfies the following differential equation in the sense of $\mathcal{D}(A^{-1})$ -valued distributions:

$$w''(t) + Aw(t) = -ADg(D^*w'(t)) = 0, \quad (58)$$

where we used (57). Then, the isomorphism properties of A yield

$$z \in \mathcal{C}(\mathbb{R}^+, H_0^1(\Omega)) \cap W^{1,\infty}(0, +\infty; L^2(\Omega)); \quad (59)$$

with $z' = [A^{-1}w']' = A^{-1}w'' = -w$, where we used (58). Formal differentiations are justified by the following vector-valued integrals: for all $t \geq 0$, $z(t) = A^{-1}w'(t) = A^{-1}w'(0) + A^{-1} \int_0^t w''(s) \, ds = z(0) + \int_0^t A^{-1}w''(s) \, ds$. Similarly, it follows from $z' = -w$ that z satisfies the standard variational formulation of the wave equation with homogeneous Dirichlet boundary condition, namely

$$z''(t) + Az(t) = 0, \quad (60)$$

in the sense of $H^{-1}(\Omega)$ -valued distributions. It can be deduced from (57) and (60) that z satisfies the following boundary-value problem:

$$\partial_{tt} z - \Delta z = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (61a)$$

$$z|_{\Gamma} = 0 \quad \text{on } \Gamma \times (0, +\infty), \quad (61b)$$

$$\partial_{\nu} z = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (61c)$$

which is in fact verified in a classical, pointwise sense. Indeed, since $[w, w']$ is a strong solution, z takes values in $\mathcal{D}(A)$. Now, a classic unique continuation argument using Holmgren's theorem – as in the proof of (Lasiecka and Triggiani, 1992, Lemma 4.3) – yields $z(t) = A^{-1}w'(t) = 0$, hence $w'(t) = 0$ for all $t \geq 0$. Thus, w is in fact a stationary solution, which satisfies $Aw = 0$, i.e. $w = 0$. \square

4. NUMERICAL SIMULATIONS

In this section, we illustrate Theorem 2 with some simulations of the closed-loop system (1)-(6) in space dimension two, using the FreeFEM finite element software – see Hecht (2012). At each time step, two finite-dimensional variational problems are solved: first, to compute the boundary feedback, and then, to solve the wave equation with the updated boundary condition. The wave is solved using an implicit scheme, while the feedback is updated according to an explicit scheme. Here, Ω is a two-dimensional ring-shaped domain, with Γ_0 being the circle of radius $1/2$

and Γ_1 being the circle of radius 1. It is discretized into $N = 60$ triangles. For technical reasons, we consider a smooth nonlinearity given by $g = \arctan$. The time step is $dt = 0.01$. Figure 2 shows the position variable u at the initial time $t = 0$, while Figures 3 represent the solution after some time under the feedback action. Figure 4 illustrates the decay of the energy \mathcal{E} along the solution. The simulation code for this example can be found at <https://github.com/nvanspra/dirichlet-wave>.

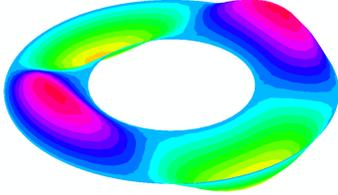


Fig. 2. Smooth initial data given by $u_0(x, y) = xy \sin(2\pi(x^2 + y^2)^{1/2})$ and $v_0(x, y) = 0$.

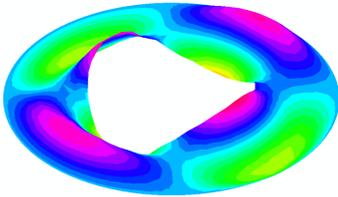


Fig. 3. Representation of u at the time $t = 0.35$. Note that only the inner boundary is actuated.

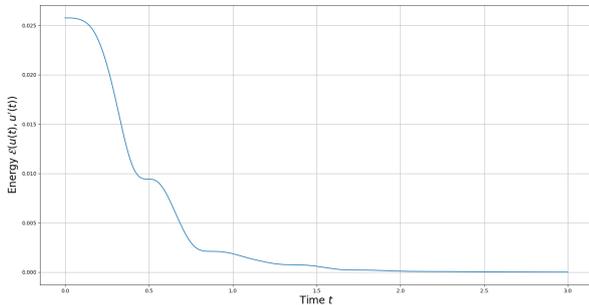


Fig. 4. Evolution of the energy along the trajectory.

5. CONCLUDING REMARKS

In this paper, asymptotic stability of a high-dimensional wave equation with a nonlinear Dirichlet boundary feedback has been proved. Since the analysis has been carried out in the framework of nonlinear semigroups, a natural extension would concern *multivalued* (but still dissipative) nonlinearities in the feedback loop, as in Chitour et al. (2020) where a one-dimensional wave equation is considered in the Neumann boundary conditions. Non-smooth nonlinearities of interest in control include *quantizers*, as in Ferrante et al. (2020), where the effect of sensor quantization on the stability of finite-dimensional linear systems is studied.

ACKNOWLEDGEMENTS

This work has been partially supported by MIAI @ Grenoble Alpes, (ANR-19-P3IA-0003).

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