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# The Impact of Geometry on Monochrome Regions in the Flip Schelling Process

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## Abstract

Schelling’s classical segregation model gives a coherent explanation for the wide-spread phenomenon of residential segregation. We introduce an agent-based saturated open-city variant, the Flip Schelling Process (FSP), in which agents, placed on a graph, have one out of two types and, based on the predominant type in their neighborhood, decide whether to change their types; similar to a new agent arriving as soon as another agent leaves the vertex.

We investigate the probability that an edge  $\{u, v\}$  is monochrome, i.e., that both vertices  $u$  and  $v$  have the same type in the FSP, and we provide a general framework for analyzing the influence of the underlying graph topology on residential segregation. In particular, for two adjacent vertices, we show that a highly decisive common neighborhood, i.e., a common neighborhood where the absolute value of the difference between the number of vertices with different types is high, supports segregation and, moreover, that large common neighborhoods are more decisive.

As an application, we study the expected behavior of the FSP on two common random graph models with and without geometry: (1) For random geometric graphs, we show that the existence of an edge  $\{u, v\}$  makes a highly decisive common neighborhood for  $u$  and  $v$  more likely. Based on this, we prove the existence of a constant  $c > 0$  such that the expected fraction of monochrome edges after the FSP is at least  $1/2 + c$ . (2) For Erdős–Rényi graphs we show that large common neighborhoods are unlikely and that the expected fraction of monochrome edges after the FSP is at most  $1/2 + o(1)$ . Our results indicate that the cluster structure of the underlying graph has a significant impact on the obtained segregation strength.

**2012 ACM Subject Classification** Theory of computation → Network formation; Theory of computation → Random network models

**Keywords and phrases** Agent-based Model, Schelling Segregation, Spin System

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## 1 Introduction

Residential segregation is a well-known sociological phenomenon [49] where different groups of people tend to separate into largely homogeneous neighborhoods. Studies, e.g., [18], show that individual preferences are the driving force behind present residential patterns and bear much to the explanatory weight. Local choices therefore lead to a global phenomenon [47]. A simple model for analyzing residential segregation was introduced by Schelling [46, 47] in



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44 the 1970s. In his model, two types of agents, placed on a grid, act according to the following  
 45 threshold behavior, with  $\tau \in (0, 1)$  as the *intolerance threshold*: agents are *content* with  
 46 their current position on the grid if at least a  $\tau$ -fraction of neighbors is of their own type.  
 47 Otherwise they are *discontent* and want to move, either via swapping with another random  
 48 discontent agent or via jumping to a vacant position. Schelling demonstrated via simulations  
 49 that, starting from a uniform random distribution, the described process drifts towards strong  
 50 segregation, even if agents are tolerant and agree to live in mixed neighborhoods, i.e., if  $\tau \leq \frac{1}{2}$ .  
 51 Many empirical studies have been conducted to investigate the influence of various parameters  
 52 on the obtained segregation, see [8, 9, 25, 41, 45]. On the theoretical side, Schelling's model  
 53 started recently gaining traction within the algorithmic game theory and artificial intelligence  
 54 communities [1, 11, 16, 17, 21, 22, 33], with focus on core game theoretic questions, where  
 55 agents strategically select locations. Henry et al. [31] described a simple model of graph  
 56 clustering motivated by Schelling where they showed that segregated graphs always emerge.  
 57 Variants of the random Schelling segregation process were analyzed by a line of work that  
 58 showed that residential segregation occurs with high probability [5, 7, 10, 13, 32, 51].

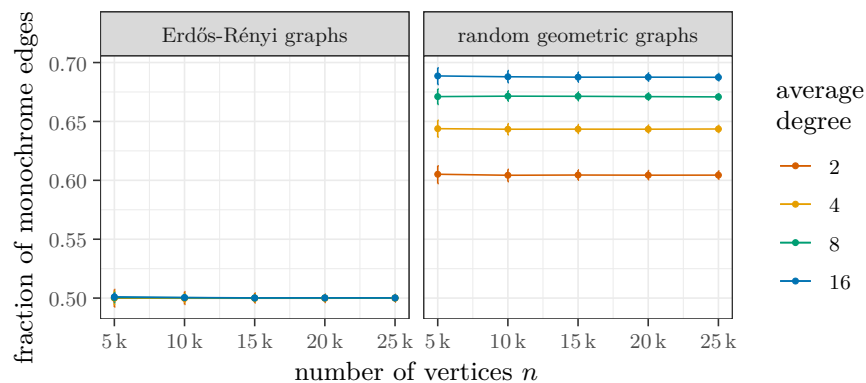
59 We initiate the study of an agent-based model, called the *Flip Schelling Process (FSP)*,  
 60 which can be understood as the Schelling model in a *saturated open city*. In contrast to *closed*  
 61 *cities* [7, 13, 32, 51], which require fixed populations, open cities [4, 5, 10, 27] allow resident  
 62 to move away. In saturated city models, also known as voter models [20, 35, 36], vertices are  
 63 not allowed to be unoccupied, hence, a new agent enters as soon as one agent vacates a vertex.  
 64 In general, in voter models, two types of agents are placed on a graph. Agents examine their  
 65 neighbors and, if a certain threshold is of another type, they change their types. Also in  
 66 this model, segregation is visible. There is a line of work, mainly in physics, that studies  
 67 the voting dynamics on several types of graphs [3, 14, 37, 43, 50]. Related to voter models,  
 68 Granovetter [30] proposed another threshold model treating binary decisions and spurred a  
 69 number of research, which studied and motivated variants of the model, see [2, 34, 38, 44].

70 In the FSP, agents have binary types. An agent is content if the fraction of agents in  
 71 its neighborhood with the same type is larger than  $\frac{1}{2}$ . Otherwise, if the fraction is smaller  
 72 than  $\frac{1}{2}$ , an agent is discontent and is willing to flip its type to become content. If the fraction  
 73 of same type agents in its neighborhood is exactly  $\frac{1}{2}$ , an agent flips its type with probability  $\frac{1}{2}$ .  
 74 Starting from an initial configuration where the type of each agent is chosen uniformly at  
 75 random, we investigate a simultaneous-move, one-shot process and bound the number of  
 76 monochrome edges, which is a popular measurement for segregation strength [19, 26].

77 Close to our model is the work by Omidvar and Franceschetti [39, 40], who initiated an  
 78 analysis of the size of monochrome regions in the so called *Schelling Spin Systems*. Agents of  
 79 two different types are placed on a grid [39] and a geometric graph [40], respectively. Then  
 80 independent and identical Poisson clocks are assigned to all agents and, every time a clock  
 81 rings, the state of the corresponding agent is flipped if and only if the agent is discontent w.r.t.  
 82 a certain intolerance threshold  $\tau$  regarding the neighborhood size. The model corresponds to  
 83 the Ising model with zero temperature with Glauber dynamics [15, 48].

84 The commonly used underlying topology for modeling the residential areas are (toroidal)  
 85 grid graphs [11, 32, 39], regular graphs [11, 17, 21], paths [11, 33], cycles [4, 6, 7, 13, 51]  
 86 and trees [1, 11, 22, 33]. Considering the influence of the given topology that models the  
 87 residential area regarding, e.g., the existence of stable states and convergence behavior  
 88 leads to phenomena like non-existence of stable states [21, 22], non-convergence to stable  
 89 states [11, 17, 21], and high-mixing times in corresponding Markov chains [10, 28].

90 To avoid such undesirable characteristics, we suggest to investigate *random geometric*  
 91 *graphs* [42], like in [40]. Random geometric graphs demonstrate, in contrast to other random



■ **Figure 1** The fraction of monochrome edges after the Flip Schelling Process (FSP) in Erdős-Rényi graphs and random geometric graphs for different graph sizes (number of vertices  $n$ ) and different expected average degrees. Each data point shows the average over 1000 generated graphs with one simulation of the FSP per graph. The error bars show the interquartile range, i.e., 50% of the measurements lie between the top and bottom end of the error bar.

92 graphs without geometry, such as *Erdős-Rényi graphs* [23, 29], community structures, i.e.,  
 93 densely connected clusters of vertices. An effect observed by simulating the FSP is that the  
 94 fraction of monochrome edges is significantly higher in random geometric graphs compared  
 95 to Erdős-Rényi graphs, where the fraction stays almost stable around  $\frac{1}{2}$ , cf. Fig 1.

96 We set out for rigorously proving this phenomenon. In particular, we prove for random  
 97 geometric graphs that there exists a constant  $c$  such that, given an edge  $\{u, v\}$ , the probability  
 98 that  $\{u, v\}$  is monochrome is lower-bounded by  $\frac{1}{2} + c$ , cf. Theorem 6. In contrast, we show  
 99 for Erdős-Rényi graphs that segregation is not likely to occur and that the probability that  
 100  $\{u, v\}$  is monochrome is upper-bounded by  $\frac{1}{2} + o(1)$ , cf. Theorem 17.

101 We introduce a general framework to deepen the understanding of the influence of the  
 102 underlying topology on residential segregation. To this end, we first show that a highly decisive  
 103 common neighborhood supports segregation, cf. Section 3.1. In particular, we provide a lower  
 104 bound on the probability that an edge  $\{u, v\}$  is monochrome based on the probability that  
 105 the difference between the majority and the minority regarding both types in the common  
 106 neighborhood, i.e., the number of agents which are adjacent to  $u$  and  $v$ , is larger than their  
 107 exclusive neighborhoods, i.e., the number of agents which are adjacent to either  $u$  or  $v$ . Next,  
 108 we show that large sets of agents are more decisive, cf. Section 3.2. This implies that a large  
 109 common neighborhood, compared to the exclusive neighborhood, is likely to be more decisive,  
 110 i.e., makes it more likely that the absolute value of the difference between the number of  
 111 different types in the common neighborhood is larger than in the exclusive ones. These  
 112 considerations hold for arbitrary graphs. Hence, we reduce the question concerning a lower  
 113 bound for the fraction of monochrome edges in the FSP to the probability that, given  $\{u, v\}$ ,  
 114 the common neighborhood is larger than the exclusive neighborhoods of  $u$  and  $v$ , respectively.

115 For random geometric graphs, we prove that a large geometric region, i.e., the intersecting  
 116 region that is formed by intersecting disks, leads to a large vertex set, cf. Section 3.3, and  
 117 that random geometric graphs have enough edges that have sufficiently large intersecting  
 118 regions, cf. Section 3.4, such that segregation is likely to occur. In contrast, for Erdős-Rényi  
 119 graphs, we show that the common neighborhood between two vertices  $u$  and  $v$  is with high  
 120 probability empty and therefore segregation is not likely to occur, cf. Section 4.

121 Overall, we shed light on the influence of the structure of the underlying graph and

122 discovered the significant impact of the community structure as an important factor on the  
 123 obtained segregation strength. We reveal for random geometric graphs that already after  
 124 one round a provable tendency is apparent and a strong segregation occurs.

## 125 2 Model and Preliminaries

126 Let  $G = (V, E)$  be an unweighted and undirected graph, with vertex set  $V$  and edge set  $E$ . For  
 127 any vertex  $v \in V$ , we denote the *neighborhood* of  $v$  in  $G$  by  $N_v = \{u \in V : \{u, v\} \in E\}$  and  
 128 the degree of  $v$  in  $G$  by  $\delta_v = |N_v|$ . We consider *random geometric graphs* and *Erdős–Rényi*  
 129 *graphs* with a total of  $n \in \mathbf{N}^+$  vertices and an *expected average degree*  $\bar{\delta} > 0$ .

130 For a given  $r \in \mathbf{R}^+$ , a random geometric graph  $G \sim \mathcal{G}(n, r)$  is obtained by distributing  $n$   
 131 vertices uniformly at random in some geometric ground space and connecting vertices  $u$  and  $v$   
 132 if and only if  $\text{dist}(u, v) \leq r$ . We use a two-dimensional toroidal Euclidean space with total  
 133 area 1 as ground space. More formally, each vertex  $v$  is assigned to a point  $(v_1, v_2) \in [0, 1]^2$   
 134 and the distance between  $u = (u_1, u_2)$  and  $v$  is  $\text{dist}(u, v) = \sqrt{|u_1 - v_1|_0^2 + |u_2 - v_2|_0^2}$  for  
 135  $|u_i - v_i|_0 = \min\{|u_i - v_i|, 1 - |u_i - v_i|\}$ . We note that using a torus instead of, e.g., a unit  
 136 square, has the advantage that we do not have to consider edge cases, for vertices that are  
 137 close to the boundary. In fact, a disk of radius  $r$  around any point has the same area  $\pi r^2$ .  
 138 Since we consider a ground space with total area 1,  $r \leq \frac{1}{\sqrt{\pi}}$ . As every vertex  $v$  is connected  
 139 to all vertices in the disk of radius  $r$  around it, its expected average degree is  $\bar{\delta} = (n - 1)\pi r^2$ .

140 For a given  $p \in [0, 1]$ , let  $\mathcal{G}(n, p)$  denote an Erdős–Rényi graph. Each edge  $\{u, v\}$  is  
 141 included with probability  $p$ , independently from every other edge. It holds that  $\bar{\delta} = (n - 1)p$ .

142 Consider two different vertices  $u$  and  $v$ . Let  $N_{u \cap v} := |N_u \cap N_v|$  be the number of vertices  
 143 in the *common neighborhood*, let  $N_{u \setminus v} := |N_u \setminus N_v|$  be the number of vertices in the *exclusive*  
 144 *neighborhood* of  $u$ , and let  $N_{v \setminus u} := |N_v \setminus N_u|$  be the number of vertices in the exclusive  
 145 neighborhood of  $v$ . Furthermore, with  $N_{\overline{u \cup v}} := |V \setminus (N_u \cup N_v)|$ , we denote the number of  
 146 vertices that are neither adjacent to  $u$  nor to  $v$ .

147 Let  $G$  be a graph where each vertex represents an agent of type  $t^+$  or  $t^-$ . The type of  
 148 each agent is chosen independently and uniformly at random. An edge  $\{u, v\}$  is *monochrome*  
 149 if and only if  $u$  and  $v$  are of the same type. The *Flip Schelling Process* (FSP) is defined as  
 150 follows: an agent  $v$  whose type is aligned with the type of more than  $\delta_v/2$  of its neighbors  
 151 keeps its type. If more than  $\delta_v/2$  neighbors have a different type, then agent  $v$  changes its  
 152 type. In case of a tie, i.e., if exactly  $\delta_v/2$  neighbors have a different type, then  $v$  changes its  
 153 type with probability  $\frac{1}{2}$ . FSP is a simultaneous-move, one-shot process, i.e., all agents make  
 154 their decision at the same time and, moreover, only once.

155 For  $x, y \in \mathbf{N}$ , we define  $[x..y] = [x, y] \cap \mathbf{N}$  and for  $x \in \mathbf{N}^+$ , we define  $[x] = [1..x]$ .

### 156 2.1 Useful Technial Lemmas

157 In this section, we state several lemmas that we will use in order to prove our results in the  
 158 next sections.

159 ► **Lemma 1.** *Let  $X \sim \text{Bin}(n, p)$  and  $Y \sim \text{Bin}(n, q)$  with  $p \geq q$  be independent. Then*  
 160  $\Pr[X \geq Y] \geq \frac{1}{2}$ .

161 **Proof.** Let  $Y_1, \dots, Y_n$  be the individual Bernoulli trials for  $Y$ , i.e.,  $Y = \sum_{i \in [n]} Y_i$ . Define new  
 162 random variables  $Y'_1, \dots, Y'_n$  such that  $Y_i = 1$  implies  $Y'_i = 1$  and if  $Y_i = 0$ , then  $Y'_i = 1$  with  
 163 probability  $(p - q)/(1 - q)$  and  $Y'_i = 0$  otherwise. Note that for each individual  $Y'_i$ , we have  
 164  $Y'_i = 1$  with probability  $p$ , i.e.,  $Y' = \sum_{i \in [n]} Y'_i \sim \text{Bin}(n, p)$ . Moreover, as  $Y' \geq Y$  for every  
 165 outcome, we have  $\Pr[X \geq Y] \geq \Pr[X \geq Y']$ . It remains to show that  $\Pr[X \geq Y'] \geq \frac{1}{2}$ .

166 As  $X$  and  $Y'$  are equally distributed, we have  $\Pr[X \geq Y'] = \Pr[X \leq Y']$ . Moreover, as  
 167 one of the two inequalities holds in any event, we get  $\Pr[X \geq Y'] + \Pr[X \leq Y'] \geq 1$ , and  
 168 thus equivalently  $2\Pr[X \geq Y'] \geq 1$ , which proves the claim. ◀

169 ▶ **Lemma 2.** *Let  $n \in \mathbf{N}^+$ ,  $p \in [0, 1)$ , and let  $X \sim \text{Bin}(n, p)$ . Then, for all  $i \in [0..n]$ , it holds*  
 170 *that  $\Pr[X = i] \leq \Pr[X = \lfloor p(n + 1) \rfloor]$ .*

171 **Proof.** We interpret the distribution of  $X$  as a finite series and consider the sign of the  
 172 differences of two neighboring terms. A maximum of the distribution of  $X$  is located at  
 173 the position at which the difference switches from positive to negative. To this end, let  
 174  $b: [0, n - 1] \rightarrow [-1, 1]$  be defined such that, for all  $i \in [0, n - 1] \cap \mathbf{N}$ , it holds that

$$175 \quad b(d) = \binom{n}{d+1} p^{d+1} (1-p)^{n-d-1} - \binom{n}{d} p^d (1-p)^{n-d}.$$

176 We are interested in the sign of  $b$ . In more detail, for any  $d \in [0, n - 2] \cap \mathbf{N}$ , if  $\text{sgn}(b(d)) \geq 0$   
 177 and  $\text{sgn}(b(d + 1)) \leq 0$ , then  $d + 1$  is a local maximum. If the sign is always negative, then  
 178 there is a global maximum in the distribution of  $X$  at position 0.

179 In order to determine the sign of  $b$ , for all  $i \in [0..n - 1]$ , we rewrite

$$180 \quad b(i) = \frac{n!}{i!(n-i-1)!} p^i (1-p)^{n-i-1} \frac{p}{i+1} - \frac{n!}{i!(n-i-1)!} p^d (1-p)^{n-i-1} \frac{1-p}{n-i}$$

$$181 \quad = \frac{n!}{i!(n-i-1)!} p^i (1-p)^{n-i-1} \left( \frac{p}{i+1} - \frac{1-p}{n-i} \right).$$

183 Since the term  $n! p^i (1-p)^{n-i-1}$  is always non-negative, the sign of  $b(i)$  is determined by the  
 184 sign of  $p/(i + 1) - (1 - p)/(n - i)$ .

185 Solving for  $i$ , we get that

$$186 \quad \frac{p}{i+1} - \frac{1-p}{n-i} \geq 0 \Leftrightarrow i \leq p(n+1) - 1.$$

188 Note that  $p(n + 1) - 1$  may not be integer. Further note that the distribution of  $X$  is  
 189 unimodal, as the sign of  $b$  changes at most once. Thus, each local maximum is also a global  
 190 maximum. As discussed above, the largest value  $d \in [0, n - 2] \cap \mathbf{N}$  such that  $\text{sgn}(b(d)) \geq 0$   
 191 and  $\text{sgn}(b(d + 1)) \leq 0$  then results in a global maximum at position  $d + 1$ . Since  $d$  needs to  
 192 be integer, the largest value that satisfies this constraint is  $\lfloor p(n + 1) - 1 \rfloor$ . If the sign of  $b$   
 193 is always negative ( $p \leq 1/(n + 1)$ ), then the distribution of  $X$  has a global maximum at 0,  
 194 which is also satisfied by  $\lfloor p(n + 1) - 1 \rfloor + 1$ , which concludes the proof. ◀

195 ▶ **Theorem 3** (Stirling's Formula [24, page 54]). *For all  $n \in \mathbf{N}^+$ , it holds that*

$$196 \quad \sqrt{2\pi n}^{n+1/2} e^{-n} \cdot e^{(12n+1)^{-1}} < n! < \sqrt{2\pi n}^{n+1/2} e^{-n} \cdot e^{(12n)^{-1}}.$$

198 ▶ **Corollary 4.** *For all  $n \geq 2$  with  $n \in \mathbf{N}$ , it holds that*

$$199 \quad n! > \sqrt{2\pi n}^{n+1/2} e^{-n} \quad \text{and} \tag{1}$$

$$200 \quad n! < e n^{n+1/2} e^{-n}. \tag{2}$$

202 **Proof.** For both inequalities, we aim at using Theorem 3.

203 Equation (1): Note that  $e^{(12n+1)^{-1}} > 1$ , since  $\frac{1}{12n+1} > 0$ . Hence,

$$204 \quad \sqrt{2\pi n}^{n+1/2} e^{-n} < \sqrt{2\pi n}^{n+1/2} e^{-n} \cdot e^{(12n+1)^{-1}}.$$

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205 Equation (2): We prove this case by showing that

$$206 \quad \sqrt{2\pi}e^{(12n)^{-1}} < e. \quad (3)$$

208 Note, that  $e^{(12n)^{-1}}$  is strictly decreasing. Hence, we only have to check whether Equation (3)  
209 holds for  $n = 2$ .

$$210 \quad \sqrt{2\pi}e^{(12n)^{-1}} \leq \sqrt{2\pi}e^{\frac{1}{24}} < 2.7 < e. \quad \blacktriangleleft$$

211 **► Lemma 5.** *Let  $A$ ,  $B$ , and  $C$  be random variables such that  $\Pr[A > C \wedge B > C] > 0$  and*  
212  *$\Pr[A > C \wedge B \leq C] > 0$ . Then  $\Pr[A > B \wedge A > C] \geq \Pr[A > B] \cdot \Pr[A > C]$ .*

213 **Proof.** Using the definition of conditional probability, we obtain

$$214 \quad \Pr[A > B \wedge A > C] = \Pr[A > B \mid A > C] \cdot \Pr[A > C].$$

216 Hence, we are left with bounding  $\Pr[A > B \mid A > C] \geq \Pr[A > B]$ . Partitioning the sample  
217 space into the two events  $B > C$  and  $B \leq C$  and using the law of total probability, we obtain

$$218 \quad \Pr[A > B \mid A > C] = \Pr[B > C \mid A > C] \cdot \Pr[A > B \mid A > C \wedge B > C] \\ 219 \quad \quad \quad + \Pr[B \leq C \mid A > C] \cdot \Pr[A > B \mid A > C \wedge B \leq C].$$

221 Note that the condition  $A > C \wedge B \leq C$  already implies  $A > B$  and thus the last probability  
222 equals to 1. Moreover, using the definition of conditional probability, we obtain

$$223 \quad \Pr[A > B \mid A > C] = \Pr[B > C \mid A > C] \cdot \frac{\Pr[A > B \wedge A > C \wedge B > C]}{\Pr[A > C \wedge B > C]} \\ 224 \quad \quad \quad + \Pr[B \leq C \mid A > C].$$

226 Using that  $\Pr[B > C \mid A > C] \geq \Pr[A > C \wedge B > C]$ , that  $A > B \wedge B > C$  already  
227 implies  $A > C$ , that  $\Pr[B \leq C \mid A > C] \geq \Pr[A > B \wedge B \leq C]$ , and finally the law of total  
228 probability, we obtain

$$229 \quad \Pr[A > B \mid A > C] \geq \Pr[A > B \wedge A > C \wedge B > C] + \Pr[B \leq C \mid A > C] \\ 230 \quad \quad \quad = \Pr[A > B \wedge B > C] + \Pr[B \leq C \mid A > C] \\ 231 \quad \quad \quad \geq \Pr[A > B \wedge B > C] + \Pr[A > B \wedge B \leq C] \\ 232 \quad \quad \quad = \Pr[A > B]. \quad \blacktriangleleft$$

### 234 **3 Monochrome Edges in Geometric Random Graphs**

235 In this section, we prove the following main theorem.

236 **► Theorem 6.** *Let  $G \sim \mathcal{G}(n, r)$  be a random geometric graph with expected average degree*  
237  *$\bar{\delta} = o(\sqrt{n})$ . The expected fraction of monochrome edges after the FSP is at least*

$$238 \quad \frac{1}{2} + \frac{9}{800} \cdot \left( \frac{1}{2} - \frac{1}{\sqrt{2\pi\lceil \bar{\delta}/2 \rceil}} \right)^2 \cdot \left( 1 - e^{-\bar{\delta}/2} \left( 1 + \frac{\bar{\delta}}{2} \right) \right) \cdot (1 - o(1)).$$

239 Note that the bound in Theorem 6 is bounded away from  $\frac{1}{2}$  for all  $\bar{\delta} \geq 2$ . Moreover, the two  
240 factors depending on  $\bar{\delta}$  go to  $\frac{1}{2}$  and 1, respectively, for a growing  $\bar{\delta}$ .

241 Given an edge  $\{u, v\}$ , we prove the above lower bound on the probability that  $\{u, v\}$  is  
242 monochrome in the following four steps.

- 243 1. For a vertex set, we introduce the concept of *decisiveness* that measures how much the  
244 majority is ahead of the minority in the FSP. With this, we give a lower bound on  
245 the probability that  $\{u, v\}$  is monochrome based on the probability that the common  
246 neighborhood of  $u$  and  $v$  is more decisive than their exclusive neighborhoods.
- 247 2. We show that large neighborhoods are likely to be more decisive than small neighborhoods.  
248 To this end, we give bounds on the likelihood that two similar random walks behave  
249 differently. This step reduces the question of whether the common neighborhood is more  
250 decisive than the exclusive neighborhoods to whether the former is larger than the latter.
- 251 3. Turning to geometric random graphs, we show that the common neighborhood is suf-  
252 ficiently likely to be larger than the exclusive neighborhoods if the geometric region  
253 corresponding to the former is sufficiently large. We do this by first showing that the ac-  
254 tual distribution of the neighborhood sizes is well approximated by independent binomial  
255 random variables. Then, we give the desired bounds for these random variables.
- 256 4. We show that the existence of the edge  $\{u, v\}$  in the geometric random graph makes it  
257 sufficiently likely that the geometric region hosting the common neighborhood of  $u$  and  $v$   
258 is sufficiently large.

### 259 3.1 Monochrome Edges via Decisive Neighborhoods

260 Let  $\{u, v\}$  be an edge of a given graph. To formally define the above mentioned decisiveness,  
261 let  $N_{u \cap v}^+$  and  $N_{u \cap v}^-$  be the number of vertices in the common neighborhood of  $u$  and  $v$  that  
262 are occupied by agents of type  $t^+$  and  $t^-$ , respectively. Then  $D_{u \cap v} := |N_{u \cap v}^+ - N_{u \cap v}^-|$  is the  
263 *decisiveness* of the common neighborhood of  $u$  and  $v$ . Analogously, we define  $D_{u \setminus v}$  and  $D_{v \setminus u}$   
264 for the exclusive neighborhoods of  $u$  and  $v$ , respectively.

265 The following theorem bounds the probability for  $\{u, v\}$  to be monochrome based on the  
266 probability that the common neighborhood is more decisive than each of the exclusive ones.

267 ► **Theorem 7.** *In the FSP, let  $\{u, v\} \in E$  be an edge and let  $D$  be the event  $\{D_{u \cap v} >$   
268  $D_{u \setminus v} \wedge D_{u \cap v} > D_{v \setminus u}\}$ . Then  $\{u, v\}$  is monochrome with probability at least  $1/2 + \Pr[D]/2$ .*

269 **Proof.** If  $D$  occurs, then the types of  $u$  and  $v$  after the FSP coincide with the predominant  
270 type before the FSP in the shared neighborhood. Thus,  $\{u, v\}$  is monochrome.

271 Otherwise, assuming  $\bar{D}$ , w.l.o.g., let  $D_{u \cap v} \leq D_{u \setminus v}$  and assume further the type of  $v$  has  
272 already been determined. If  $D_{u \cap v} = D_{u \setminus v}$ , then  $u$  chooses a type uniformly at random,  
273 which coincides with the type of  $v$  with probability  $\frac{1}{2}$ . Otherwise,  $D_{u \cap v} < D_{u \setminus v}$  and thus  $u$   
274 takes the type that is predominant in  $u$ 's exclusive neighborhood, which is  $t^+$  and  $t^-$  with  
275 probability  $\frac{1}{2}$ , each. Moreover, this is independent from the type of  $v$  as  $v$ 's neighborhood is  
276 disjoint to  $u$ 's exclusive neighborhood.

277 Thus, for the event  $M$  that  $\{u, v\}$  is monochrome, we get  $\Pr[M \mid D] = 1$  and  $\Pr[M \mid \bar{D}] =$   
278  $\frac{1}{2}$ . Hence, we get  $\Pr[M] > \Pr[D] + \frac{1}{2}(1 - \Pr[D]) = \frac{1}{2} + \Pr[D]/2$ . ◀

### 279 3.2 Large Neighborhoods are More Decisive

280 The goal of this section is to reduce the question of how decisive a neighborhood is to the  
281 question of how large it is. To be more precise, assume we have a set of vertices of size  $a$  and  
282 give each vertex the type  $t^+$  and  $t^-$ , respectively, each with probability  $\frac{1}{2}$ . Let  $A_i$  for  $i \in [a]$   
283 be the random variable that takes the value  $+1$  and  $-1$  if the  $i$ -th vertex in this set has type  
284  $t^+$  and  $t^-$ , respectively. Then, for  $A = \sum_{i \in [a]} A_i$ , the decisiveness of the vertex set is  $|A|$ . In  
285 the following, we study the decisiveness  $|A|$  depending on the size  $a$  of the set. Note that  
286 this can be viewed as a random walk on the integer line: Starting at 0, in each step, it moves



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287 one unit either to the left or to the right with equal probabilities. We are interested in the  
 288 distance from 0 after  $a$  steps.

289 Assume for the vertices  $u$  and  $v$  that we know that  $b$  vertices lie in the common neigh-  
 290 borhood and  $a$  vertices lie in the exclusive neighborhood of  $u$ . Moreover, let  $A$  and  $B$  be  
 291 the positions of the above random walk after  $a$  and  $b$  steps, respectively. Then the event  
 292  $D_{u \cap v} > D_{u \setminus v}$  is equivalent to  $|B| > |A|$ . Motivated by this, we study the probability of  
 293  $|B| > |A|$ , assuming  $b \geq a$ . The core difficulty here comes from the fact that we require  $|B|$   
 294 to be strictly larger than  $|A|$ . Also note that  $a + b$  corresponds to the degree of  $u$  in the  
 295 graph. Thus, we have to study the random walks also for small numbers of  $a$  and  $b$ . We note  
 296 that all results in this section are independent from the specific application to the FSP, and  
 297 thus might be of independent interest.

298 Before we give a lower bound on the probability that  $|B| > |A|$ , we need the following  
 299 technical lemma. It states that doing more steps in the random walk only makes it more  
 300 likely to deviate further from the starting position.

301 ► **Lemma 8.** *For  $i \in [a]$  and  $j \in [b]$  with  $0 \leq a \leq b$ , let  $A_i$  and  $B_j$  be independent random  
 302 variables that are  $-1$  and  $1$  each with probability  $\frac{1}{2}$ . Let  $A = \sum_{i \in [a]} A_i$  and  $B = \sum_{j \in [b]} B_j$ .  
 303 Then  $\Pr[|A| < |B|] \geq \Pr[|A| > |B|]$ .*

304 **Proof.** Let  $\Delta_k$  be the event that  $|B| - |A| = k$ . First note that

$$305 \quad \Pr[|A| < |B|] = \sum_{k \in [b]} \Pr[\Delta_k] \quad \text{and} \quad \Pr[|A| > |B|] = \sum_{k \in [a]} \Pr[\Delta_{-k}].$$

306 To prove the statement of the lemma, it thus suffices to prove the following claim.

307 ▷ **Claim 9.** For  $k \geq 0$ ,  $\Pr[\Delta_k] \geq \Pr[\Delta_{-k}]$ .

308 We prove this claim via induction on  $b - a$ . For the base case  $a = b$ ,  $A$  and  $B$  are equally  
 309 distributed and thus Claim 9 clearly holds.

310 For the induction step, let  $B^+$  be the random variable that takes the values  $B + 1$  and  
 311  $B - 1$  with probability  $\frac{1}{2}$  each. Note that  $B^+$  represents the same type of random walk as  $A$   
 312 and  $B$  but with  $b + 1$  steps. Moreover  $B^+$  is coupled with  $B$  to make the same decisions in  
 313 the first  $b$  steps. Let  $\Delta_k^+$  be the event that  $|B^+| - |A| = k$ . It remains to show that Claim 9  
 314 holds for these  $\Delta_k^+$ . For this, first note that the claim trivially holds for  $k = 0$ . For  $k \geq 1$ ,  
 315 we can use the definition of  $\Delta_k^+$  and the induction hypothesis to obtain

$$316 \quad \Pr[\Delta_k^+] = \frac{\Pr[\Delta_{k-1}]}{2} + \frac{\Pr[\Delta_{k+1}]}{2}$$

$$317 \quad \geq \frac{\Pr[\Delta_{-k+1}]}{2} + \frac{\Pr[\Delta_{-k-1}]}{2} = \Pr[\Delta_{-k}^+]. \quad \blacktriangleleft$$

319 Using Lemma 8, we now prove the following general bound for the probability that  $|A| < |B|$ ,  
 320 depending on certain probabilities for binomially distributed variables.

321 ► **Lemma 10.** *For  $i \in [a]$  and  $j \in [b]$  with  $0 \leq a \leq b$ , let  $A_i$  and  $B_j$  be independent random  
 322 variables that are  $-1$  and  $1$  each with probability  $\frac{1}{2}$ . Let  $A = \sum_{i \in [a]} A_i$  and  $B = \sum_{j \in [b]} B_j$ .  
 323 Moreover, let  $X \sim \text{Bin}(a, \frac{1}{2})$ ,  $Y \sim \text{Bin}(b, \frac{1}{2})$ , and  $Z \sim \text{Bin}(a + b, \frac{1}{2})$ . Then*

$$324 \quad \Pr[|A| < |B|] \geq \frac{1}{2} - \Pr\left[Z = \frac{a+b}{2}\right] + \frac{\Pr[X = \frac{a}{2}] \cdot \Pr[Y = \frac{b}{2}]}{2}.$$

325 **Proof.** Using that  $\Pr[|A| < |B|] \geq \Pr[|A| > |B|]$  (see Lemma 8), we obtain

$$\begin{aligned}
 326 \quad & \Pr[|A| < |B|] + \Pr[|A| > |B|] + \Pr[|A| = |B|] = 1 \\
 327 \quad \Rightarrow \quad & 2\Pr[|A| < |B|] + \Pr[|A| = |B|] \geq 1 \\
 328 \quad \Leftrightarrow \quad & \Pr[|A| < |B|] \geq \frac{1}{2} - \frac{\Pr[|A| = |B|]}{2}. \quad (4) \\
 329
 \end{aligned}$$

330 Thus, it remains to give an upper bound for  $\Pr[|A| = |B|]$ .

331 Using the inclusion–exclusion principle and the fact that  $B$  is symmetric around 0, i.e.,  
 332  $\Pr[B = x] = \Pr[B = -x]$  for any  $x$ , we obtain

$$\begin{aligned}
 333 \quad \Pr[|A| = |B|] &= \Pr[A = B \vee A = -B] \\
 334 \quad &= \Pr[A = B] + \Pr[A = -B] - \Pr[A = B = 0] \\
 335 \quad &= 2\Pr[A = -B] - \Pr[A = B = 0]. \quad (5) \\
 336
 \end{aligned}$$

337 We estimate  $\Pr[A = -B]$  and  $\Pr[A = B = 0]$  using bounds for binomially distributed vari-  
 338 ables. To this end, define new random variables  $X_i = \frac{A_i+1}{2}$  for  $i \in [a]$  and let  $X = \sum_{i \in [a]} X_i$ .  
 339 Note that the  $X_i$  are independent and take values 0 and 1, each with probability  $\frac{1}{2}$ . Thus,  
 340  $X \sim \text{Bin}(a, \frac{1}{2})$ . Moreover,  $A = 2X - a$ . Analogously, we define  $Y$  with  $Y \sim \text{Bin}(b, \frac{1}{2})$  and  
 341  $B = 2Y - b$ . Note that  $X$  and  $Y$  are independent and thus  $Z = X + Y \sim \text{Bin}(a + b, \frac{1}{2})$ .  
 342 With this, we get

$$\begin{aligned}
 343 \quad \Pr[A = -B] &= \Pr[2X - a = -2Y + b] = \Pr\left[Z = \frac{a+b}{2}\right], \text{ and} \\
 344 \quad \Pr[A = B = 0] &= \Pr[A = 0] \cdot \Pr[B = 0] = \Pr\left[X = \frac{a}{2}\right] \cdot \Pr\left[Y = \frac{b}{2}\right]. \\
 345
 \end{aligned}$$

346 This, together with Equations (4) and (5) yield the claim. ◀

347 The bound in Lemma 10 becomes worse for smaller values of  $a$  and  $b$ . Considering this worst  
 348 case, we obtain the following specific bound.

349 **► Theorem 11.** For  $i \in [a]$  and  $j \in [b]$  with  $0 \leq a \leq b$ , let  $A_i$  and  $B_j$  be independent random  
 350 variables that are  $-1$  and  $1$  each with probability  $\frac{1}{2}$ . Let  $A = \sum_{i \in [a]} A_i$  and  $B = \sum_{j \in [b]} B_j$ .  
 351 If  $a = b = 0$  or  $a = b = 1$ , then  $\Pr[|A| < |B|] = 0$ . Otherwise  $\Pr[|A| < |B|] \geq \frac{3}{16}$ .

352 **Proof.** Clearly, if  $a = b = 0$ , then  $A = B = 0$  and thus  $\Pr[|A| < |B|] = 0$ . Similarly, if  
 353  $a = b = 1$ , then  $|A| = |B| = 1$  and thus  $\Pr[|A| < |B|] = 0$ . For the remainder, assume that  
 354 neither  $a = b = 0$  nor  $a = b = 1$ , and let  $X$ ,  $Y$ , and  $Z$  be defined as in Lemma 10, i.e.,  
 355  $X \sim \text{Bin}(a, \frac{1}{2})$ ,  $Y \sim \text{Bin}(b, \frac{1}{2})$ , and  $Z \sim \text{Bin}(a + b, \frac{1}{2})$ .

356 If  $a + b$  is odd, then  $\Pr[Z = \frac{a+b}{2}] = 0$ . Thus, by Lemma 10, we have  $\Pr[|A| < |B|] \geq \frac{1}{2}$ .  
 357 If  $a + b$  is even and  $a + b \geq 6$ , then

$$358 \quad \Pr\left[Z = \frac{a+b}{2}\right] = \binom{a+b}{\frac{a+b}{2}} \left(\frac{1}{2}\right)^{a+b} \leq \binom{6}{3} \left(\frac{1}{2}\right)^6 = \frac{5}{16}.$$

359 Hence, by Lemma 10, we have  $\Pr[|A| < |B|] \geq \frac{1}{2} - \frac{5}{16} = \frac{3}{16}$ .

360 If  $a + b < 6$  (and  $a + b$  is even), there are four cases:  $a = 0, b = 2$ ;  $a = 0, b = 4$ ;  
 361  $a = 1, b = 3$ ;  $a = 2, b = 2$ . If  $a = 0$  and  $b = 2$ , then  $A = 0$  with probability 1 and  $|B| = 2$   
 362 with probability  $\frac{1}{2}$ . Thus,  $\Pr[|A| < |B|] = \frac{1}{2}$ . If  $a = 0$  and  $b = 4$ , then  $|A| < |B|$  unless  
 363  $B = 0$ . As  $\Pr[B = 0] = \binom{4}{2} \cdot \left(\frac{1}{2}\right)^4 = \frac{3}{8}$ , we get  $\Pr[|A| < |B|] = 1 - \frac{3}{8} = \frac{5}{8}$ . If  $a = 1$  and  $b = 3$ ,

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364 then  $|A| = 1$  with probability 1 and  $|B| = 3$  with probability  $\frac{1}{4}$  (either  $B_1 = B_2 = B_3 = 1$   
 365 or  $B_1 = B_2 = B_3 = -1$ ). Thus,  $\Pr[|A| < |B|] = \frac{1}{4}$ . If  $a = b = 2$ , then  $|A| = 0$  with  
 366 probability  $\frac{1}{2}$  and  $|B| = 2$  with probability  $\frac{1}{2}$ . Thus  $\Pr[|A| < |B|] = \frac{1}{4}$ .

367 We note that the bound of  $\Pr[|A| < |B|] = \frac{3}{16}$  is tight for  $a = b = 3$ . ◀

### 368 3.3 Large Common Regions Yield Large Common Neighborhoods

369 To be able to apply Theorem 11 to an edge  $\{u, v\}$ , we need to make sure that the size of their  
 370 common neighborhood (corresponding to  $b$  in the theorem) is at least the size of the exclusive  
 371 neighborhoods (corresponding to  $a$  in the theorem). In the following, we give bounds for the  
 372 probability that this happens. Note that this is the first time we actually take the graph  
 373 into account. Thus, all above considerations hold for arbitrary graphs.

374 Recall that we consider random geometric graphs  $\mathcal{G}(n, r)$  and  $u$  and  $v$  are each connected  
 375 to all vertices that lie within a disk of radius  $r$  around them. As  $u$  and  $v$  are adjacent, their  
 376 disks intersect, which separates the ground space into four regions; cf. Fig 2a. Let  $R_{u \cap v}$  be  
 377 the intersection of the two disks. Let  $R_{u \setminus v}$  be the set of points that lie in the disk of  $u$  but  
 378 not in the disk of  $v$ , and analogously, let  $R_{v \setminus u}$  be the disk of  $v$  minus the disk of  $u$ . Finally,  
 379 let  $R_{\overline{u \cup v}}$  be the set of points outside both disks. Then, each of the  $n - 2$  remaining vertices  
 380 ends up in exactly one of these regions with a probability equal to the corresponding measure.  
 381 Let  $\mu(\cdot)$  be the area of the respective region and  $p = \mu(R_{u \cap v})$  and  $q = \mu(R_{u \setminus v}) = \mu(R_{v \setminus u})$   
 382 be the probabilities for a vertex to lie in the common and exclusive regions, respectively.  
 383 The probability for  $R_{\overline{u \cup v}}$  is then  $1 - p - 2q$ .

384 We are now interested in the sizes  $N_{u \cap v}$ ,  $N_{u \setminus v}$ , and  $N_{v \setminus u}$  of the common and the exclusive  
 385 neighborhoods, respectively. As each of the  $n - 2$  remaining vertices ends up in  $N_{u \cap v}$  with  
 386 probability  $p$ , we have  $N_{u \cap v} \sim \text{Bin}(n - 2, p)$ . For  $N_{u \setminus v}$  and  $N_{v \setminus u}$ , we already know that  $v$  is a  
 387 neighbor of  $u$  and vice versa. Thus,  $(N_{u \setminus v} - 1) \sim \text{Bin}(n - 2, q)$  and  $(N_{v \setminus u} - 1) \sim \text{Bin}(n - 2, q)$ .  
 388 Moreover, the three random variables are not independent, as each vertex lies in only exactly  
 389 one of the four neighborhoods, i.e.,  $N_{u \cap v}$ ,  $(N_{u \setminus v} - 1)$ ,  $(N_{v \setminus u} - 1)$ , and the number of vertices  
 390 in neither neighborhood together follow a multinomial distribution  $\text{Multi}(n - 2, \mathbf{p})$  with  
 391  $\mathbf{p} = (p, q, q, 1 - p - 2q)$ .

392 The following lemma shows that these dependencies are small if  $p$  and  $q$  are sufficiently  
 393 small. This lets us assume that  $N_{u \cap v}$ ,  $(N_{u \setminus v} - 1)$ ,  $(N_{v \setminus u} - 1)$  are independent random  
 394 variables following binomial distributions if the expected average degree is not too large.

395 ▶ **Lemma 12.** *Let  $X = (X_1, X_2, X_3, X_4) \sim \text{Multi}(n, \mathbf{p})$  with  $\mathbf{p} = (p, q, q, 1 - p - 2q)$ . Then  
 396 there exist independent random variables  $Y_1 \sim \text{Bin}(n, p)$ ,  $Y_2 \sim \text{Bin}(n, q)$ , and  $Y_3 \sim \text{Bin}(n, q)$   
 397 such that  $\Pr[(X_1, X_2, X_3) = (Y_1, Y_2, Y_3)] \geq 1 - 3n \cdot \max(p, q)^2$ .*

398 **Proof.** Let  $Y_1 \sim \text{Bin}(n, p)$ , and  $Y_2, Y_3 \sim \text{Bin}(n, q)$  be independent random variables. We  
 399 define the event  $B$  to hold, if each of the  $n$  individual trials increments at most one of  
 400 the random variables  $Y_1, Y_2$ , or  $Y_3$ . More formally, for  $i \in [3]$  and  $j \in [n]$ , let  $Y_{i,j}$  be the  
 401 individual Bernoulli trials of  $Y_i$ , i.e.,  $Y_i = \sum_{j \in [n]} Y_{i,j}$ . For  $j \in [n]$ , we define the event  $B_j$  to  
 402 be  $Y_{1,j} + Y_{2,j} + Y_{3,j} \leq 1$ , and the event  $B = \bigcap_{j \in [n]} B_j$ .

403 Based on this, we now define the random variables  $X_1, X_2, X_3$ , and  $X_4$  as follows.  
 404 If  $B$  holds, we set  $X_i = Y_i$  for  $i \in [3]$  and  $X_4 = n - X_1 - X_2 - X_3$ . Otherwise, if  $\overline{B}$ ,  
 405 we draw  $X = (X_1, X_2, X_3, X_4) \sim \text{Multi}(n, \mathbf{p})$  independently from  $Y_1, Y_2$ , and  $Y_3$  with  
 406  $\mathbf{p} = (p, q, q, 1 - p - 2q)$ . Note that  $X$  clearly follows  $\text{Multi}(n, \mathbf{p})$  if  $\overline{B}$ . Moreover, conditioned  
 407 on  $B$ , each individual trial increments exactly one of the variables  $X_1, X_2, X_3$ , or  $X_4$  with  
 408 probabilities  $p, q, q$ , and  $1 - p - 2q$ , respectively, i.e.,  $X \sim \text{Multi}(n, \mathbf{p})$ .

409 Thus, we end up with  $X \sim \text{Multi}(n, \mathbf{p})$ . Additionally, we have three independent random  
 410 variables  $Y_1 \sim \text{Bin}(n, p)$ , and  $Y_2, Y_3 \sim \text{Bin}(n, q)$  with  $(X_1, X_2, X_3) = (Y_1, Y_2, Y_3)$  if  $B$  holds.  
 411 Thus, to prove the lemma, it remains to show that  $\Pr[B] \geq 1 - 3n \max(p, q)^2$ . For  $j \in [n]$ ,  
 412 the probability that the  $j$ th trial goes wrong is

$$413 \Pr[\overline{B}_j] = 1 - ((1-p)(1-q)^2) - (p(1-q)^2) - 2(q(1-p)(1-q)) \\ 414 = 2pq - 2pq^2 + q^2 \leq 2pq + q^2 \leq 3 \cdot \max(p, q)^2. \\ 415$$

416 Using the union bound it follows that  $\Pr[\overline{B}] \leq \sum_{j \in [n]} \Pr[\overline{B}_j] \leq 3n \cdot \max(p, q)^2$ . ◀

417 As mentioned before, we are interested in the event  $N_{u \cap v} \geq N_{u \setminus v}$  (and likewise  $N_{u \cap v} \geq N_{v \setminus u}$ ),  
 418 in order to apply Theorem 11. Moreover, due to Lemma 12, we know that  $N_{u \cap v}$  and  $(N_{u \setminus v} - 1)$   
 419 almost behave like independent random variables that follow  $\text{Bin}(n-2, p)$  and  $\text{Bin}(n-2, q)$ ,  
 420 respectively. The following lemma helps to bound the probability for  $N_{u \cap v} \geq N_{u \setminus v}$ . Note  
 421 that it gives a bound for the probability of achieving strict inequality (instead of just  $\geq$ ),  
 422 which accounts for the fact that  $(N_{u \setminus v} - 1)$  and not  $N_{u \setminus v}$  itself follows a binomial distribution.

423 ▶ **Lemma 13.** *Let  $n \in \mathbf{N}$  with  $n \geq 2$ , and let  $p \geq q > 0$ . Further, let  $X \sim \text{Bin}(n, p)$   
 424 and  $Y \sim \text{Bin}(n, q)$  be independent, let  $d = \lfloor p(n+1) \rfloor$ , and assume  $d = o(\sqrt{n})$ , then  
 425  $\Pr[X > Y] \geq (\frac{1}{2} - 1/\sqrt{2\pi d})(1 - o(1))$ .*

426 **Proof.** By Lemma 1, we get  $\Pr[X \geq Y] \geq \frac{1}{2}$ , and we bound

$$427 \Pr[X > Y] = \Pr[X \geq Y] - \Pr[X = Y] \geq \frac{1}{2} - \Pr[X = Y], \\ 428$$

429 leaving us to bound  $\Pr[X = Y]$  from above. By independence of  $X$  and  $Y$ , we get

$$430 \Pr[X = Y] = \sum_{i \in [n]} \Pr[X = i] \cdot \Pr[Y = i]. \quad (6) \\ 431$$

432 Note that, by Lemma 2, for all  $i \in [0..n]$ , it holds that  $\Pr[X = i] \leq \Pr[X = d]$ . Assume that  
 433 we have a bound  $B$  such that  $\Pr[X = d] \leq B$ . Substituting this into Equation (6) yields

$$434 \Pr[X = Y] \leq B \sum_{i \in [n]} \Pr[Y = i] = B,$$

435 resulting in  $\Pr[X > Y] \geq \frac{1}{2} - B$ . Thus, we now derive such a bound for  $B$  and apply the  
 436 inequality that for all  $x \in \mathbf{R}$ , it holds that  $1 + x \leq e^x$ , as well as Equation (1). We get

$$437 \binom{n}{d} p^d (1-p)^{n-d} \leq \frac{n^d}{d!} \left(\frac{d}{n}\right)^d \left(1 - \frac{d}{n}\right)^n \left(1 - \frac{d}{n}\right)^{-d} \\ 438 \leq \frac{d^d}{d!} e^{-d} \left(1 - \frac{d}{n}\right)^{-d} \\ 439 \leq \frac{d^d}{\sqrt{2\pi d}^{d+1/2} e^{-d}} e^{-d} \left(1 - \frac{d}{n}\right)^{-d} \\ 440 = \frac{1}{\sqrt{2\pi d}} \frac{1}{(1 - d/n)^d}. \quad (7) \\ 441$$

442 By Bernoulli's inequality, we bound  $(1 - d/n)^d \geq 1 - d^2/n = 1 - o(1)$  by the assumption  
 443  $d = o(\sqrt{n})$ . Substituting this back into Equation (7) concludes the proof. ◀

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444 Finally, in order to apply Theorem 11, we have to make sure not to end up in the special  
 445 case where  $a = b \leq 1$ , i.e., we have to make sure that the common neighborhood includes at  
 446 least two vertices. The probability for this to happen is given by the following lemma.

447 ► **Lemma 14.** *Let  $X \sim \text{Bin}(n, p)$  and let  $c = np \in o(n)$ . Then it holds that  $\Pr[X > 1] \geq$   
 448  $(1 - e^{-c}(1 + c))(1 - o(1))$ .*

449 **Proof.** As  $X > 1$  holds if and only if  $X \neq 0$  and  $X \neq 1$ , we get

$$450 \quad \Pr[X > 1] = 1 - \Pr[X = 0] - \Pr[X = 1] = 1 - (1 - p)^n - n \cdot p \cdot (1 - p)^{n-1}.$$

452 Using that for all  $x \in \mathbf{R}$  it holds that  $1 - x \leq e^{-x}$ , we get

$$453 \quad \begin{aligned} \Pr[X > 1] &\geq 1 - e^{-pn} - n \cdot p \cdot e^{-p(n-1)} \\ 454 \quad &= 1 - e^{-c} - c \cdot e^{c/n} \cdot e^{-c} \\ 455 \quad &= 1 - e^{-c} \left(1 + c \cdot e^{c/n}\right). \end{aligned}$$

457 As  $e^{c/n}$  goes to 1 for  $n \rightarrow \infty$ , we get the claimed bound. ◀

### 458 3.4 Many Edges Have Large Common Regions

459 In Section 3.3, we derived a lower bound on the probability that  $N_{u \cap v} \geq N_{u \setminus v}$  provided that  
 460 the probability for a vertex to end up in the shared region  $R_{u \cap v}$  is sufficiently large compared  
 461 to  $R_{u \setminus v}$ . In the following, we estimate the measures of these regions depending on the distance  
 462 between  $u$  and  $v$ . Then, we give a lower bound on the probability that  $\mu(R_{u \cap v}) \geq \mu(R_{u \setminus v})$ .

463 ► **Lemma 15.** *Let  $G \sim \mathcal{G}(n, r)$  be a random geometric graph with expected average degree  $\bar{\delta}$ ,  
 464 let  $\{u, v\} \in E$  be an edge, and let  $\tau := \frac{\text{dist}(u, v)}{r}$ . Then,*

$$465 \quad \mu(R_{u \cap v}) = \frac{\bar{\delta}}{(n-1)\pi} \left( 2 \arccos\left(\frac{\tau}{2}\right) - \sin\left(2 \arccos\left(\frac{\tau}{2}\right)\right) \right) \text{ and} \quad (8)$$

$$466 \quad \mu(R_{u \setminus v}) = \mu(R_{v \setminus u}) = \frac{\bar{\delta}}{n-1} - \mu(R_{u \cap v}). \quad (9)$$

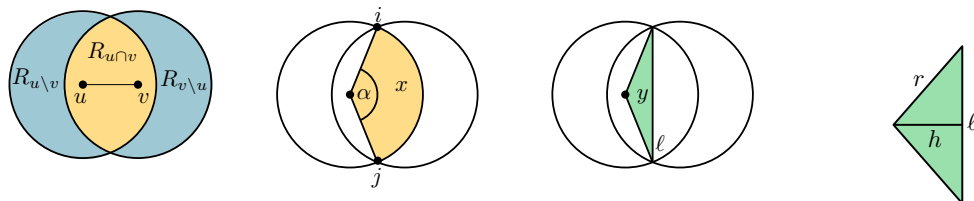
468 **Proof.** We start with proving Equation (8). Let  $i$  and  $j$  be the two intersection points of the  
 469 disks of  $u$  and  $v$ , let  $\alpha$  be the central angle enclosed by  $i$  and  $j$ , and let  $x$  be the corresponding  
 470 circular sector, cf. Fig 2b. Moreover, let the triangle  $y$  be a subarea of  $x$  determined by  $\alpha$   
 471 and the radical axis  $\ell$ , cf. Fig 2c. Let  $h$  denote the height of the triangle  $y$ , cf. Fig 2d.  
 472 For our calculations, we restrict the length of  $\ell$  by the intersection points  $i$  and  $j$ . Since  
 473 we consider the intersection between disks and thus  $\ell$  divides the area  $\mu(R_{u \cap v})$  into two  
 474 subareas of equal sizes, it holds that  $\mu(R_{u \cap v}) = 2(\mu(x) - \mu(y))$ . Considering the two areas  
 475  $\mu(x)$  and  $\mu(y)$ , it holds that

$$476 \quad \mu(x) = \frac{\alpha}{2} r^2 \quad \text{and} \quad \mu(y) = h \cdot \frac{\ell}{2} = \cos\left(\frac{\alpha}{2}\right) r \cdot \sin\left(\frac{\alpha}{2}\right) r = \frac{\sin(\alpha)}{2} r^2. \quad (10)$$

478 For the central angle  $\alpha$  we know  $\cos(\alpha/2) = h/r = \tau/2$  and therefore  $\alpha = 2 \arccos(\tau/2)$ .  
 479 Together with Equation (10), we obtain

$$480 \quad \mu(R_{u \cap v}) = 2(\mu(x) - \mu(y)) = 2 \left( \frac{2 \arccos(\tau/2)}{2} r^2 - \frac{\sin(2 \arccos(\tau/2))}{2} r^2 \right). \quad (11)$$

481



(a) The geometric regions corresponding to the common and exclusive neighborhoods, respectively, with yellow illustrating  $R_{u \cap v}$  and blue illustrating  $R_{u \setminus v}$  and  $R_{v \setminus u}$ .  
 (b) Let  $\alpha$  be the central angle determined by the intersection points  $i$  and  $j$ , and let  $x$  be the corresponding circular sector (illustrated in yellow).  
 (c) Let  $y$  be a triangle in the intersection (illustrated in green) determined by the radical axis  $\ell$  and the central angle  $\alpha$ , cf. Fig 2b.  
 (d) The height  $h$  divides the area  $\mu(y)$  (illustrated in green) of the triangle  $y$ , cf. Fig 2c, into two sub-areas of equal size, since adjacent and opposite legs have the same length  $r$ .

■ **Figure 2** The neighborhood of two adjacent vertices  $u$  and  $v$  in a random geometric graph.

482 The area of a general circle is equal to  $\pi r^2$ . Since we consider a ground space with total  
 483 area 1, the area of one disk in the random geometric graph equals  $\frac{\bar{\delta}}{n-1}$ , i.e.,  $r^2 = \frac{\bar{\delta}}{(n-1)\pi}$ .  
 484 Together with Equation (11), we obtain Equation (8).

485 Equation (9): We get the claimed equality by noting that  $\mu(R_{u \cap v}) + \mu(R_{u \setminus v}) = \pi r^2$ . ◀

486 ▶ **Lemma 16.** Let  $G \sim \mathcal{G}(n, r)$  be a random geometric graph, and let  $\{u, v\} \in E$  be an edge.  
 487 Then  $\Pr [\mu(R_{u \cap v}) \geq \mu(R_{u \setminus v})] \geq (\frac{4}{5})^2$ .

488 **Proof.** Let  $\tau = \frac{\text{dist}(u,v)}{r}$ . By Lemma 15 with  $\mu(R_{u \cap v}) \geq \mu(R_{v \setminus u})$ , we get

489 
$$\left( 2 \arccos \left( \frac{\tau}{2} \right) - \sin \left( 2 \arccos \left( \frac{\tau}{2} \right) \right) \right) \geq \frac{\pi}{2},$$
  
 490

491 which is true for  $\tau \geq \frac{4}{5}$ . The area of a disk of radius  $\frac{4}{5}r$  is  $(\pi(\frac{4}{5}r)^2) / (\pi r^2) = (\frac{4}{5})^2$  times  
 492 the area of a disk of radius  $r$ . Hence, the fraction of edges with distance at most  $\frac{4}{5}r$  is at  
 493 least  $(\frac{4}{5})^2$ , concluding the proof. ◀

494 **3.5 Proof of Theorem 6**

495 By Theorem 7, the probability that a random edge  $\{u, v\}$  is monochrome is at least  $\frac{1}{2} +$   
 496  $\Pr [D] / 2$ , where  $D$  is the event that the common neighborhood of  $u$  and  $v$  is more decisive  
 497 than each exclusive neighborhood. It remains to bound  $\Pr [D]$ .

498 **Existence of an edge yields a large shared region.** Let  $R$  be the event that  $\mu(R_{u \cap v}) \geq$   
 499  $\mu(R_{u \setminus v})$ . Note that this also implies  $\mu(R_{u \cap v}) \geq \mu(R_{v \setminus u})$  as  $\mu(R_{u \setminus v}) = \mu(R_{v \setminus u})$ . Due to the  
 500 law of total probability, we have

501 
$$\Pr [D] \geq \Pr [R] \cdot \Pr [D \mid R].$$

502 Due to Lemma 16, we have  $\Pr [R] \geq (\frac{4}{5})^2$ . By conditioning on  $R$  in the following, we can  
 503 assume that  $\mu(R_{u \cap v}) \geq \frac{\bar{\delta}}{2n} \geq \mu(R_{u \setminus v}) = \mu(R_{v \setminus u})$ , where  $\bar{\delta}$  is the expected average degree.

504 **Neighborhood sizes are roughly binomially distributed.** The next step is to go from the  
 505 size of the regions to the number of vertices in these regions. Each of the remaining  $n' = n - 2$   
 506 vertices is sampled independently to lie in one of the regions  $R_{u \cap v}$ ,  $R_{u \setminus v}$ ,  $R_{v \setminus u}$ , or  $R_{\overline{u \cup v}}$ .  
 507 Denote the resulting numbers of vertices with  $X_1$ ,  $X_2$ ,  $X_3$ , and  $X_4$ , respectively. Then  
 508  $(X_1, X_2, X_3, X_4)$  follows a multinomial distribution with parameter  $\mathbf{p} = (p, q, q, 1 - p - 2q)$   
 509 for  $p = \mu(R_{u \cap v})$  and  $q = \mu(R_{u \setminus v}) = \mu(R_{v \setminus u})$ . Note that  $N_{u \cap v} = X_1$ ,  $N_{u \setminus v} = X_2 + 1$ , and  
 510  $N_{v \setminus u} = X_3 + 1$  holds for the sizes of the common and exclusive neighborhoods, where the  $+1$   
 511 comes from the fact that  $v$  is always a neighbor of  $u$  and vice versa.

512 We apply Lemma 12 to obtain independent binomially distributed random variables  $Y_1$ ,  
 513  $Y_2$ , and  $Y_3$  that are likely to coincide with  $X_1 = N_{u \cap v}$ ,  $X_2 = N_{u \setminus v} - 1$ , and  $X_3 = N_{v \setminus u} - 1$ ,  
 514 respectively. Let  $B$  denote the event that  $(N_{u \cap v}, N_{u \setminus v} - 1, N_{v \setminus u} - 1) = (Y_1, Y_2, Y_3)$ . Again,  
 515 using the law of total probabilities and due to the fact that  $R$  and  $B$  are independent, we get

$$516 \quad \Pr[D \mid R] \geq \Pr[B \mid R] \cdot \Pr[D \mid R \cap B] = \Pr[B] \cdot \Pr[D \mid R \cap B].$$

517 Note that  $p, q \leq \frac{\bar{\delta}}{n}$  for the expected average degree  $\bar{\delta}$ . Thus, Lemma 12 implies that  
 518  $\Pr[B] \geq \left(1 - 3\bar{\delta}^2/n\right)$ . Conditioning on  $B$  makes it correct to assume that  $N_{u \cap v} \sim \text{Bin}(n', p)$ ,  
 519  $(N_{u \setminus v} - 1) \sim \text{Bin}(n', q)$ ,  $(N_{v \setminus u} - 1) \sim \text{Bin}(n', q)$  are independently distributed. Additionally  
 520 conditioning on  $R$  gives us  $p \geq \frac{\bar{\delta}}{2n} \geq q$ .

521 **A large shared region yields a large shared neighborhood.** In the next step, we consider  
 522 an event that makes sure that the number  $N_{u \cap v}$  of vertices in the shared neighborhood is  
 523 sufficiently large. Let  $N_1$ ,  $N_2$ , and  $N_3$  be the events that  $N_{u \cap v} \geq N_{u \setminus v}$ ,  $N_{u \cap v} \geq N_{v \setminus u}$ , and  
 524  $N_{u \cap v} > 1$ , respectively. Let  $N$  be the intersection of  $N_1$ ,  $N_2$ , and  $N_3$ . We obtain

$$525 \quad \Pr[D \mid R \cap B] \geq \Pr[N \mid R \cap B] \cdot \Pr[D \mid R \cap B \cap N]$$

$$526 \quad \geq \Pr[N_1 \mid R \cap B] \cdot \Pr[N_2 \mid R \cap B] \cdot \Pr[N_3 \mid R \cap B] \cdot \Pr[D \mid R \cap B \cap N],$$

528 where the last step follows from Lemma 5 as the inequalities in  $N_1$ ,  $N_2$ , and  $N_3$  all go in  
 529 the same direction. Note that  $N_{u \cap v} \geq N_{u \setminus v}$  is equivalent to  $N_{u \cap v} > N_{u \setminus v} - 1$ . Due to the  
 530 condition on  $B$ ,  $N_{u \cap v}$  and  $N_{u \setminus v} - 1$  are independent random variables following  $\text{Bin}(n', p)$   
 531 and  $\text{Bin}(n', q)$ , respectively, with  $p \geq q$  due to the condition on  $R$ . Thus, we can apply  
 532 Lemma 13, to obtain

$$533 \quad \Pr[N_1 \mid R \cap B] = \Pr[N_2 \mid R \cap B] \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi[\bar{\delta}/2](1 - o(1))}},$$

534 and Lemma 14 gives the bound

$$535 \quad \Pr[N_3 \mid R \cap B] \geq 1 - e^{-\bar{\delta}/2} \left(1 + \frac{\bar{\delta}}{2} \cdot (1 + o(1))\right).$$

536 Note that both of these probabilities are bounded away from 0 for  $\bar{\delta} \geq 2$ . Conditioning on  $N$   
 537 lets us assume that the shared neighborhood of  $u$  and  $v$  contains at least two vertices and  
 538 that it is at least as big as each of the exclusive neighborhoods.

539 **A large shared neighborhood yields high decisiveness.** The last step is to actually bound  
 540 the remaining probability  $\Pr[D \mid R \cap B \cap N]$ . Note that, once we know the number of vertices  
 541 in the shared and exclusive neighborhoods, the decisiveness no longer depends on  $R$  or  $B$ , i.e.,  
 542 we can bound  $\Pr[D \mid N]$  instead. For this, let  $D_1$  and  $D_2$  be the events that  $D_{u \cap v} > D_{u \setminus v}$

543 and  $D_{u \cap v} > D_{v \setminus u}$ , respectively. Note that  $D$  is their intersection. Moreover, due to Lemma 5,  
 544 we have  $\Pr[D \mid N] \geq \Pr[D_1 \mid N] \cdot \Pr[D_2 \mid N]$ . To bound  $\Pr[D_1 \mid N] = \Pr[D_2 \mid N]$ , we use  
 545 Theorem 11. Note that the  $b$  and  $a$  in Theorem 11 correspond to  $N_{u \cap v}$  and  $N_{u \setminus v} + 1$  (the  
 546  $+1$  coming from the fact that  $N_{u \setminus v}$  does not count the vertex  $v$ ). Moreover conditioning on  
 547  $N$  implies that  $a \leq b$  and  $b > 1$ . Thus, Theorem 11 implies  $\Pr[D_1 \mid N] \geq \frac{3}{16}$ .

548 **Conclusion.** The above arguments give us that the fraction of monochrome edges is

$$549 \quad \frac{1}{2} + \frac{\Pr[D]}{2} \geq \frac{1}{2} + \frac{1}{2} \cdot \underbrace{\Pr[R]}_{\geq (\frac{4}{5})^2} \cdot \underbrace{\Pr[B]}_{1-o(1)} \cdot \underbrace{(\Pr[N_1 \mid R \cap B])^2}_{\geq \frac{1}{2} - \frac{1}{\sqrt{2\pi \lfloor \bar{\delta}/2 \rfloor}}} \cdot \underbrace{\Pr[N_3 \mid R \cap B]}_{\geq 1 - e^{-\bar{\delta}/2} (1 + \frac{\bar{\delta}}{2})} \cdot \underbrace{(\Pr[D_1 \mid N])^2}_{\geq \frac{3}{16}},$$

550 where we omitted the  $o(1)$  terms for  $\Pr[N_1 \mid R \cap B]$  and  $\Pr[N_3 \mid R \cap B]$ , as they are already  
 551 covered by the  $1 + o(1)$  coming from  $\Pr[B]$ . This yields the bound stated in Theorem 6:

$$552 \quad \frac{1}{2} + \frac{9}{800} \cdot \left( \frac{1}{2} - \frac{1}{\sqrt{2\pi \lfloor \bar{\delta}/2 \rfloor}} \right)^2 \cdot \left( 1 - e^{-\bar{\delta}/2} \left( 1 + \frac{\bar{\delta}}{2} \right) \right) \cdot (1 - o(1)).$$

#### 553 4 Monochrome Edges in Erdős–Rényi Graphs

554 In the following, we are interested in the probability that an edge  $\{u, v\}$  is monochrome  
 555 after the FSP on Erdős–Rényi graphs. In contrast to geometric random graphs, we prove  
 556 an upper bound. To this end, we show that it is likely that the common neighborhood is  
 557 empty and therefore  $u$  and  $v$  choose their types to be the predominant type in their exclusive  
 558 neighborhood, which is  $t^+$  and  $t^-$  with probability  $\frac{1}{2}$ , each.

559 **► Theorem 17.** *Let  $G \sim \mathcal{G}(n, p)$  be an Erdős–Rényi graph with expected average degree*  
 560  *$\bar{\delta} = o(\sqrt{n})$ . The expected fraction of monochrome edges after the FSP is at most  $\frac{1}{2} + o(1)$ .*

561 **Proof.** Given an edge  $\{u, v\}$ , let  $M$  be the event that  $\{u, v\}$  is monochrome. We first split  $M$   
 562 into disjoint sets with respect to the size of the common neighborhood and apply the law of  
 563 total probability and get

$$564 \quad \Pr[M] = \Pr[M \mid N_{u \cap v} = 0] \cdot \Pr[N_{u \cap v} = 0] + \Pr[M \mid N_{u \cap v} > 0] \cdot \Pr[N_{u \cap v} > 0]$$

$$565 \quad \leq \Pr[M \mid N_{u \cap v} = 0] \cdot 1 + 1 \cdot \Pr[N_{u \cap v} > 0].$$

567 We bound each of the summands separately. For estimating  $\Pr[M \mid N_{u \cap v} = 0]$ , we note  
 568 that the types of  $u$  and  $v$  are determined by the predominant type in disjoint vertex sets. By  
 569 definition of the FSP this implies that the probability of a monochrome edge is equal to  $\frac{1}{2}$ .

570 We are left with bounding  $\Pr[N_{u \cap v} > 0]$ . Note that  $N_{u \cap v} \sim \text{Bin}(n, p^2)$ . Thus, by  
 571 Bernoulli's inequality we get  $\Pr[N_{u \cap v} > 0] = 1 - \Pr[N_{u \cap v} = 0] = 1 - (1 - p^2)^n \leq np^2$ .  
 572 Noting that  $np^2 = o(1)$  holds due to our assumption on  $\bar{\delta}$ , concludes the proof. ◀



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