



HAL
open science

Low-rank tensor recovery for Jacobian-based Volterra identification of parallel Wiener-Hammerstein systems

Konstantin Usevich, Philippe Dreesen, Mariya Ishteva

► To cite this version:

Konstantin Usevich, Philippe Dreesen, Mariya Ishteva. Low-rank tensor recovery for Jacobian-based Volterra identification of parallel Wiener-Hammerstein systems. 19th IFAC Symposium on System Identification, SYSID 2021, Jul 2021, Padova (virtual), Italy. hal-03349340

HAL Id: hal-03349340

<https://hal.science/hal-03349340>

Submitted on 20 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Low-rank tensor recovery for Jacobian-based Volterra identification of parallel Wiener-Hammerstein systems

Konstantin Usevich* Philippe Dreesen** Mariya Ishteva***

* *Université de Lorraine, CNRS, CRAN, Nancy, France (e-mail: konstantin.usevich@univ-lorraine.fr).*

** *KU Leuven, Dept. Electrical Engineering (ESAT), STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics, Belgium (e-mail: philippe.dreesen@gmail.com)*

*** *KU Leuven, Department of Computer Science, ADVISE-NUMA, campus Geel, Belgium (e-mail: mariya.ishteva@kuleuven.be)*

Abstract: We consider the problem of identifying a parallel Wiener-Hammerstein structure from Volterra kernels. Methods based on Volterra kernels typically resort to coupled tensor decompositions of the kernels. However, in the case of parallel Wiener-Hammerstein systems, such methods require nontrivial constraints on the factors of the decompositions. In this paper, we propose an entirely different approach: by using special sampling (operating) points for the Jacobian of the nonlinear map from past inputs to the output, we can show that the Jacobian matrix becomes a linear projection of a tensor whose rank is equal to the number of branches. This representation allows us to solve the identification problem as a tensor recovery problem.

Keywords: Block structured system identification, parallel Wiener-Hammerstein systems, Volterra kernels, low-rank tensor recovery, canonical polyadic decomposition

1. INTRODUCTION

Nonlinear identification methods that go beyond the well-established linear system identification tools (Pintelon and Schoukens, 2012; Ljung, 1999; Katayama, 2005), are steadily gaining research attention in recent years. Advances in nonlinear modeling tools, combined with the ever increasing computing power allows for the exploration of nonlinear models that account for nonlinear effects that occur when pushing systems outside of their linear operating regions. There is a host of procedures that range from simple extensions of linear models, over nonlinear state space modeling (possibly using particle filtering), to variations on neural network architectures, each of which typically require tailored nonconvex optimization methods. While such models may provide satisfactory prediction results, their internal workings are often hard to assess, which makes them difficult to use and interpret.

The current paper considers a combination of two promising nonlinear models (block-oriented models and Volterra series), and aims at combining their advantages while avoiding the drawbacks. Block-oriented systems are composed as interconnections of linear time-invariant system blocks and static nonlinearities such as the well-known Wiener, Hammerstein, Wiener-Hammerstein and Hammerstein-Wiener systems (Giri and Bai, 2010). A block-oriented system description strikes a balance between flexibility and model interpretability: the model accounts for (strong) nonlinearities in its description, but stays close to the familiar linear world and allows for a transparent understanding of its workings. Nev-

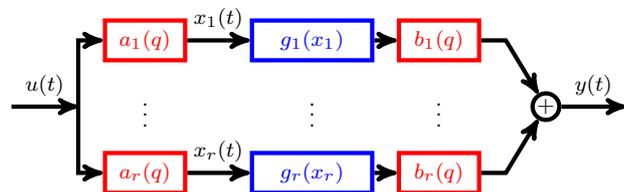


Fig. 1. Parallel Wiener-Hammerstein system.

ertheless, block-oriented system identification methods rely on heuristics and nonconvex optimization routines (Schoukens and Tiels, 2017) to find the parameters, which may cause difficulties. Volterra series models, on the other hand, can be viewed as nonlinear extensions of the well-known convolution operation of the input signal with the (finite) impulse response: in the Volterra description, the output is defined as a polynomial function of (delayed) inputs (as opposed to the output being a linear function of delayed inputs in the case of linear systems). A major advantage is that the Volterra model is linear in the parameters and its identification can be posed as a least-squares problem (Birpoutsoukis et al., 2017). Unfortunately, due to the polynomial structure, the number of coefficients grows very quickly as the polynomial degree increases. In addition, the model does not allow for an intuitive understanding of its internal operation.

In this article, we are interested in identification of discrete-time parallel Wiener-Hammerstein systems, see Fig. 1. Each branch of such a system has a Wiener-

Hammerstein structure, i.e., a static nonlinearity sandwiched in between two linear time-invariant (LTI) blocks. Parallel Wiener-Hammerstein models have improved approximation properties as opposed to single branch Wiener-Hammerstein models (Palm, 1979). However, identification of a parallel Wiener-Hammerstein structure is particularly challenging, see Schoukens and Tiels (2017). For example, the frequency-domain methods (Schoukens and Tiels, 2017) suffer from the pole/zero mixing of the Wiener and Hammerstein filters, and thus require a computationally heavy pole/zero splitting procedure.

The method that we present in this article starts from estimating Volterra kernels, which can be readily viewed as higher-order symmetric tensors containing the polynomial coefficients. Existing methods that aim at finding block-oriented models from the Volterra kernels resort to coupled tensor decompositions of Volterra kernels (Kibangou and Favier, 2007) and require nontrivial constraints on the factors of the tensor decomposition for parallel Wiener-Hammerstein case (Dreesen et al., 2017; Westwick et al., 2017; Dreesen and Ishteva, 2021). In this paper, we propose an entirely different approach: by choosing special sampling points, we can show that the Jacobian matrix becomes a linear projection of a certain low-rank tensor whose rank is equal to the number of parallel branches in the model. This representation allows us to solve the identification problem as a tensor recovery problem, which may be approached by an alternating least squares (ALS) solution strategy.

2. PRELIMINARIES

2.1 Tensor and vector notation

In this paper we mainly follow Comon (2014) in what concerns tensor notation (see also Kolda and Bader (2009)). We use lowercase (a) or uppercase (A) plain font for scalars, boldface lowercase (\mathbf{a}) for vectors, uppercase boldface (\mathbf{A}) for matrices, calligraphic font (\mathcal{A}) for N -D arrays (tensors) and script (\mathcal{P}) for operators. Vectors are, by convention, one-column matrices. The elements of vectors/matrices/tensors are accessed as a_i , $A_{i,j}$ and $\mathcal{A}_{i_1,\dots,i_N}$ respectively. We use $\text{vec}\{\cdot\}$ for the standard column-major vectorization of a tensor or a matrix. Operator \bullet_p denotes the contraction on the p th index of a tensor, i.e.,

$$[\mathcal{A} \bullet_1 \mathbf{u}]_{jk} = \sum_i \mathcal{A}_{ijk} u_i.$$

For a matrix \mathbf{A} , \mathbf{A}^\top and \mathbf{A}^\dagger denotes its transpose and Moore-Penrose pseudoinverse respectively. The notation \mathbf{I}_M is used for the $M \times M$ identity matrix and $\mathbf{0}_{L \times K}$ for the $L \times K$ matrix of zeroes. We use the symbol \boxtimes for the Kronecker product of matrices (in order to distinguish it from the tensor product \otimes), and \odot for the (column-wise) Khatri-Rao product of matrices: i.e., the Khatri-Rao product of

$$\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_r] \quad \text{and} \quad \mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_r]$$

is defined as

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \boxtimes \mathbf{b}_1 \cdots \mathbf{a}_r \boxtimes \mathbf{b}_r].$$

We use the notation $\text{Diag}\{\mathbf{v}\}$ for the diagonal matrix built from the vector \mathbf{v} .

A polyadic decomposition (PD) is a decomposition of a tensor into a sum of rank-one terms, i.e., for $\mathcal{Y} \in \mathbb{R}^{I \times J \times K}$,

$$\mathcal{Y} = \sum_{\ell=1}^r \mathbf{a}_\ell \otimes \mathbf{b}_\ell \otimes \mathbf{c}_\ell \quad (1)$$

is a polyadic decomposition. It is called canonical polyadic (CPD) if the number r in (1) is minimal among all possible PDs of \mathcal{Y} ; in that case r is called the tensor rank of \mathcal{Y} .

By grouping vectors into matrices

$$\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_r], \quad \mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_r], \quad \mathbf{C} = [\mathbf{c}_1 \cdots \mathbf{c}_r]$$

we can use a more compact notation

$$\mathcal{Y} = \llbracket \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket, \quad \mathcal{Y}_{ijk} = \sum_{\ell=1}^r A_{i\ell} B_{j\ell} C_{k\ell};$$

for a PD (or a CPD).

Finally, for a (possibly finite) sequence

$$(\dots, x(1), \dots, x(T), \dots)$$

its convolution with a vector $\mathbf{a} \in \mathbb{R}^K$ is defined as

$$(\mathbf{a} * x)(t) = \sum_{i=1}^K x(t-i+1) a_i.$$

2.2 Volterra kernels

The Volterra series (Schetzen, 1980) is a classical model for nonlinear systems, and is similar to the Taylor expansion for multivariate maps. In the discrete-time case, Volterra series can be interpreted as a power series expansion of the output of a system as a function of past inputs:

$$y(t) = f^{(0)} + \sum_{s=1}^{\infty} \left(\sum_{\tau_1=0}^{\infty} \cdots \sum_{\tau_s=0}^{\infty} H^{(s)}(\tau_1, \dots, \tau_s) u(t-\tau_1) \cdots u(t-\tau_s) \right),$$

where $H^{(s)}(\cdot)$ is the s -th order Volterra kernel. In the special case when the output depends only on a finite number L of past inputs, i.e., is defined by $f: \mathbb{R}^L \rightarrow \mathbb{R}$

$$y(t) = f(u(t), u(t-1), \dots, u(t-L+1)), \quad (2)$$

we can consider truncated the Volterra kernels $\mathcal{H}^{(s)}$ (which are $L \times \cdots \times L$ tensors). By denoting for convenience the vector of past inputs as

$$\mathbf{u} = [u(t) \ u(t-1) \ \dots \ u(t-L+1)]^\top \quad (3)$$

we can write the function expansion as

$$y(t) = f(\mathbf{u}) = f^{(0)} + f^{(1)}(\mathbf{u}) + \cdots + f^{(d)}(\mathbf{u}) + \cdots, \quad (4)$$

where the degree- s term is given by

$$f^{(s)}(\mathbf{u}) = \sum_{i_1, \dots, i_s=1}^{L, \dots, L} \mathcal{H}_{i_1, \dots, i_s}^{(s)} u(t-i_1+1) \cdots u(t-i_s+1),$$

with

$$\mathcal{H}_{i_1, \dots, i_s}^{(s)} = H^{(s)}(i_1-1, \dots, i_s-1).$$

Order- s terms can be compactly expressed using the multiple contraction

$$f^{(s)}(\mathbf{u}) = \mathcal{H}^{(s)} \bullet_1 \mathbf{u} \bullet_2 \mathbf{u} \cdots \bullet_s \mathbf{u}.$$

2.3 Parallel Wiener-Hammerstein model

In this paper, we consider the case when the LTI blocks in Fig. 1 are given by finite impulse response (FIR) filters of

where the vector $\mathbf{v}_\mu \in \mathbb{C}^L$ is defined as

$$\mathbf{v}_\mu = \mathbf{V} \begin{bmatrix} b_1 \\ b_2 \mu^{(s-1)} \\ \vdots \\ b_{L_2} \mu^{(L_2-1)(s-1)} \end{bmatrix} \\ = \begin{bmatrix} a_1 b_1 \\ a_2 b_1 + a_1 b_2 \mu^{(s-1)} \\ a_3 b_1 + a_2 b_2 \mu^{(s-1)} + a_1 b_3 \mu^{2(s-1)} \\ \vdots \\ a_{L_1} b_{L_2-1} \mu^{(L_2-2)(s-1)} + a_{L_1-1} b_{L_2} \mu^{(L_2-1)(s-1)} \\ a_{L_1} b_{L_2} \mu^{(L_2-1)(s-1)} \end{bmatrix},$$

where \mathbf{V} is defined in (6).

3.4 Gradient as a projection of a rank-one term

We are going to show that \mathbf{v}_μ from the previous subsection can be conveniently written as a linear projection (sampling) of a rank-one matrix. First of all, we introduce the diagonal summation (“Hankelization”) operator $\mathcal{H} : \mathbb{C}^{L_1 \times L_2} \rightarrow \mathbb{C}^L$, which takes the sums on the antidiagonals

$$\mathcal{H}(\mathbf{E}) = \begin{bmatrix} E_{1,1} \\ E_{2,1} + E_{1,2} \\ E_{3,1} + E_{2,2} + E_{1,3} \\ \vdots \\ E_{L_1, L_2-1} + E_{L_1-1, L_2} \\ E_{L_1, L_2} \end{bmatrix}.$$

Next, it is easy to see that \mathbf{v}_μ can be obtained by applying the projection operator $\mathcal{P}_{\mu,s} : \mathbb{C}^{L_1 \times L_2} \rightarrow \mathbb{C}^L$, which is a composition of the diagonal summation with the scaling of columns by powers of μ :

$$\mathcal{P}_{\mu,s}(\mathbf{E}) = \mathcal{H}(\mathbf{E} \text{Diag}\{[1 \ \mu^{(s-1)} \ \dots \ \mu^{(L_2-1)(s-1)}]\}),$$

i.e., $\mathbf{v}_\mu = \mathcal{P}_{\mu,s}(\mathbf{a}\mathbf{b}^\top)$. After that, we get that the gradient in (8) can be expressed as follows

$$\nabla f^{(s)}(\mathbf{u}_\mu) = \mathcal{P}_{\mu,s}((cs(a(\mu))^{s-1})\mathbf{a}\mathbf{b}^\top).$$

Finally, in the next subsections we are going to evaluate the gradients at different operating points and collect information from several kernels.

3.5 Combining several kernels and points

Now consider a set of N points in the complex plane

$$\{\mu_1, \dots, \mu_N\} \subset \mathbb{C}$$

at which we are evaluating the gradients of the homogeneous parts, and collecting them into one single vector:

$$\mathbf{y} = \begin{bmatrix} \nabla f^{(1)}(\mathbf{u}_1) \\ \nabla f^{(2)}(\mathbf{u}_{\mu_1}) \\ \vdots \\ \nabla f^{(2)}(\mathbf{u}_{\mu_N}) \\ \vdots \\ \nabla f^{(d)}(\mathbf{u}_{\mu_1}) \\ \vdots \\ \nabla f^{(d)}(\mathbf{u}_{\mu_N}) \end{bmatrix} \in \mathbb{C}^{((d-1)N+1)L}. \quad (9)$$

Unlike the previous section, we now consider a general polynomial nonlinearity:

$$g(t) = c_1 t + c_2 t^2 + \dots + c_d t^d.$$

By using the results of the previous subsection, we can show that \mathbf{y} is a projection of a rank-1 tensor:

$$\mathbf{y} = \mathcal{P}(\mathcal{T}),$$

where the rank-one tensor is

$$\mathcal{T} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{h},$$

vectors \mathbf{a} , \mathbf{b} are as before, and $\mathbf{h}^\top =$

$$[c_1 \ 2c_2 a(\mu_1) \dots 2c_2 a(\mu_N) \dots dc_d(a(\mu_1))^{d-1} \dots dc_d(a(\mu_N))^{d-1}].$$

The sampling operator is defined as a concatenation of sampling operators of tensor slices

$$\mathcal{P}(\mathcal{T}) = \begin{bmatrix} \mathcal{P}_{1,1}(\mathcal{T}_{::,1}) \\ \mathcal{P}_{\mu_1,2}(\mathcal{T}_{::,2}) \\ \vdots \\ \mathcal{P}_{\mu_N,2}(\mathcal{T}_{::,1+N}) \\ \vdots \\ \mathcal{P}_{\mu_1,d}(\mathcal{T}_{::,2+N(d-2)}) \\ \vdots \\ \mathcal{P}_{\mu_N,d}(\mathcal{T}_{::,1+N(d-1)}) \end{bmatrix}.$$

4. IDENTIFICATION AS TENSOR RECOVERY

4.1 Several branches and overall algorithm

We saw in the previous section that in the case of a single branch, the vector of the gradients \mathbf{y} evaluated at the Vandermonde evaluation points \mathbf{u}_{μ_k} , $k = 1, \dots, N$, is a projection of a rank-one tensor. This implies that for a sum of r branches the vector \mathbf{y} is a projection of a tensor having polyadic decomposition with r terms:

$$\mathbf{y} = \mathcal{P}(\mathcal{T}), \quad \mathcal{T} = \llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket = \sum_{\ell=1}^r \mathbf{a}_\ell \otimes \mathbf{b}_\ell \otimes \mathbf{h}_\ell,$$

where \mathbf{a}_ℓ , \mathbf{b}_ℓ are the coefficients of the corresponding filters, and \mathbf{h}_ℓ are the vectors for the nonlinearities constructed as previously. Thus the identification problem can be reformulated as a low-rank tensor recovery of the tensor \mathcal{T} from the samples \mathbf{y} . Low-rank tensor recovery is a generalization of the tensor completion problem to the case of arbitrary sampling operators (and not just selection of the elements as in a typical tensor completion problem).

This leads us to the following algorithm.

Algorithm 1. Input: number of branches r , filter sizes L_1, L_2 , Volterra kernels up to order d .

- (1) Choose sampling points $\mu_1, \dots, \mu_N \in \mathbb{C}$.
- (2) Evaluate the gradients of the homogeneous parts of f at \mathbf{u}_{μ_k} via contractions with the Volterra kernels (see (5)).
- (3) Build \mathbf{y} as in (9) by evaluating the gradients via Volterra kernels.
- (4) Find the rank- r tensor $\mathcal{T} = \llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket$ such that $\mathbf{y} \approx \mathcal{P}(\mathcal{T})$.
- (5) Recover the filter coefficients from \mathbf{a}_ℓ , \mathbf{b}_ℓ .
- (6) Recover the coefficients of the polynomials

$$g_\ell(t) = c_{\ell,1} t + c_{\ell,2} t^2 + \dots + c_{\ell,d} t^d$$

by solving

$$\mathbf{h}_\ell \approx \begin{bmatrix} c_{\ell,1} \\ 2c_{\ell,2}(a_\ell(\mu_1)) \\ \vdots \\ 2c_{\ell,2}(a_\ell(\mu_N)) \\ \vdots \\ dc_{\ell,d}(a_\ell(\mu_1))^{d-1} \\ \vdots \\ dc_{\ell,d}(a_\ell(\mu_N))^{d-1} \end{bmatrix}$$

Remark 2. In order to avoid numerical issues we restrict the sampling points to the unit circle

$$\mathbb{T} = \{\mu \in \mathbb{C} : |\mu| = 1\}.$$

Also, in Algorithm 1, we allow for approximations of \mathbf{y} in order to account for modelling errors or noise. While the estimation of $g_\ell(t)$ is a simple least squares problem, the most difficult part becomes the CPD of a partially observed tensor, which we detail in the next section.

4.2 Partially observed CPD

In order to find the rank- r tensor from its projection, we are going to solve the following tensor recovery problem in the least squares sense:

$$\min_{\mathbf{A}, \mathbf{B}, \mathbf{H}} \|\mathcal{P}(\llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket) - \mathbf{y}\|_2^2,$$

where $\mathcal{P} : \mathbb{C}^{L_1 \times L_2 \times L_3} \rightarrow \mathbb{C}^M$ is a sampling operator.

We are going to use a well-known alternating least squares (block coordinate descent) strategy Kolda and Bader (2009). This strategy consists in alternate minimization with respect to each variable with fixing other variables, and can be summarized in the following algorithm.

Algorithm 2. (Partial ALS). Input: initializations $\mathbf{A}_0, \mathbf{B}_0, \mathbf{H}_0$.

- (1) For $k=1,2,\dots$ until a stopping criterion is satisfied
- (2) $\mathbf{A}_k \leftarrow \arg \min_{\mathbf{A}} \|\mathcal{P}(\llbracket \mathbf{A}, \mathbf{B}_{k-1}, \mathbf{H}_{k-1} \rrbracket) - \mathbf{y}\|_2^2$;
- (3) $\mathbf{B}_k \leftarrow \arg \min_{\mathbf{B}} \|\mathcal{P}(\llbracket \mathbf{A}_k, \mathbf{B}, \mathbf{H}_{k-1} \rrbracket) - \mathbf{y}\|_2^2$;
- (4) $\mathbf{H}_k \leftarrow \arg \min_{\mathbf{H}} \|\mathcal{P}(\llbracket \mathbf{A}_k, \mathbf{B}_k, \mathbf{H} \rrbracket) - \mathbf{y}\|_2^2$.
- (5) End for

Each update in Algorithm 2 is a linear least squares problem, which explains the name ‘‘alternating least squares’’. Note that the overall cost function is nonconvex, and thus the algorithm may suffer from local minima and other convergence problems (Comon et al., 2009). However, this is one of the most popular and practically successful strategies. In what follows, we provide details on implementation of updates for recovery of partially observed low-rank tensors, which we did not find in the literature.

We assume that the operator $\mathcal{P} : \mathbb{C}^{L_1 \times L_2 \times L_3} \rightarrow \mathbb{C}^M$ has the matrix representation $\mathbf{P} \in \mathbb{C}^{M \times (L_1 L_2 L_3)}$, i.e.,

$$\mathcal{P}(\mathcal{T}) = \mathbf{P} \text{vec}\{\mathcal{T}\}.$$

Then the updates of ALS can be derived as follows:

- Updating \mathbf{A} : $\text{vec}\{\mathbf{A}\} = (\mathbf{Z}_\mathbf{A})^\dagger \mathbf{y}$, where $\mathbf{Z}_\mathbf{A} = \mathbf{P}((\mathbf{C} \odot \mathbf{B}) \boxtimes \mathbf{I}_{L_1})$.
- Updating \mathbf{B} : $\text{vec}\{\mathbf{B}\} = (\mathbf{Z}_\mathbf{B})^\dagger \mathbf{y}$, where $\mathbf{Z}_\mathbf{B} = \mathbf{P}[c_1 \boxtimes \mathbf{I}_{L_2} \boxtimes \mathbf{a}_1 \cdots c_r \boxtimes \mathbf{I}_{L_2} \boxtimes \mathbf{a}_r]$.

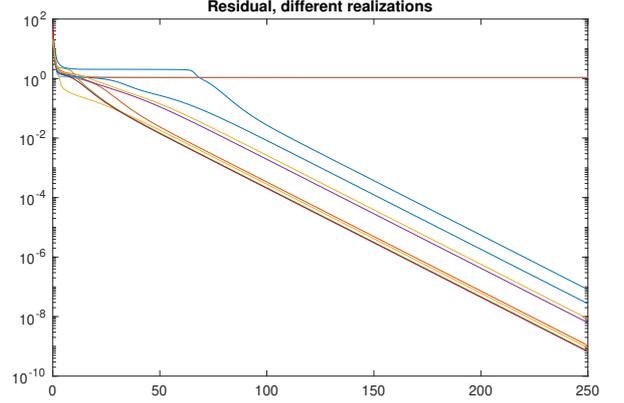


Fig. 2. Evolution of the residual ($\|\mathcal{P}(\llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket) - \mathbf{y}\|_2$) with respect to the number of cycles of ALS.

- Updating \mathbf{C} : $\text{vec}\{\mathbf{C}\} = (\mathbf{Z}_\mathbf{C})^\dagger \mathbf{y}$, where $\mathbf{Z}_\mathbf{C} = \mathbf{P}[\mathbf{I}_{L_3} \boxtimes \mathbf{b}_1 \boxtimes \mathbf{a}_1 \cdots \mathbf{I}_{L_3} \boxtimes \mathbf{b}_r \boxtimes \mathbf{a}_r]$.

For the practical implementation, we take advantage of the sparsity of the matrix \mathbf{P} : an easy inspection reveals that \mathbf{P} is block-diagonal with banded blocks.

5. EXPERIMENTS

Here we present an example that illustrates our approach. The algorithms were implemented in MATLAB R2019b on MacBook Air (2014, 1.4 GHz Intel i5, 4GB RAM).

We consider $r = 2$ branches and filter lengths $L_1 = 3, L_2 = 3$ with the following coefficients:

$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.6 \\ -0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \\ 0.1 & 0.01 \end{bmatrix},$$

and nonlinearities

$$g_1(x_1) = 3x_1^3 - x_1^2 + 5, \quad g_2(x_2) = -5x_2^3 + 3x_2 - 7. \quad (10)$$

We use $N = 30$ operating points generated randomly on the unit circle. We run Algorithm 1 for 10 different starting points (i.i.d. Gaussian distributed), maximum 250 iterations, and show the convergence plots in Fig. 2. We see that the algorithm converges linearly for all but one initialization, which is reasonable due to nonconvexity of the problem. For one of the realizations, the final residual is $\|\mathcal{P}(\llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket) - \mathbf{y}\|_2 = 8.48 \cdot 10^{-9}$, and the estimated factor $\hat{\mathbf{A}}$ is (with the first row normalized to 1 and shown with 4 fractional digits of the mantissa),

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & 1 \\ -1.3333 - i0.8305 \cdot 10^{-8} & 0.3333 + i0.4203 \cdot 10^{-9} \\ 0.3333 - i0.2210 \cdot 10^{-8} & 0.4999 + i0.7007 \cdot 10^{-10} \end{bmatrix},$$

which is complex-valued, but recovers quite accurately the true \mathbf{A} (the same holds for \mathbf{B} , not shown here).

In order to illustrate the reconstruction of the nonlinearities, instead of solving the least squares problem in Algorithm 1, we apply the idea similar the visualization of nonlinearities in Dreesen et al. (2015). In fact, the elements of \mathbf{H} can be combined in such a way to yield the values of the derivatives of $g_\ell(\cdots)$ at the points $a(\mu_k)$. We perform polynomial regression for degree 2, take the real parts and

obtain the following polynomials (with leading coefficient normalized to 1), rounded to the 4 fractional digits

$$h_1(t) = t^2 - 0.2222t, \quad h_2(t) = t^2 - 0.2.$$

after inspecting (10), we obtain that these are (up to numerical errors) the derivatives of the original nonlinearities, (i.e., $h_1(t) = \alpha g'_1(t)$, $h_2(t) = \alpha g'_2(t)$ with $\alpha = 9$).

6. CONCLUSION

We developed a novel promising algorithm for identification of Wiener-Hammerstein systems from Volterra kernels. Our approach has the following advantages:

- It is based on tensor recovery, rather than CPD with structured factors, and can be solved with a simple alternating least squares scheme.
- It does not need all the coefficients of the Volterra kernels to be estimated: we just need to compute contractions with Vandermonde-structured vectors for a fixed number of operating points.

Furthermore, we believe that our method may have an interesting interpretation from the frequency-domain identification perspective. Note that the operating points that we use are typically taken on the unit circle, i.e., an operating point is chosen as $\mu_k = e^{2\pi i \omega_k}$. Viewed from a frequency-domain point of view (Pintelon and Schoukens, 2012), contraction of Volterra kernels with Vandermonde-structured vectors is somewhat similar to an “excitation” of the first-order derivative at a frequency ω_k . However, for such an interpretation, we would potentially need to consider the framework of the Volterra kernel identification with complex valued inputs (Bouvier et al., 2019).

ACKNOWLEDGEMENTS

This research was supported by the ANR (Agence Nationale de Recherche) grant LeaFleT (ANR-19-CE23-0021); KU Leuven start-up-grant STG/19/036 ZKD7924; KU Leuven Research Fund; Fonds Wetenschappelijk Onderzoek - Vlaanderen (EOS Project 30468160 (SeLMA), SBO project S005319N, Infrastructure project I013218N, TBM Project T001919N, Research projects G028015N, G090117N, PhD grants SB/1SA1319N, SB/1S93918, and SB/151622); Flemish Government (AI Research Program); European Research Council under the European Union’s Horizon 2020 research and innovation programme (ERC AdG grant 885682). P. Dreesen is affiliated to Leuven.AI – KU Leuven institute for AI, Leuven, Belgium. Part of this work was performed while P. Dreesen and M. Ishteva were with Dept. ELEC of Vrije Universiteit Brussel, and P. Dreesen was with CoSys-lab at Universiteit Antwerpen, Belgium. The authors would like to thank the three anonymous reviewers for their useful comments that helped to improve the presentation of the results.

REFERENCES

Birpoutsoukis, G., Marconato, A., Lataire, J., and Schoukens, J. (2017). Regularized nonparametric Volterra kernel estimation. *Automatica*, 82, 324–327.

Bouvier, D., Hélie, T., and Roze, D. (2019). Phase-based order separation for Volterra series identification. *International Journal of Control*. doi: 10.1080/00207179.2019.1694175.

Comon, P. (2014). Tensors : A brief introduction. *IEEE Signal Processing Magazine*, 31(3), 44–53.

Comon, P., Luciani, X., and De Almeida, A.L. (2009). Tensor decompositions, alternating least squares and other tales. *Journal of Chemometrics*, 23(7-8), 393–405.

Dreesen, P. and Ishteva, M. (2021). Parameter estimation of parallel Wiener-Hammerstein systems by decoupling their Volterra representations. In *19th IFAC Symposium on System Identification (SYSID 2021)*.

Dreesen, P., Ishteva, M., and Schoukens, J. (2015). Decoupling multivariate polynomials using first-order information and tensor decompositions. *SIAM Journal on Matrix Analysis and Applications*, 36(2), 864–879.

Dreesen, P., Westwick, D.T., Schoukens, J., and Ishteva, M. (2017). *Modeling Parallel Wiener-Hammerstein Systems Using Tensor Decomposition of Volterra Kernels*, volume 10169 of *Lecture Notes on Computer Science*, 16–25. Springer International Publishing, Cham.

Dreesen, P., De Geeter, J., and Ishteva, M. (2018). Decoupling multivariate functions using second-order information and tensors. In Y. Deville, S. Gannot, R. Mason, M.D. Plumley, and D. Ward (eds.), *Latent Variable Analysis and Signal Separation*, 79–88. Springer International Publishing, Cham.

Giri, F. and Bai, E. (2010). *Block-oriented Nonlinear System Identification*. Lecture Notes in Control and Information Sciences. Springer.

Hollander, G. (2017). *Multivariate polynomial decoupling in nonlinear system identification*. Ph.D. thesis, Vrije Universiteit Brussel.

Katayama, T. (2005). *Subspace Methods for System Identification*. Springer.

Kibangou, A. and Favier, G. (2007). Toeplitz–Vandermonde matrix factorization with application to parameter estimation of Wiener–Hammerstein systems. *IEEE Signal Processing Letters*, 14, 141–144.

Kolda, T. and Bader, B. (2009). Tensor decompositions and applications. *SIAM Review*, 51(3), 455–500.

Ljung, L. (1999). *System identification*. Wiley.

Palm, G. (1979). On representation and approximation of nonlinear systems. *Biological Cybernetics*, 34(1), 49–52.

Pintelon, R. and Schoukens, J. (2012). *System Identification: A Frequency Domain Approach*. Wiley, 2nd edition.

Schetzen, M. (1980). *The Volterra and Wiener Theories of Nonlinear Systems*. Wiley, New York.

Schoukens, M. and Tiels, K. (2017). Identification of block-oriented nonlinear systems starting from linear approximations: A survey. *Automatica*, 85, 272–292.

Usevich, K. (2014). Decomposing multivariate polynomials with structured low-rank approximation. In *21th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014)*.

Van Mulders, A., Vanbeylen, L., and Usevich, K. (2014). Identification of a block-structured model with several sources of nonlinearity. In *2014 European Control Conference (ECC)*, 1717–1722. doi: 10.1109/ECC.2014.6862455.

Westwick, D., Ishteva, M., Dreesen, P., and Schoukens, J. (2017). Tensor factorization based estimates of parallel Wiener Hammerstein models. In *Proc. 20th IFAC World Congress (IFAC 2017)*, volume 50(1) of *IFAC-PapersOnLine*, 9468–9473. Toulouse, France.