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TOPOLOGICAL STRUCTURE OF THE SPACE OF COMPOSITION OPERATORS ON THE HARDY SPACE OF DIRICHLET SERIES

FRÉDÉRIC BAYART, MAOFA WANG AND XINGXING YAO

Abstract. The aim of this paper is to study when two composition operators on the Hilbert space of Dirichlet series with square summable coefficients belong to the same component or when their difference is compact. As a corollary we show that if a linear combination of composition operators with polynomial symbols of degree at most 2 is compact, then each composition operator is compact.

1. Introduction

Let H be a Hilbert space of analytic functions on a domain $\mathcal{U}$ of the complex plane $\mathbb{C}$ and let φ be an analytic selfmap of $\mathcal{U}$. The composition operator C_φ is the linear operator defined by

$$C_\varphi f = f \circ \varphi, \ f \in H.$$

A central problem in the investigation of composition operators is to relate function theoretic properties of the symbol φ to operator theoretic properties of C_φ, see the books [17, 27, 30].

In recent decades, the study of the topological structure, for the operator norm topology, of the set $\mathcal{C}(H)$ of composition operators acting on H attracted the attention of many mathematicians. The earliest work is Berkson's [9] isolation theorem in the setting of Hardy-Hilbert space $H^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ in $\mathbb{C}$, and then this result was generalized by MacCluer [22]. The understanding of the topological structure of the set of composition operators was linked to the problem of determining if the difference of two composition operators is compact by the Shapiro and Sundberg conjecture raised in [28]: $C_{\varphi_0}, C_{\varphi_1}$ belong to the same connected component of $\mathcal{C}(H^2(\mathbb{D}))$ if and only if $C_{\varphi_0} - C_{\varphi_1}$ is compact. Bourdon [10] and independently Moorhouse and Toews [23] disproved this conjecture but they also showed clearly why these problems are closely related.

Beyond compact differences, the compactness of linear combinations of composition operators has also attracted the attention of many mathematicians. As a sample of results it is known that composition operators induced by linear fractional maps of the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ behave quite rigidly in the following sense: the compactness of linear combinations $\sum_{j=1}^N \lambda_j C_{\varphi_j}$, for finitely many distinct linear fractional maps φ_j and nonzero complex numbers λ_j, implies that each composition operator C_{φ_j} is compact on the Hardy space $H^2(\mathbb{B}_n)$ (see [14]). It should be noticed that the

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study of the compactness of linear combinations of composition operators is a more difficult problem than the study of the compactness of linear differences, since high order cancelations may occur, and that this often requires new ideas. For example, see [15, 16, 21] and references therein.

Our aim, in the present work, is to make a thorough study of these problems on the Hardy-Hilbert space $\mathcal{H}$ of Dirichlet series. Hedenmalm-Lindqvist-Seip [20] introduced and started the study of the Hilbert space of Dirichlet series with square summable coefficients:

$$\mathcal{H} = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} < \infty \right\}.$$

By the Cauchy-Schwarz inequality, the functions in $\mathcal{H}$ are all analytic on the half-plane $\mathbb{C}_{1/2}$ (where, for $\theta \in \mathbb{R}$, $\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$ and $\mathbb{C}_+ = \mathbb{C}_0$). A characteristic feature of the space $\mathcal{H}$ is its reproducing kernel function being essentially the Riemann zeta function: $K_w(s) = \zeta(\overline{w} + s)$ for $w, s \in \mathbb{C}_{1/2}$, where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. The main results in [19, 25] show that an analytic map $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator C_φ on $\mathcal{H}$ if and only if it is a member of the Gordon-Hedenmalm class $\mathcal{G}$ defined as follows.

Definition. The **Gordon-Hedenmalm class** $\mathcal{G}$ consists of the maps $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ of the form

$$\varphi(s) = c_0 s + \psi(s),$$

where c_0 is a non-negative integer (called the **characteristic** of φ, i.e., $\operatorname{char}(\varphi) = c_0$), and $\psi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and has the following properties:

(a) If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$, then either $\psi \equiv c_1$ with $\operatorname{Re} c_1 \geq 0$ or $\psi(\mathbb{C}_0) \subseteq \mathbb{C}_0$.

From then on, several authors have studied the properties of composition operators acting on $\mathcal{H}$ or on similar spaces of Dirichlet series, see for instance [2, 3, 5, 6, 7, 12, 25]. In particular, the study of the compactness of composition operators on $\mathcal{H}$ is a difficult problem. Very recently, when the symbol has zero characteristic, Brevig and Perfekt [13] gave a characterization of compactness by introducing a mean-counting function which has to satisfy a decay condition. Nevertheless, a similar characterization is unknown for symbols with positive characteristic, and even for simple symbols with zero characteristic, subtle phenomena occur.

Let us introduce some terminology. A symbol $\varphi \in \mathcal{G}$ is said to have **unrestricted range** if

$$\inf_{s \in \mathbb{C}_0} \operatorname{Re} \varphi(s) = \begin{cases} 1/2 & \text{if } c_0 = 0, \\ 0 & \text{if } c_0 \geq 1. \end{cases}$$

Correspondingly, if $\varphi(\mathbb{C}_0)$ is strictly contained in any smaller half-plane, we say that φ has **restricted range**. It is well-known that the composition operator C_φ is compact on $\mathcal{H}$ when φ has restricted range [5, Theorem 21]. We call φ a **Dirichlet**

polynomial symbol, if

$$\varphi(s) = c_0 s + c_1 + \sum_{k=2}^{n} c_k k^{-s}.$$

In this work, we will be interested in polynomial symbols satisfying some conditions on their degree. Recall the definition of the arithmetic function $\Omega(n) := \alpha_1 + \cdots + \alpha_d$ provided $n = p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ where $\mathcal{P} = (p_i)_{i \geq 1}$ denotes the set of prime numbers. If $\psi(s) = \sum_{k=1}^{n} c_n k^{-s}$ is a Dirichlet polynomial, its **degree** is the infimum of the integers N such that $c_n = 0$ provided $\Omega(n) > N$.

We will also be interested in composition operators C_φ generated by *linear symbols*

$$\varphi(s) = c_0 s + c_1 + \sum_{k=1}^{d} c_{q_k} q_k^{-s},$$

where $\{q_k\}_{k=1}^{d}$ is a sequence of finitely multiplicatively independent positive integers, and $q_k \neq 1$, $c_{q_k} \neq 0$ for each $k = 1, \cdots, d$. It should be observed that a linear symbol is not necessary a polynomial symbol of degree less than or equal to 1. Following [6] and [18], it is known that the composition operator C_φ induced by the linear symbol φ with $c_0 = 0$ is compact if and only if φ has restricted range or $d \geq 2$; when $c_0 \geq 1$, C_φ is compact if and only if φ has restricted range. For a Dirichlet polynomial symbol φ with $c_0 \geq 1$, Bayart-Brevig [7] showed that C_φ is compact if and only if φ has restricted range.

Let us come back to our initial problems of the topological structure of $\mathcal{C}(\mathcal{H})$ and of the compactness of linear combinations of composition operators on $\mathcal{H}$. The first result in this direction was obtained by the authors in [8]: if φ_1 and φ_2 are linear symbols and $\lambda_1,\ \lambda_2$ are nonzero complex numbers, then $\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2}$ is compact if and only if both C_{φ_1} and C_{φ_2} are compact. We extend this result in two directions, allowing any finite linear symbols, and allowing polynomial symbols of degree at most 2 (see Theorems 4.5, 4.6 and 4.8).

To prove this, we will develop a machinery, inspired by [23], allowing to get lower bounds for the essential norm of a linear combination of composition operators. Many problems arise in the setting of $\mathcal{H}$: the lack of reproducing kernels defined on $\mathbb{C}_+$, the infinite polydisc $\mathbb{D}^\infty$ hidden behind $\mathbb{C}_+$, the characteristic of the symbol which forces to adopt different proofs following its value. For instance, we will show that a basic result of Shapiro and Sundberg (the set of compact composition operators on $H^2(\mathbb{D})$ is connected) does not extend to $\mathcal{C}(\mathcal{H})$, but that the set of compact composition operators with prescribed characteristic is connected.

Most of these results are necessary conditions for the difference of two composition operators to be compact (or for two operators to be in the same connected component). In Section 5 we turn to sufficient conditions. We succeed (see Theorem 5.5 for instance) to give sufficient conditions that are easy to testify to prove that $C_{\varphi_0} - C_{\varphi_1}$ is compact, or that C_{φ_0} and C_{φ_1} belong to the same connected component. These results involve sufficient conditions of independent interest to prove that some weighted composition operators are bounded or compact. Again the cases of zero and nonzero characteristics require quite different approaches, one based on

Carleson measures, one based on Nevanlinna counting functions. Finally we mention that in some cases, the necessary and sufficient conditions coincide, giving a characterization of the compactness of $C_{\varphi_0} - C_{\varphi_1}$.

Throughout this paper we use the notation $X \lesssim Y$ or $Y \gtrsim X$ for non-negative functions X and Y to mean that there exists $C > 0$ such that $X \leq CY$, where C does not depend on the associated variables. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. We will sometimes write $X \lesssim_a Y$ if we want to point out that the involved constant depend on a. Given two positive functions $f(x)$ and $g(x)$ defined on $(a, +\infty)$, we write $f(x) \sim_{x\to a} g(x)$ (respectively, $f(x) \sim_{x\to+\infty} g(x)$), provided $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$ (respectively, $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$).

2. Preliminaries

This section first introduces some background material associated to Bohr lifts, vertical limit functions, reproducing and partial reproducing kernels.

2.1. **Bohr lifts.** At some places, it will be useful to work on the polydisc $\mathbb{D}^d$ instead of $\mathbb{C}_0$. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series and let $\mathcal{Q} = \{q_1, \dots, q_d\}$ be a sequence of multiplicatively independent integers such that for each n with $a_n \neq 0$, we have $n = q^\alpha := q_1^{\alpha_1} \cdots q_d^{\alpha_d}$ with $\alpha \in \mathbb{N}_0^d$. We will say that f has d multiplicative generators when $d < +\infty$, and that $\mathcal{Q}$ is compatible with f. The associated Bohr lift $\mathcal{B}_{\mathcal{Q}} f(z)$ is the holomorphic function defined on $\mathbb{D}^d$ by

$$\mathcal{B}_{\mathcal{Q}} f(z) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

By Kronecker's theorem (see for example [24]), the map $\mathbb{R} \to \mathbb{T}^d$, $t \mapsto (q_1^{it}, \dots, q_d^{it})$ has dense range. In particular, if f maps $\mathbb{C}_0$ into $\mathbb{C}_0$, then $\mathcal{B}_{\mathcal{Q}} f$ maps $\mathbb{D}^d$ into $\mathbb{C}_0$, and $\overline{f(i\mathbb{R})} = \overline{\mathcal{B}_{\mathcal{Q}} f(\mathbb{T}^d)}$.

For any Dirichlet series f having finitely multiplicative generators, for all d sufficiently large, the sequence of the first d prime numbers $\{p_1, \dots, p_d\}$ is always compatible with f. This Bohr lift will be simply denoted by $\mathcal{B}$. It induces a unitary map from $\mathcal{H}$ onto the Hardy space $H^2(\mathbb{D}^\infty)$ over the infinite polydisc $\mathbb{D}^\infty$. It should be observed that the degree of a Dirichlet polynomial f is exactly the degree of its Bohr lift $\mathcal{B}f$.

2.2. **Vertical limit functions.** Following [20], let Ξ be the dual group of $\mathbb{Q}_+$, where $\mathbb{Q}_+$ denotes the multiplicative discrete group of strictly positive rational numbers. Consequently, Ξ is the set of all functions $\chi : \mathbb{Q}_+ \to \mathbb{C}$, such that

(a) $\chi(mn) = \chi(m)\chi(n)$ for all m, n in $\mathbb{Q}_+$.
(b) $|\chi(n)| = 1$.

Ξ can be identified with $\mathbb{T}^\infty$, the Cartesian product of countably many copies of the unit circle. Indeed, given any point $z = (z_1, z_2, \dots) \in \mathbb{T}^\infty$, we define the value of χ at the primes through

$$\chi(2) = z_1, \chi(3) = z_2, \dots, \chi(p_m) = z_m, \dots$$

and extend the definition multiplicatively. This then yields a character, and clearly all characters can be obtained in this procedure. Also the unique normalized Haar measure on Ξ can be identified with the ordinary product measure m on $\mathbb{T}^\infty$.

For $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}$ and $\chi \in \mathbb{T}^\infty$, the vertical limit function f_χ is defined by

$$f_\chi(s) := \sum_{n=1}^\infty a_n \chi(n) n^{-s}.$$

Note that the vertical translate function $T_\tau f(s) := f(s+i\tau)$, $\tau \in \mathbb{R}$, corresponds to f_χ with $\chi(n) = n^{-i\tau}$. The name vertical limit function is justified by [20, Lemma 2.4], which asserts that the functions f_χ are precisely those obtained from the Dirichlet series f by taking a limit of vertical translations,

$$f_\chi(s) = \lim_{k\to\infty} T_{\tau_k} f(s).$$

Here the convergence is uniform on compact subsets of the half-plane where f converges uniformly. The vertical limit function f_χ sometimes has better properties than the original function f. As explained in [20, Section 4.2], if f is in $\mathcal{H}$, the Dirichlet series f_χ converges in $\mathbb{C}_0$ for almost every $\chi \in \mathbb{T}^\infty$, and the non-tangential boundary value

$$f_\chi^*(it) := \lim_{\sigma\to 0^+} f_\chi(\sigma + it)$$

exists for almost every $\chi \in \mathbb{T}^\infty$ and almost all $t \in \mathbb{R}$. By [20, Theorem 4.1], we can compute the norm of f in terms of the function f_χ: if μ is a finite Borel measure on $\mathbb{R}$, then

$$\|f\|^2 \mu(\mathbb{R}) = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^2 d\mu(t) dm(\chi). \tag{2.1}$$

For any symbol $\varphi(s) = c_0 s + \psi(s)$, define φ_χ as $\varphi_\chi(s) = c_0 s + \psi_\chi(s)$ for $\chi \in \mathbb{T}^\infty$, a vertical limit of the functions $\varphi_\tau(s) = c_0 s + \psi(s+i\tau)$. As is clarified in [19], for any such symbol $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$, and $f \in \mathcal{H}$, $\chi \in \mathbb{T}^\infty$, the following relation holds:

$$(f \circ \varphi)_\chi(s) = f_{\chi^{c_0}} \circ \varphi_\chi(s), \quad s \in \mathbb{C}_{1/2}.$$

Especially, for any Dirichlet series $\varphi \in \mathcal{G}$ and $f \in \mathcal{H}$, $C_\varphi f$ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$, and $(C_\varphi f)_\chi(s) = C_{\varphi_\chi} f_{\chi^{c_0}}(s)$. This implies that $\|C_{\varphi_\chi} f\| = \|C_\varphi f\|$, and thus φ_χ belongs to $\mathcal{G}$ for every $\chi \in \mathbb{T}^\infty$.

2.3. **Reproducing and partial reproducing kernels.** As recalled in the introduction, the reproducing kernel at $w \in \mathbb{C}_{1/2}$ in $\mathcal{H}$ is defined by $K_w(s) = \zeta(\overline{w} + s)$. Reproducing kernels are often useful in the study of composition operators (especially in the study of their compactness) since $C_\varphi^*(K_w) = K_{\varphi(w)}$. Moreover, we know that

$$\|K_w\|^2 = \zeta(2\mathrm{Re}\, w) \sim_{\mathrm{Re}\, w \to 1/2} \frac{1}{2\mathrm{Re}\, w - 1}$$

and that if $\{w_k\}$ is any sequence of $\mathbb{C}_{1/2}$ such that $\mathrm{Re}\, w_k \to 1/2$, then $K_{w_k}/\|K_{w_k}\|$ tends weakly to 0.

It turns out that in the specific case of composition operators acting on $\mathcal{H}$, the reproducing kernels K_w have a serious drawback: they are limited to $\mathbb{C}_{1/2}$ whereas the symbol φ may be extended to $\mathbb{C}_0$. This motivates the introduction of partial

reproducing kernels (refer to [6] for more details). For $w \in \mathbb{C}_0$, and $d \geq 1$, the partial reproducing kernel of order d at w is defined by

$$K_{d,w}(s) = \prod_{j=1}^{d} \left(\sum_{n \geq 1} p_j^{-n(\overline{w}+s)} \right) = \prod_{j=1}^{d} \frac{1}{1 - p_j^{-(\overline{w}+s)}} = \sum_{n \geq 1,\ p^+(n) \leq p_d} n^{-\overline{w}} n^{-s}$$

where $p^+(n)$ denotes the biggest prime factor of n. The function $K_{d,w}$ reproduces partially $\mathcal{H}$: if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$, then

$$\langle f, K_{d,w} \rangle = \sum_{n \geq 1,\ p^+(n) \leq p_d} a_n n^{-w}.$$

By Euler's identity, we have

$$\|K_{d,w}\| = \prod_{j=1}^{d} \left(\frac{1}{1 - p_j^{-2\mathrm{Re}\, w}} \right)^{1/2} =: \zeta_d(2\mathrm{Re}\, w)^{1/2}.$$

Moreover, it is clear that $k_{d,w_k} := K_{d,w_k}/\|K_{d,w_k}\|$ converges weakly to zero for any sequence $\{w_k\} \subseteq \mathbb{C}_0$ satisfying $\mathrm{Re}\, w_k \to 0$ and that

$$\|K_{d,w}\|^2 \sim_{\mathrm{Re}\, w \to 0} \frac{A_d}{(\mathrm{Re}\, w)^d}$$

for some positive constant A_d depending only on d.

The interest of using partial reproducing kernels in the study of composition operators comes from [6, Proposition 5]: if $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ belongs to $\mathcal{G}$ with $c_n = 0$ for $p^+(n) > p_d$, then $C_\varphi^* K_{d,w} = K_{d,\varphi(w)}$ when $c_0 \geq 1$, and $C_\varphi^* K_{d,w} = K_{\varphi(w)}$ when $c_0 = 0$. We will see later (Lemma 3.6) how to generalize this formula to any symbol.

We shall also use a variant of these partial reproducing kernels. For an integer $q \geq 2$ and $w \in \mathbb{C}_0$, we set $K_w^{(q)}(s) = (1 - q^{-\overline{w}} q^{-s})^{-1}$, $s \in \mathbb{C}_{1/2}$. Clearly, $\|K_w^{(q)}\| \sim C_q(\mathrm{Re}\, w)^{-1/2}$ as $\mathrm{Re}\, w \to 0$ and, setting $k_w^{(q)} = K_w^{(q)}/\|K_w^{(q)}\|$, any sequence $(k_{w_k}^{(q)})$ converges weakly to zero in $\mathcal{H}$ as soon as $\mathrm{Re}\, w_k$ goes to zero. The following lemma comes from [8].

Lemma 2.1. *Let $r \geq 2$ be an integer and $\varphi(s) = c_1 + c_r r^{-s} \in \mathcal{G}$.*

(i) Assume that $r = q^n$ for some positive integer n. Then $C_\varphi^*(K_w^{(q)}) = K_{\varphi(w)}$.

(ii) Assume that $r^m \neq q^n$ for all positive integers n and m. Then $C_\varphi^*(K_w^{(q)}) = K_{c_1}$.

3. Connected components in the space of composition operators

3.1. **The role of the characteristic.** It is a result of Shapiro and Sundberg [28] that the collection of compact composition operators on $H^2(\mathbb{D})$ is arcwise connected. This result breaks down for composition operators on $\mathcal{H}$ because of the characteristic: two composition operators with different characteristics cannot belong to the same component.

Theorem 3.1. *The map $\mathcal{C}(\mathcal{H}) \to \mathbb{N}$, $C_\varphi \mapsto \mathrm{char}\,(\varphi)$ is continuous.*

We prove this result by a reasoning on the coefficients of $p^{-\varphi(s)}$ which can be adapted to a much more general context than $\mathcal{H}$. We need to introduce the following definition.

Definition 3.2. Let $l \geq 1$. A **weighted partition** of l is a triple (r, i, γ) with $r \geq 1$, $i \in \{1, \dots, l\}^r$ satisfying $i_1 < i_2 < \cdots < i_r$, $\gamma \in \mathbb{N}^r$ and $\gamma_1 i_1 + \cdots + \gamma_r i_r = l$. The set of weighted partitions of l will be denoted by $\mathcal{WP}_l$.

Lemma 3.3. *Let $l \geq 1$ and let p be a prime number. For each (r, i, γ) a weighted partition of l, there exists a real number $u(p, l, r, i, \gamma)$ such that, for all sequences $c = (c_k)_{k\geq 2}$ with $\sigma_a\left(\sum_k c_k k^{-s}\right) < +\infty$, for all prime numbers q,*

$$p^{-\sum_{k\geq 2} c_k k^{-s}} = \sum_{k=1}^{+\infty} \alpha_k(c) k^{-s}$$

with

$$\alpha_{q^l}(c) = -c_{q^l} \log p + \sum_{\substack{(r,i,\gamma)\in\mathcal{WP}_l \\ i_r < l}} u(p, l, r, i, \gamma)(c_{q^{i_1}})^{\gamma_1} \cdots (c_{q^{i_r}})^{\gamma_r}.$$

Proof. We can write formally

$$p^{-\sum_{k\geq 2} c_k k^{-s}} = \prod_{k\geq 2} \left(\sum_{n=0}^{+\infty} \frac{(-1)^n (c_k)^n}{n!} (\log p)^n (k^n)^{-s} \right).$$

If we expand the infinite product and since q is prime, the term $(q^l)^{-s}$ can only be obtained as the sum of products of the terms $\left((q^{i_1})^{\gamma_1} \cdots (q^{i_r})^{\gamma_r}\right)^{-s}$ with $\gamma_1 i_1 + \cdots + \gamma_r i_r = l$. More precisely,

$$u(p, l, r, i, \gamma) = \frac{(-1)^{\gamma_1 + \cdots + \gamma_r}}{\gamma_1! \cdots \gamma_r!} (\log p)^{\gamma_1 + \cdots + \gamma_r}.$$

That this formal computation leads to a convergent Dirichlet series follows from the absolute convergence of the involved Dirichlet series in some half-plane. □

Proof of Theorem 3.1. Let $\varphi(s) = c_0 s + c_1 + \sum_{k\geq 2} c_k k^{-s} \in \mathcal{C}(\mathcal{H})$ and let $(C_{\varphi_j}) \in \mathcal{C}(\mathcal{H})$ converging to C_φ in the operator norm topology. In order to proceed by contradiction, we assume that $(\mathrm{char}\,(\varphi_j))$ does not converge to c_0. Extracting if necessary, we may assume that $\mathrm{char}\,(\varphi_j) \neq c_0$ for all j. We first handle the easiest case $c_0 = 0$. Since, for all j,

$$\begin{aligned} \|2^{-\varphi} - 2^{-\varphi_j}\| &\geq \lim_{\sigma\to+\infty} |2^{-\varphi(\sigma)} - 2^{-\varphi_j(\sigma)}| \\ &\geq 2^{-c_1} > 0, \end{aligned}$$

we get that (C_{φ_j}) cannot converge to C_φ. Hence, we assume $c_0 \geq 1$.

We write $\varphi_j(s) = d_0(j)s + d_1(j) + \sum_{k\geq 2} d_k(j) k^{-s}$. Let p, q be two prime numbers and $l \geq 0$. We write

$$p^{-\varphi(s)} = \sum_{k\geq 1} a_k k^{-s} \text{ and } p^{-\varphi_j(s)} = \sum_{k\geq 1} b_k(j) k^{-s}.$$

Let us first assume that $\mathrm{char}\,(\varphi_j) > c_0$ for an infinite number of integers j, hence for all j by extraction. Then $a_{p^{c_0}} = p^{-c_1}$ whereas $b_{p^{c_0}}(j) = 0$. This contradicts that $p^{-\varphi_j}$ tends to $p^{-\varphi}$ in $\mathcal{H}$.

Therefore it remains to handle the case $\mathrm{char}\,(\varphi_j) < c_0$ for all j. Upon another extraction we may and shall assume that $d_0(j) = d_0 \in \{0, \dots, c_0 - 1\}$. When $p \neq q$, for all $l \geq 1$, $a_{p^{d_0} q^l} = 0$ whereas $b_{p^{d_0} q^l}(j) = p^{-d_1(j)} \alpha_{q^l}(d(j))$. Hence

$$p \neq q,\ l \geq 1 \implies p^{-d_1(j)} \alpha_{q^l}(d(j)) \xrightarrow{j \to +\infty} 0. \tag{3.1}$$

When $p = q$, setting $m = c_0 - d_0$, then $a_{q^{c_0}} = q^{-c_1}$ whereas $b_{q^{c_0}}(j) = q^{-d_1(j)} \alpha_{q^m}(d(j))$. Thus

$$q^{-d_1(j)} \alpha_{q^m}(d(j)) \xrightarrow{j \to +\infty} q^{-c_1}. \tag{3.2}$$

We then fix an increasing sequence of prime numbers $(q_l)_{1 \leq l \leq m+1}$ such that, for all $l = 1, \dots, m$, $q_{l+1} > (q_l)^{l+1}$. We show by a finite induction that, for all $l = 1, \dots, m$,

$$q_l^{-d_1(j)} d_{q_{m+1}^l}(j) \xrightarrow{j \to +\infty} 0. \tag{3.3}$$

For $l = 1$, this is (3.1) applied with $p = q_1$, $q = q_{m+1}$ and we use Lemma 3.3 to compute the value of $\alpha_{q_{m+1}}(d(j))$. Let us assume that (3.3) has been obtained for $l - 1$ and let us prove it for l. We apply again (3.1) with $p = q_l$ and $q = q_{m+1}$ so that

$$q_l^{-d_1(j)} \Bigg(- d_{q_{m+1}^l}(j) \log(q_l) + \sum_{\substack{(r,i,\gamma) \in \mathcal{WP}_l \\ i_r < l}} u(q_l, l, r, i, \gamma) (d_{q_{m+1}^{i_1}}(j))^{\gamma_1} \cdots (d_{q_{m+1}^{i_r}}(j))^{\gamma_r} \Bigg) \xrightarrow{j \to +\infty} 0.$$

Now since $q_l > q_{i_1}^{\gamma_1} \cdots q_{i_r}^{\gamma_r}$ for all $(r, i, \gamma) \in \mathcal{WP}_l$ with $i_r < l$, we may write

$$q_l^{-d_1(j)} |d_{q_{m+1}^{i_1}}(j)|^{\gamma_1} \cdots |d_{q_{m+1}^{i_r}}(j)|^{\gamma_r} \leq |q_{i_1}^{-d_1(j)} d_{q_{m+1}^{i_1}}(j)|^{\gamma_1} \cdots |q_{i_r}^{-d_1(j)} d_{q_{m+1}^{i_r}}(j)|^{\gamma_r}$$

and using the induction hypothesis we obtain that

$$q_l^{-d_1(j)} (d_{q_{m+1}^{i_1}}(j))^{\gamma_1} \cdots (d_{q_{m+1}^{i_r}}(j))^{\gamma_r} \xrightarrow{j \to +\infty} 0.$$

This yields (3.3) at rank l.

Let us now conclude. Property (3.2) for $q = q_{m+1}$ tells us that

$$q_{m+1}^{-d_1(j)} \Bigg(- d_{q_{m+1}^m}(j) \log(q_{m+1}) + \sum_{\substack{(r,i,\gamma) \in \mathcal{WP}_m \\ i_r < m}} u(q_{m+1}, m, r, i, \gamma) (d_{q_{m+1}^{i_1}}(j))^{\gamma_1} \cdots (d_{q_{m+1}^{i_r}}(j))^{\gamma_r} \Bigg) \xrightarrow{j \to +\infty} q_{m+1}^{-c_1}.$$

But since $q_{m+1} \geq q_m$, (3.3) implies that $q_{m+1}^{-d_1(j)} d_{q_{m+1}^m}(j)$ tends to zero. Furthermore we can repeat the argument of the inductive step to get

$$q_{m+1}^{-d_1(j)} \sum_{\substack{(r,i,\gamma) \in \mathcal{WP}_m \\ i_r < m}} u(q_{m+1}, m, r, i, \gamma) (d_{q_{m+1}^{i_1}}(j))^{\gamma_1} \cdots (d_{q_{m+1}^{i_r}}(j))^{\gamma_r} \xrightarrow{j \to +\infty} 0.$$

This yields the contradiction we are looking for. □

Remark 3.4. In fact our proof shows that the map $\mathcal{C}(\mathcal{H}) \to \mathbb{N}$, $C_\varphi \mapsto \mathrm{char}(\varphi)$ is continuous even if we endow $\mathcal{C}(\mathcal{H})$ with the weak operator topology.

3.2. **Compact composition operators.** We now show that the set of compact composition operators with prescribed characteristic is connected. The proof is necessary different from the proof on $H^2(\mathbb{D})$ since we cannot connect any compact composition operator to a composition operator with constant symbol.

Theorem 3.5. *Let $c_0 \in \mathbb{N}_0$. The set of compact composition operators with characteristic equal to c_0 is arcwise connected.*

Proof. We fix $\varphi \in \mathcal{G}$ with $\mathrm{char}\,(\varphi) = c_0$ such that C_φ is compact. We first show that there exists an arc of compact composition operators between C_φ and $C_{\tilde\varphi}$ for some $\tilde\varphi \in \mathcal{G}$ with $\tilde\varphi(\mathbb{C}_+) \subseteq \mathbb{C}_{\frac12+\varepsilon}$ for some $\varepsilon > 0$, $\mathrm{char}\,(\tilde\varphi) = c_0$ and, writing $\tilde\varphi = c_0 s + \tilde\psi$, $\tilde\psi(\mathbb{C}_+)$ is bounded. Define $\varphi_\sigma(\cdot) = \varphi(\cdot + \sigma)$. We claim that $\sigma \mapsto C_{\varphi_\sigma}$ is continuous (for the operator norm topology). The choice $\tilde\varphi = \varphi_1$ will answer the problem: for $c_0 = 0$, φ_1 has restricted range, so that $\varphi_1(\mathbb{C}_+) \subseteq \mathbb{C}_{\frac12+\varepsilon}$; for $c_0 \geq 1$, we even have $\varphi(\mathbb{C}_+) \subseteq \mathbb{C}_1$. Moreover, if $\varphi = c_0 s + \psi$, since the abscissa of uniform convergence of ψ is nonpositive, its abscissa of absolute convergence does not exceed $1/2$. Hence, $\psi(\cdot + 1)$ is an absolutely convergent Dirichlet series in $\overline{\mathbb{C}_+}$ which implies that $\tilde\psi(\mathbb{C}_+)$ is bounded. Finally, since for any $\sigma > 0$, φ_σ has restricted range, C_{φ_σ} is compact.

Let T_σ be the composition operator induced by the map $s \mapsto s + \sigma$, $\sigma \geq 0$, so that $C_{\varphi_\sigma} = T_\sigma \circ C_\varphi$. Now observe that for a fixed $f \in \mathcal{H}$ and a fixed $\sigma_0 \geq 0$, $T_\sigma f \to T_{\sigma_0} f$ provided $\sigma \to \sigma_0$. Furthermore, $\{C_\varphi(f) : \|f\| \leq 1\}$ is relatively compact in $\mathcal{H}$ and the family $(T_\sigma)_{\sigma\geq 0}$ is equicontinuous since $\|T_\sigma\| \leq 1$ for all $\sigma \geq 0$. It follows from a standard compactness argument that the pointwise convergence of $T_\sigma f$ to $T_{\sigma_0} f$ when σ tend to σ_0 is uniform on $\{C_\varphi(f) : \|f\| \leq 1\}$. Namely for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f\| \leq 1 \text{ and } |\sigma - \sigma_0| < \delta \implies \|C_{\varphi_\sigma}(f) - C_{\varphi_{\sigma_0}}(f)\| \leq \varepsilon.$$

This proves the continuity of $\sigma \mapsto C_{\varphi_\sigma}$.

To finish the proof, we only need to prove that, given $\varepsilon > 0$ and $C > 0$, for two symbols $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$ such that $\varphi_0(\mathbb{C}_+), \varphi_1(\mathbb{C}_+)$ are contained in $\mathbb{C}_{\frac12+\varepsilon}$ and $|\psi_0(s)|, |\psi_1(s)| \leq C$ for all $s \in \mathbb{C}_+$, there exists an arc in $\mathcal{C}(\mathcal{H})$ between C_{φ_0} and C_{φ_1} consisting in compact composition operators. Define, for $\lambda \in (0,1)$, $\varphi_\lambda = c_0 s + (1-\lambda)\psi_0 + \lambda\psi_1$. Again it suffices to show that $\lambda \mapsto C_{\varphi_\lambda}$ is continuous since each φ_λ has restricted range. Pick $f \in \mathcal{H}$ with $\|f\| \leq 1$ and write

$$\begin{aligned}\left\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\right\|^2 &= \int_{\mathbb{T}^\infty} |(f\circ\varphi_\lambda)_\chi(0) - (f\circ\varphi_{\lambda'})_\chi(0)|^2 dm(\chi)\\ &= \int_{\mathbb{T}^\infty} |f_{\chi^{c_0}}((\varphi_\lambda)_\chi(0)) - f_{\chi^{c_0}}((\varphi_{\lambda'})_\chi(0))|^2 dm(\chi).\end{aligned}$$

Since $(\varphi_\lambda)_\chi$ is a vertical limit of φ_λ, for all $\chi \in \mathbb{T}^\infty$, $(\varphi_\lambda)_\chi(0)$ and $(\varphi_{\lambda'})_\chi(0)$ belong to $\overline{\mathbb{C}_{\frac12+\varepsilon}}$. Since $\|f_{\chi^{c_0}}\| \leq 1$, there exists some $M > 0$, depending only on ε, such

that, for all $\chi \in \mathbb{T}^\infty$,

$$\sup_{s\in\overline{\mathbb{C}_{\frac{1}{2}+\varepsilon}}} |f'_{\chi^{c_0}}(s)| \leq M.$$

Therefore

$$\begin{aligned}\left\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\right\|^2 &\leq M^2 \int_{\mathbb{T}^\infty} |(\varphi_\lambda)_\chi(0)) - (\varphi_{\lambda'})_\chi(0)|^2 dm(\chi) \\ &\lesssim |\lambda - \lambda'|^2 \int_{\mathbb{T}^\infty} \left(|(\psi_0)_\chi(0)| + |(\psi_1)_\chi(0)|\right)^2 dm(\chi) \\ &\lesssim |\lambda - \lambda'|^2\end{aligned}$$

since for all $\chi \in \mathbb{T}^\infty$, $|(\psi_0)_\chi(0)| \leq C$ and $|(\psi_1)_\chi(0)| \leq C$. □

3.3. **Essential norm of the difference and connected components.** The aim of this subsection is to give a necessary condition for two composition operators to be in the same connected component. Inspired by [22], we will deduce it from a minoration of the essential norm of the difference of two composition operators. Usually, this can been done using reproducing kernels. As pointed out in Section 2.3, reproducing kernels are less efficient for composition operators on $\mathcal{H}$ and they have been replaced by partial reproducing kernels to study the compactness of composition operators with polynomial symbols. However, Proposition 5 of [6] is not sufficient in our context, even if we intend to prove that two composition operators with polynomial symbols are not in the same component: an arc between these two operators could involve non polynomial symbols.

Therefore, we need supplementary material about partial reproducting kernels. We fix $\mathcal{Q}$ a finite subset of $\mathcal{P}$, the set of prime numbers. Let $\mathbb{N}_\mathcal{Q}$ be the set of integers which can be factorized using prime numbers in $\mathcal{Q}$. For $\varphi(s) = c_0 s + c_1 + \sum_{n\geq 1} c_n n^{-s}$, we let

$$\varphi_\mathcal{Q}(s) = c_0 s + c_1 + \sum_{n\in\mathbb{N}_\mathcal{Q}} c_n n^{-s} =: c_0 s + \psi_\mathcal{Q}(s).$$

We first observe that $\psi_\mathcal{Q}$, which is well-defined on $\mathbb{C}_{1/2}$, where $\psi_\mathcal{Q}$ converges absolutely, extends to $\mathbb{C}_+$, and that $\psi_\mathcal{Q}(\mathbb{C}_+) \subseteq \overline{\mathbb{C}_+}$. Indeed, let us write any $z \in \mathbb{T}^\infty$ as $z = (z_\mathcal{Q}, w)$ where $z_\mathcal{Q} = (z_{i_1}, \dots, z_{i_d})$, $\mathcal{Q} = \{p_{i_1}, \dots, p_{i_d}\}$. We may define F a.e. on $\mathbb{T}^d$ by

$$F(z_\mathcal{Q}) = \int_{\mathbb{T}^\infty} \frac{\mathcal{B}\psi(z_\mathcal{Q}, w) - 1}{\mathcal{B}\psi(z_\mathcal{Q}, w) + 1} dm(w)$$

which clearly belongs to $H^\infty(\mathbb{T}^d)$ with $\|F\|_\infty \leq 1$. By looking at the Taylor coefficients, $F = (\mathcal{B}\psi_\mathcal{Q} - 1)/(\mathcal{B}\psi_\mathcal{Q} + 1)$ in a neighbourhood of zero. Therefore $\mathcal{B}\psi_\mathcal{Q}$ extends holomorphically to $\mathbb{D}^d$ with $\mathrm{Re}\,(\mathcal{B}\psi_\mathcal{Q}) \geq 0$. This yields that $\psi_\mathcal{Q}$ extends to $\mathbb{C}_+$ with $\psi_\mathcal{Q}(\mathbb{C}_+) \subseteq \overline{\mathbb{C}_+}$.

By the open mapping theorem, $\psi_\mathcal{Q}(\mathbb{C}_+) \subseteq \mathbb{C}_+$ excepts if $\psi_\mathcal{Q}$ is constant. Obviously when $c_0 = 0$, if $\psi_\mathcal{Q}$ is constant, then $\psi_\mathcal{Q} = c_1$ and $\mathrm{Re}\,(c_1) > 1/2$. Hence, in all cases, for all $\mathcal{Q} \subseteq \mathcal{P}$, $\varphi_\mathcal{Q} \in \mathcal{G}$.

Let us also introduce a variant of partial reproducing kernels. We set, for $w \in \mathbb{C}_+$,

$$K_{\mathcal{Q},w}(s) = \prod_{p\in\mathcal{Q}} \frac{1}{1 - p^{-(\bar{w}+s)}}$$

which is a well-defined function on $\mathbb{C}_+$, belonging to $\mathcal{H}$, with

$$\|K_{\mathcal{Q},w}\|^2 = \prod_{p\in\mathcal{Q}} \frac{1}{1-p^{-2\mathrm{Re}\, w}}.$$

Moreover, it is easy to check for any sequence $(s_k) \subseteq \mathbb{C}_+$ with $\mathrm{Re}\,(s_k) \to 0$, $K_{\mathcal{Q},s_k}/\|K_{\mathcal{Q},s_k}\|$ converges weakly to zero.

The following lemma links C_φ, $\varphi_\mathcal{Q}$ and $K_{\mathcal{Q},w}$. Its proof follows exactly that of Proposition 5 in [6].

Lemma 3.6. *Let $\varphi \in \mathcal{G}$ and let $\mathcal{Q} \subseteq \mathcal{P}$ finite. Then, for all $w \in \mathbb{C}_+$,*

$$C_\varphi^*(K_{\mathcal{Q},w}) = \begin{cases} K_{\mathcal{Q},\varphi_\mathcal{Q}(w)} & \text{provided } c_0 \geq 1 \\ K_{\varphi_\mathcal{Q}(w)} & \text{provided } c_0 = 0. \end{cases}$$

Let us finally recall the Julia-Caratheodory theorem on the right half-plane $\mathbb{C}_+$.

Definition 3.7. Let $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ be analytic and let $\alpha \in \mathbb{R}$. We say that φ admits a **finite angular derivative** at $i\alpha$ provided

$$\liminf_{s\to i\alpha} \frac{\mathrm{Re}\,\varphi(s)}{\mathrm{Re}\, s} < +\infty.$$

Lemma 3.8. *Let $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ be analytic and let $\alpha \in \mathbb{R}$. Assume that φ admits a finite angular derivative at $i\alpha$ and let*

$$d := \liminf_{s\to i\alpha} \frac{\mathrm{Re}\,\varphi(s)}{\mathrm{Re}\, s} < +\infty.$$

Then both φ and φ' have respective nontangential limits $\varphi(i\alpha)$ and $\varphi'(i\alpha)$, with $\varphi(i\alpha) \in i\mathbb{R}$ and $\varphi'(i\alpha) = d$. Moreover, $d \in (0, +\infty)$, d is the nontangential limit of $\mathrm{Re}\,\varphi(s)/\mathrm{Re}\, s$ and $(\varphi(s) - \varphi(i\alpha))/(s - i\alpha)$ converges nontangentially to $\varphi'(i\alpha)$.

We are now ready to give a lower bound for the essential norm of the difference of two composition operators (see [22] for the corresponding result on the disk).

Theorem 3.9. *Let $\varphi_0, \varphi_1 \in \mathcal{G}$ with positive characteristic. Let $\mathcal{Q} \subseteq \mathcal{P}$ finite, $d = \mathrm{card}(\mathcal{Q})$. Assume that there exists $\alpha \in i\mathbb{R}$ such that $\varphi_{0,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$. Then either $\varphi_{1,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$ with $\varphi_{1,\mathcal{Q}}(i\alpha) = \varphi_{0,\mathcal{Q}}(i\alpha)$ and $\varphi'_{1,\mathcal{Q}}(i\alpha) = \varphi'_{0,\mathcal{Q}}(i\alpha)$ or*

$$\|C_{\varphi_0} - C_{\varphi_1}\|_e \geq \frac{1}{\varphi'_{0,\mathcal{Q}}(i\alpha)^{d/2}}.$$

Proof. Without loss of generality, we may and shall assume that $i\alpha = 0$ and $\varphi_{0,\mathcal{Q}}(i\alpha) = 0$. We assume that either $\varphi_{1,\mathcal{Q}}$ does not admit a finite angular derivative at 0 or that either $\varphi_{1,\mathcal{Q}}(0) \neq 0$ or $\varphi'_{1,\mathcal{Q}}(0) \neq \varphi'_{0,\mathcal{Q}}(0)$. For a sequence $(s_k) \subseteq \mathbb{C}_+$ with $\mathrm{Re}\,(s_k)$ tending to zero, we set $k_{\mathcal{Q},s_k} = K_{\mathcal{Q},s_k}/\|K_{\mathcal{Q},s_k}\|$ and observe that $k_{\mathcal{Q},s_k}$ converges weakly to zero. Therefore, by Lemma 3.6,

$$\begin{aligned}
\|C_{\varphi_0} - C_{\varphi_1}\|_e^2 &= \|C_{\varphi_0}^* - C_{\varphi_1}^*\|_e^2 \\
&\geq \limsup_{k\to+\infty} \|C_{\varphi_0}^*(k_{\mathcal{Q},s_k}) - C_{\varphi_1}^*(k_{\mathcal{Q},s_k})\|^2 \\
&\geq \limsup_{k\to+\infty} \frac{\|K_{\mathcal{Q},\varphi_{0,\mathcal{Q}}(s_k)}\|^2 + \|K_{\mathcal{Q},\varphi_{1,\mathcal{Q}}(s_k)}\|^2 - 2\mathrm{Re}\, K_{\mathcal{Q},\varphi_{0,\mathcal{Q}}(s_k)}(\varphi_{1,\mathcal{Q}}(s_k))}{\|K_{\mathcal{Q},s_k}\|^2}.
\end{aligned}$$

If (s_k) converges nontangentially to zero, using the computation of the norm of the partial reproducing kernel and Julia-Caratheodory theorem, we get

$$\frac{\|K_{\mathcal{Q},\varphi_{0,\mathcal{Q}}(s_k)}\|^2}{\|K_{\mathcal{Q},s_k}\|^2} \sim \left(\frac{\operatorname{Re} s_k}{\operatorname{Re}\varphi_{0,\mathcal{Q}}(s_k)}\right)^d \sim \frac{1}{\varphi'_{0,\mathcal{Q}}(0)^d}.$$

Hence, to conclude, we just need to find a sequence $(s_k) \subseteq \mathbb{C}_+$ tending nontangentially to zero and such that

$$\lim_{k\to+\infty} \frac{K_{\mathcal{Q},\varphi_{0,\mathcal{Q}}(s_k)}(\varphi_{1,\mathcal{Q}}(s_k))}{\|K_{\mathcal{Q},s_k}\|^2} = \lim_{k\to+\infty} \prod_{p\in\mathcal{Q}} \frac{1-p^{-2\operatorname{Re}(s_k)}}{1-p^{-(\overline{\varphi_{0,\mathcal{Q}}(s_k)}+\varphi_{1,\mathcal{Q}}(s_k))}} \tag{3.4}$$

is as small as possible. First of all, if $\lim_{\sigma\to 0^+}\varphi_{1,\mathcal{Q}}(\sigma) \neq 0$, we consider a sequence (σ_n) of positive real numbers tending to zero and such that $|\varphi_{1,\mathcal{Q}}(\sigma_n)| \geq \delta$ for some $\delta > 0$. Then, setting $s_k = \sigma_k$, it is clear that the limit appearing in (3.4) is equal to zero. Thus we may assume that $\varphi_{1,\mathcal{Q}}$ has radial limit 0 at 0. Let $M > 0$ and set $s_k = \frac{1}{k} + i\frac{M}{k}$. We observe that

$$\begin{aligned}\prod_{p\in\mathcal{Q}} \frac{1-p^{-2\operatorname{Re}(s_k)}}{1-p^{-(\overline{\varphi_{0,\mathcal{Q}}(s_k)}+\varphi_{1,\mathcal{Q}}(s_k))}} &\sim_{+\infty} \left(\frac{\overline{\varphi_{0,\mathcal{Q}}(s_k)}+\varphi_{1,\mathcal{Q}}(s_k)}{2\operatorname{Re} s_k}\right)^{-d} \\ &\sim_{+\infty} \left(\frac{s_k}{2\operatorname{Re}(s_k)}\right)^{-d} \times \left(\frac{\overline{\varphi_{0,\mathcal{Q}}(s_k)}}{\overline{s_k}} \times \frac{\overline{s_k}}{s_k} + \frac{\varphi_{1,\mathcal{Q}}(s_k)}{s_k}\right)^{-d}.\end{aligned}$$

Provided M is large enough, $|s_k|/\operatorname{Re}(s_k)$ could become as large as we want when k tends to $+\infty$. On the other hand,

$$\frac{\overline{\varphi_{0,\mathcal{Q}}(s_k)}}{\overline{s_k}} \times \frac{\overline{s_k}}{s_k} \xrightarrow{k\to+\infty} \varphi'_{0,\mathcal{Q}}(0) \times \frac{1-iM}{1+iM}.$$

If $\varphi_{1,\mathcal{Q}}$ does not have finite angular derivative at 0, then $\varphi_{1,\mathcal{Q}}(s_k)/s_k$ tends to $+\infty$. If $\varphi_{1,\mathcal{Q}}$ has a finite angular derivative at 0 with $\varphi'_{1,\mathcal{Q}}(0) \neq \varphi'_{0,\mathcal{Q}}(0)$, then

$$\begin{aligned}\lim_{k\to+\infty}\left|\frac{\overline{\varphi_{0,\mathcal{Q}}(s_k)}}{\overline{s_k}} \times \frac{\overline{s_k}}{s_k} + \frac{\varphi_{1,\mathcal{Q}}(s_k)}{s_k}\right| &= \left|\varphi'_{0,\mathcal{Q}}(0) \times \frac{1-iM}{1+iM} + \varphi'_{1,\mathcal{Q}}(0)\right| \\ &\geq \frac{1}{2}\left|\varphi'_{0,\mathcal{Q}}(0) - \varphi'_{1,\mathcal{Q}}(0)\right|\end{aligned}$$

provided M is sufficiently large. Therefore, in both cases, for any $\varepsilon > 0$, we have shown that we may choose $M > 0$ such that

$$\lim_{k\to+\infty}\left|\prod_{p\in\mathcal{Q}} \frac{1-p^{-2\operatorname{Re}(s_k)}}{1-p^{-(\overline{\varphi_{0,\mathcal{Q}}(s_k)}+\varphi_{1,\mathcal{Q}}(s_k))}}\right| \leq \varepsilon.$$

This yields the result. □

Remark 3.10. Theorem 3.9 is only interesting when φ_0 and φ_1 have unrestricted range. However, even if φ_0 is continuous, it could be possible that there does not exist $\alpha \in \mathbb{R}$ with $\operatorname{Re}\varphi(i\alpha) = 0$. Nevertheless, even in that case, one can apply Theorem 3.9 to some φ_χ as in the proof of the fothcoming Theorem 4.5.

Corollary 3.11. *Let $\varphi_0,\ \varphi_1 \in \mathcal{G}$ with* $\mathrm{char}\,(\varphi_0) = \mathrm{char}\,(\varphi_1) > 0$. *Assume that C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$. Then for all $\mathcal{Q} \subseteq \mathcal{P}$ finite and all $\alpha \in \mathbb{R}$ such that $\varphi_{0,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$, $\varphi_{1,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$ with $\varphi_{0,\mathcal{Q}}(i\alpha) = \varphi_{1,\mathcal{Q}}(i\alpha)$ and $\varphi'_{0,\mathcal{Q}}(i\alpha) = \varphi'_{1,\mathcal{Q}}(i\alpha)$.*

Proof. Let $\mathcal{Q} \subseteq \mathcal{P}$ finite and let $\alpha \in \mathbb{R}$ be such that $\varphi_{0,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$. Assume that C_{φ_1} belongs to the connected component $\mathcal{C}_0$ of C_{φ_0} in $\mathcal{C}(\mathcal{H})$. For $C_\tau \in \mathcal{C}_0$, let U_τ be the ball (in $\mathcal{C}(\mathcal{H})$) of center C_τ and radius $1/2\varphi'_{0,\mathcal{Q}}(i\alpha)^{d/2}$, with $d = \mathrm{card}(\mathcal{Q})$. There exists a simple chain $U_{\tau_1}, \dots, U_{\tau_n}$ extracted from $(U_\tau)_{C_\tau \in \mathcal{C}_0}$ with $C_{\varphi_0} \in U_{\tau_1}$, $C_{\varphi_1} \in U_{\tau_n}$ and $U_{\tau_i} \cap U_{\tau_j} \neq \varnothing$ for $i \neq j$ if and only if $|i-j| \leq 1$. Let $C_{\tau_i} \in U_{\tau_i} \cap U_{\tau_{i+1}}$. Then setting $\tilde{\varphi}_0 = \varphi_0$, $\tilde{\varphi}_i = \tau_i$ for $i = 1, \dots, n-1$, and $\tilde{\varphi}_n = \varphi_1$, we have

$$\|C_{\tilde{\varphi}_i} - C_{\widetilde{\varphi_{i-1}}}\| < \frac{1}{\varphi'_{0,\mathcal{Q}}(i\alpha)^{d/2}}$$

for all $i = 1, \dots, n$. By an immediate induction and Theorem 3.9, we get that for all $i = 0, \dots, n$, $\tilde{\varphi}_i$ admits a finite angular derivative at $i\alpha$ with $\tilde{\varphi}_i(i\alpha) = \varphi_{0,\mathcal{Q}}(i\alpha)$ and $\tilde{\varphi}_i{}'(i\alpha) = \varphi'_{0,\mathcal{Q}}(i\alpha)$. This is exactly the desired result for $i = n$. □

Example 3.12. Let $\varphi_0(s) = s + (1 - 2^{-s})$ and $\varphi_1(s) = s + (1 - 3^{-s})$. Then C_{φ_0} and C_{φ_1} are not in the same component of $\mathcal{C}(\mathcal{H})$.

We just need to apply Corollary 3.11 with $\mathcal{Q} = \{2\}$.

Example 3.13. Let $\varphi_0(s) = s$. Then C_{φ_0} is isolated in $\mathcal{C}(\mathcal{H})$.

Indeed, if C_{φ_1} belongs to the same component of C_{φ_0}, then for all $\mathcal{Q} \subseteq \mathcal{P}$, for all $\alpha \in \mathbb{R}$, $\varphi_{1,\mathcal{Q}}$ admits an angular derivative at $i\alpha$ with $\varphi_{1,\mathcal{Q}}(i\alpha) = \varphi_{0,\mathcal{Q}}(i\alpha) = \varphi_0(i\alpha) = i\alpha$. Hence, $\varphi_{1,\mathcal{Q}}(s) = s$ which in turn implies $\varphi_1(s) = s$.

The use of the angular derivative is not as efficient when $c_0 = 0$. The reason is simple: it is possible that $\varphi_{\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$ and that C_φ is compact. This is for instance the case of

$$\varphi(s) = \frac{1}{2} + (1 - 2^{-s}) + (1 - 3^{-s}).$$

Nevertheless, we still get a result when $\mathrm{card}(\mathcal{Q}) = 1$. Observe that at a boundary point $i\alpha$, we now have $\varphi_{\mathcal{Q}}(i\alpha) \in \frac{1}{2} + i\mathbb{R}$ and that we are now interested in $(\mathrm{Re}\,\varphi_{\mathcal{Q}}(s) - 1/2)/\mathrm{Re}\,s$.

Theorem 3.14. *Let $\varphi_0, \varphi_1 \in \mathcal{G}$ with zero characteristic. Let $\mathcal{Q} \subseteq \mathcal{P}$ with* $\mathrm{card}(\mathcal{Q}) = 1$. *Assume that there exists $\alpha \in i\mathbb{R}$ such that $\varphi_{0,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$. Then either $\varphi_{1,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$ with $\varphi_{1,\mathcal{Q}}(i\alpha) = \varphi_{0,\mathcal{Q}}(i\alpha)$ and $\varphi'_{1,\mathcal{Q}}(i\alpha) = \varphi'_{0,\mathcal{Q}}(i\alpha)$ or*

$$\|C_{\varphi_0} - C_{\varphi_1}\|_e \geq \frac{C_{\mathcal{Q}}}{\varphi'_{0,\mathcal{Q}}(i\alpha)^{1/2}}.$$

Proof. We only point out the differences with the proof of Theorem 3.9. We have to replace everywhere $K_{\mathcal{Q},\varphi_{0,\mathcal{Q}}(s_k)}$ by $K_{\varphi_{0,\mathcal{Q}}(s_k)}$. One now has

$$\|K_w\|^2 = \zeta(2\mathrm{Re}\,(w))$$

and

$$K_{\varphi_{0,\mathcal{Q}}(s_k)}(\varphi_{1,\mathcal{Q}}(s_k)) = \zeta\left(\overline{\varphi_{0,\mathcal{Q}}(s_k)} + \varphi_{1,\mathcal{Q}}(s_k)\right).$$

The estimate of this last term follows that done in Theorem 3.9, except that we have to use $\zeta(s) \sim 1/(s-1)$ if $s \to 1$ instead of $1 - p^{-s} \sim -(\log p)s$ if $s \to 0$. □

Corollary 3.15. *Let $\varphi_0,\ \varphi_1 \in \mathcal{G}$ with* $\mathrm{char}\,(\varphi_0) = \mathrm{char}\,(\varphi_1) = 0$. *Assume that C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$. Then for all $\mathcal{Q} \subseteq \mathcal{P}$ with* $\mathrm{card}(\mathcal{Q}) = 1$ *and all $\alpha \in \mathbb{R}$ such that $\varphi_{0,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$, $\varphi_{1,\mathcal{Q}}$ admits a finite angular derivative at $i\alpha$ with $\varphi_{0,\mathcal{Q}}(i\alpha) = \varphi_{1,\mathcal{Q}}(i\alpha)$ and $\varphi'_{0,\mathcal{Q}}(i\alpha) = \varphi'_{1,\mathcal{Q}}(i\alpha)$.*

Example 3.16. Let $\varphi_0(s) = \frac{1}{2} + (1 - 2^{-s})$ and $\varphi_1(s) = \frac{1}{2} + (1 - 3^{-s})$. Then C_{φ_0} and C_{φ_1} do not belong to the same component of $\mathcal{C}(\mathcal{H})$.

Corollary 3.11 already gives interesting examples of composition operators that do not belong to the same component of $\mathcal{C}(\mathcal{H})$. However, it does not help us to solve the following simple problem: does C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$ where

$$\varphi_0(s) = s + (2\log 2 + \log 3) - \log 3 \cdot 2^{-s} - 2\log 2 \cdot 3^{-s}$$
$$\varphi_1(s) = s + (\log 2 + 2\log 3) - 2\log 3 \cdot 2^{-s} - \log 2 \cdot 3^{-s}?$$

A plausible conjecture would be that Corollary 3.11 could be extended to the angular partial derivatives of the Bohr lifts $\mathcal{B}\psi_{0,\mathcal{Q}}$ and $\mathcal{B}\psi_{1,\mathcal{Q}}$. Nevertheless, the curve $\sigma \mapsto (2^{-\sigma}, \dots, p_d^{-\sigma})$ is not a special curve in $\mathbb{D}^d$, which prevents the use of the Julia-Caratheodory theorem of the polydisc (see [1]).

However, inspecting the partial derivatives of the Bohr lifts will be fruitful for studying compact linear combinations of composition operators with regular symbols, as indicated by the results of the next section.

4. Compact linear combinations of polynomial symbols

4.1. **Lower bound for the esssential norm from second order data.** We start with a result saying that if a linear combination $\lambda_1 C_{\varphi_1} + \cdots + \lambda_N C_{\varphi_N}$ is compact, then the coefficients λ_j should satisfy some properties depending on the behaviour of the $\mathcal{B}\psi_j$ at their boundary points. We need first to introduce some terminology inspired by [21].

Definition 4.1. Let $F, G : \overline{\mathbb{D}^d} \to \overline{\mathbb{C}_+}$ be of class $\mathcal{C}^2$ and let $\xi \in \mathbb{T}^d$. We say that F and G have the **same second order boundary data** at ξ if $F(\xi) = G(\xi)$ and $\partial_\alpha F(\xi) = \partial_\alpha G(\xi)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2$. We also denote by $\Gamma(F) = \{z \in \mathbb{T}^d : \mathrm{Re}\, F(z) = 0\}$.

We will need a couple of lemmas.

Lemma 4.2. *Let $d \geq 1$ and let Q be a non-zero quadratic polynomial in d variables (with complex coefficients). Then $Z(Q) = \{x \in \mathbb{R}^d : \ Q(x) = 0\}$ has Lebesgue measure zero.*

Proof. The proof is done by induction on d. For $d = 1$, the result is of course trivial. Assume that the result has been done until d and let us consider Q a quadratic polynomial in $d+1$ variables. Fix $x_{d+1} \in \mathbb{R}$ and let $Q_{x_{d+1}}(x_1, \dots, x_d) = Q(x_1, \dots, x_{d+1})$. Then either $Q_{x_{d+1}}$ is identically zero or its zero set has Lebesgue measure zero. Writing

$$Q(x) = a + \sum_{l=1}^{d+1} b_l x_l + \sum_{1 \le l \le m \le d+1} c_{l,m} x_l x_m,$$

the former case implies that

$$\begin{cases} a + b_{d+1} x_{d+1} + c_{d+1,d+1} x_{d+1}^2 & = & 0 \\ \forall l = 1, \dots, d, \ b_l + c_{l,d+1} x_{d+1} & = & 0. \end{cases}$$

Since Q itself is assumed to be non-zero, we can suppose, permuting the variables if necessary, that either $a \neq 0$ or $b_{d+1} \neq 0$ or $c_{l,d+1} \neq 0$ for some $l = 1, \dots, d+1$. Therefore there are at most two values of x_{d+1} such that $Q_{x_{d+1}}$ is identically zero. Hence, for almost all $x_{d+1} \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \mathbf{1}_{Z(Q)}(x_1, \dots, x_{d+1}) dx_1 \dots dx_d = 0.$$

The lemma follows from an application of Fubini's theorem. □

Our second lemma is a result about functions which vanish along $(0, +\infty)$.

Lemma 4.3. *Let $\mathcal{C} \subseteq \mathbb{N}$ be finite, let $\alpha > 0$, let $(\lambda_j)_{j \in \mathcal{C}}$ be complex numbers, let $(\mu_j)_{j \in \mathcal{C}}$ be pairwise distinct complex numbers and let $d \geq 1$. Assume that, for all $M > 0$,*

$$\sum_{j,j' \in \mathcal{C}} \frac{\lambda_j \overline{\lambda_{j'}}}{(M\alpha + \mu_j + \overline{\mu_{j'}})^d} = 0.$$

Then $\lambda_j = 0$ for all $j \in \mathcal{C}$.

Proof. Let us consider

$$F(z) = \sum_{j,j' \in \mathcal{C}} \frac{\lambda_j \overline{\lambda_{j'}}}{(\alpha + z(\mu_j + \overline{\mu_{j'}}))^d}.$$

Then F is an analytic function defined in a neighbourhood of the origin and the assumption tells us that the origin is an accumulation point of the zeros of F. The result now follows from [14, Corollary 3.4]. □

Theorem 4.4. *Let $\varphi_j(s) = c_{0,j} s + \psi_j$ be a finite sequence of symbols in $\mathcal{G}$ with ψ_j involving only the first d prime numbers, $j = 1, \dots, N$. Assume that each $\mathcal{B}\psi_j$ extends to a $\mathcal{C}^2$-function on $\overline{\mathbb{D}^d}$ and that* $\operatorname{char}(\varphi_1) > 0$. *Let $(\lambda_j)_{j=1,\dots,N}$ be a finite sequence of complex numbers such that $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact. Let $z \in \Gamma(\mathcal{B}\psi_1)$ and let $\mathcal{J}$ be the set of $j \in \{1, \dots, N\}$ such that $\mathcal{B}\psi_j$ and $\mathcal{B}\psi_1$ have the same second order boundary data at z. Then*

$$\sum_{j \in \mathcal{J}} \lambda_j = 0.$$

Proof. Without loss of generality, we may and shall assume that $z = \mathbf{e} := (1, \dots, 1)$ and that $\mathcal{B}\psi_1(\mathbf{e}) = 0$. Let us write for $j = 1, \dots, N$,

$$\mathcal{B}\psi_j(z) = A^{(j)} + \sum_{l=1}^{d} B_l^{(j)}(z_l - 1) + \sum_{1 \le l \le m \le d} C_{l,m}^{(j)}(z_l - 1)(z_m - 1) + o(\|z - \mathbf{e}\|^2).$$

It is a standard fact (see [7, Lemma 7] that each $B_l^{(j)} \in (-\infty, 0]$. We set $c_0 = c_{0,1}$ and introduce the following subsets of $\{1, \dots, N\}$:

$$\begin{aligned} I &= \{j : \ c_{0,j} = c_0 \text{ and } \varphi_j(0) = 0\} = \left\{j : \ c_{0,j} = c_0 \text{ and } A^{(j)} = 0\right\} \\ J_1 &= \{j : \ c_{0,j} = c_0 \text{ and } \varphi_j(0) \neq 0\} \\ J_2 &= \{j : \ c_{0,j} \neq c_0 \text{ and } c_{0,j} > 0\} =: \{j_1, \dots, j_p\} \\ J_3 &= \{j : \ c_{0,j} = 0\}. \end{aligned}$$

For $j \in \{1, \dots, N\}$, let us introduce

$$\begin{aligned} L_j(x_1, \dots, x_d) &= \sum_{l=1}^{d} B_l^{(j)} \log(p_l) x_l \\ Q_j(x_1, \dots, x_d) &= \frac{1}{2} \sum_{l=1}^{d} B_l^{(j)} (\log p_l)^2 x_l^2 + \sum_{1 \le l \le m \le d} C_{l,m}^{(j)} \log(p_l) \log(p_m) x_l x_m. \end{aligned}$$

We define an equivalence relation on I by $j \sim j' \iff L_j = L_{j'}$. In other words, $j \sim j'$ if and only if $\mathcal{B}\psi_j$ and $\mathcal{B}\psi_{j'}$ have the same first order boundary data at $\mathbf{e}$. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_q$ the equivalence classes for this relation with $1 \in \mathcal{C}_1$. We define another equivalence relation on $\mathcal{C}_1$ by $j \mathcal{R} j' \iff Q_j = Q_{j'}$ and we denote by $\mathcal{C}_{1,1}, \dots, \mathcal{C}_{1,q'}$ the equivalence classes for this relation, with $1 \in \mathcal{C}_{1,1}$. In other words, $j \mathcal{R} j'$ if and only if $\mathcal{B}\psi_j$ and $\mathcal{B}\psi_{j'}$ have the same second order boundary data. In particular, $\mathcal{J}$ is nothing else than $\mathcal{C}_{1,1}$. For $i \neq i' \in \{1, \dots, q\}$, let

$$E_{i,i'} = \left\{\beta \in \mathbb{R}^d : \ \forall j \in \mathcal{C}_i, \ \forall j' \in \mathcal{C}_{i'}, \ L_j(\beta) \neq L_{j'}(\beta)\right\}$$

(we can replace $\forall$ by $\exists$ in the previous definition). Similarly for $i \neq i' \in \{1, \dots, q'\}$, let

$$F_{i,i'} = \left\{\beta \in \mathbb{R}^d : \ \forall j \in \mathcal{C}_{1,i}, \ \forall j' \in \mathcal{C}_{1,i'}, \ Q_j(\beta) \neq Q_{j'}(\beta)\right\}.$$

By Lemma 4.2, there exists $\beta \in \bigcap_{i \neq i'} E_{i,i'} \cap \bigcap_{i \neq i'} F_{i,i'}$. Let finally, for $1 \le l \le d$, $k \ge 1$ and $M > 0$,

$$z_l(k) = p_l^{-\frac{M}{k} + i\frac{\beta_l}{\sqrt{k}}}$$

so that

$$z_l(k) = 1 + \frac{i \log(p_l)\beta_l}{\sqrt{k}} - \frac{2M \log(p_l) + \log^2(p_l)\beta_l^2}{2k} + o\left(\frac{1}{k}\right).$$

Therefore,

$$\mathcal{B}\psi_j(z(k)) = A^{(j)} + \frac{i}{\sqrt{k}} \sum_{l=1}^{d} B_l^{(j)} \log(p_l)\beta_l - \frac{1}{k}\Bigg(M \sum_{l=1}^{d} B_l^{(j)} \log(p_l)$$

$$+\frac{1}{2}\sum_{l=1}^{d} B_l^{(j)}\log^2(p_l)\beta_l^2+\sum_{1\le l\le m\le d} C_{l,m}^{(j)}\log(p_l)\log(p_m)\beta_l\beta_m\Bigg)+o\left(\frac{1}{k}\right)$$
$$=A^{(j)}+\frac{i}{\sqrt{k}}L_j(\beta_1,\dots,\beta_d)-\frac{1}{k}\left(ML_j(\mathbf{e})+Q_j(\beta_1,\dots,\beta_d)\right)+o\left(\frac{1}{k}\right).$$

We now choose a sequence of real numbers (t_k) such that

(1) for $1\le l\le d$, $p_l^{-it_k}$ is very close to $p_l^{\frac{i\beta_l}{\sqrt{k}}}$, so close that, setting $w_k=\frac{M}{k}+it_k$,
$$\mathcal{B}\psi_j(p_1^{-w_k},\dots,p_d^{-w_k})=\mathcal{B}\psi_j(z(k))+o\left(\frac{1}{k}\right)$$
for all $j=1,\dots,N$.

(2) For all $j\in J_2$, writing $j=j_r$ with $1\le r\le p$, $p_{d+r}^{-it_k(c_{0,j_r}-c_0)}p_{d+r}^{-A^{(j_r)}}$ tends to some $\omega(j_r)\neq 1$.

These two conditions may be simultaneously satisfied because of Kronecker's theorem and since $c_{0,j}\neq c_0$ for all $j\in J_2$. We set $d'=d+p$ and use that

$$\left\|\sum_{j=1}^{N}\lambda_j C_{\varphi_j}\right\|_e\ge\limsup_{k\to+\infty}\left\|\sum_{j=1}^{N}\overline{\lambda_j}C_{\varphi_j}^*(k_{d',w_k})\right\|^2$$
$$\ge\limsup_{k\to+\infty}\sum_{1\le j,j'\le N}\lambda_j\overline{\lambda_{j'}}\langle C_{\varphi_{j'}}^*(k_{d',w_k}),C_{\varphi_j}^*(k_{d',w_k})\rangle.$$

We claim that if $j\in I$ and $j'\notin I$, then
$$\langle C_{\varphi_{j'}}^*(k_{d',w_k}),C_{\varphi_j}^*(k_{d',w_k})\rangle\xrightarrow{k\to+\infty}0.$$
First assume that $j'\in J_3$. Then
$$\langle C_{\varphi_{j'}}^*(k_{d',w_k}),C_{\varphi_j}^*(k_{d',w_k})\rangle=\frac{\langle K_{\varphi_{j'}(w_k)},K_{d',\varphi_j(w_k)}\rangle}{\|K_{d',w_k}\|^2}$$
$$=\frac{\langle K_{d',\varphi_{j'}(w_k)},K_{d',\varphi_j(w_k)}\rangle}{\|K_{d',w_k}\|^2}$$
and this goes to zero since $\|K_{d',\varphi_{j'}(w_k)}\|$ is bounded because $\operatorname{Re}\varphi_{j'}(w_k)\ge 1/2$. If $j\in I$ and $j'\notin J_3$, then
$$\langle C_{\varphi_{j'}}^*(k_{d',w_k}),C_{\varphi_j}^*(k_{d',w_k})\rangle=\prod_{l=1}^{d'}\frac{1-p_l^{-2\operatorname{Re}(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}}.$$
Pick $j'\in J_1$. Then
$$\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k)\to\overline{A^{(j')}}\neq 0$$
so that
$$\prod_{l=1}^{d'}\frac{1-p_l^{-2\operatorname{Re}(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}}\xrightarrow{k\to+\infty}0. \tag{4.1}$$
Pick finally $j'\in J_2$ and write $j'=j_r$ for $1\le r\le p$. Then
$$\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k)=(c_0-c_{0,j_r})it_k+A^{(j_r)}+o(1)$$

so that

$$p_{d+r}^{-(\varphi_{j'}(w_k)+\varphi_j(w_k)} \xrightarrow{k\to+\infty} \omega(j') \neq 1.$$

Therefore,

$$\frac{1-p_{d+r}^{-2\mathrm{Re}\,(w_k)}}{1-p_{d+r}^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \xrightarrow{k\to+\infty} 0.$$

On the other hand, since $\mathrm{Re}\,\varphi_j(w_k), \mathrm{Re}\,\varphi_{j'}(w_k) \geq \mathrm{Re}\,(w_k)$, we also have

$$\prod_{1\leq l\leq d',\ l\neq d+r}^{d'} \left| \frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \right| \leq 1.$$

Hence, (4.1) remains true for these values of j and j'. This implies that

$$0 \geq \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e \geq \limsup_{k\to+\infty} \left(\sum_{j,j'\in I} \lambda_j \overline{\lambda_{j'}} \prod_{l=1}^{d'} \frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \right.$$
$$\left. + \sum_{j,j'\notin I} \lambda_j \overline{\lambda_{j'}} \prod_{l=1}^{d'} \frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \right).$$

Since each term in the previous sum is nonnegative (it is equal to $\left\| \sum_{j\in E} \lambda_j C_{\varphi_j}^*(k_{d,w_k}) \right\|^2$ for either $E = I$ or $E = \{1,\dots,N\}\backslash I$), this yields

$$\limsup_{k\to+\infty} \sum_{j,j'\in I} \lambda_j \overline{\lambda_{j'}} \prod_{l=1}^{d'} \frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} = 0.$$

Consider now $j, j' \in I$, and observe that

$$\begin{aligned}\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k) &= \frac{2c_0M}{k} + \overline{\mathcal{B}\psi_{j'}(z(k))} + \mathcal{B}\psi_j(z(k)) + o\left(\frac{1}{k}\right) \\ &= \frac{i(L_j(\beta)-L_{j'}(\beta))}{\sqrt{k}} + \frac{2c_0M}{k} - \frac{M(L_j(\mathbf{e})+L_{j'}(\mathbf{e}))}{k} \\ &\quad - \frac{\overline{Q_{j'}(\beta)}+Q_j(\beta)}{k} + o\left(\frac{1}{k}\right).\end{aligned}$$

If $j \in \mathcal{C}_i$ and $j' \in \mathcal{C}_{i'}$ with $i \neq i'$, so that $L_j(\beta) \neq L_{j'}(\beta)$, then

$$\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k) \sim_{k\to+\infty} \frac{i(L_j(\beta)-L_{j'}(\beta))}{\sqrt{k}}$$

and this implies easily that, for all $l \geq 1$,

$$\frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \xrightarrow{k\to+\infty} 0.$$

If $j \neq j' \in \mathcal{C}_1$, one now gets that

$$\frac{1-p_l^{-2\mathrm{Re}\,(w_k)}}{1-p_l^{-(\overline{\varphi_{j'}(w_k)}+\varphi_j(w_k))}} \xrightarrow{k\to+\infty} \frac{M}{2M(c_0-L_1(\mathbf{e})) - (\overline{Q_{j'}(\beta)}+Q_j(\beta))}.$$

Arguing as above, we finally find that, for all $M > 0$,

$$\sum_{j,j'\in\mathcal{C}_1} \frac{\lambda_j\overline{\lambda_{j'}}M^{d'}}{(2M(c_0 - L_1(\mathbf{e})) - (\overline{Q_{j'}(\beta)} + Q_j(\beta))^{d'}} = 0. \tag{4.2}$$

Let for $k = 1, \dots, q'$, $\nu_k = \sum_{j\in\mathcal{C}_{1,k}} \lambda_j$ and let $\mu_k = Q_j(\beta)$ for some (all) $j \in \mathcal{C}_{1,k}$, $1 \le k \le q'$. Then (4.2) may be rewritten

$$\sum_{k,k'=1}^{q'} \frac{\nu_k\overline{\nu_{k'}}M^{d'}}{\left(2M(c_0 - L_1(\mathbf{e})) - (\overline{\mu_{k'}} + \mu_k)\right)^{d'}} = 0.$$

Lemma 4.3 tells us that $\nu_k = 0$ for all $k = 1, \dots, q'$. The case $k = 1$ is exactly the required statement. □

4.2. **Linear combinations of polynomial symbols with degree at most 2.** We are now ready for the proof of our extension of the main result of [8].

Theorem 4.5. *Let $(\varphi_j)_{j=1,\dots,N} \subseteq \mathcal{G}$ be a finite sequence of distinct symbols. Assume that each φ_j is a polynomial symbol with degree at most 2. Let also $(\lambda_j)_{j=1,\dots,N}$ be a finite sequence of non-zero complex numbers. Then $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on $\mathcal{H}$ if and only if each C_{φ_j} is compact.*

Proof. We proceed by induction on N, the case $N = 1$ being trivial. Assume that the result has been obtained for $N-1$ symbols and let us prove it for N symbols. By contradiction, assume that no C_{φ_j} is compact. If one symbol, say φ_1, has a positive characteristic, then we can conclude immediately using Theorem 4.4. Indeed, since φ_1 has unrestricted range (it is assumed to be noncompact), $\Gamma(\mathcal{B}\psi_1) \neq \varnothing$ (the polycircle $\mathbb{T}^d$ is compact). Now, for $j \neq 1$, $\mathcal{B}\psi_1$ and $\mathcal{B}\psi_j$ cannot have the same second order boundary data at some point, since φ_1 and φ_j are distinct polynomials of degree at most 2. Thus, we get $\lambda_1 = 0$, a contradiction.

Therefore, we are reduced to the case where all symbols φ_j have zero characteristic. Let us fix $j \in \{1, \dots, N\}$. In the terminology of [7], the complex dimension of φ_j cannot be greater than or equal to 2: since the degree of φ_j is at most 2, Theorem 3 of [7] would imply that C_{φ_j} is compact. Thus we have two possibilities for φ_j: either

$$\varphi_j(s) = c_{1,j} + d_j q_{1,j}^{-s} q_{2,j}^{-s} \text{ with } q_{1,j} \neq q_{2,j} \in \mathcal{P},\ d_j \neq 0 \tag{4.3}$$

which corresponds to the case where the complex dimension and the degree of φ_2 are equal to 2, or

$$\varphi_j(s) = c_{1,j} + e_j q_j^{-s} + f_j q_j^{-2s} \text{ with } q_j \in \mathcal{P} \tag{4.4}$$

which means that φ_j has complex dimension 1 and degree 1 or 2 (provided $f_j = 0$ or not). If all symbols may be written as in (4.3), then we face to linear symbols and this case will be handled in the forthcoming Theorem 4.8. Therefore we assume that at least one symbol, say φ_1, has a form like in (4.4). Let $\chi \in \mathbb{T}^\infty$. Since

$(f\circ\varphi_j)_\chi = f\circ(\varphi_j)_\chi$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\sum_j \lambda_j C_{\varphi_j}} & \mathcal{H} \\ U\big\downarrow & & \big\downarrow V \\ \mathcal{H} & \xrightarrow{\sum_j \lambda_j C_{(\varphi_j)_\chi}} & \mathcal{H} \end{array}$$

where U and V are the unitary maps of $\mathcal{H}$ defined by $U(f)=f$ and $V(f)=f_\chi$. Therefore, the compactness of $\sum\lambda_j C_{\varphi_j}$ is equivalent to that of $\sum_j \lambda_j C_{(\varphi_j)_\chi}$ which implies that we may assume that $\mathrm{Re}\,(\varphi_1(0))=1/2$. By vertical translations, we may even assume that $\varphi_1(0)=1/2$. Let

$$I=\Big\{j:\ \varphi_j \text{ may be written as in (4.4) with } q_j=q_1,\ \varphi_j(0)=\frac{1}{2},\ \varphi_j'(0)=\varphi_1'(0)\Big\}$$
$$J_1=\Big\{j:\ \varphi_j \text{ may be written as in (4.4) with } q_j=q_1,\ \varphi_j(0)=\frac{1}{2},\ \varphi_j'(0)\neq\varphi_1'(0)\Big\}$$
$$J_2=\Big\{j:\ \varphi_j \text{ may be written as in (4.4) with } q_j=q_1,\ \varphi_j(0)\neq\frac{1}{2}\Big\}$$
$$J_3=\Big\{j:\ \varphi_j \text{ may be written as in (4.4) with } q_j\neq q_1\Big\}$$
$$J_4=\Big\{j:\ \varphi_j \text{ may be written as in (4.3)}\Big\}.$$

Let finally $s_k=\frac{1}{k}+i\frac{M}{\sqrt{k}}$ for some $M>0$ and consider the normalized partial reproducing kernels $k_{s_k}=K_{s_k}^{(q_1)}/\|K_{s_k}^{(q_1)}\|$; the sequence (k_{s_k}) goes weakly to zero. By Lemma 3.6, if $j\in I\cup J_1\cup J_2$,

$$C_{\varphi_j}^*(k_{s_k})=\frac{K_{\varphi_j(s_k)}}{\|K_{s_k}^{(q_1)}\|}$$

whereas, if $j\in J_3\cup J_4$,

$$C_{\varphi_j}^*(k_{s_k})=\frac{K_{c_{1,j}}}{\|K_{s_k}^{(q_1)}\|}\xrightarrow{k\to+\infty}0.$$

Using this last result, one gets

$$\begin{aligned}\Big\|\sum_{j=1}^N\lambda_j C_{\varphi_j}\Big\|_e^2 &\geq \limsup_{k\to+\infty}\Big\|\sum_{j=1}^N\overline{\lambda_j}C_{\varphi_j}^*(k_{s_k})\Big\|^2\\ &\geq \limsup_{k\to+\infty}\sum_{j,j'\in I\cup J_1\cup J_2}\frac{\overline{\lambda_j}\lambda_{j'}\langle K_{\varphi_j(s_k)},K_{\varphi_{j'}(s_k)}\rangle}{\|K_{s_k}^{(q_1)}\|^2}\\ &\geq \limsup_{k\to+\infty}\sum_{j,j'\in I\cup J_1\cup J_2}\frac{\overline{\lambda_j}\lambda_{j'}\zeta(\overline{\varphi_{j'}(s_k)}+\varphi_j(s_k))}{\|K_{s_k}^{(q_1)}\|^2}.\end{aligned}$$

We claim that, for all $j\in I$ and all $j'\in J_1\cup J_2$,

$$\frac{\zeta(\overline{\varphi_{j'}(s_k)}+\varphi_j(s_k))}{\|K_{s_k}^{(q_1)}\|^2}\xrightarrow{k\to+\infty}0. \tag{4.5}$$

Let us observe that, for all $j \in \{1, \dots, N\}$,

$$\varphi_j(s_k) = \varphi_j(0) + \frac{iM\varphi_j'(0)}{\sqrt{k}} + \frac{2\varphi_j'(0) - M^2\varphi_j''(0)}{2k} + o\left(\frac{1}{k}\right)$$

so that if $j \in I$ and $j' \in I \cup J_1 \cup J_2$,

$$\overline{\varphi_{j'}(s_k)} + \varphi_j(s_k) = \frac{1}{2} + \overline{\varphi_{j'}(0)} + i\frac{\varphi_j'(0) - \varphi_{j'}'(0)}{\sqrt{k}}M + \frac{2\varphi_j'(0) + 2\varphi_{j'}'(0) - M^2(\overline{\varphi_{j'}''(0)} + \varphi_j''(0))}{2k} + o\left(\frac{1}{k}\right).$$

Hence, (4.5) follows now easily from the properties of ζ and $\|K_{s_k}^{(q_1)}\|^2 \sim Ak$. On the other hand, when $j, j' \in I$, one gets

$$\frac{\zeta(\overline{\varphi_{j'}(s_k)} + \varphi_j(s_k))}{\|K_{s_k}^{(q_1)}\|^2} \xrightarrow{k\to+\infty} \frac{2A}{4\varphi_1'(0) - M^2(\overline{\varphi_{j'}''(0)} + \varphi_j''(0))}.$$

Therefore, arguing as in the proof of Theorem 4.4, we get that for all $M > 0$,

$$\sum_{j,j'\in I} \frac{\lambda_j\overline{\lambda_{j'}}}{4\varphi_1'(0) - M^2(\overline{\varphi_{j'}''(0)} + \varphi_j''(0))} = 0.$$

Now, for $j \neq j' \in I$, one has $\varphi_j''(0) \neq \varphi_{j'}''(0)$. Indeed, if $j, j' \in I$ with $\varphi_j''(0) = \varphi_{j'}''(0)$, then

$$\varphi_j'(0) = -(\log q_1)e_j - 2(\log q_1)f_j = -(\log q_1)e_{j'} - 2(\log q_1)f_{j'} = \varphi_{j'}(0)$$
$$\varphi_j''(0) = (\log q_1)^2 e_j + 4(\log q_1)^2 f_j = (\log q_1)^2 e_{j'} + 4(\log q_1)^2 f_{j'} = \varphi_{j'}''(0).$$

which implies $e_j = e_{j'}$ and $f_j = f_{j'}$, thus $\varphi_j = \varphi_{j'}$ and $j = j'$. Finally, Lemma 4.3 gives $\lambda_j = 0$ for $j \in I$, a contradiction. □

4.3. **Linear combinations of linear symbols.** Theorem 4.5 does not encompass completely the main result of [8]. Indeed, for instance, $\varphi(s) = s + 1 - 30^{-s}$ is a linear symbol which is not a polynomial symbol of degree at most 2. Nevertheless, we are also able to handle these symbols, thanks to the two forthcoming theorems.

Theorem 4.6. *Let $(\varphi_j)_{j=1,\dots,N} \subseteq \mathcal{G}$ be a finite sequence of distinct linear symbols such that at least one of them has a positive characteristic. Let $(\lambda_j)_{j=1,\dots,N}$ be a finite sequence of non-zero complex numbers. Then $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on $\mathcal{H}$ if and only if each C_{φ_j} is compact.*

Proof. Arguing as at the beginning of the proof of Theorem 4.5, we can assume that φ_1 has a positive characteristic, and $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact whereas no C_{φ_j} is compact. Then it will be shown that $\lambda_1 = 0$, which leads to a contradiction with the hypotheses. We can also assume that $\varphi_1(0) = 0$. We write each φ_j as

$$\varphi_j(s) = c_{0,j}s + c_{1,j} + \sum_{l=1}^{d^{(j)}} c_{q_l^{(j)}}(q_l^{(j)})^{-s} = c_{0,j}s + \psi_j(s)$$

where the integers $(q_l^{(j)})_{j=1,\dots,d^{(j)}}$ are multiplicatively independent and the complex numbers $c_{q_l^{(j)}}$ are non-zero. Since φ_j is supposed to have unrestricted range and

the $(q_l^{(j)})$ are multiplicatively independent, we know that $\mathrm{Re}\,(c_{1,j}) = \sum_{l=1}^{d^{(j)}} |c_{q_l^{(j)}}|$ if $c_{0,j} \neq 0$, and $\mathrm{Re}\,(c_{1,j}) = \frac{1}{2} + \sum_{l=1}^{d^{(j)}} |c_{q_l^{(j)}}|$ otherwise. To simplify the notation, we set $c_0 = c_{0,1}$, $d = d^{(1)}$, $q_l = q_l^{(1)}$, $c_{q_l} = c_{q_l^{(1)}}$ and $\mathcal{Q} = \{q_1, \dots, q_d\}$. We set $\mathcal{Q}_{\mathbb{N}} = \{q \in \mathbb{N} : \ \exists \alpha \in \mathbb{Q}^d, \ q = q_1^{\alpha_1} \cdots q_d^{\alpha_d}\}$ and observe that if $q \notin \mathcal{Q}_{\mathbb{N}}$, then the $(d+1)$ integers $q_1, \dots, q_d, q$ are multiplicatively independent.

Inspired by the proof of Theorem 4.5, we set

$$\begin{aligned}
I &= \left\{ j : \ c_{0,j} = c_0, \ \varphi_j(0) = 0, \ q_l^{(j)} \in \mathcal{Q}_{\mathbb{N}}, \ l = 1, \dots, d^{(j)} \right\} \\
J_1 &= \left\{ j : \ c_{0,j} = c_0, \ \varphi_j(0) \neq 0, \ q_l^{(j)} \in \mathcal{Q}_{\mathbb{N}}, \ l = 1, \dots, d^{(j)} \right\} \\
J_2 &= \left\{ j : \ c_{0,j} = c_0, \ \exists l \in \{1, \dots, d^{(j)}\}, \ q_l^{(j)} \notin \mathcal{Q}_{\mathbb{N}} \right\} \\
J_3 &= \{ j : \ c_{0,j} \neq c_0 \text{ and } c_{0,j} > 0 \} =: \{ j_1, \dots, j_p \} \\
J_4 &= \{ j : \ c_{0,j} = 0 \} .
\end{aligned}$$

We start by working with indices in I. Therefore, let $j \in I$. The map φ_j may be written

$$\varphi_j(s) = c_0 s + \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \big((q^\alpha)^{-s} - 1 \big)$$

with $\Gamma_j \subseteq \mathbb{Q}^d \backslash \{(0, \dots, 0)\}$ and $c_\alpha^{(j)} < 0$ for all $\alpha \in \Gamma_j$. In particular

$$\mathcal{B}_{\mathcal{Q}} \psi_j(z) = \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \left(\prod_{l=1}^{d} z_l^{\alpha_l} - 1 \right).$$

For $j \in I$, let us introduce

$$\begin{aligned}
L_j(x_1, \dots, x_d) &= \sum_{l=1}^{d} \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \alpha_l \log(p_l) x_l \\
Q_j(x_1, \dots, x_d) &= \frac{1}{2} \sum_{1 \le l, m \le d} \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \alpha_l \alpha_m \log(p_l) \log(p_m) x_l x_m .
\end{aligned}$$

Let also $\mathcal{C} = \{ j \in I : \ L_j = L_1 \}$ and pick $j \in I$, $j \neq 1$. If $Q_j = Q_1$, then

$$\text{for all } 1 \le l \neq m \le d, \ \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \alpha_l \alpha_m = 0 \tag{4.6}$$

$$\text{for all } 1 \le l \le d, \ \sum_{\alpha \in \Gamma_j} c_\alpha^{(j)} \alpha_l^2 = c_{q_l}. \tag{4.7}$$

Since each $c_\alpha^{(j)}$ is negative, (4.6) tells us that for any $\alpha \in \Gamma_j$, at most one of its component is non-zero. Moreover, since the integers q^α, $\alpha \in \Gamma_j$, are multiplicatively independent, if $l \in \{1, \dots, d\}$, at most one $\alpha \in \Gamma_j$ can have its non-zero component equal to its l-th component. Therefore, the sum appearing in (4.7) reduces to at most one element. Since $c_{q_l} \neq 0$, the sum cannot be empty and we have shown that $\mathrm{card}(\Gamma_j) = d$, that Γ_j may be written $\{\alpha(1), \dots, \alpha(d)\}$ with

$\alpha(l) = (0, \dots, 0, \alpha_l(l), 0, \dots, 0)$, the non-zero term $\alpha_l(l)$ appearing at the $l-$th position. Now, if we compare (4.7) with $\alpha_l(l) d^{(j)}_{\alpha(l)} = \sum_{\alpha \in \Gamma_j} \alpha_l c^{(j)}_\alpha = c_{q_l}$ (which comes from $L_j = L_1$), we get that each $\alpha_l(l)$ has to be equal to 1. Finally, we have shown that $\Gamma_j = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ which in turn yields $\varphi_j = \varphi_1$, a contradiction.

From this, we deduce using an argument similar to that of the proof of Theorem 4.5 the existence of $\beta \in \mathbb{R}^d$ such that, for all $j \in I$, $j \neq 1$, either $L_j(\beta) \neq L_1(\beta)$ or $Q_j(\beta) \neq Q_1(\beta)$. We now set, for $1 \leq l \leq d$, $k \geq 1$ and $M > 0$,

$$z_l(k) = p_l^{-\frac{M}{k} + \frac{i\beta_l}{\sqrt{k}}}$$

so that

$$z_l(k)^{\alpha_l} = 1 + i\frac{\alpha_l \log(p_l)\beta_l}{\sqrt{k}} - \frac{M\alpha_l \log(p_l) + \frac{1}{2}\alpha_l^2 \log^2(p_l)\beta_l^2}{k} + o\left(\frac{1}{k}\right).$$

This yields for any $j \in I$,

$$\begin{aligned}
\mathcal{B}_{\mathcal{Q}}\psi_j(z(k)) &= \frac{i}{\sqrt{k}} \sum_{\alpha \in \Gamma_j} c^{(j)}_\alpha \sum_{l=1}^{d} \alpha_l \log(p_l)\beta_l - \frac{1}{k} \sum_{\alpha \in \Gamma_j} c^{(j)}_\alpha \Bigg(\sum_{l=1}^{d} M\alpha_l \log(p_l) \\
&\quad + \frac{1}{2} \sum_{l=1}^{d} \alpha_l^2 \log^2(p_l)\beta_l^2 + \sum_{1 \leq l < m \leq d} c^{(j)}_\alpha \alpha_l \alpha_m \log(p_l)\log(p_m)\beta_l\beta_m \Bigg) \\
&\quad + o\left(\frac{1}{k}\right) \\
&= \frac{iL_j(\beta)}{\sqrt{k}} - \frac{L_j(\mathbf{e}) + Q_j(\beta)}{k} + o\left(\frac{1}{k}\right).
\end{aligned}$$

Suppose now that $j \in J_1$ which means that $\mathcal{B}_{\mathcal{Q}}\psi_j(\mathbf{e}) \neq 0$. Hence, $\mathcal{B}_{\mathcal{Q}}\psi_j(z(k))$ tends to some $a_j \neq 0$. To handle J_2, we first introduce integers $q_{d+1}, \dots, q_e$ such that $q_1, \dots, q_d, q_{d+1}, \dots, q_e$ is a maximal sequence of multiplicatively independent integers extracted from $\{q_l^{(j)} : \ 1 \leq j \leq N, \ 1 \leq l \leq d^{(j)}\}$. For $j \in J_2$, each $q_l^{(j)}$ may be written $q_1^{\alpha_1} \cdots q_d^{\alpha_d} q_{d+1}^{\alpha_{d+1}} \cdots q_e^{\alpha_e}$ with $\alpha_i \in \mathbb{Q}$. Furthermore, there exists some $l(j)$ such that the family $(\alpha_{d+1}, \dots, \alpha_e)$ is not identically zero (otherwise j would belong to $I \cup J_1$). Denote this family by $(\alpha_{d+1}(j), \dots, \alpha_e(j))$. Then

$$E_j := \left\{ (\theta_{d+1}, \dots, \theta_e) \in \mathbb{R}^{e-d} : \ e^{i(\theta_{d+1}\alpha_{d+1}(j) + \dots + \theta_e \alpha_e(j)} c_{q_j^{(l)}} = |c_{q_l^{(j)}}| \right\}$$

has Lebesgue measure zero. Therefore we can pick $(\theta_{d+1}, \dots, \theta_e) \in \mathbb{R}^{e-d} \setminus \bigcup_{j \in J_2} E_j$. Denoting $\mathcal{Q}' = \{q_{d+1}, \dots, q_e\}$, since $\mathrm{Re}\,(c_{1,j}) = \sum_{l=1}^{d^{(j)}} |c_{q_l^{(j)}}|$, this implies that

$$\mathcal{B}_{\mathcal{Q} \cup \mathcal{Q}'}\psi_j(1, \dots, 1, e^{i\theta_{d+1}}, \dots, e^{i\theta_e}) =: a_j \neq 0.$$

To proceed with the conclusion, we finally fix some positive integers $q_{e+1}, \dots, q_{e+p}$ such that $q_1, \dots, q_{e+p}$ are multiplicatively independent (recall that $p = \mathrm{card}(J_3)$). We then choose, by Kronecker's theorem, a sequence of real numbers (t_k) such that

(a) for $1 \le l \le d$, $q_l^{-it_k}$ is very close to $q_l^{-\frac{i\beta_l}{\sqrt{k}}}$, so close that, setting $w_k = \frac{-M}{k} + it_k$,

$$\mathcal{B}_{\mathcal{Q}}\psi_j(q_1^{-w_k}, \dots, q_d^{-w_k}) = \mathcal{B}_{\mathcal{Q}}\psi_j(z(k)) + o\left(\frac{1}{k}\right)$$

for all $j \in I \cup J_1$.

(b) for $d+1 \le l \le e$, $q_l^{-it_k}$ tends to $e^{i\theta_l}$, so that

$$\mathcal{B}_{\mathcal{Q}\cup\mathcal{Q}'}\psi_j(q_1^{-w_k}, \dots, q_e^{-w_k}) = a_j + o\left(\frac{1}{k}\right)$$

for all $j \in J_2$.

(c) for all $j \in J_3$, writing $j = j_r$ with $1 \le r \le p$,

$$q_{e+r}^{-it_k(c_{0,j_r}-c_0)} q_{e+r}^{-\mathcal{B}_{\mathcal{Q}\cup\mathcal{Q}'}\psi_j(1,\dots,1,e^{i\theta_{d+1}},\dots,e^{i\theta_e})} \text{ tends to some } \omega(j_r) \neq 1.$$

Let d' be such that $p^+(q_l) \le p_{d'}$ for all $l = 1, \dots, e+r$ and $p^+(q_l^{(j)}) \le p_{d'}$ for all $j \in J_4$ and all $1 \le l \le d^{(j)}$. We first observe that, for $j \in \mathcal{C}$ and $j' \notin \mathcal{C}$,

$$\langle C^*_{\varphi_{j'}}(k_{d',w_k}), C^*_{\varphi_j}(k_{d',w_k})\rangle \xrightarrow{k\to+\infty} 0.$$

When $j' \notin J_4$, this follows from the choice of the sequence t_k, and the fact that for $l \in \{1, \dots, N\}$, $l \notin J_4$,

$$\varphi_l(w_k) = c_{0,l}w_k + \mathcal{B}_{\mathcal{Q}\cup\mathcal{Q}'}\psi_l(q_1^{-w_k}, \dots, q_e^{-w_k}).$$

When $j' \in J_4$, this has already been observed during the proof of Theorem 4.4.

We now argue exactly as in the proof of Theorem 4.4. to conclude that, for all $M > 0$,

$$\sum_{j,j'\in\mathcal{C}} \frac{\lambda_j\overline{\lambda_{j'}}}{\left(2M(c_0 - L_1(\mathbf{e})) - (\overline{Q_{j'}(\beta)} + Q_j(\beta))\right)^{d'}} = 0. \tag{4.8}$$

Finally we consider the following equivalence relation on $\mathcal{C}$: $j\mathcal{R}j' \iff Q_j(\beta) = Q_{j'}(\beta)$. Let $\mathcal{C}_1, \dots, \mathcal{C}_\kappa$ be the equivalent classes for this relation, with $\mathcal{C}_1 = \{1\}$, let $\nu_k = \sum_{j\in\mathcal{C}_k}\lambda_j$ and let $\mu_k = Q_j(\beta)$ for some (all) $j \in \mathcal{C}_k$, $1 \le k \le \kappa$. Then (4.8) may be rewritten

$$\sum_{k,k'=1}^{\kappa} \frac{\nu_k\overline{\nu_{k'}}}{\left(2M(c_0 - L_1(\mathbf{e})) - (\overline{\mu_{k'}} + \mu_k)\right)^{d'}} = 0.$$

Lemma 4.3 tells us that, for all $k = 1, \dots, \kappa$, $\nu_k = 0$. In particular, $\lambda_1 = \nu_1 = 0$, a contradiction! □

To tackle the case where all symbols have zero characteristic, we need one elementary lemma:

Lemma 4.7. *Let $\mathcal{Q}$ be a finite set of positive integers. Assume that for any $n, m \in \mathcal{Q}$, there exist $\alpha, \beta \in \mathbb{N}$ such that $n^\alpha = m^\beta$. Then there exists $q \in \mathbb{N}$ such that, for any $n \in \mathcal{Q}$, there exists $\alpha \in \mathbb{N}$ with $n = q^\alpha$.*

Proof. We proceed by induction on the cardinal N of $\mathcal{Q}$, the result being trivial for $N = 1$ and $N = 2$. Let us assume that it has been proved until N and let $\mathcal{Q} = \{q_1, \dots, q_{N+1}\}$ be a set with $N+1$ elements. We apply the inductive step to

find some q' working with $\mathcal{Q}' = \{q_1, \dots, q_N\}$ and then we apply the result for $N = 2$ to $\{q', q_{N+1}\}$ to conclude. $\square$

Theorem 4.8. *Let $(\varphi_j)_{j=1,\dots,N} \subseteq \mathcal{G}$ be a finite sequence of distinct linear symbols with zero characteristic. Let $(\lambda_j)_{j=1,\dots,N}$ be a finite sequence of non-zero complex numbers. Then $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact on $\mathcal{H}$ if and only if each C_{φ_j} is compact.*

Proof. Arguing as above, we start with N linear symbols $\varphi_1, \dots, \varphi_N$ having zero characteristic and with N non-zero complex numbers $\lambda_1, \dots, \lambda_N$ such that $\sum_{j=1}^N \lambda_j C_{\varphi_j}$ is compact whereas no C_{φ_j} is compact. In particular, one may write $\varphi_j(s) = c_j + d_j q_j^{-s}$ where $|d_j| = \mathrm{Re}\,(c_j) - \frac{1}{2}$. Without loss of generality, we shall assume that $\varphi_1(0) = \frac{1}{2}$.

As before, we split $\{1, \dots, N\}$ into disjoint subsets:

$$\begin{aligned} I &= \Big\{ j : \ \varphi_j(0) = \varphi_1(0) \text{ and } q_j^\alpha = q_1^\beta \text{ for some } \alpha, \beta \in \mathbb{N} \Big\} \\ J_1 &= \Big\{ j : \ \varphi_j(0) \neq \varphi_1(0) \text{ and } q_j^\alpha = q_1^\beta \text{ for some } \alpha, \beta \in \mathbb{N} \Big\} \\ J_2 &= \Big\{ j : \ q_j^\alpha \neq q_1^\beta \text{ for all } \alpha, \beta \in \mathbb{N} \Big\}. \end{aligned}$$

Let q be given by Lemma 4.7 with $\mathcal{Q} = \{q_j : \ j \in I \cup J_1\}$, for $j \in I$, let $\alpha_j \in \mathbb{N}$ be such that $q_j = q^{\alpha_j}$ and, for $M > 0$, consider $w_k = \frac{M}{k} + \frac{i}{\sqrt{k}}$. When $j \in I$,

$$\begin{aligned} \varphi_j(w_k) &= c_j + d_j\big(q^{-\alpha_j w_k} - 1\big) + d_j \\ &= \frac{1}{2} - \frac{i\alpha_j d_j \log q}{\sqrt{k}} - \frac{2M\alpha_j d_j \log q + \alpha_j^2 d_j \log^2 q}{2k} + o\left(\frac{1}{k}\right) \end{aligned}$$

and let us recall that $d_j < 0$. Moreover, when $j \in J_2$, applying Lemma 2.1,

$$C^*_{\varphi_j}(k^{(q)}_{w_k}) = \frac{K_{c_j}}{\|K^{(q)}_{w_k}\|} \xrightarrow{k \to +\infty} 0.$$

Therefore, one gets

$$\begin{aligned} 0 &\geq \left\| \sum_{j=1}^N \lambda_j C_{\varphi_j} \right\|_e^2 \\ &\geq \limsup_{k \to +\infty} \left\| \sum_{j=1}^N \overline{\lambda_j} C^*_{\varphi_j}(k^{(q)}_{w_k}) \right\|^2 \\ &\geq \limsup_{k \to +\infty} \left\| \sum_{j \in I \cup J_1} \overline{\lambda_j} C^*_{\varphi_j}(k^{(q)}_{w_k}) \right\|^2 \\ &\geq \limsup_{k \to +\infty} \sum_{j, j' \in I \cup J_1} \frac{\lambda_j \overline{\lambda_{j'}} K_{\varphi_{j'}(w_k)}(\varphi_j(w_k))}{\|K^{(q)}_{w_k}\|^2} \\ &\geq \limsup_{k \to +\infty} \sum_{j, j' \in I \cup J_1} \lambda_j \overline{\lambda_{j'}} \frac{\zeta\big(\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k)\big)}{\|K^{(q)}_{w_k}\|^2}. \end{aligned}$$

Now, if $j \in I$ and $j' \in J_1$, $\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k)$ tends to some complex number in $\overline{\mathbb{C}_1}$ which is not equal to 1. Hence,

$$\frac{\zeta\big(\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k)\big)}{\|K_{w_k}^{(q)}\|^2} \xrightarrow{k\to+\infty} 0$$

so that, arguing as in the proof of Theorem 4.5,

$$\limsup_{k\to+\infty} \sum_{j,j'\in I} \lambda_j \overline{\lambda_{j'}} \frac{\zeta\big(\overline{\varphi_{j'}(w_k)} + \varphi_j(w_k)\big)}{\|K_{w_k}^{(q)}\|^2} = 0.$$

Let finally $\mathcal{C} = \{j \in I : \ \alpha_j d_j = \alpha_1 d_1\}$. Using $\zeta(s) \sim_{s\to 1} \frac{1}{s-1}$ and $\|K_w^{(q)}\|^2 \sim_{\operatorname{Re} w\to 0} \frac{C_q}{\operatorname{Re} w}$, we finally get

$$\sum_{j,j'\in\mathcal{C}} \frac{\lambda_j \overline{\lambda_{j'}}}{4M\alpha_1 d_1 \log q + \alpha_j^2 d_j \log^2 q + \alpha_{j'}^2 d_{j'} \log^2 q} = 0.$$

Since for all $j \neq j' \in \mathcal{C}$, it is clear that $\alpha_j^2 d_j \neq \alpha_{j'}^2 d_{j'}$ (otherwise $\alpha_j = \alpha_{j'}$ and $d_{j'} = d_j$) we conclude by Lemma 4.3 that all λ_j, $j \in \mathcal{C}$, are equal to zero, a contradiction. □

5. Weigthed composition operators, connected components and compact differences

5.1. **Introduction.** Our aim in this section is to give sufficient conditions to prove that two composition operators C_{φ_0} and C_{φ_1} belong to the same connected component of $\mathcal{C}(\mathcal{H})$, or that their difference is compact. We shall see that this could be deduced from results on weighted composition operators (see [23] where this idea seems to appear for the first time). We start with $\varphi_0,\ \varphi_1 \in \mathcal{G}$ with the same characteristic c_0. The simplest way to draw an arc between C_{φ_0} and C_{φ_1} is to consider $\lambda \in [0,1] \mapsto C_{\varphi_\lambda}$, where $\varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1$. To prove that $\lambda \mapsto C_{\varphi_\lambda}$ is continuous, we start as in the proof of Theorem 3.5: let μ be a probability measure on $\mathbb{R}$, $f \in \mathcal{H}$ with $\|f\| \leq 1$ and let us write

$$\big\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\big\|^2 = \int_{\mathbb{T}^\infty}\int_{\mathbb{R}} |f_{\chi^{c_0}} \circ (\varphi_\lambda)_\chi(it) - f_{\chi^{c_0}} \circ (\varphi_{\lambda'})_\chi(it)|^2 d\mu(t) dm(\chi).$$

At this point we do not use the mean value theorem but instead we write

$$\int_\lambda^{\lambda'} f'_{\chi^{c_0}}((\varphi_r)_\chi(it))dr = \frac{1}{(\varphi_1)_\chi(it) - (\varphi_0)_\chi(it)} \left(f_{\chi^{c_0}} \circ (\varphi_\lambda)_\chi(it) - f_{\chi^{c_0}} \circ (\varphi_{\lambda'})_\chi(it)\right)$$

so that, using Jensen's inequality,

$$\begin{aligned}\big\|C_{\varphi_\lambda}(f) - C_{\varphi_{\lambda'}}(f)\big\|^2 \ \leq \ |\lambda' - \lambda|^2 \int_\lambda^{\lambda'}\int_{\mathbb{T}^\infty}\int_0^1 |(\varphi_1)_\chi(it) - (\varphi_0)_\chi(it)|^2 \qquad (5.1)\\ \times |f'_{\chi^{c_0}}((\varphi_r)_\chi(it))|^2 d\mu(t) dm(\chi) dr.\end{aligned}$$

This inequality leads us to the following lemma.

Lemma 5.1. *Let* $\varphi_0,\ \varphi_1 \in \mathcal{G}$ *with* $\operatorname{char}(\varphi_0) = \operatorname{char}(\varphi_1)$ *and let* $\varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1$, $\lambda \in [0,1]$.

(i) Suppose that for all $\lambda \in [0,1]$, $f \mapsto (\varphi_0 - \varphi_1)C_{\varphi_\lambda}(f')$ acts boundedly on $\mathcal{H}$ with uniformly bounded norms. Then there is a constant $A > 0$ such that, for all $\lambda, \lambda' \in [0,1]$,

$$\left\|C_{\varphi_\lambda} - C_{\varphi_{\lambda'}}\right\| \leq A|\lambda - \lambda'|.$$

(ii) Suppose moreover that for each $\lambda \in [0,1]$, $f \mapsto (\varphi_0 - \varphi_1)C_{\varphi_\lambda}(f')$ is a compact operator on $\mathcal{H}$. Then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

Proof. (i) is an immediate consequence of (5.1) whereas (ii) follows from an application of Lebesgue's dominated convergence theorem. □

Hence we are reduced to study the boundedness and the compactness of a weighted composition and derivation operator. By working with a Bergman space instead of a Hardy space as initial space, we will be reduced to study the same problems for weighted composition operators acting between different spaces. At this point, we will divide the proof into two cases, following the value of c_0. For $c_0 > 0$ we will stay on the halfplane (and on the hidden polydisc) and use Nevanlinna counting functions. For $c_0 = 0$ we will work on the disk and use Carleson measures.

5.2. **The case of positive characteristic.**

5.2.1. *Boundedness and compactness of weighted composition operators.* We shall need the following lemma which explains how to compute the norm of an element of $\mathcal{H}$ using its derivative.

Lemma 5.2. *Let μ be a probability measure on $\mathbb{R}$ and $f = \sum_n a_n n^{-s}$. Then the following assertions are equivalent:*

(i) $f \in \mathcal{H}$;

(ii) for almost all $\chi \in \mathbb{T}^\infty$, f_χ extends to $\mathbb{C}_+$ and

$$\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^1 |f'_\chi(\sigma + it)|^2 \sigma d\sigma d\mu(t) dm(\chi) < +\infty.$$

Moreover, provided $f \in \mathcal{H}$, then

$$\|f\|^2 \approx |a_1|^2 + \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^1 |f'_\chi(\sigma + it)|^2 \sigma d\sigma d\mu(t) dm(\chi). \tag{5.2}$$

Proof. We just need to prove (5.2) for a Dirichlet polynomial. Now by (2.1) and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^1 |f'_\chi(\sigma + it)|^2 \sigma d\sigma d\mu(t) dm(\chi) &= \sum_{n\geq 2} |a_n|^2 \log^2(n) \int_0^1 \sigma n^{-2\sigma} d\sigma \\ &\approx \sum_{n\geq 2} |a_n|^2.\end{aligned}$$

□

This leads us to introduce the following Banach space of Dirichlet series: $\mathcal{A}$ is the completion of the space of Dirichlet polynomials for the norm

$$\|f\|_{\mathcal{A}}^2 \;:=\; \int_0^1 \|f(\cdot + \sigma)\|_{\mathcal{H}}^2 \sigma d\sigma$$

$$= \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^1 |f_\chi(\sigma+it)|^2\sigma d\sigma d\mu(t)dm(\chi). \tag{5.3}$$

This space, a particular example of the Bergman spaces introduced in [4], is a space of Dirichlet series defined on $\mathbb{C}_{1/2}$. Moreover, if $f\in\mathcal{A}$, for almost all $\chi\in\mathbb{T}^\infty$, f_χ converges on $\mathbb{C}_+$ and the norm in $\mathcal{A}$ can be computed using (5.3).

Lemma 5.2 says that $f\in\mathcal{H}$ if and only if $f'\in\mathcal{A}$ and that up to the constant term, the norms are comparable. Moreover, it is easy to check that if a sequence (f_n) converges weakly to 0 in $\mathcal{H}$, then (f_n') converges weakly to 0 in $\mathcal{A}$. Therefore we are led to the study of the boundedness and of the compacness of weighted composition operators from $\mathcal{A}$ to $\mathcal{H}$. This is the content of the next theorem, of independent interest.

Theorem 5.3. *Let $\varphi\in\mathcal{G}$ with* $\mathrm{char}\,(\varphi)>0$ *and write it $\varphi=c_0s+\psi$. Let u be a Dirichlet series convergent in $\mathbb{C}_+$. Assume that there exists $C>0$ such that*

- $|u'|\le C,\ |\psi|\le C$;
- $|u|\le C\mathrm{Re}\,\varphi$.

Then $\|uC_\varphi\|_{\mathcal{A}\to\mathcal{H}}\lesssim_{c_0,C}1$.

The proof will need the following analogue of Lemma 5.2 for $\mathcal{A}$, which has a completely similar proof: for all $f\in\mathcal{A}$, for all $c_0\ge 1$,

$$\|f\|_{\mathcal{A}}^2\approx|a_1|^2+\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^{+\infty}|f'_{\chi^{c_0}}(\sigma+it)|^2\sigma^3d\sigma d\mu(t)dm(\chi).$$

Proof of Theorem 5.3. Let $f\in\mathcal{A}$. We choose for μ the probability measure $d\mu=\mathbf{1}_{[0,1]}dt$. We intend to show that $\|uC_\varphi(f)\|_{\mathcal{H}}\lesssim_{c_0,C}\|f\|_{\mathcal{A}}$. By Lemma 5.2, it suffices to show that $\|(uC_\varphi(f))'\|_{\mathcal{A}}\lesssim_{c_0,C}\|f\|_{\mathcal{A}}$. Now,

$$(uC_\varphi f)'=u'f\circ\varphi+u\varphi'f'\circ\varphi.$$

Therefore we have to study the following integrals:

$$I_1=\int_{\mathbb{T}^\infty}\int_0^1\int_0^1|u'_\chi(\sigma+it)|^2|f_{\chi^{c_0}}\circ\varphi_\chi(\sigma+it)|^2\sigma d\sigma dtdm(\chi)$$

$$I_2=\int_{\mathbb{T}^\infty}\int_0^1\int_0^1|u_\chi(\sigma+it)|^2|\varphi'_\chi(\sigma+it)|^2|f'_{\chi^{c_0}}\circ\varphi_\chi(\sigma+it)|^2\sigma d\sigma dtdm(\chi).$$

We first handle I_1 and observe that, u'_χ being a vertical translate of u', for all $\chi\in\mathbb{T}^\infty$, for all $s\in\mathbb{C}_+$, $|u'_\chi(s)|\le C$. On the other hand, we appeal to [2] to ensure that

$$I_1\le C\|f\circ\varphi\|_{\mathcal{A}}\le C\|f\|_{\mathcal{A}}.$$

Let us turn to I_2. Using again vertical translates, we know that, for all $s\in\mathbb{C}_+$, $|u_\chi(s)|\le C\mathrm{Re}\,\big(\varphi_\chi(s)\big)$. Since we also know that $\mathrm{Re}\,(s)\le\mathrm{Re}\,\big(\varphi_\chi(s)\big)/c_0$, we obtain

$$I_2\lesssim_{c_0,C}\int_{\mathbb{T}^\infty}\int_0^1\int_0^1|\varphi'_\chi(\sigma+it)|^2|f'_{\chi^{c_0}}\circ\varphi_\chi(\sigma+it)|^2\mathrm{Re}^{\,3}\big(\varphi_\chi(\sigma+it)\big)d\sigma dtdm(\chi).$$

We then do, for a fixed $\chi\in\mathbb{T}^\infty$, the nonunivalent change of variables $w=\varphi_\chi(s)$. We denote by $\mathcal{N}_{\varphi_\chi,3}$ the appropriated Nevanlinna counting function, namely

$$\mathcal{N}_{\varphi_\chi,3}(w)=\sum_{\varphi_\chi(w)=s}\mathrm{Re}^{\,3}(s).$$

Observing that our assumptions ensure that for all $\chi \in \mathbb{T}^\infty$, $\varphi_\chi([0,1]^2) \subseteq [0, c_0 + C] \times [-c_0 - C, c_0 + C]$, we then get

$$I_2 \lesssim_{c_0,C} \int_{\mathbb{T}^\infty} \int_{-c_0-C}^{c_0+C} \int_0^{c_0+C} |f'_{\chi^{c_0}}(\sigma + it)|^2 \mathcal{N}_{\varphi_\chi,3}(\sigma + it) d\sigma dt dm(\chi).$$

Now, since

$$\begin{aligned} N_{\varphi_\chi,3}(w) &\leq \left(\sum_{\varphi_\chi(w)=s} \mathrm{Re}\,(s) \right)^3 \\ &\leq c_0^3 \mathrm{Re}^{\,3}(w) \end{aligned}$$

by [6, Proposition 3], we can finally conclude that

$$I_2 \lesssim_{c_0,C} \int_{\mathbb{T}^\infty} \int_{-C-c_0}^{C+c_0} \int_0^{c_0+C} |f'_{\chi^{c_0}}(\sigma + it)|^2 \sigma^3 d\sigma dt dm(\chi) \lesssim_{c_0,C} \|f\|_{\mathcal{A}}^2.$$

□

Let us now turn to compactness.

Theorem 5.4. *Let $\varphi \in \mathcal{G}$ with* $\mathrm{char}\,(\varphi) > 0$ *and write it $\varphi = c_0 s + \psi$. Let u be a Dirichlet series convergent in $\mathbb{C}_+$. Assume that there exists $C > 0$ such that*

- $|u'| \leq C,\ |\psi| \leq C$;
- $|u| \leq C \mathrm{Re}\,\varphi$;
- $|u'| \to 0$ *and* $|u| = o\big(\mathrm{Re}\,(\varphi)\big)$ *as* $\mathrm{Re}\,(\varphi) \to 0$.

Then uC_φ is a compact operator from $\mathcal{A}$ to $\mathcal{H}$.

Proof. Let (f_n) be a sequence in $\mathcal{A}$ which converges weakly to 0 and satisfies $\|f_n\|_{\mathcal{A}} \leq 1$. Arguing as in the proof of Theorem 5.3, we have to show that $I_{n,1}$ and $I_{n,2}$ go to zero where

$$I_{n,1} = \int_{\mathbb{T}^\infty} \int_0^1 \int_0^1 |u'_\chi(\sigma + it)|^2 |f_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi),$$

$$I_{n,2} = \int_{\mathbb{T}^\infty} \int_0^1 \int_0^1 |u_\chi(\sigma + it)|^2 |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi).$$

We first handle $I_{n,1}$. Let $\varepsilon > 0$ and $\theta \in (0,1)$ be such that, for all $\chi \in \mathbb{T}^\infty$ and for all $s \in \mathbb{C}_+$ with $\mathrm{Re}\,\varphi_\chi(s) \in (0,\theta)$, $|u'_\chi(s)| \leq \varepsilon$. Then, setting $\Omega_\chi = \{(\sigma,t) \in [0,1]^2 :\ \mathrm{Re}\,(\varphi_\chi(s)) \in (0,\theta)\}$,

$$\begin{aligned} &\int_{\mathbb{T}^\infty} \iint_{\Omega_\chi} |u'_\chi(\sigma + it)|^2 |f_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi) \\ &\qquad \leq \varepsilon^2 \int_{\mathbb{T}^\infty} \int_0^1 \int_0^1 |f_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi) \\ &\qquad \leq \varepsilon^2. \end{aligned}$$

On the other hand, for all $\chi \in \mathbb{T}^\infty$, for all $(\sigma,t) \in [0,1]^2 \backslash \Omega_\chi$, setting $s = \sigma + it$, $\mathrm{Re}\,(\varphi_\chi(s)) \geq \theta$ and $|\mathrm{Im}\,(\varphi_\chi(s))| \leq C + c_0$. Therefore, there exists $C_\theta > 0$ such that,

for all $\chi \in \mathbb{T}^\infty$ and all such (σ, t),

$$|f_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \leq C_\theta \|f_{n,\chi^{c_0}}(\cdot + \theta/2)\|^2_{H^2_i(\mathbb{C}_+)},$$

where $H^2_i(\mathbb{C}_+)$ is the invariant Hardy space on the right halfplane, namely the set of analytic functions $g : \mathbb{C}_+ \to \mathbb{C}_+$ satisfying

$$\|g\|^2_{H^2_i(\mathbb{C}_+)} := \frac{1}{\pi} \int_{-\infty}^{+\infty} |g(it)|^2 \frac{dt}{1+t^2} < +\infty.$$

This yields

$$\begin{aligned}
&\int_{\mathbb{T}^\infty} \iint_{[0,1]^2 \setminus \Omega_\chi} |u'_\chi(\sigma + it)|^2 |f_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi) \\
&\quad \lesssim_\theta \int_{\mathbb{T}^\infty} \int_0^1 \int_0^1 \int_{\mathbb{R}} \left| f_{n,\chi^{c_0}} \left(iu + \frac{\theta}{2} \right) \right|^2 \frac{du}{1+u^2} \sigma d\sigma dt dm(\chi) \\
&\quad \lesssim_\theta \sum_{k \geq 1} |a_{n,k}|^2 k^{-\theta},
\end{aligned}$$

where we have written each f_n as $f_n = \sum_k a_{n,k} k^{-s}$ and we have used (2.1) for $d\mu(u) = \frac{du}{1+u^2}$. Since (f_n) tends weakly to zero and $\theta > 0$, it is easy to prove that $\sum_k |a_{n,k}|^2 k^{-\theta}$ goes to zero as n tends to $+\infty$. Hence we have shown that for all $\varepsilon > 0$, $\limsup_{n \to +\infty} I_{n,1} \leq \varepsilon$ which implies that $\lim_{n \to +\infty} I_{n,1} = 0$.

Let us turn to $I_{n,2}$. Again for arbitrary $\varepsilon > 0$ there exists $\theta \in (0,1)$ such that $\mathrm{Re}\,(\varphi_\chi(s)) < \theta$ implies

$$|u_\chi(s)| \leq \varepsilon \mathrm{Re}\,\big(\varphi_\chi(s)\big).$$

Keeping the notation $\Omega_\chi = \{(\sigma, t) \in [0,1]^2 : \ \mathrm{Re}\,(\varphi_\chi(s)) \in (0, \theta)\}$ and using $\mathrm{Re}\,(s) \leq \mathrm{Re}\,\big(\varphi_\chi(s)\big)/c_0$, We obtain

$$\begin{aligned}
&\int_{\mathbb{T}^\infty} \iint_{\Omega_\chi} |u_\chi(\sigma + it)|^2 |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi) \\
&\leq \frac{1}{c_0} \varepsilon^2 \int_{\mathbb{T}^\infty} \int_0^1 \int_0^1 |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \mathrm{Re}^3\big(\varphi_\chi(\sigma + it)\big) d\sigma dt dm(\chi) \\
&\lesssim_{c_0, C} \varepsilon^2
\end{aligned}$$

by the same argument as that of Theorem 5.3. On the other hand, arguing as above,

$$\begin{aligned}
&\int_{\mathbb{T}^\infty} \iint_{[0,1]^2 \setminus \Omega_\chi} |u_\chi(\sigma + it)|^2 |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi) \\
&\lesssim_{c_0, C} \int_{\mathbb{T}^\infty} \iint_{[0,1]^2 \setminus \Omega_\chi} |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \mathrm{Re}^3(\varphi_\chi(\sigma + it)) d\sigma dt dm(\chi).
\end{aligned}$$

For a fixed $\chi \in \mathbb{T}^\infty$, we do the nonunivalent change of variables $w = \varphi_\chi(s)$. Observe that we now have, for all $\chi \in \mathbb{T}^\infty$, $\varphi_\chi([0,1]^2 \setminus \Omega_\chi) \subseteq [\theta, c_0 + C] \times [-c_0 - C, c_0 + C]$. We get, using again the Nevanlinna counting function $\mathcal{N}_{\varphi_\chi, 3}$ as in the proof of Theorem 5.3,

$$\int_{\mathbb{T}^\infty} \iint_{[0,1]^2 \setminus \Omega_\chi} |u_\chi(\sigma + it)|^2 |\varphi'_\chi(\sigma + it)|^2 |f'_{n,\chi^{c_0}} \circ \varphi_\chi(\sigma + it)|^2 \sigma d\sigma dt dm(\chi)$$

$$\lesssim_{c_0,C} \int_{\mathbb{T}^\infty}\int_{-c_0-C}^{c_0+C}\int_{\theta}^{\sigma_0} |f'_{n,\chi^{c_0}}(\sigma+it)|^2\sigma^3 d\sigma dt dm(\chi)$$
$$\lesssim_{c_0,C} \sum_{k=2}^{+\infty}(\log k)|a_{k,n}|^2 k^{-2\theta}$$

and again it is easy to prove that this last quantity goes to zero as n goes to $+\infty$ by weak convergence of (f_n) to zero. This concludes the proof that uC_φ is a compact operator from $\mathcal{A}$ to $\mathcal{H}$. □

5.2.2. *Applications to connected components and compact differences.* Lemmas 5.1 and 5.2, Theorems 5.3 and 5.4 lead us to the following sufficient conditions for two composition operators to be in the same component or for their difference to be compact.

Theorem 5.5. *Let φ_0 and $\varphi_1 \in \mathcal{G}$ with* $\mathrm{char}\,(\varphi_0) = \mathrm{char}\,(\varphi_1) =: c_0 \geq 1$ *and write them $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. Assume moreover that there exists $C > 0$ such that*

- $|\varphi_0 - \varphi_1| \leq C \min(\mathrm{Re}\,\varphi_0, \mathrm{Re}\,\varphi_1)$;
- $|\psi_0|$, $|\psi_1| \leq C$;
- $|\varphi_0' - \varphi_1'| \leq C$.

Then C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$. If moreover we assume that

$$|\varphi_0 - \varphi_1| = o\big(\min(\mathrm{Re}\,\varphi_0, \mathrm{Re}\,\varphi_1)\big) \text{ and } \varphi_0' - \varphi_1' \to 0 \text{ as } \min(\mathrm{Re}\,\varphi_0, \mathrm{Re}\,\varphi_1) \to 0$$

then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

We can deduce from this a result about non isolation of composition operators.

Corollary 5.6. *Let $\varphi = c_0 s + \psi \in \mathcal{G}$ with positive characteristic. Assume that ψ and ψ' are bounded on $\mathbb{C}_+$ and that there exist $C > 0$, $k \geq 1$, $t_0 \in \mathbb{R}$ such that, for all $s \in \mathbb{C}_+$,*

$$|\mathrm{Im}\,(\psi(s)) - t_0| \leq C\big(\mathrm{Re}\,(\psi(s))\big)^{1/k}.$$

Then C_φ is not isolated in $\mathcal{C}(\mathcal{H})$.

Proof. Without loss of generality, we may assume that $t_0 = 0$. Let us denote $\varphi_0 = \varphi$ and let, for $\delta > 0$, $\varphi_1 = c_0 s + \psi + \delta\psi^k$. We claim that, provided δ is small enough, φ_0 and φ_1 satisfy the assumptions of Theorem 5.5. For $s \in \mathbb{C}_+$, let us write

$$\mathrm{Re}\,\varphi_1(s) = c_0 \mathrm{Re}\,(s) + \mathrm{Re}\,\psi(s) + \delta \mathrm{Re}\,(\psi^k(s)).$$

Now,

$$\mathrm{Re}\,(\psi^k(s)) = \sum_{j=0}^{k} a_j \big(\mathrm{Re}\,\psi(s)\big)^j \big(\mathrm{Im}\,\psi(s)\big)^{k-j}$$

for some coefficients a_j. Using our assumption on $\mathrm{Im}\,\psi$ and the boundedness of $\mathrm{Re}\,\psi$, there exist $C_0 > 0$ such that, for all $s \in \mathbb{C}_+$,

$$\mathrm{Re}\,(\psi^k(s)) \geq -C_0 \mathrm{Re}\,(\psi(s)).$$

We set $\delta = 1/2C_0$. We get for all $s \in \mathbb{C}_+$

$$\operatorname{Re}\varphi_1(s) \geq c_0 \operatorname{Re}(s) + \frac{1}{2}\operatorname{Re}\psi(s) \geq \frac{1}{2}\operatorname{Re}\varphi_0(s).$$

Therefore $\varphi_1 \in \mathcal{G}$. Furthermore, for all $s \in \mathbb{C}_+$, with involved constants independent of s,

$$\begin{aligned}|\varphi_0(s) - \varphi_1(s)|^2 &= \delta^2 |\psi(s)|^{2k} \\ &\lesssim \big(\operatorname{Re}\psi(s)\big)^{2k} + \big(\operatorname{Im}\psi(s)\big)^{2k} \\ &\lesssim \big(\operatorname{Re}\psi(s)\big)^{2} \\ &\lesssim \big(\min(\operatorname{Re}\varphi_0(s), \operatorname{Re}\varphi_1(s))\big)^2,\end{aligned}$$

where, again, we used our assumption on $\operatorname{Im}\psi$ and the boundedness of $\operatorname{Re}\psi$. □

Corollary 5.7. *Let $\varphi_0(s) = c_0 s + c_1 + \sum_{j=1}^d c_{q_j} q_j^{-s}$ be a linear symbol with* $\operatorname{char}(\varphi_0) \geq 1$. *Then C_{φ_0} is not isolated in $\mathcal{C}(\mathcal{H})$. Moreover, if φ_0 has unrestricted range, there exists a continuous map $\lambda \in [0, \delta] \mapsto \varphi_\lambda \in \mathcal{G}$ such that, for $\lambda \neq \lambda'$, $C_{\varphi_\lambda} - C_{\varphi_{\lambda'}}$ is never compact.*

Proof. We only have to prove the statement when C_{φ_0} is not compact, namely when φ_0 has unrestricted range. In that case, we may assume that $c_{q_j} = -|c_{q_j}|$ and $c_1 = \sum_{j=1}^d |c_{q_j}|$. Since $s \mapsto q_j^{-s}$ maps $\mathbb{C}_+$ into $\mathbb{D}$, ψ_0 maps $\mathbb{C}_+$ into some disk in $\mathbb{C}_+$ which is internally tangent to the imaginary axis at 0. Therefore the assumptions of Corollary 5.6 are satisfied with $k = 2$. Moreover, a look at the proof of Theorem 5.5 shows that we can choose a continuous path $\lambda \mapsto \varphi_\lambda$ given by

$$\varphi_\lambda(s) = c_0 s + c_1 + \sum_{j=1}^d c_{q_j} q_j^{-s} + \lambda \left(c_1 + \sum_{j=1}^d c_{q_j} q_j^{-s} \right)^2.$$

By Theorem 4.5, for $\lambda \neq \lambda'$, $C_{\varphi_\lambda} - C_{\varphi_{\lambda'}}$ is never compact. □

In particular, the Shapiro-Sundberg conjecture keeps false in $\mathcal{C}(\mathcal{H})$. We now show the optimality of Theorem 4.5, by proving that we cannot extend it to polynomial symbols of degree 3.

Corollary 5.8. *Let $\varphi_0 = c_0 s + \psi_0 \in \mathcal{G}$ with positive characteristic. Assume that ψ_0 and ψ_0' are bounded on $\mathbb{C}_+$ and that there exist $C > 0$, $k \geq 1$, $t_0 \in \mathbb{R}$ such that, for all $s \in \mathbb{C}_+$,*

$$|\operatorname{Im}(\psi_0(s)) - t_0| \leq C\big(\operatorname{Re}(\psi_0(s))\big)^{1/k}.$$

Let $\delta > 0$ and $\varphi_1 = c_0 s + \psi_0 + \delta \psi_0^{k+1}$. Then, for $\delta > 0$ sufficiently small, $\varphi_1 \in \mathcal{G}$ and $C_{\varphi_0} - C_{\varphi_1}$ is compact.

Proof. The proof follows that of Corollary 5.6. With the same computations, still assuming $t_0 = 0$, one now gets

$$|\varphi_0(s) - \varphi_1(s)|^2 \lesssim \big(\min(\operatorname{Re}\varphi_0(s), \operatorname{Re}\varphi_1(s))\big)^{2(k+1)/k}$$

which is enough to ensure that $|\varphi_0 - \varphi_1| = o\big(\min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1)\big)$ as $\min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1)$ tends to zero. We conclude by applying Theorem 5.5 since $\varphi_1' - \varphi_0' = \delta(k+1)\psi_0'\psi_0^k$

and since the assumptions imply that ψ_0 tends to zero as $\min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1)$ tend to zero. □

Example 5.9. Let $\varphi_0(s) = s + (1 - 2^{-s})$ and $\varphi_1(s) = s + (1 - 2^{-s}) + \delta(1 - 2^{-s})^3$ for some sufficiently small $\delta > 0$. Then $C_{\varphi_1} - C_{\varphi_0}$ is compact whereas neither C_{φ_0} nor C_{φ_1} is compact. In particular, Theorem 4.5 is optimal.

5.3. **The case of zero characteristic.**

5.3.1. *Weighted composition operators from Bergman spaces to Hardy spaces.* In this section, we give a sufficient condition for a weighted composition operator uC_φ to induce a bounded, respectively compact, operator from the Bergman space $A^2_1(\mathbb{D})$ to the Hardy space $H^2(\mathbb{D})$. Here $A^2_1(\mathbb{D})$ is the set of analytic functions f on the disk such that

$$\|f\|^2_{A^2_1(\mathbb{D})} := \int_{\mathbb{D}} |f(z)|^2(1 - |z|^2)dz < +\infty.$$

The continuity of $uC_\varphi : A^2_1(\mathbb{D}) \to H^2(\mathbb{D})$ has been characterized in terms of Carleson measures in [23] or using integral operators in [29]. Inspired by [26], we shall deduce from the results of [23] a sufficient condition which will be easy to testify. Let us introduce some terminology. For a point ξ in the boundary of the unit disk and $\delta > 0$, we define the Carleson window $S(\xi, \delta) = \{z \in \mathbb{D} : |z - \xi| < \delta\}$.

Let φ be an analytic self-map of $\mathbb{D}$ and let u be a bounded analytic function on $\mathbb{D}$. We define a positive Borel measure $|u|^2 m_1\varphi^{-1}$ on the closed unit disk $\overline{\mathbb{D}}$ by assigning to each Borel set E the value:

$$|u|^2 m_1\varphi^{-1}(E) = \int_{\varphi^{-1}(E)} |u(\xi)|^2 dm_1(\xi),$$

where we still denote by φ and u the radial limit functions of φ and u and where m_1 is the normalized Lebesgue measure on $\mathbb{T}$.

The following lemma is [23, Proposition 2].

Lemma 5.10. *Let φ be an analytic self-map of $\mathbb{D}$ and let u be a bounded analytic function on $\mathbb{D}$ not identically equal to zero. Then $uC_\varphi : A^2_1 \to H^2$ is bounded if and only if φ has radial limits of modulus strictly less than 1 almost everywhere and*

$$\sup_{S(\xi,\delta)} \frac{|u|^2 m_1\varphi^{-1}(S(\xi, \delta))}{\delta^3} < +\infty.$$

Moreover

$$\|uC_\varphi\|_{A^2_1 \to H^2} \approx \left(\sup_{S(\xi,\delta)} \frac{|u|^2 m_1\varphi^{-1}(S(\xi, \delta))}{\delta^3} \right)^{1/2}.$$

If $\sup_{S(\xi,\delta)} |u|^2 m_1\varphi^{-1}(S(\xi, \delta))/\delta^3$ goes to zero uniformly in ξ as δ tends to zero, then uC_ϕ is a compact operator from A^2_1 to H^2.

Remark 5.11. In the statement of [23], the exponent 1/2 in the last display does not appear. Nevertheless a look at the proof as well as homogeneity considerations show that it has to appear.

Corollary 5.12. *Let φ be an analytic self-map of $\mathbb{D}$ which has radial limits of modulus strictly less than 1 almost everywhere and let u be a bounded analytic function on $\mathbb{D}$ not identically equal to zero. Then*

$$\|uC_\varphi\|_{A^2_1\to H^2} \lesssim \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2} \sup_{\xi\in\mathbb{T}} \frac{|u(\xi)|}{1-|\varphi(\xi)|}.$$

If moreover

$$\sup_{|\varphi(\xi)|\geq 1-\delta} \frac{|u(\xi)|}{1-|\varphi(\xi)|} \xrightarrow{\delta\to 0} 0$$

then uC_φ is a compact operator from A^2_1 to H^2.

Proof. We simply write, for $\xi\in\mathbb{T}$ and $\delta>0$,

$$\begin{aligned}\frac{|u|^2 m_1\varphi^{-1}(S(\xi,\delta))}{\delta^3} &= \frac{1}{\delta^3}\int_{\varphi^{-1}(S(\xi,\delta))} |u|^2 dm_1\\ &\leq \left(\sup_{\xi_0\in\mathbb{T}} \frac{|u(\xi_0)|}{1-|\varphi(\xi_0)|}\right)^2 \times \frac{1}{\delta^3}\int_{\varphi^{-1}(S(\xi,\delta))} (1-|\varphi|)^2 dm_1\\ &\leq \left(\sup_{\xi_0\in\mathbb{T}} \frac{|u(\xi_0)|}{1-|\varphi(\xi_0)|}\right)^2 \times \frac{1}{\delta} m_1\big(\varphi^{-1}(S(\xi,\delta))\big).\end{aligned}$$

It follows from the Carleson measure characterisation of the continuity of composition operators on $H^2(\mathbb{D})$ that $\frac{1}{\delta}m_1\big(\varphi^{-1}(S(\xi,\delta))\big) \lesssim \|C_\varphi\|^2_{H^2\to H^2} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}$. □

We translate this statement to get a result on the invariant Hardy space $H^2_i(\mathbb{C}_+)$.

Lemma 5.13. *Let ϕ be an analytic self-map of $\mathbb{C}_+$ which has radial limits of positive real part almost everywhere and let U be a bounded analytic function on $\mathbb{C}_+$ not identically equal to zero. Then for all $F\in H^2_i(\mathbb{C}_+)$,*

$$\|U\cdot F'\circ\phi\|_{H^2_i(\mathbb{C}_+)} \lesssim \left(\frac{|1+\phi(1)|^2}{\operatorname{Re}\phi(1)}\right)^{1/2} \sup_{t\in\mathbb{R}} \frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)} \|F\|_{H^2_i(\mathbb{C}_+)}.$$

If, moreover,

$$\sup\left\{\frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)} : \frac{\operatorname{Re}\phi(it)}{|1+\phi(it)|^2}\leq\delta\right\} \xrightarrow{\delta\to 0} 0$$

then the operator $F\mapsto U\cdot F'\circ\phi$ is a compact operator of $H^2_i(\mathbb{C}_+)$.

Proof. Let ω be the Cayley map, $\omega(z)=\frac{1+z}{1-z}$ and let us set $f=F\circ\omega$, $u=U\circ\omega$, $\varphi=\omega^{-1}\circ\phi\circ\omega$. Observe that $UF'\circ\phi = g\circ\omega^{-1}$ where

$$g = u\times(\omega^{-1})'\circ\omega\times f'\circ\varphi = \frac{(z-1)^2}{2}u\times f'\circ\varphi.$$

A small computation shows that

$$\sup_{\xi\in\mathbb{T}} \frac{|u(\xi)|}{1-|\varphi(\xi)|^2} \approx \sup_{t\in\mathbb{R}} \frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)}$$

and

$$\frac{1+|\varphi(0)|}{1-|\varphi(0)|} \approx \frac{|1+\phi(1)|^2}{\operatorname{Re}\phi(1)}.$$

Therefore

$$\begin{aligned}\|U\cdot F'\circ\phi\|_{H_i^2(\mathbb{C}_+)} &= \|g\|_{H^2(\mathbb{D})}\\ &\lesssim \left(\frac{|1+\phi(1)|^2}{\operatorname{Re}\phi(1)}\right)^{1/2}\sup_{t\in\mathbb{R}}\frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)}\|f'\|_{A_1^2(\mathbb{D})}\\ &\lesssim \left(\frac{|1+\phi(1)|^2}{\operatorname{Re}\phi(1)}\right)^{1/2}\sup_{t\in\mathbb{R}}\frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)}\|f\|_{H^2(\mathbb{D})}\\ &\lesssim \left(\frac{|1+\phi(1)|^2}{\operatorname{Re}\phi(1)}\right)^{1/2}\sup_{t\in\mathbb{R}}\frac{|U(it)|\times|1+\phi(it)|^2}{\operatorname{Re}\phi(it)}\|F\|_{H_i^2(\mathbb{C}_+)}.\end{aligned}$$

The statement about compactness follows from the second half of Corollary 5.12. □

Remark 5.14. We could improve the statement of Lemma 5.13 by replacing everywhere the term $|U(it)|\times|1+\phi(it)|^2/\operatorname{Re}\phi(it)$ by $|U(it)|\times|1+\phi(it)|^2/(\operatorname{Re}\phi(it)\times(1+t^2))$. Indeed, we apply Corollary 5.12 not to u but to $(z-1)^2u/2$ which allows this better inequality. Nevertheless, since we plan to apply this result to vertical limits, with a uniform bound, the term $1+t^2$ would not be helpful to us.

5.3.2. *Applications to connected components and compact differences.*

Theorem 5.15. *Let $\varphi_0,\ \varphi_1\in\mathcal{G}$ with $\operatorname{char}(\varphi_0)=\operatorname{char}(\varphi_1)=0$. Assume that there exists $C>0$ such that $|\varphi_0-\varphi_1|\le C\min\left(\frac{\operatorname{Re}\varphi_0-1/2}{|1+\varphi_0|^2},\frac{\operatorname{Re}\varphi_1-1/2}{|1+\varphi_1|^2}\right)$. Then C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$. If, moreover, $|\varphi_0-\varphi_1|=o\left(\min\left(\frac{\operatorname{Re}\varphi_0-1/2}{|1+\varphi_0|^2},\frac{\operatorname{Re}\varphi_1-1/2}{|1+\varphi_1|^2}\right)\right)$ as $\min\left(\frac{\operatorname{Re}\varphi_0-1/2}{|1+\varphi_0|^2},\frac{\operatorname{Re}\varphi_1-1/2}{|1+\varphi_1|^2}\right)$ tends to zero, then $C_{\varphi_0}-C_{\varphi_1}$ is compact.*

Proof. We start as in the proof of Theorem 5.5 but now we choose the measure $d\mu=\frac{dt}{1+t^2}$. Taking into account that the symbols have zero characteristic, we know that for all $\lambda,\ \lambda'\in[0,1]$,

$$\begin{aligned}\left\|C_{\varphi_\lambda}(f)-C_{\varphi_{\lambda'}}(f)\right\|^2 \ \le\ |\lambda'-\lambda|^2\int_\lambda^{\lambda'}\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}|(\varphi_1)_\chi(it)-(\varphi_0)_\chi(it)|^2\\ \times|f'((\varphi_r)_\chi(it))|^2\frac{dt}{1+t^2}dm(\chi)dr.\end{aligned}$$

Now $\mathcal{H}$ embeds into $H_i^2(\mathbb{C}_+)$ via the following inequality ([11]): for all $g\in\mathcal{H}$,

$$\|g\|_{H_i^2(\mathbb{C}_{1/2})}\le\sqrt{2}\|g\|_{\mathcal{H}}.$$

Therefore we just need to prove that there exists $A>0$ such that, for all $\chi\in\mathbb{T}^\infty$, for all $r\in[0,1]$ and for all $g\in H_i^2(\mathbb{C}_{1/2})$,

$$\|(\varphi_1-\varphi_0)_\chi\cdot g'\circ(\varphi_r)_\chi\|_{H_i^2(\mathbb{C}_+)}\le A\|g\|_{H_i^2(\mathbb{C}_{1/2})}.$$

For a fixed $a>0$, the condition $\frac{\operatorname{Re}z-1/2}{|1+z|^2}\ge a$ is equivalent to say that z belongs to some disk. Since every disk is convex, we get that for all $r\in[0,1]$,

$$|\varphi_0-\varphi_1|\le C\frac{\operatorname{Re}\varphi_r-1/2}{|1+\varphi_r|^2}.$$

By vertical translation, we also have

$$|(\varphi_0)_\chi - (\varphi_1)_\chi| \leq C \frac{\mathrm{Re}\,(\varphi_r)_\chi - 1/2}{|1 + (\varphi_r)_\chi|^2}.$$

Moreover, since φ_0 and φ_1 converge uniformly on $\mathbb{C}_{1/2}$, there exists some compact subset $K \subseteq \mathbb{C}_{1/2}$ such that, for all $\chi \in \mathbb{T}^\infty$ and all $r > 0$, $(\varphi_r)_\chi(1)$ belongs to K. Therefore, Lemma 5.13 gives the desired result. The proof of the compactness of $C_{\varphi_0} - C_{\varphi_1}$ follows the same lines and is left to the reader. □

We then get a result similar to Corollary 5.6 for composition operators with zero characteristic.

Corollary 5.16. *Let $\varphi \in \mathcal{G}$ with zero characteristic and such that $|\varphi|$ is bounded. Assume that there exist $C > 0$, $k \geq 1$, $t_0 \in \mathbb{R}$ such that, for all $s \in \mathbb{C}_+$,*

$$|\mathrm{Im}\,(\varphi(s)) - t_0| \leq C \left(\mathrm{Re}\,(\varphi(s)) - \frac{1}{2}\right)^{1/k}.$$

Then C_φ is not isolated in $\mathcal{C}(\mathcal{H})$.

Proof. We set $\varphi_0 = \varphi$ and let, for $\delta > 0$, $\varphi_1 = \varphi + \delta\left(\varphi - \frac{1}{2}\right)^k$. As in the proof of Corollary 5.6, $\varphi_1 \in \mathcal{G}$ provided δ is small enough. Moreover, the assumptions of Corollary 5.16 are satisfied since

$$\begin{aligned}
|\varphi_0(s) - \varphi_1(s)|^2 &= \delta^2 \left|\varphi(s) - \frac{1}{2}\right|^{2k} \\
&\lesssim \left(\mathrm{Re}\,\varphi(s) - \frac{1}{2}\right)^{2k} + \left(\mathrm{Im}\,\varphi(s)\right)^{2k} \\
&\lesssim \left(\mathrm{Re}\,\varphi(s) - \frac{1}{2}\right)^{2} \\
&\lesssim \frac{\left(\mathrm{Re}\,\varphi(s) - \frac{1}{2}\right)^2}{|\varphi + 1|^4}
\end{aligned}$$

since $|\varphi|$ is bounded. □

We deduce from this result that the Shapiro-Sundberg conjecture remains false for composition operators with zero characteristic.

Example 5.17. Let $\varphi_0(s) = \frac{1}{2} + (1 - 2^{-s})$ and $\varphi_1(s) = \varphi_0(s) + \delta(1 - 2^{-s})^2$. Then, for $\delta > 0$ sufficiently small, $\varphi_0,\ \varphi_1 \in \mathcal{G}$, C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$ but $C_{\varphi_0} - C_{\varphi_1}$ is not compact.

Proof. The first half of the statement comes from Corollary 5.16 (and its proof). The second half is a consequence of Theorem 4.5 since neither φ_0 nor φ_1 is compact by [7, Theorem 2]. □

5.4. **Order of contact, connected components and compact differences.** In this final section, we show how the conditions of Theorem 5.5 can be satisfied from conditions on the derivatives of $\mathcal{B}\psi$. We need a notion of order of contact, which is maybe less natural than in one variable (see [21]).

Let $\varphi \in \mathcal{G}$ be a Dirichlet polynomial symbol with $\mathrm{char}\,(\varphi) \geq 1$. Let d be the smallest integer such that

$$\varphi(s) = c_0 s + \sum_{p^+(n) \leq p_d} c_n n^{-s}.$$

Then $\mathcal{B}\psi$ maps $\mathbb{D}^d$ into $\mathbb{C}_+$. Denote by $J_\varphi : \mathbb{R}^d \to [0, +\infty[$, $J_\varphi(\theta_1, \dots, \theta_d) = \mathrm{Re}\,\mathcal{B}\psi(e^{i\theta_1}, \cdots, e^{i\theta_d})$. Recall also that $\Gamma(\mathcal{B}\psi)$ is defined by $\Gamma(\mathcal{B}\psi) = \{z \in \mathbb{T}^d : \mathrm{Re}\,\mathcal{B}\psi(z) = 0\}$ (this set is nonempty provided φ has unrestricted range). Let $z = (e^{i\theta_1^0}, \dots, e^{i\theta_d^0}) \in \Gamma(\mathcal{B}\psi)$. Then J_φ attains its minimum at $(\theta_1^0, \dots, \theta_d^0)$.

Definition 5.18. We say that φ has a **Dirichlet contact of order** $2n$ **at** $z = (e^{i\theta_1^0}, \dots, e^{i\theta_d^0})$ provided there exists $c > 0$ and a neighbourhood $\mathcal{U}$ of $(\theta_1^0, \dots, \theta_d^0)$ in $\mathbb{R}^d$ such that, for all $(\theta_1, \dots, \theta_d) \in \mathcal{U}$,

$$J_\varphi(\theta_1, \dots, \theta_d) \geq c\big((\theta_1 - \theta_1^0)^2 + \cdots + (\theta_d - \theta_d^0)^2\big)^n.$$

Let us explain this terminology. Without loss of generality, we may assume that $\theta_1^0 = \cdots = \theta_d^0 = 0$ and $\mathcal{B}\psi(1, \dots, 1) = 0$. The Taylor expansion of $\mathcal{B}\psi$ at z may be written

$$\mathcal{B}\psi(e^{i\theta_1}, \dots, e^{i\theta_d}) = \sum_{j=1}^{d} a_j(1 - e^{i\theta_j}) + O\left(\theta_1^2 + \cdots + \theta_d^2\right)$$

with each $a_j \geq 0$ (see [7]). In particular, there exists a neighbourhood $\mathcal{V}$ of $(0, \dots, 0)$ in $\mathbb{R}^d$ such that, for all $(\theta_1, \dots, \theta_d) \in \mathcal{V}$,

$$|\mathrm{Im}\,\mathcal{B}\psi(e^{i\theta_1}, \dots, e^{i\theta_d})| \leq C(\theta_1^2 + \cdots + \theta_d^2)^{1/2}.$$

Therefore, for $(\theta_1, \dots, \theta_d) \in \mathcal{U} \cap \mathcal{V}$,

$$|\mathrm{Im}\,\mathcal{B}\psi(e^{i\theta_1}, \dots, e^{i\theta_d})|^{2n} \lesssim \mathrm{Re}\,\mathcal{B}\psi(e^{i\theta_1}, \dots, e^{i\theta_d}).$$

Hence, if φ has a Dirichlet contact of order $2n$ at z, the Bohr lift $\mathcal{B}\psi$ has a contact of order $2n$ at z.

For $w \in \mathbb{T}^d$ and $\sigma > 0$, we denote by $w_\sigma = (2^{-\sigma} w_1, \dots, p_d^{-\sigma} w_d)$.

Lemma 5.19. *Let $\varphi_0, \varphi_1 \in \mathcal{G}$ be Dirichlet polynomial symbols with* $\mathrm{char}\,(\varphi_0) = \mathrm{char}\,(\varphi_1) \geq 1$. *Let $z \in \Gamma(\mathcal{B}\psi_0) \cap \Gamma(\mathcal{B}\psi_1)$ such that*

- *$\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;*
- *φ_0 and φ_1 have a Dirichlet contact of order $2n$ at z;*
- *for $|\alpha| \leq 2n - 1$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$.*

Then there exist a neighbourhood $\mathcal{U}$ of z in $\mathbb{T}^d$, $\sigma_0 > 0$ and $C > 0$ such that, for all $w \in \mathcal{U}$, for all $\sigma \in [0, \sigma_0]$,

$$|\mathcal{B}\psi_0(w_\sigma) - \mathcal{B}\psi_1(w_\sigma)| \leq C \min\big(\mathrm{Re}\,\mathcal{B}\psi_0(w_\sigma), \mathrm{Re}\,\mathcal{B}\psi_1(w_\sigma)\big).$$

Moreover, if we assume that $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$ for $|\alpha| \leq 2n$, then

$$|\mathcal{B}\psi_0(w_\sigma) - \mathcal{B}\psi_1(w_\sigma)| = o\left(\min\left(\mathrm{Re}\,\mathcal{B}\psi_0(w_\sigma), \mathrm{Re}\,\mathcal{B}\psi_1(w_\sigma)\right)\right)$$

as $w_\sigma \to z$.

Proof. We may assume that $z = \mathbf{e}$ and that $\mathcal{B}\psi_0(\mathbf{e}) = \mathcal{B}\psi_1(\mathbf{e}) = 0$. We set $m = 2n-1$ or $m = 2n$, following the assumptions that we have made. These assumptions imply that, looking at some $w = (e^{i\theta_1}, \dots, e^{i\theta_d})$,

$$\begin{aligned}|\mathcal{B}\psi_0(w_\sigma) - \mathcal{B}\psi_1(w_\sigma)| &= |\mathcal{B}\psi_0(w_\sigma) - 1 + 1 - \mathcal{B}\psi_1(w_\sigma)| \\ &= \Bigg| \sum_{|\alpha|=m+1} a_\alpha (1 - 2^{-\sigma} e^{i\theta_1})^{\alpha_1} \cdots (1 - p_d^{-\sigma} e^{i\theta_d})^{\alpha_d} \\ &\qquad + o(\|w_\sigma - \mathbf{e}\|^{m+1}) \Bigg| \\ &\le C_1 \left(\sigma^{m+1} + |\theta_1|^{m+1} + \cdots + |\theta_d|^{m+1}\right)\end{aligned}$$

for some constant $C_1 > 0$.

On the other hand, let $i = 0, 1$ and let us write the Taylor expansion of $\mathcal{B}\psi_i$ at w as

$$\mathcal{B}\psi_i(w_\sigma) = \mathcal{B}\psi_i(w) + \sum_{j=1}^{d} a_{i,j}(w)(1 - p_j^{-\sigma}) + F_i(w, \sigma)$$

where the functions $a_{i,j}$ are continuous and $|F_i(w, \sigma)| \le C_2 \sigma^2$. Expanding $1 - p_j^{-\sigma}$, this can been rewritten as

$$\mathcal{B}\psi_i(w_\sigma) = \mathcal{B}\psi_i(w) + \left(\sum_{j=1}^{d} a_{i,j}(w) \log p_j\right) \sigma + G_i(w, \sigma)$$

with $|G_i(w, \sigma)| \le C_3 \sigma^2$. Now, we know (see [7]) that $a_{i,j}(\mathbf{e}) \ge 0$ and $a_{i,j_0}(\mathbf{e}) > 0$ for at least one $j_0 \in \{1, \dots, d\}$. From the continuity of the functions $a_{i,j}$, we deduce the existence of a neighbourhood $\mathcal{U}$ of $\mathbf{e}$ in $\mathbb{T}^d$, of $C_4 > 0$ and of $\sigma_0 > 0$ such that, for all $w \in \mathcal{U}$ and for all $\sigma \in [0, \sigma_0]$,

$$\begin{aligned}\operatorname{Re} \mathcal{B}\psi_i(w_\sigma) &\ge \operatorname{Re} \mathcal{B}\psi_i(w) + C_4 \sigma \\ &\ge C_4 \sigma + C_5(\theta_1^{2n} + \cdots + \theta_d^{2n})\end{aligned}$$

where the last inequality holds because φ_i has a Dirichlet contact of order $2n$ at $\mathbf{e}$. This yields the following lemma. □

Lemma 5.20. *Let* $\varphi_0, \varphi_1 \in \mathcal{G}$ *be Dirichlet polynomial symbols with* $\operatorname{char}(\varphi_0) = \operatorname{char}(\varphi_1) \ge 1$. *Assume that* $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ *and that, for all* $z \in \Gamma(\mathcal{B}\psi_0)$, *there exists* $n \in \mathbb{N}$ *such that*

- $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;
- φ_0 *and* φ_1 *have a Dirichlet contact of order* $2n$ *at* z;
- *for* $|\alpha| \le 2n - 1$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$.

Then there exists $C > 0$ *such that, for all* $w \in \mathbb{T}^d$ *and all* $\sigma \ge 0$,

$$|\mathcal{B}\psi_0(w_\sigma) - \mathcal{B}\psi_1(w_\sigma)| \le C \min\left(\operatorname{Re} \mathcal{B}\psi_0(w_\sigma), \operatorname{Re} \mathcal{B}\psi_1(w_\sigma)\right).$$

Moreover, if we assume that $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$ *for* $|\alpha| \le 2n$, *then*

$$|\mathcal{B}\psi_0(w_\sigma) - \mathcal{B}\psi_1(w_\sigma)| = o\left(\min\left(\operatorname{Re} \mathcal{B}\psi_0(w_\sigma), \operatorname{Re} \mathcal{B}\psi_1(w_\sigma)\right)\right)$$

as $\min\big(\operatorname{Re}\mathcal{B}\psi_0(w_\sigma), \operatorname{Re}\mathcal{B}\psi_1(w_\sigma)\big)$ *goes to zero.*

Proof. This follows from the previous lemma and a compactness argument. □

Corollary 5.21. *Let* $\varphi_0, \varphi_1 \in \mathcal{G}$ *be Dirichlet polynomial symbols with* $\operatorname{char}(\varphi_0) = \operatorname{char}(\varphi_1) \geq 1$. *Assume that* $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ *and that, for all* $z \in \Gamma(\mathcal{B}\psi_0)$, *there exists* $n \in \mathbb{N}$ *such that*

- $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$;
- φ_0 *and* φ_1 *have a Dirichlet contact of order* $2n$ *at* z;
- *for* $|\alpha| \leq 2n-1$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$.

Then C_{φ_0} *and* C_{φ_1} *belong to the same component of* $\mathcal{C}(\mathcal{H})$. *Moreover, if we assume that* $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$ *for* $|\alpha| \leq 2n$, *then* $C_{\varphi_0} - C_{\varphi_1}$ *is compact.*

Proof. The previous lemma implies that the assumptions of Theorem 5.5 are satisfied. □

Combining Theorem 4.4 and the previous corollary, we finally get:

Theorem 5.22. *Let* $\varphi_0, \varphi_1 \in \mathcal{G}$ *be Dirichlet polynomial symbols with* $\operatorname{char}(\varphi_0) = \operatorname{char}(\varphi_1) \geq 1$. *Assume that for all* $z \in \Gamma(\mathcal{B}\psi_0)$ *(resp.* $z \in \Gamma(\mathcal{B}(\psi_1))$*),* φ_0 *(resp.* φ_1*) has Dirichlet order of contact* 2 *at* z. *Then the following assertions are equivalent:*

(i) $C_{\varphi_0} - C_{\varphi_1}$ *is compact.*
(ii) $\Gamma(\mathcal{B}\psi_0) = \Gamma(\mathcal{B}\psi_1)$ *and, for all* $z \in \Gamma(\mathcal{B}\psi_0)$, $\mathcal{B}\psi_0(z) = \mathcal{B}\psi_1(z)$, $\partial_\alpha \mathcal{B}\psi_0(z) = \partial_\alpha \mathcal{B}\psi_1(z)$ *for* $|\alpha| \leq 2$.

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