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# WHEN THE ROBIN INEQUALITY DOES NOT HOLD

FRANK VEGA

ABSTRACT. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality  $\sigma(n) < e^\gamma \times n \times \log \log n$  holds for all sufficiently large  $n$ , where  $\sigma(n)$  is the sum-of-divisors function and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all  $n > 5040$  if and only if the Riemann Hypothesis is true. Let  $n > 5040$  be  $n = r \times q$ , where  $q$  denotes the largest prime factor of  $n$ . If  $n > 5040$  is the smallest number such that Robin inequality does not hold, then we show the following inequality is also satisfied:  $\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2$ .

## 1. INTRODUCTION

As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$  [Cho+07]:

$$\sum_{d|n} d.$$

Define  $f(n)$  to be  $\frac{\sigma(n)}{n}$ . Say  $\text{Robins}(n)$  holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant  $\gamma$  is the Euler-Mascheroni constant, and  $\log$  is the natural logarithm. The importance of this property is:

**Theorem 1.1.** [RH] *If  $\text{Robins}(n)$  holds for all  $n > 5040$ , then the Riemann Hypothesis is true [Rob84].*

There are several known results about the possible counterexamples of  $\text{Robins}(n)$  when  $n > 5040$  [Cho+07]. In addition, we show that

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**Theorem 1.2.** [\[counterexample\]](#) *Let  $n > 5040$  be  $n = r \times q$ , where  $q$  denotes the largest prime factor of  $n$ . If  $n > 5040$  is the smallest number such that  $\text{Robins}(n)$  does not hold, then*

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

## 2. SOME USEFUL LEMMAS

The following lemma is a very helpful inequality:

**Lemma 2.1.** [\[ineq\]](#) *We have*

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}$$

where  $y = 1 - x$ .

*Proof.* We know  $1 + x \leq e^x$  [Koz21]. Therefore,

$$\frac{x}{1-x} \leq \frac{e^{x-1}}{1-x} = \frac{1}{(1-x) \times e^{1-x}} = \frac{1}{y \times e^y}.$$

However, for every real number  $y \in \mathbb{R}$  [Koz21]:

$$y \times e^y \geq y + y^2 + \frac{y^3}{2}$$

and this can be transformed into

$$\frac{1}{y \times e^y} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

Consequently, we show

$$\frac{x}{1-x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.$$

□

Here, it is another practical result:

**Lemma 2.2.** [\[prop\]](#) *Suppose that  $n > 5040$  and let  $n = r \times q$ , where  $q$  denotes the largest prime factor of  $n$ . We have*

$$f(n) \leq \left(1 + \frac{1}{q}\right) \times f(r).$$

*Proof.* Suppose that  $n$  is the form of  $m \times q^k$  where  $q \nmid m$  and  $m$  and  $k$  are natural numbers. We have that

$$f(n) = f(m \times q^k) = f(m) \times f(q^k)$$

since  $f$  is multiplicative and  $m$  and  $q$  are coprimes [Voj20]. However, we know that

$$f(q^k) \leq f(q^{k-1}) \times f(q)$$

because of we notice that  $f(a \times b) \leq f(a) \times f(b)$  when  $a, b \geq 2$  [Voj20]. In this way, we obtain that

$$f(q^{k-1}) \times f(q) = f(q^{k-1}) \times \left(1 + \frac{1}{q}\right)$$

according to the value of  $f(q)$  [Voj20]. In addition, we analyze that

$$f(m) \times f(q^{k-1}) = f(m \times q^{k-1}) = f(r)$$

because  $f$  is multiplicative and  $m$  and  $q$  are coprimes [Voj20]. Finally, we obtain that

$$f(n) = f(m) \times f(q^k) \leq f(m) \times f(q^{k-1}) \times f(q) = f(r) \times \left(1 + \frac{1}{q}\right)$$

and as a consequence, the proof is completed.  $\square$

### 3. PROOF OF MAIN THEOREM

**Theorem 3.1.** *Let  $n > 5040$  be  $n = r \times q$ , where  $q$  denotes the largest prime factor of  $n$ . If  $n > 5040$  is the smallest number such that  $\text{Robins}(n)$  does not hold, then*

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

*Proof.* Suppose that  $n$  is the smallest integer exceeding 5040 that does not satisfy the Robin's inequality. Let  $n = r \times q$ , where  $q$  denotes the largest prime factor of  $n$ . In this way, the following inequality

$$f(n) \geq e^\gamma \times \log \log n$$

should be true. We know that

$$\left(1 + \frac{1}{q}\right) \times f(r) \geq f(q \times r) \geq f(n) \geq e^\gamma \times \log \log n$$

due to lemma 2.2 [prop]. Besides, this shows that

$$\left(1 + \frac{1}{q}\right) \times e^\gamma \times \log \log r > e^\gamma \times \log \log n$$

should be true as well. Certainly, if  $n$  is the smallest counterexample exceeding 5040 of the Robin's inequality, then  $\text{Robins}(r)$  holds [Cho+07]. That is the same as

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log \log n.$$

We have that

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log(\log r + \log q)$$

where we notice that  $\log(a + c) = \log a + \log\left(1 + \frac{c}{a}\right)$ . This follows

$$\left(1 + \frac{1}{q}\right) \times \log \log r > \log \log r + \log\left(1 + \frac{\log q}{\log r}\right)$$

which is equal to

$$(1 + q) \times \log \log r > q \times \log \log r + q \times \log\left(1 + \frac{\log q}{\log r}\right)$$

and thus,

$$\log \log r > q \times \log\left(1 + \frac{\log q}{\log r}\right).$$

This implies that

$$\begin{aligned} \frac{\log \log r}{\log\left(1 + \frac{\log q}{\log r}\right)} &= \\ \frac{\log \log r}{\log \frac{\log r + \log q}{\log r}} &= \\ \frac{\log \log r}{\log \frac{\log n}{\log r}} &= \\ \frac{\log \log r}{\log \log n - \log \log r} &= \\ \frac{\log \log r}{\log \log n \times \left(1 - \frac{\log \log r}{\log \log n}\right)} &= \\ \frac{\frac{\log \log r}{\log \log n}}{\left(1 - \frac{\log \log r}{\log \log n}\right)} &> q \end{aligned}$$

should be true. If we assume that  $y = 1 - \frac{\log \log r}{\log \log n}$ , then we analyze that

$$\frac{1}{y + y^2 + \frac{y^3}{2}} \geq \frac{\frac{\log \log r}{\log \log n}}{\left(1 - \frac{\log \log r}{\log \log n}\right)}$$

because of lemma 2.1 [\[ineq\]](#). As result, we have that

$$\frac{1}{y + y^2 + \frac{y^3}{2}} > q$$

and therefore,

$$\frac{1}{1 + y + \frac{y^2}{2}} > q \times y.$$

Since we have

$$1 + y + \frac{y^2}{2} > 1$$

then

$$\frac{1}{1 + y + \frac{y^2}{2}} < 1.$$

Consequently, we obtain that

$$1 > q \times y$$

which is the same as

$$e > e^{q \times y}.$$

Because of we have that  $1 + y \leq e^y$  [Koz21], then

$$e > e^{q \times y} \geq (1 + y)^q = \left(2 - \frac{\log \log r}{\log \log n}\right)^q$$

that is

$$\sqrt[q]{e} > \left(2 - \frac{\log \log r}{\log \log n}\right)$$

and finally,

$$\sqrt[q]{e} + \frac{\log \log r}{\log \log n} > 2.$$

□

#### REFERENCES

- [Cho+07] YoungJu Choie et al. “On Robin’s criterion for the Riemann hypothesis”. In: *Journal de Théorie des Nombres de Bordeaux* 19.2 (2007), pp. 357–372. DOI: 10.5802/jtnb.591.
- [Koz21] László Kozma. *Useful Inequalities*. [http://www.lkozma.net/inequalities\\_cheat\\_sheet/ineq.pdf](http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf). Accessed on 2021-09-18. 2021.
- [Rob84] Guy Robin. “Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann”. In: *J. Math. pures appl* 63.2 (1984), pp. 187–213.
- [Voj20] Robert Vojak. “On numbers satisfying Robin’s inequality, properties of the next counterexample and improved specific bounds”. In: *arXiv preprint arXiv:2005.09307* (2020).

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