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# The asymmetric splitting of the wave operator

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## Abstract

In this article, we present a new way to split the wave operator asymmetrically. We find two solutions of minimal dimension 2 involving the Pauli matrices and leading to new first-order wave equations. The physical interpretation of these new wave equations in terms of the weak interaction is discussed.

*Keywords:* Dirac equation, wave operator, asymmetry

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## 1. Introduction

In his paper [1], Dirac presented the relativistic quantum equation of the electron in 1928. Since then, this equation named after him has been widely used in physics to describe electromagnetism, see for instance [2]. In our article, we go back to the origin of the Dirac equation for the electron in order to see if there is another way to split the wave operator also called the d'Alembertian. We found a new way to do this which is asymmetrical in a sense defined below. There are two solutions of minimal dimension 2 involving the Pauli matrices for the asymmetric splitting of the wave equation and they lead to new first-order wave equations. The physical interpretation of these new first-order equations in terms of the weak interaction is discussed.

The article is organized as follows. After recalling the symmetric splitting of the wave equation at the beginning of Section 2, we develop the asymmetric splitting of the wave equation and provide two solutions of minimal dimension 2 involving the Pauli matrices. In Section 3, we give the new first-order equations associated with the asymmetric splitting of the wave equation and we discuss the physical interpretation of these equations in terms of the weak interaction. The problem of Lorentz invariance of the solutions of the new first-order equations is tackled in Section 4. Finally, a conclusion is addressed in Section 5.

## 2. The wave operator

### 2.1. Symmetric splitting

When Dirac looked for the relativistic quantum equation for the electron, he looked for the square root of the wave operator defined by

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

in order to obtain a first-order wave equation. He split the wave operator symmetrically as follows

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \left( A_1 \partial_x + A_2 \partial_y + A_3 \partial_z + \frac{i}{c} A_4 \partial_t \right) \left( A_1 \partial_x + A_2 \partial_y + A_3 \partial_z + \frac{i}{c} A_4 \partial_t \right) \quad (1)$$

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under the rules

$$\begin{aligned} A_j A_k + A_k A_j &= 0, & 1 \leq j, k \leq 4, j \neq k \\ A_j^2 &= I, & 1 \leq j \leq 4. \end{aligned} \quad (2)$$

The solution of minimal dimension 4 for the symmetric splitting of the wave operator (1) under the rules (2) is given by the following  $4 \times 4$  Dirac matrices

$$A_1 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad (3)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the  $2 \times 2$  Pauli matrices. It is well known that it is not possible to build  $n \times n$  matrices satisfying the rules (2) below the dimension  $n = 4$ , see for instance [3, Theorem 12.5.2]

## 2.2. Asymmetric splitting

It is possible to split the wave operator asymmetrically as follows

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \left( A_1 \partial_x + A_2 \partial_y + A_3 \partial_z + \frac{i}{c} A_4 \partial_t \right) \left( B_1 \partial_x + B_2 \partial_y + B_3 \partial_z + \frac{i}{c} B_4 \partial_t \right) \quad (4)$$

with the rules

$$\begin{aligned} A_j B_k + A_k B_j &= 0, & 1 \leq j, k \leq 4, j \neq k \\ A_j B_j &= I, & 1 \leq j \leq 4. \end{aligned} \quad (5)$$

### 2.2.1. First solution of minimal dimension 2

Consider the following  $2 \times 2$  matrices

$$A_1 = i\sigma_1, \quad A_2 = i\sigma_2, \quad A_3 = i\sigma_3, \quad A_4 = I_2 \quad (6)$$

and the following  $2 \times 2$  matrices

$$B_1 = -i\sigma_1, \quad B_2 = -i\sigma_2, \quad B_3 = -i\sigma_3, \quad B_4 = I_2. \quad (7)$$

Let us check that the matrices given by (6) and (7) are solution of the asymmetric splitting of the wave operator (4) under the rules (5). For  $1 \leq j, k \leq 3$  with  $j \neq k$ , we have

$$\begin{aligned} A_j B_k + A_k B_j &= (i\sigma_j)(-i\sigma_k) + (i\sigma_k)(-i\sigma_j) \\ &= \sigma_j \sigma_k + \sigma_k \sigma_j \\ &= 0 \end{aligned}$$

and

$$A_j B_j = (i\sigma_j)(-i\sigma_j) = \sigma_j^2 = I_2$$

because  $\sigma_j$  are the Pauli matrices. Moreover, it is clear that  $A_4 B_4 = I_2$ . Finally, for  $1 \leq j \leq 3$ , we have

$$\begin{aligned} A_j B_4 + A_4 B_j &= (i\sigma_j)I_2 + I_2(-i\sigma_j) = 0, \\ A_4 B_j + A_j B_4 &= I_2(-i\sigma_j) + (i\sigma_j)I_2 = 0. \end{aligned}$$

The solution given by the matrices (6) and (7) is the first solution of minimal dimension 2 for the asymmetric splitting of the wave operator (4) under the rules (5). Indeed, it is clear that there is no solution in dimension 1.

### 2.2.2. Second solution of minimal dimension 2

Consider the following  $2 \times 2$  matrices

$$\bar{A}_1 = -i\sigma_1, \quad \bar{A}_2 = -i\sigma_2, \quad \bar{A}_3 = -i\sigma_3, \quad \bar{A}_4 = -I_2 \quad (8)$$

and the following  $2 \times 2$  matrices

$$\bar{B}_1 = i\sigma_1, \quad \bar{B}_2 = i\sigma_2, \quad \bar{B}_3 = i\sigma_3, \quad \bar{B}_4 = -I_2. \quad (9)$$

By using calculations similar to those above, we obtain a second solution of minimal dimension 2 for the asymmetric splitting of the wave operator (4) with opposite signs to those of the previous solution. This is the reason why we use of the notation  $\bar{A}$  and  $\bar{B}$  which is related to the notion of particle and antiparticle.

### 3. First-order wave equations

By using the relativistic expression for the energy, we obtain the following new first-order wave equation of the particle  $A$  associated with the matrices (6)

$$i \left( \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z + \frac{1}{c} I_2 \partial_t \right) \psi = \frac{m_A c}{\hbar} \psi \quad (10)$$

where  $m_A$  is the rest mass of the particle  $A$  and the new first-order wave equation of the particle  $B$  associated with the matrices (7)

$$i \left( -\sigma_1 \partial_x - \sigma_2 \partial_y - \sigma_3 \partial_z + \frac{1}{c} I_2 \partial_t \right) \psi = \frac{m_B c}{\hbar} \psi \quad (11)$$

where  $m_B$  is the rest mass of the particle  $B$ . We may wonder if these new first-order wave equations have a possible physical interpretation. The process of creation of these particles  $A$  and  $B$  is not symmetric in the sense that  $B$  is not the antiparticle of  $A$ . So it could be related to asymmetric physical processes of creation of particles, in the sense that the particles created do not involve particle/antiparticle pairs, as those appearing in the weak interaction. The first-order wave equation (10) with the matrices (6) could be associated with the electron  $e^-$  and the first-order wave equation (11) with the matrices (7) could be associated with the electron antineutrino  $\bar{\nu}_e$  in the  $W^-$  decay process

$$W^- \rightarrow e^- + \bar{\nu}_e.$$

The new first-order wave equation of the particle  $\bar{A}$  associated with the matrices (8) given by

$$-i \left( \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z + \frac{1}{c} I_2 \partial_t \right) \psi = \frac{m_{\bar{A}} c}{\hbar} \psi \quad (12)$$

where  $m_{\bar{A}}$  is the rest mass of the particle  $\bar{A}$ , could be associated with the positron  $e^+$  and the new first-order wave equation of the particle  $\bar{B}$  associated with the matrices (9) given by

$$-i \left( -\sigma_1 \partial_x - \sigma_2 \partial_y - \sigma_3 \partial_z + \frac{1}{c} I_2 \partial_t \right) \psi = \frac{m_{\bar{B}} c}{\hbar} \psi \quad (13)$$

where  $m_{\bar{B}}$  is the rest mass of the particle  $\bar{B}$ , could be associated with the electron neutrino  $\nu_e$  in the  $W^+$  decay process

$$W^+ \rightarrow e^+ + \nu_e.$$

The fact that the antiparticle  $\bar{A}$  of  $A$  and the antiparticle  $\bar{B}$  of  $B$  have also new first-order wave equations reinforces the idea that the two solutions of minimal dimension 2 for the asymmetric splitting of the wave operator could have a physical interpretation related to the weak interaction. Thus, it might help us to understand why the observed asymmetric process of creation of particles for the weak interaction is possible, as the Dirac equation has explained the observed symmetric process of pair productions for the electromagnetism.

#### 4. Lorentz invariance

In the previous sections, we emphasised the fact that the asymmetric splitting of the wave operator has two solutions of minimal dimension 2 contrary to the symmetric splitting whose solution of minimal dimension is of dimension 4. However, the dimension 2 is not well suited to tackle the Lorentz invariance. It is possible to group the previous  $2 \times 2$  matrices given by (6), (7), (8) and (9) into  $4 \times 4$  matrices as follows

$$\mathbf{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} A_2 & 0 \\ 0 & \bar{A}_2 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} A_3 & 0 \\ 0 & \bar{A}_3 \end{pmatrix}, \quad \mathbf{A}_4 = \begin{pmatrix} A_4 & 0 \\ 0 & \bar{A}_4 \end{pmatrix}, \quad (14)$$

and

$$\mathbf{B}_1 = \begin{pmatrix} B_1 & 0 \\ 0 & \bar{B}_1 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} B_2 & 0 \\ 0 & \bar{B}_2 \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} B_3 & 0 \\ 0 & \bar{B}_3 \end{pmatrix}, \quad \mathbf{B}_4 = \begin{pmatrix} B_4 & 0 \\ 0 & \bar{B}_4 \end{pmatrix}. \quad (15)$$

The matrices (14) and (15) provide a solution of dimension 4 for the asymmetric splitting of the wave operator (2) under the rules (5). It is possible to define two first-order wave equations associated with the matrices (14) and with the matrices (15) involving two wave functions  $\psi$  having four components. However, it is not possible to use directly the Pauli theorem in [4] to prove that  $\psi$  are invariant under a Lorentz transformation because the matrices (14) and (15) are different from the Dirac matrices (3) even if they are all made up of Pauli matrices. Thus, this requires further mathematical study.

#### 5. Conclusion

In this article, we develop a new asymmetric splitting of the wave operator. It is possible to obtain two solutions of minimal dimension 2 involving the Pauli matrices which provide new first-order wave equations. The physical interpretation of these equations in terms of the weak interaction seems possible.

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