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An Arrovian impossibility in combining ranking and evaluation*

Justin Kruger[†] M. Remzi Sanver[‡]

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Abstract

In a world where voters not only rank the alternatives but also qualify them as “approved” or “disapproved”, we observe that majoritarianism in preferences and majoritarianism in approvals are logically incompatible. We show that this observation generalises to the following result: every aggregation rule that respects unanimity and decomposes the aggregation of preferences and approvals is dictatorial. Our result implies an incompatibility between ordinal and evaluative approaches to social choice theory under two weak assumptions: respect for unanimity and independence of evaluation of each alternative. We describe possibilities when the latter assumption is relaxed. On the other hand, our impossibility generalises to the case where there are more than the two evaluative levels of “approved” and “disapproved”.

JEL Classifications: D71.

Keywords: preference-approval, majoritarian judgment, approval voting, preference aggregation, evaluation.

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1 Introduction

The traditional approach in social choice theory involves aggregating ordinal preferences over alternatives. We will refer to this as the ranking approach. This can be contrasted with what we call the evaluative approach, which involves aggregating evaluations of the alternatives made by the voters. For example, approval voting, as pioneered by Brams and Fishburn (1978), is evaluative; it affords two possible evaluations for each of the alternatives: approved or disapproved. In this paper we prove an impossibility in combining the ranking and evaluative approaches. Our result implies that there is an incompatibility between the two.

Suppose that a committee is tasked with selecting a candidate for a position that can be left vacant if no suitable candidate is found. The committee has identified the best candidate and considers her suitable, so offers this candidate the position. If this first choice candidate rejects the offer, the position can then be offered to the second best candidate, and so on until the next candidate is considered unsuitable. Overall, the committee’s task can be divided into two: creating a ranking over the candidates— the ordinal part; and determining which candidates are suitable for the position—the evaluative part. It is these two different parts that are the subject of this paper.

In the above task, candidates are individually evaluated as being suitable, or not, for a position. A more neutral presentation of the evaluation part, which does not assign it such a specific meaning, occurs in the preference-approval framework— originally conceived by Brams and Sanver (2009). Let us describe the smallest non-trivial version of this framework. This involves two alternatives, a and b , which are ranked with respect to each other and also individually evaluated as approved or disapproved. The ordering over the two possible evaluations brings about natural restrictions upon how rankings and evaluations should be combined. For example, if a is approved and b is disapproved, it would be unreasonable to also rank the two alternatives as indifferent. A consistent combination of ranking and approvals is known as a preference-approval.

A preference-approval can be represented as a complete preorder over the alternatives that is also equipped with a “zero-line” to demarcate which alternatives are approved. For the two alternatives a and b there are eight possible preference-approvals. These are displayed below. The preference-approvals read from left to right. When the two alternatives are within parenthesis, the voter is indifferent between them. Otherwise, an alternative is strictly preferred to the other if and only if it is at the left of the other; and alternatives are approved if and only if they are at the left of the horizontal line:

1st	<i>a</i>	<i>b</i>	
2nd	<i>a</i>		<i>b</i>
3rd		<i>a</i>	<i>b</i>
4th	<i>b</i>	<i>a</i>	
5th	<i>b</i>		<i>a</i>
6th		<i>b</i>	<i>a</i>
7th	(<i>ab</i>)		
8th		(<i>ab</i>)	

The following collection of preference-approvals may set off alarm bells. Suppose that there are only three voters and that they have, respectively, the first, third and fifth preference-approvals displayed above. Amongst these voters *a* is ranked above *b* by a majority—if we aggregate the ranking according to the majority method of May (1952), *a* will be ranked above *b* in the social ordering. At the same time, *a* is disapproved by a majority and *b* is approved by a majority—if we aggregate the evaluations by majority, thereby complying with Sanver’s (2010) axiom of majoritarian approval, *b* should be uniquely approved in the output. Thus applying majority on the decomposed components of these preference-approvals results in an inconsistent outcome.

The collection of preference-approvals described above effectively recreates the Condorcet paradox with the zero-line as an implicit third alternative. Compare the following profile of preference-approvals and profile of rankings.

<i>Voter 1</i>	<i>a</i>	<i>b</i>		<i>Voter 1'</i>	<i>a</i>	<i>b</i>	<i>z</i>
<i>Voter 2</i>		<i>a</i>	<i>b</i>	<i>Voter 2'</i>	<i>z</i>	<i>a</i>	<i>b</i>
<i>Voter 3</i>	<i>b</i>		<i>a</i>	<i>Voter 3'</i>	<i>b</i>	<i>z</i>	<i>a</i>

This Condorcet-like example presages a more general phenomenon, which we prove as an impossibility theorem. This impossibility states that if aggregation of preference-approvals is performed in a decomposed manner and the approval aggregation part satisfies unanimity, then the only possibility is a dictatorship. Thus we extend the example into a more general result, mirroring the connection between Condorcet’s paradox and Arrow’s (1950) impossibility theorem.

Unanimity is a weak condition; the important condition is decomposability, which takes the place of the well-known independence condition of Arrow (1950). Decomposability requires that the social evaluation for each alternative only depends upon the individuals’ evaluations for that alternative, and that the social ranking of the alternatives only depends upon the individuals’ rankings over the alternatives. Decomposability can be weakened in a variety of ways. We identify various possibilities when decomposability is weakened within the evaluative part, but the ordinal and evaluative parts are kept separate—these possibilities are stated as Theorems 4.1, 4.2 and 4.3. We also identify four specific possibilities that arise when the decomposability between the ordinal and evaluative parts is relaxed.

To give a brief overview of the rest of the paper: in Section 2 we define the basic model of preference-approvals. We show our central impossibility concerning the aggregation of preference-approvals in Section 3. Possibilities arise if

one does not require that preference-approvals are aggregated in a decomposed manner, which we explore in Section 4. We prove an extended version of the impossibility, applicable for more than two evaluation levels, in Section 5. Final remarks are provided in Section 6.

2 The preference-approval model

The basic building blocks of our model are a finite set of alternatives $A = \{a_1, \dots, a_m\}$ with $m \geq 2$, and a set of voters $N = \{1, \dots, n\}$ with $n \geq 2$. We write $W(A)$ for the set of complete preorders over A . A complete preorder is linear if it is antisymmetric; we write $L(A)$ for the set of linear orders over A . We use $R_i \in W(A)$ for the preference of $i \in N$ over A , and use P_i for the strict part of R_i . We write 2^A for the set of subsets of A and we use $B_i \in 2^A$ for the set of alternatives approved by $i \in N$.

A *preference-approval* of voter $i \in N$ is a pair $p_i = (R_i, B_i) \in W(A) \times 2^A$, composed of a complete preorder and set of approved alternatives, with the extra consistency requirement that $\forall x, y \in A, (x R_i y \text{ and } y \in B_i) \implies x \in B_i$.

We write Π for the set of all preference-approvals. A profile of preference approvals is some $\mathbf{p} = (p_1, \dots, p_n) \in \Pi^N$. As a general rule we write vectors in boldface.

A *preference-approval aggregator* is a function $\pi : D^N \rightarrow \Pi$, where $D \subseteq \Pi$. A preference-approval aggregator π is *dictatorial* if there is a voter d whose strict preference and approval line are reproduced in the output preference-approval; i.e., $d \in N$ is a dictator if for all $\mathbf{p} = ((R_1, B_1), \dots, (R_n, B_n)) \in D^N$ we have $\pi(\mathbf{p}) = (R, B_d)$ where R is some preference such that $x P y$ if $x P_d y$.

A *social welfare function* is some $f : D^N \rightarrow W(A)$, where $D \subseteq W(A)$. We write $f^*(\mathbf{R})$ for the strict part of $f(\mathbf{R})$, for all $\mathbf{R} \in D^N$. We say that f is *Pareto optimal* if, for all $x, y \in A$ and for all $\mathbf{R} \in D^N$ with $x P_i y$ for all $i \in N$, we have $x f^*(\mathbf{R}) y$. We say that f is *dictatorial* if there is some $d \in N$ such that for all $x, y \in A$ and for all $\mathbf{R} \in D^N$ with $x P_d y$ we have $x f^*(\mathbf{R}) y$.

We also consider aggregation in the approval voting tradition. For $x \in A$, an *elementary approval aggregator* is a function $\alpha_x : \{0, 1\}^N \rightarrow \{0, 1\}$. We refer to the elementary approval aggregator of a subscripted alternative, such as a_j , directly with the subscript, i.e., $\alpha_j = \alpha_{a_j}$. An elementary approval aggregator should be interpreted as a map from a profile where each voter either approves x (assigns it 1) or disapproves x (assigns it 0) into a situation where either x is globally approved (the result is 1) or x is globally disapproved (the result is 0). Such a function satisfies *unanimity* if $\alpha_x(0, \dots, 0) = 0$ and $\alpha_x(1, \dots, 1) = 1$. An *approval aggregator* is a function $\alpha : (2^A)^N \rightarrow 2^A$. We say that α satisfies (*alternative-wise*) *unanimity* if, for any $x \in A$ and any $\mathbf{B} \in (2^A)^N$,

(i) if $x \in B_i$ for all $i \in N$ then $x \in \alpha(\mathbf{B})$ and (ii) if $x \in B_i$ for no $i \in N$ then

$x \notin \alpha(\mathbf{B})$.¹

Our final definitions concern splitting up various aggregators into sub-functions. An approval aggregator α is *decomposable* if, for each alternative $x \in A$ there is an elementary approval aggregator α_x such that, for all $\mathbf{B} \in (2^A)^N$,

$$x \in \alpha(\mathbf{B}) \iff \alpha_x(1_{B_1}(x), \dots, 1_{B_n}(x)) = 1$$

where $1_X : A \rightarrow \{0, 1\}$ is the indicator function of $X \subseteq A$, defined for each $x \in A$ as $1_X(x) = 1 \iff x \in X$.

In such a case we write $\alpha = (\alpha_x)_{x \in A}$. Similarly, π can be decomposed into (f, α) if for all $((R_1, B_1), \dots, (R_n, B_n)) \in D^N$, we have $\pi((R_1, B_1), \dots, (R_n, B_n)) = (f(R_1, \dots, R_n), \alpha(B_1, \dots, B_n))$, and further into $(f, (\alpha_x)_{x \in A})$ if also $\alpha = (\alpha_x)_{x \in A}$. In such cases we write, respectively, $\pi = (f, \alpha)$ and $\pi = (f, (\alpha_x)_{x \in A})$.

3 An impossibility in aggregating preference-approvals

Theorem 3.1. *If $D = \Pi$ then any preference-approval aggregator $\pi = (f, \alpha_1, \dots, \alpha_m)$ such that α_x satisfies unanimity for all $x \in A$ is dictatorial.*

Proof: Take a preference-approval aggregator $\pi = (f, \alpha_1, \dots, \alpha_m) : \Pi^N \rightarrow \Pi$ such that each α_j satisfies unanimity. We first show that f is Pareto optimal. Take any $a, b \in A$ and any \mathbf{R} with $a P_i b$ for all $i \in N$. To establish $a f^*(\mathbf{R}) b$, define $\mathbf{B} \in (2^A)^N$ such that for all $i \in N$ and for all $x \in A$, $x \in B_i$ if and only if $x R_i a$. For each $i \in N$, the ordered pair (R_i, B_i) forms a preference-approval. By unanimity, $x \in \alpha(\mathbf{B})$ and $y \notin \alpha(\mathbf{B})$. By consistency of the output, $a f^*(\mathbf{R}) b$, thus f satisfies Pareto optimality.

To show that π is dictatorial, we define a social welfare function \tilde{f} over an extended set of alternatives $A \cup \{z\} = X$. The function \tilde{f} corresponds to π , while the alternative z represents the “zero-line”. Using two results from the literature on social welfare functions, we show that \tilde{f} is dictatorial, which in turn implies that π is dictatorial.

We say that a social welfare function $g : D^N \rightarrow W(X)$, $D \subseteq W(X)$, satisfies *binary independence* if for every pair $\{x, y\} \subseteq X$, for all $\mathbf{R}, \mathbf{R}' \in D^N$, if $\mathbf{R}|_{\{x, y\}} = \mathbf{R}'|_{\{x, y\}}$ then $g(\mathbf{R})|_{\{x, y\}} = g(\mathbf{R}')|_{\{x, y\}}$.² Note that in order for binary independence to not be trivially satisfied it is necessary that $|X| > 2$. We say a domain is *Arrovian* if every Pareto optimal social welfare function that satisfies

¹The term “alternative-wise” indicates a distinction from a weaker version of unanimity which requires that if all voters have exactly the same evaluations for all the alternatives, then the output evaluations are identical. We do not consider the weaker version of unanimity in this paper.

²By $g(\mathbf{R})|_{\{x, y\}}$, we mean the restriction of $g(\mathbf{R})$ over $\{x, y\}$, i.e., the ranking of $\{x, y\}$ as in $g(\mathbf{R})$. In a similar vein, $\mathbf{R}|_{\{x, y\}}$ is the restriction of \mathbf{R} over $\{x, y\}$, i.e., the preference profile over $\{x, y\}$ where each $i \in N$ orders $\{x, y\}$ as in R_i .

binary independence is dictatorial. Kelly (1994) gives conditions for a domain to be Arrovian.

For a preference-approval $p = (R, B)$, define a complete preorder $\bar{p} \in W(A \cup \{z\})$ by, for $\{a, b\} \subseteq A$, $a \bar{p} b \iff aRb$, $b \bar{p} a \iff bRa$. Moreover, for $x \in \{a, b\}$, $x \bar{p} z \iff x \in B$ and, $z \bar{p} x \iff x \notin B$. It can be verified that \bar{p} is complete and transitive. Write \bar{p}^* for the strict part of \bar{p} ; the above definitions imply that, for any $\{a, b\} \subseteq A$, $aPb \iff a\bar{p}^*b$ while for any $x \in \{a, b\}$, $x \in B \iff x\bar{p}^*z$.

Define $\bar{D} = \{\bar{p} : p \in \Pi\} \subset W(A \cup \{z\})$. By definition $p \mapsto \bar{p}$ is a bijection from Π to \bar{D} . This means there is also a bijection from preference approval profiles $\mathbf{p} \in \Pi^N$ to $\bar{\mathbf{p}} = (\bar{p}_i)_{i \in N} \in \bar{D}^N$. The domain \bar{D}^N is a (Cartesian) product domain that contains all possible linear orders and yet does not contain the complete preorder where all pairs of alternatives are considered indifferent. Thus this domain is Arrovian according to Kelly's (1994) "Theorem 1".

Define the social welfare function $\bar{f} : \bar{D}^N \rightarrow W(A \cup \{z\})$ as, for an arbitrary $\bar{\mathbf{p}} \in \bar{D}^N$, $\bar{f}(\bar{\mathbf{p}}) = \overline{\pi(\mathbf{p})}$. It may be noted that \bar{f} is defined in terms of a decomposable preference-approval aggregator $\pi = (f, (\alpha_x)_{x \in A})$, and thus that the output ranking over $a, b \in A$ can be determined by the output of f , and also that the output ranking over $a \in A$ and z can be determined by the output of α_a . We now show that \bar{f} is Pareto optimal and satisfies binary independence.

For Pareto optimality, take an arbitrary $\bar{\mathbf{p}}$. We split into two cases. First we consider pairs $\{a, b\} \subseteq A$. If $a \bar{p}_i^* b$ for all $i \in N$ then the result follows directly from the fact that f is Pareto optimal. Second, we consider pairs $\{a, z\}$ with $a \in A$. As a first subcase, suppose $a \bar{p}_i^* z$ for all $i \in N$. This implies that all voters approve a in \mathbf{p} . By unanimity of α_a , a is approved in $\pi(\mathbf{p})$, thus $a \bar{f}^*(\bar{\mathbf{p}})z$ as required. For the subcase where $z \bar{p}_i^* a$ for all $i \in N$, replace approve with disapprove in the preceding sentence.

For the proof of binary independence, we utilise results due to Campbell and Kelly (2000), who establish the notion of *relevant sets*. A relevant set is defined with respect to a Pareto optimal social welfare function $g : D^N \rightarrow W(X)$, where $D \subseteq W(X)$. A relevant set for a pair $\{x, y\} \subseteq X$ is the minimal (by inclusion) set of alternatives $Y \supseteq \{x, y\}$, $Y \subseteq X$, such that for all profiles $\mathbf{R}, \mathbf{R}' \in D^N \subseteq W(X)^N$, if $\mathbf{R}|_Y = \mathbf{R}'|_Y$ then $g(\mathbf{R})|_{\{x, y\}} = g(\mathbf{R}')|_{\{x, y\}}$. Campbell and Kelly's "Theorem 2 part (I)", paraphrased, states: for any Pareto optimal social welfare function g and distinct alternatives x, y, w , if the relevant set of $\{x, y\}$ is $\{x, y\}$ then the relevant set of $\{y, w\}$ either contains x or is identical to $\{y, w\}$.

We know already that \bar{f} is Pareto optimal. Take arbitrary $\{a, b\} \subseteq A$. By the decomposability of π , and by the definition of relevant sets as minimal by inclusion, the relevant set of $\{z, a\}$ is $\{z, a\}$ and the relevant set of $\{a, b\}$ is a subset of A that does not contain z . Thus, by "Theorem 2 part (I)", the relevant set of $\{a, b\}$ must be $\{a, b\}$, which establishes binary independence.

Altogether this implies that \bar{f} is dictatorial, with a dictator d . We claim that d must also be a dictator for π . Take an arbitrary profile $\mathbf{p} = (\mathbf{R}, \mathbf{B})$ and

write $\pi(\mathbf{p}) = (R, B)$.³ Take any $a, b \in A$. Let $a P_d b$ which, by definition of \bar{p} , implies $a \bar{p}_d^* b$. As \bar{f} is dictatorial, we have $a \bar{f}^*(\bar{\mathbf{p}})b$ which, by construction of \bar{f} , gives $a \bar{\pi}(\bar{\mathbf{p}})^* b$, implying $a P b$, as desired. Now, let $a \in B_d$ which, by definition of \bar{p} , implies $a \bar{p}_d^* z$. As \bar{f} is dictatorial, we have $a \bar{f}^*(\bar{\mathbf{p}})z$ which, by construction of \bar{f} , gives $a \bar{\pi}(\bar{\mathbf{p}})^* z$, implying $a \in B$, proving that π is dictatorial. Q.E.D.

Remark 1: There are two subtle requirements in this proof, concerning properties of the created domain and the informational restrictions of the created social welfare function. Concerning the first, the presence of all linear orders and the absence of the preference of complete indifference is sufficient for a domain to be Arrovian. This means that for a possibility based on a restricted domain it will be necessary to remove at least one preference-approval that corresponds to a linear order. Concerning the second, it is crucial that the relevant set of an alternative and the “zero-line” alternative was exactly that pair: if we do not decompose the aggregation of approvals, then this need no longer be the case, and the theorem of Campbell and Kelly (2000) no longer applies.

The impossibility vanishes for certain interesting restricted domains of preference-approvals. Denote by Π^- the domain where all alternatives below the line must be considered indifferent.⁴ Hence, a preference approval (R, B) is in Π^- if and only if $x R y$ for all $x, y \in A \setminus B$. For $A = \{a, b\}$, Π^- contains exactly the following six preference-approvals:

a	b	
a		b
b	a	
b		a
(ab)		
	(ab)	

For any $I \subseteq N$, $I \neq \emptyset$, let α_x^{ud} be the “unanimous disapproval rule” that disapproves $x \in A$ if and only if x is disapproved by all voters in N . Note that this elementary approval aggregator satisfies unanimity. Given that all alternatives below the line must be considered indifferent, the unanimous disapproval rule will only disapprove alternatives that all voters are indifferent between. Any approved alternative will be weakly preferred by all voters, and strongly preferred by at least one voter. It is a somewhat natural strengthening of Pareto optimality to require that in such cases the sometimes preferred alternative is ranked above the others. Formally, we say that a social welfare function f is

³To ease presentation, we transpose the Cartesian product without comment, i.e., when we write (\mathbf{R}, \mathbf{B}) , where $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{B} = (B_1, \dots, B_n)$, we are technically referring to the profile $((R_1, B_1), \dots, (R_n, B_n))$.

⁴The restriction to the domain Π^- is reasonable for many applications of the preference-approval framework. Suppose we need to select a date for a meeting; attendees may have preferences over those dates they can attend, but it seems reasonable to assume that they would be indifferent between those dates that they are unavailable.

strongly Pareto optimal if, for all $x, y \in A$ and for all $\mathbf{R} \in D^N$ with $x R_i y$ for all $i \in N$ while $x P_i y$ for some $i \in N$, we have $x f^*(\mathbf{R}) y$. Obviously, if a social welfare function is strongly Pareto optimal, it is also Pareto optimal. Note that although for the example of Π^- given above $|A| = 2$, the following result holds for $|A| \geq 2$.

Theorem 3.2. *If f is a strongly Pareto optimal social welfare function with domain $W(A)^N$ then $(f, (\alpha_x^{ud})_{x \in A})$ is a preference-approval aggregator with domain Π^- .*

Proof: We need to verify that for all $(\mathbf{R}, \mathbf{B}) \in (\Pi^-)^N$, $(f(\mathbf{R}), (\alpha_x^{ud})_{x \in A}(\mathbf{B})) = (R, B) \in \Pi$. For any $x, y \in A$, let $x R y$, without loss of generality. If $x \in B$, then (R, B) is a preference-approval whether $y \in B$ or not. Now let $x \notin B$. For (R, B) to be a preference-approval, we need to show that $y \notin B$. As $x \notin B$, by definition of α_x^{ud} , all voters disapprove x . Given this, the definition of Π^- implies that each voter must be of one of two types: first, a voter can approve y and disapprove x ; second, a voter can disapprove both and rank them as indifferent. But if any voter is of the first type, then $y P x$ by the strong Pareto criterion, contradicting $x R y$. Thus all voters disapprove y , and thus $y \notin B$ by definition of α_y^{ud} . Q.E.D.

4 Possibilities from relaxing decomposability

Our notion of decomposing a preference-approval aggregator takes place at two different levels. Theorem 3.1 requires:

1. that determining whether a specific alternative is approved is independent of the approvals of the other alternatives in the profile.
2. that the aggregation of rankings and approvals are independent of each other.

Relaxing either of these requirements leads to possibilities.

4.1 Relaxing the decomposability of the approval aggregation

We describe two families of rules that arise when we relax the internal decomposability of the approval part of a preference-approval aggregator. Our descriptions apply to the two alternative case: *well-behaved* approval aggregators that work alongside any Pareto optimal social welfare function; and *maximally discriminating* approval aggregators that distinguish between alternatives as often as possible. However, for three or more alternatives, these descriptions can no longer be applied. We nevertheless show that there are still possibilities in the three alternative case.

4.1.1 Well-behaved approval aggregators

There are approval aggregators over $\{a, b\}$ that create a preference-approval aggregator no matter what Pareto optimal social welfare function they are paired

with. Let \mathcal{N} be the set of non-empty proper subsets of N . Take any function $g : \mathcal{N} \times \mathcal{N} \rightarrow \{0, 1\}$. Define the approval aggregator α^g as follows:

1. If a is approved by all voters and b approved by none, approve only a .
2. Else if b is approved by all voters and a approved by none, approve only b .
3. Else:
 - (a) if a is approved by no voters or b is approved by none, approve neither.
 - (b) if a is approved by all voters or b is approved by all, approve both.
 - (c) otherwise, let N_a be the set of voters that approve a and N_b be the set of voters that approve b ; if $g(N_a, N_b) = 1$ then approve both alternatives, otherwise disapprove both.

Note that α^g satisfies unanimity. Let G be the set containing all functions of type $g : \mathcal{N} \times \mathcal{N} \rightarrow \{0, 1\}$. One may consider an approval aggregator α as *well-behaved* if (f, α) is well-defined for any Pareto optimal f . That is to say, well-behaved approval aggregators are compatible with any reasonable social welfare function, if one takes Pareto optimality as a minimal requirement for a social welfare function to be reasonable. Under such an interpretation, the theorem below characterises the well-behaved approval aggregators.

Theorem 4.1. *Let $m = 2$. If f is Pareto optimal and has domain $W(A)^N$ then for any $g \in G$, (f, α^g) defines a preference-approval aggregator with domain Π^N . In the other direction, if $\alpha = \alpha^g$ for no $g \in G$ then there is some Pareto optimal f with domain $W(A)^N$ such that (f, α) does not define a preference-approval aggregator with domain Π^N .*

Proof: Take f and α^g as described above and let $\pi = (f, \alpha^g)$. Take a profile $\mathbf{p} = (\mathbf{R}, \mathbf{B})$. Suppose $x \in \pi_2(\mathbf{p})$ and $y \notin \pi_2(\mathbf{p})$ for $\{x, y\} = \{a, b\}$.⁵ It suffices to show that $x \pi_1^*(\mathbf{p}) y$. As we must be in case 1 or case 2, x is approved by all voters and y is approved by none, thus by consistency $x P_i y$ for all $i \in N$, thus by Pareto optimality $x \pi_1^*(\mathbf{p}) y$, as desired.

Now take an arbitrary approval aggregator $\alpha' \neq \alpha^g$ that satisfies unanimity. This implies that the outputs given in cases 1 and 2 still apply, thus the output in one or other of 3. (a), (b) or (c) must be different. Suppose that (a) is not the case; the proof for the others is similar. This implies that there is a non-empty proper subset N_x of voters such that when x is approved only by all voters in N_x and y is approved by no voters, x is approved in the output and y disapproved in the output. Consider the function f' such that $w f'(\mathbf{R}) z$ if and only if $w P_i z$ for some $i \in N$. It is not hard to see that f' is a well-defined Pareto optimal social welfare function, however (f', α') is not admissible. Q.E.D.

⁵For preference-approval aggregators, and other functions that return pairs, sometimes we will want to isolate the first or second coordinate of the returned value. We do this by subscripting the function with 1 or 2. Generally, for a function of type $h : X \rightarrow Y \times Z$, we write h_1 for the projection of the function onto the first coordinate and h_2 for the projection of the function onto the second coordinate, i.e., for $h(x) = (y, z)$ we have $h_1(x) = y$ and $h_2(x) = z$.

4.1.2 Maximally discriminating approval aggregators

The approval aggregators of the previous subsection are very undiscriminating, which is precisely what is needed to make them compatible with many different Pareto optimal social welfare functions. On the other hand, if we start with a social welfare function, we can choose an approval aggregator that tends to discriminate between the alternatives, in terms of approvals, as much as possible.

We start with a Pareto optimal social welfare function f over $\{a, b\}$. Consider a linear order profile $\mathbf{R}' \in L(A)^{N'}$ over the two alternatives with a reduced electorate $N' \subseteq N$. We say that \mathbf{R}' is *sufficient* for $x \in A$ if for all profiles $\mathbf{R} \in L(A)^N$ with $R_i = R'_i$ for all $i \in N'$, we have $x f^*(\mathbf{R}) y$ where $y \in A \setminus \{x\}$.

We now define an aggregator α^f in terms of f . Take an arbitrary approval profile \mathbf{B} . The *total approval count* is $\sum_{i \in N} |B_i|$. Let I be the set of voters who approve a and disapprove b , and let J be the set of voters who approve b and disapprove a . Let $\mathbf{R}' \in L(A)^{I \cup J}$ be such that $a P'_i b$ if $i \in I$ and $b P'_i a$ if $i \in J$. If \mathbf{R}' is sufficient for $x \in A$ then $\alpha^f(\mathbf{B}) = \{x\}$. Otherwise, if the total approval count is greater than or equal to $n = |N|$, approve both alternatives; if the total approval count is less than n disapprove both alternatives.

One may consider an approval aggregator *maximally discriminating* if it classifies different alternatives into different classes as often as possible. The theorem below shows that α^f is maximally discriminating if one wants to be consistent with f .

Theorem 4.2. *Let $m = 2$. If f is Pareto optimal and has domain $W(A)^N$ then (f, α^f) is a preference-approval aggregator with domain Π^N . Further, if α is an approval aggregator such that for some profile \mathbf{B} and alternative $x \in \{a, b\}$, $\alpha(\mathbf{B}) = \{x\}$ but $\alpha^f(\mathbf{B}) = \emptyset$ or $\alpha^f(\mathbf{B}) = A$, then (f, α) does not define a preference-approval aggregator with domain Π^N .*

Proof: We first show that, for an arbitrary profile (\mathbf{R}, \mathbf{B}) , the output of $(f, \alpha^f)(\mathbf{R}, \mathbf{B}) = (R, B)$ is a preference-approval. Supposing that $x \in B$ and $y \notin B$, we need to show that $x P y$. Because $\alpha^f(\mathbf{B}) = \{x\}$, by the definition of α^f there is a reduced electorate N' and a profile $\mathbf{R}' \in L(A)^{N'}$ over this reduced electorate that is sufficient for x . Furthermore, every voter in this electorate must either approve x and disapprove y ; or disapprove x and approve y . Formally, for each $i \in N'$, either $B_i = \{x\}$ or $B_i = \{y\}$. So, for each $i \in N'$, because (R_i, B_i) is a preference-approval and thus R_i must be consistent with B_i , we must have $R'_i = R_i$. Because \mathbf{R}' is sufficient for x , this implies that $x P y$, as desired.

Now suppose α is an approval aggregator such that for some profile \mathbf{B} and alternative $x \in \{a, b\}$, $\alpha(\mathbf{B}) = \{x\}$ but $\alpha^f(\mathbf{B}) = \emptyset$ or $\alpha^f(\mathbf{B}) = A$. First, consider the case $\alpha^f(\mathbf{B}) = \emptyset$. This means that the total approval count is less than n , and that $\mathbf{R}' \in L(A)^{N'}$, $N' \subseteq N$, is not sufficient for x , which in turn implies that there is some \mathbf{R} such that for all $i \in N'$, $R_i = R'_i$ but $y f(\mathbf{R}) x$. Applying (f, α) to the profile (\mathbf{R}, \mathbf{B}) will not produce a preference-approval. Similar arguments reach the same conclusion for the case $\alpha^f(\mathbf{B}) = A$. Q.E.D.

4.1.3 Relaxing the decomposability of the approval aggregation with three or more alternatives

For three or more alternatives, our analysis is complicated by the fact that there are Pareto optimal social welfare functions which cannot be used to create a consistent preference-approval aggregator. Ranking by the sum of Borda scores provides an example—we take the Borda score of an alternative to be the number of other alternatives it is weakly preferred to. Consider the two following preference-approval profiles, each with two voters and three alternatives.

Voter 1	a	b		c	Voter 1'	b	a		c
Voter 2	a		c	b	Voter 2'	a		b	c

Note that in both profiles a must be approved and c must be disapproved, by the unanimity condition. Using Borda scoring, b is considered indifferent to c in the left profile, and indifferent to a in the right profile. This then means that b must be disapproved in the left profile and approved in the right profile, however b has the same approvals for both, thus no approval aggregator is consistent with ranking by Borda scores.

This means that for three or more alternatives there are no well-behaved approval aggregators. Nor can we find maximally discriminating approval aggregators for any given ranking function, because no approval aggregator may exist at all. So Theorems 4.1 and 4.2 cannot be interestingly extended to the case of three or more alternatives.

Nonetheless, there are still possibilities when there are three or more alternatives. Define a “disjunctive” social welfare function f_* as ranking by Borda scores if any voter is completely indifferent between all the alternatives and as a copy of the first voter’s preferences otherwise. Similarly, define α^* such that it approves all alternatives if each alternative is approved at least once, otherwise it disapproves all alternatives if each alternative is disapproved at least once, otherwise it copies the approvals of the first voter.

Theorem 4.3. *For $m \geq 2$, (f_*, α^*) is a preference-approval aggregator with domain Π^N .*

Proof: We need to check the consistency of the output. So suppose that for two alternatives $x, y \in A$, x is approved and y is disapproved. We need to show that $x P y$. Because x and y are in different approval classes, we must be in a profile where the first voter must have had her approvals copied. As such, no voter can be indifferent between all the alternatives, so the first voter also has her preference copied, thus $x P y$ as desired. Q.E.D.

4.2 Relaxing decomposability between rankings and approvals

There are also possibilities if the output ranking is allowed to depend upon the approvals in the profile. In this subsection we describe four such preference-approval aggregators.

4.2.1 Shortlist by elementary approval aggregators then rank

We first give an example of a preference-approval aggregator π^I , where the approval aggregation is internally decomposed and independent of the ranking aggregation while the output ranking depends upon the approvals in the profile. Under π^I , the approvals are used to select an approved shortlist, and then the ranking is performed separately upon the approved and non-approved alternatives. For $x \in A$, we define α_x^{maj} for arbitrary $\mathbf{z} \in \{0, 1\}^N$ as $\alpha_x^{maj}(\mathbf{z}) = 1$ if $|\{i \in N : z_i = 1\}| \geq \frac{n}{2}$, and $\alpha_x^{maj}(\mathbf{z}) = 0$ otherwise.

Define the approval part of π^I such that $\pi_2^I(\mathbf{R}, \mathbf{B}) = \alpha_x^{maj}(\mathbf{B})$. For the ranking part, we use a local version of Borda where the score for an alternative is calculated using either only the approved or only the disapproved alternatives. First, for each $X \subseteq A$, let $\text{bscore}_X : W(A)^N \rightarrow X \rightarrow \mathbb{N}$ be defined as $\text{bscore}_X(\mathbf{R})(x) = \sum_{i \in N} |\{y \in X : x R_i y\}|$. Define $\text{borda}_X : W(A)^N \rightarrow W(X)$ by $(x, y) \in \text{borda}_X(\mathbf{R})$ if and only if $\text{bscore}_X(\mathbf{R})(x) \geq \text{bscore}_X(\mathbf{R})(y)$.

Returning to the definition of π^I , and writing $B = \pi_2^I(\mathbf{R}, \mathbf{B})$, let $R = \pi_1^I(\mathbf{R}, \mathbf{B})$ be defined by $x P y$ for $x \in B$ and $y \notin B$; $x R y$ if and only if $(x, y) \in \text{borda}_B(\mathbf{R})$ for $x, y \in B$; and $x R y$ if and only if $(x, y) \in \text{borda}_{A \setminus B}(\mathbf{R})$ for $x, y \notin B$.

If we suppose that an approval aggregator is composed of elementary approval aggregators such as α_x^{maj} , there will be many cases where either all alternatives are approved or all alternatives are disapproved. One may desire instead to try to approve half of the alternatives. It is impossible to always do so without violating unanimity, but we can create an approval aggregator that only approves all alternatives when every alternative is approved by all voters and only disapproves all alternatives when every alternative is disapproved by all voters.

4.2.2 Shortlist by non-decomposable approval aggregator then rank

We now describe a preference-approval aggregator π^{II} that typically approves around half the alternatives; under π^{II} , the approval aggregation is not anymore internally decomposed but is still independent of the ranking aggregation while the output ranking depends upon the approvals in the profile. Our informal description is iterative. First approve all alternatives that are each approved by all the voters. If more than half of the alternatives are approved, disapprove the remaining alternatives. Otherwise, from the remaining set of alternatives, approve all those alternatives with maximal approval support—repeat this step until at least half the alternatives are approved, and afterwards disapprove the remaining alternatives. Formally, define⁶ $\gamma : \mathbb{N} \rightarrow (A \rightarrow \mathbb{N}) \rightarrow 2^A$ by $x \in \gamma_j(g)$ if and only if $g(x) \geq j$.

Define $\text{ascor} : (2^A)^N \rightarrow A \rightarrow \mathbb{N}$ by $\text{ascor}(\mathbf{B})(x) = |\{i \in N : x \in B_i\}|$. Note

⁶Our definitions here are given in functional style, i.e., functions are used as arguments. We assume left associativity of expressions, i.e., $h(i)(x)$ is $(h(i))(x)$, indeed we will typically write $h_i(x)$; and that we assume right associativity of type definitions, i.e., $h : X \rightarrow Y \rightarrow Z$ is implicitly $h : X \rightarrow (Y \rightarrow Z)$.

that for $j < k$ we have $\gamma_k(\text{ascore}(\mathbf{B})) \subseteq \gamma_k(\text{ascore}(\mathbf{B}))$ and that in particular, $\gamma_0(\text{ascore}(\mathbf{B})) = A$ and $\gamma_{n+1}(\text{ascore}(\mathbf{B})) = \emptyset$. These facts mean that the following definition is well-formed: Let $\alpha^{\text{half}}(\mathbf{B}) = \gamma_j(\text{ascore}(\mathbf{B}))$ for the maximal $j \in \mathbb{N}$ such that $|\gamma_j(\text{ascore}(\mathbf{B}))| \geq \frac{m}{2}$.

Our second preference-approval aggregator is like the first, only using the non-decomposable approval aggregator defined above. Define the approval part of π^{II} such that $\pi_2^{II}(\mathbf{R}, \mathbf{B}) = \alpha^{\text{half}}(\mathbf{B})$. For the ranking part, and writing $B = \pi_2^{II}(\mathbf{R}, \mathbf{B})$, let $R = \pi_1^{II}(\mathbf{R}, \mathbf{B})$ be defined such that $x P y$ whenever $x \in B$ and $y \notin B$; $x R y$ if and only if $(x, y) \in \text{borda}_B(\mathbf{R})$ for $x, y \in B$; and $x R y$ if and only if $(x, y) \in \text{borda}_{A \setminus B}(\mathbf{R})$ for $x, y \notin B$.

4.2.3 Perform ranking aggregation then approve according to the ranking

We now consider a function π^{III} whose ranking aggregation is independent of approvals while the output approvals depend upon the rankings in the profile. There is a somewhat trivial family of examples here: perform ranking aggregation by any desired method, and approve some fixed number or proportion of alternatives. This method completely ignores the input approvals, so the output ranking is obviously independent of the approvals in the profile. For an example social welfare function, we consider Borda ranking. Formally, define the ranking part of π^{III} such that $\pi_1^{III}(\mathbf{R}, \mathbf{B}) = \text{borda}_A(\mathbf{R})$. For the approval part, let $\pi^{III} = \gamma_j(\text{bscore}(\mathbf{R}))$ for the maximal $j \in \mathbb{N}$ such that $|\gamma_j(\text{bscore}(\mathbf{R}))| \geq \frac{m}{2}$. Note that \mathbf{B} occurs nowhere in the definition of π^{III} ; the output completely ignores the input approvals.

4.2.4 Borda with a movable zero determined by the zero-line

For completeness, we also describe a natural function π^{IV} for which both the ranking and the approvals are interdependent. Suppose that each voter assigns a score to each alternative depending upon how many alternatives there are in between the alternative and the zero-line, with a positive or negative score respectively if the alternative is above or below the zero-line; for each alternative, sum their scores; rank the alternatives according to their sum, and disapprove any and only alternatives with a sum strictly less than zero. This may be thought of as Borda ranking with a movable zero—note that although the idea behind this is natural, the detailed definition requires some arbitrary decisions. For π^{IV} , first define $\text{mscore}: \Pi \rightarrow A \rightarrow \mathbb{N}$ by

$$\text{mscore}(R, B)(x) = \begin{cases} |\{y \in B : x R y\}| & \text{if } x \in B, \\ 0 - |\{y \in A \setminus B : y R x\}| & \text{if } x \notin B \end{cases}$$

Define $\pi^{IV}(\mathbf{R}, \mathbf{B}) = (R, B)$ by $x R y$ if and only if $\sum_{i \in N} \text{mscore}(R_i, B_i)(x) \geq \sum_{i \in N} \text{mscore}(R_i, B_i)(y)$ and $x \in B$ if and only if $\text{mscore}(R_i, B_i)(x) \geq 0$.

5 Extending the impossibility to more evaluation levels

The impossibility of combining the ordinal and evaluative approaches also applies to the case where there are more than two evaluation levels. In order to consider multiple evaluation levels we must extend the preference-approval framework, which requires more definitions that immediately follow.

Denote by E a set of possible evaluations, $|E| \geq 2$. A *preference-evaluation* is a pair $v = (R, s)$ where $s : A \rightarrow E$ is a *sorting function* into $|E|$ categories.⁷ We suppose that there is a linear order \succsim on E . A preference-evaluation $v = (R, s)$ is *consistent* with \succsim if $x R y$ implies $s(x) \succsim s(y)$. Let Ω be the set of consistent preference evaluations. The existence of a linear order on E is central to our interpretation of the evaluative approach. For $e, e' \in E$, the interpretation of $e \succsim e'$ is that e is a better evaluative category than e' . Consistency with ordinal preference states, roughly speaking, that it is impossible to prefer a worse category to a better category.

A *preference-evaluation aggregator* is a function $\omega : \Omega^N \rightarrow \Omega$. We say that ω is dictatorial if there is a voter d whose strict preference and evaluations are reproduced in the output preference-evaluation; i.e., $d \in N$ is a dictator if for all $\mathbf{v} = ((R_1, s_1), \dots, (R_n, s_n)) \in \Omega^N$ we have $\omega(\mathbf{v}) = (R, s_d)$ where R is some ranking such that $x P y$ if $x P_d y$ for all $x, y \in A$.

An *elementary-evaluation aggregator* for an alternative x is a function $\eta_x : E^N \rightarrow E$. For $j \in \{1, \dots, m\}$, we typically shorten η_{a_j} to η_j . Such a function satisfies *unanimity* if $\eta_x(t, \dots, t) = t$ for all $t \in E$. We write $\omega = (f, \eta_1, \dots, \eta_m)$ if for every $((R_1, s_1), \dots, (R_n, s_n)) \in \Omega^N$ we have $\omega((R_1, s_1), \dots, (R_n, s_n)) = (f(R_1, \dots, R_n), s)$ where $s(x) = \eta_x(s_1(x), \dots, s_n(x))$ for all $x \in A$. Note that this implicitly means that ω can be decomposed into a ranking part f and m evaluation parts η_j , one for each alternative $a_j \in A$.

Theorem 5.1. *Any preference-evaluation aggregator $\omega = (f, (\eta_x)_{x \in A})$ such that η_x satisfies unanimity for all $x \in A$ is dictatorial.*

Proof: Take a preference-evaluation aggregator ω as in the statement of the theorem. We first show that f satisfies Pareto optimality. We then show that every pair of evaluation categories has a “local” dictator. Such a local dictator must be a dictator on the ranking function—this implies that all such local dictators coincide. Finally, we argue that this voter is also an evaluation dictator.

Take any $a, b \in A$ and any \mathbf{R} with $a P_i b$ for all $i \in N$. To establish $a f^*(\mathbf{R}) b$, take $e_1, e_2 \in E$ with $e_1 \succ e_2$ and define s such that for all $i \in N$ and for all $x \in A$, $s_i(x) = e_1$ whenever $x R_i a$ and $s_i(x) = e_2$ otherwise. For each $i \in N$, (R_i, s_i) is a preference-evaluation consistent with \succsim . By unanimity, $\omega_2(\mathbf{R}, \mathbf{s})(\mathbf{a}) = \mathbf{e}_1$ and $\omega_2(\mathbf{R}, \mathbf{s})(\mathbf{b}) = \mathbf{e}_2$. By consistency of the output, $a f^*(\mathbf{R}) b$, thus f satisfies Pareto optimality.

⁷If one thinks of 2 as the set $\{0, 1\}$, our previous definition of preference-approvals comprises the special case where $E = 2$.

Let $\omega = (f, (\eta_x)_{x \in A}) : \Omega^N \rightarrow \Omega$ such that f is Pareto optimal and η_x satisfies unanimity for all $x \in A$. Take an arbitrary pair of evaluation categories $e, e' \in E$. Write $\Omega|_{e,e'} = \Omega \cap (W(A) \times \{e, e'\}^A)$. There is a voter $d_{e,e'} = d$ such that for all $(\mathbf{R}, \mathbf{s}) \in (\Omega|_{e,e'})^N$, for $x \in A$ and $\omega(\mathbf{v}) = (R, s)$ we have $s_d(x) = s(x)$, otherwise we can translate ω with its domain restricted to $(\Omega|_{e,e'})^N$ into a preference-approval aggregator that violates Theorem 3.1.

We now argue that d is a dictator over f . Take arbitrary $x, y \in A$ and $\mathbf{R} \in W(A)^N$ such that $x P_d y$. Without loss of generality, suppose $e \succ e'$. Consider a profile \mathbf{s} such that $s_d(z) = e$ for all $z \in A$ with $z R_d x$; $s_d(z) = e'$ for all $z \in A$ with $x R_d z$; and $s_i(z) = e$ for all $i \in N \setminus \{d\}$ and for all $z \in A$. This \mathbf{s} forms a consistent preference-evaluation profile when combined with \mathbf{R} . By consistency, for $\omega(\mathbf{R}, \mathbf{s}) = (R, s)$, $x P y$. Thus $x f^*(\mathbf{R}) y$ if and only if $x P_d y$. Note that this implies $d_{e,e'} = d_{e'',e'''}$ for any $e, e', e'', e''' \in E$.

Finally we argue that d is also an evaluation dictator. Take arbitrary $x \in A$ and $\mathbf{s} \in (E^A)^N$. We want to show that $\eta_x(\mathbf{s}) = s_d(x)$. Write $e = s_d(x)$, and pick some $y \in A \setminus \{x\}$. Consider some profile \mathbf{s}' where for all $i \in N$, $s'_i(x) = s_i(x)$ and $s'_i(y) = e$. Consider any profile \mathbf{R} such that $x P_d y$ and that is consistent with \mathbf{s}' , note that such a profile exists. By unanimity, for $\omega(\mathbf{R}, \mathbf{s}') = (R, s)$, $s(y) = e$. By dictatorship of d on f , $x P y$. By consistency, $s(x) \succeq s(y) = e$. Similar to above, consider any profile \mathbf{R}' such that $y P_d x$ and that is consistent with \mathbf{s}' , and write $\omega(\mathbf{R}', \mathbf{s}') = (R', s')$. By a similar chain of arguments, $s'(y) \succeq s'(x)$, thus $s'(x) = e$. Because for each $i \in N$, $s'_i(x) = s_i(x)$, $\eta_x(\mathbf{s}) = s_d(x)$. Q.E.D.

By allowing more than two evaluation levels, Theorem 5.1 generalizes Theorem 3.1. We nevertheless present the two theorems separately. One reason for this is because we do not see an obvious direct proof of Theorem 5.1 that does not use Theorem 3.1. Moreover, it is less obvious how the possibilities expressed in Section 4 (which hold even when the social welfare function is required to be Pareto optimal) apply when there are more than two evaluation levels—while positive results are possible, the extra technicalities are cumbersome and would clutter the results. We do not consider domain restrictions when there are more than two evaluation levels for similar reasons.

6 Final remarks

There is a view that social choice should be performed using evaluations rather than rankings. In fact, the literature contains several examples of social choice procedures that use evaluations, including approval voting (Brams and Fishburn 1978), threshold aggregation involving three-graded rankings (Aleskerov et al. 2007; Alcantud and Laruelle 2014), utilitarian voting (Hillinger 2005) and range voting (Gaertner and Xu 2012; Pivato 2014; Zahid and De Swart 2015; Macé 2018). A further example is majority judgment, introduced by Balinski and Laraki (2011), which selects the alternative with the highest median evaluation.

To be sure, the median has earlier⁸ usages as a social choice rule, for example by Bassett and Persky (1999) who apply it within the traditional Arrovian ranking framework, however it should be noted that majority judgment can choose between alternatives with tied highest median, which (depending on the setting) may significantly reduce the size of the chosen set. Balinski and Laraki's contribution goes beyond this extra tiebreaking step: they provide a whole framework within which evaluative methods can be analysed.⁹ Part of their defense of majority judgment consists of a defense of the evaluative approach as a whole, for example they claim that "the central problem becomes *how to transform many individual grades of a common language into a single collective grade* where the individuals may have unknown preferences that are too complex to be formulated. Sharing a common language of grades makes no assumptions about a voter's or a judge's utilities or preferences." (Balinski and Laraki 2011, p. xiii, their italics.)

The debate between defenders of ranking aggregation and defenders of evaluation aggregation goes back to the early days of approval voting,¹⁰ but has been reactivated by the work of Balinski and Laraki.¹¹ This literature not only notes that particular evaluative methods are incompatible with majoritarianism conditions defined on social welfare functions (such as our example in the introduction) but reflects an incommensurability between the two approaches which is substantiated by our results. In fact, Theorem 5.1 shows that when unanimous evaluation aggregators are used, dictatorship is the only consistent way to independently aggregate rankings and evaluations. Even dictatorship fails to work when the social welfare function is assumed to satisfy mild conditions such as near unanimity, where if everyone except one voter ranks an alternative top, this alternative should be ranked top in the output. In brief, caution is advised in searching for a compromise that combines the two approaches by allowing individuals to have both rankings and evaluations.¹²

The impossibilities expressed by Theorems 3.1 and 5.1 depend on decom-

⁸As noted by Jean-François Laslier, methods that implicitly select the alternative with the highest median have been rediscovered many times. As well as Bucklin voting (used under this name in the early 20th century) and majoritarian compromise (Sertel and Yilmaz 1999), such a method was proposed as early as 1793 by Condorcet (McLean et al. 1994).

⁹Later work picks up this approach and compares different evaluative methods within an evaluative framework, for example work by Brams and Potthoff (2015).

¹⁰Consider the back-and-forth between Saari and Van Newenhizen (1988), Brams et al. (1988b), Saari and Van Newenhizen (1988) and again Brams et al. (1988a).

¹¹For example: Balinski and Laraki (2007), Felsenthal and Machover (2008), Brams (2011), Edelman (2012), Balinski and Laraki (2016) and Laslier (2017).

¹²The preference-approval framework that we treat in this paper can, as Sanver (2010) discusses, be mathematically placed within the traditional literature of social welfare functionals (Sen 1977) where cardinal or interpersonal comparisons are allowed. Preference-approvals present a weak version of ordinal level comparisons (OLC) which are explored by Roberts (1980). The closest previous work of this type that we are aware of is due to List (2001), who considers a narrow informational addition that he calls OLC+0, which only allows a single level of ordinal comparability. This is almost equivalent to the preference-approval framework, but it allows for alternatives to be on the zero-line, thus there is a third evaluative category within which indifference is forced. Also, List's results concern functions that produce choice sets or ordinal rankings, not functions that output preference-approvals or their equivalent.

posing the ranking and evaluation aggregators from each other (which embodies the incommensurability between the two approaches) but also on the decomposability of the evaluation aggregation itself. This latter decomposability is satisfied by every evaluative method that we are aware of; indeed, Balinski and Laraki (2011) argue that this decomposability is the correct interpretation of the independence property.

To justify our interest in decomposability, let us expand upon the analogy to the traditional issue concerning Arrovian independence. To determine social ranking between two alternatives, one can very well use information about how voters compare these to a third alternative. In fact, we know by Arrow's Theorem that this must be the case (if one wants a sensible ranking aggregation). One may like or dislike Arrovian independence as a principle of ranking aggregation but it is important to know that ranking aggregation is not decomposable into pairs. In a similar vein, one may like or dislike decomposability between ranking and evaluation aggregation as a principle, but it is worth knowing that such a decomposition is not possible.

When decomposability is relaxed there are various possibilities, as described in Section 4. In particular, relaxing the internal decomposability of the approval aggregation leads to the possibilities given by Theorem 4.1 and Theorem 4.2. More generally, we can structure the space of possible preference-approval aggregators along two axes. One axis imposes restrictions upon the ranking part of the aggregation: the strongest restriction here is a version of binary independence applied to preference-approval aggregators, which requires that the output ranking of each pair only depends upon the input rankings of these pairs. Binary independence can be weakened to the condition that the ranking aggregation only depends upon the ranking part of the profile, i.e., that the ranking aggregation is independent of approvals. Finally, there are aggregators with no restriction on the ranking aggregation. The second axis imposes restrictions on the evaluative part of the aggregation: the evaluations of each alternative can be computed independently of any other information; or can depend upon the evaluations of all the alternatives; or can be calculated using all the information from the profile. This structure, and various possibilities and impossibilities within, is drawn in Table 1 below. The results of the table only apply to preference-approvals, not to preference-evaluations. The first column expresses the restrictions on the evaluative part of aggregation (we suppose that the evaluative part respects unanimity) and the first row expresses the restrictions on the ranking part of the aggregation.

Theorem 3.2 gives some possibilities on one natural domain restriction in the preference-approval framework. We do not know if there are other interesting domain restrictions that would lead to similar results, but believe that there is potential further work to be done concerning extensions of the single-peaked or single-crossing conditions to the preference-approval framework.

As a final note, it may be wondered: given that we consider a particular form of aggregation, can we embed our work into a more general aggregation

	Binarily independent	Independent of approvals	No restriction
Independent of rankings and completely decomposable	Dictatorial for any number of alternatives (Theorem 5.1).	Dictatorial for any number of alternatives (Theorem 5.1)	Shortlist by majority approvals then rank (π^I)
Independent of rankings	Non-dictatorial aggregators exist for two alternatives (Theorems 4.1, 4.2; May (1952) provides a Pareto optimal social welfare function.)	Non-dictatorial aggregators exist (Theorem 4.3)	Shortlist a given proportion then rank (π^{II})
No restriction	Limited possibilities concerning approvals, see appendix.	Approve k highest ranked (π^{III})	Borda with movable zero (π^{IV})

framework?¹³ Wilson (1975), Nehring (2003), or Maniquet and Mongin (2016) provide possibilities, but perhaps the most promising general framework would be that of judgment aggregation (List and Puppe (2007) and Mongin (2012) provide surveys). It is relatively easy to express preference-approvals with linear rankings in the judgment aggregation framework. One way of doing so is to follow the method used to embed Arrow’s theorem into judgment aggregation (Dietrich and List 2007): assign a two-place predicate representing strict preference, and a one-place predicate representing approval, and add rationality conditions for asymmetry, transitivity, connectedness, and consistency of approval with preference. Alternative embeddings that instead use propositional logic are equally possible. However, imposing the axiom of independence (in the judgment aggregation sense) in these settings is a much stronger restriction than the decomposability that we apply in our work. More precisely, any impossibility result that would be obtained for this embedding would only correspond to the top-left cell of Table 1. Further, it is yet more work to apply this process to weak orders. To fully recreate our results, we would need to apply a weaker version of the axiom of independence in the judgment aggregation setting, which would require something along the lines of the generalised definitions of Dietrich (2015).¹⁴

¹³In particular we thank Phillippe Mongin for considerable efforts made in this direction.

¹⁴We thank Clemens Puppe for these observations.

7 Appendix: Limited possibilities in the presence of binary independence

During the review process, we became aware that all the cases of Table 1 are not fully covered in the main text. Specifically, we do not cover the cases where there are three or more alternatives, the ranking aggregation satisfies binary independence and either (1) the approval aggregation is independent of rankings or (2) there is no restriction on the approval aggregation; i.e., the bottom-left and middle-left cells of the table. Indeed, there are limited possibilities in these cases, a fact brought to our attention by Clemens Puppe. The issue is further complicated as our Theorems 3.1 and 5.1 do not assume Pareto optimality, thus binary independence only gets us Wilson’s (1972) result as opposed to Arrow’s (1950).

We will now briefly sketch these limited possibilities. We classify these in terms of (1) and (2) given above and in terms of the orthogonal axis provided by Wilson’s theorem: either (i) the ranking part always returns complete indifference between all alternatives; or (ii) there is a ranking dictator; or (iii) there is a ranking anti-dictator.

We will first suppose that there is no restriction on the approval rankings (1). If the ranking part always returns full indifference between the alternatives (1i), then the evaluative part cannot distinguish the alternatives: for each evaluative profile, all alternatives must be approved or disapproved. Roughly speaking, if the ranking part is “null”, the evaluative part must be similarly “null”.

Now suppose that there is a ranking dictator (1ii). The evaluations must follow the ranking of this dictator, in the sense that if she evaluates x better than y , then x must be ranked better than y in the output of the aggregator. If two alternatives are placed in the same evaluative class by the dictator, then they must also be placed in the same evaluative class by the aggregator, because the dictator may rank either alternative better than the other. Finally, unanimity must be respected. Altogether, given two evaluative categories, the approval aggregator must be of the following form. For each evaluative profile, there are three possibilities, two of which require further conditions: first, the output evaluations are identical to that of the dictator. Second, all alternatives are approved, this requires that no alternative is unanimously disapproved by all voters. Third, all alternatives are disapproved, this requires that no alternative is unanimous approved by all voters. The case of a ranking anti-dictator is symmetric (1iii).

We now move to the case where there is no restriction on the approval aggregation (2). Technically, our definition of alternative-wise unanimity does not apply to non-decomposed preference-approval aggregators, but one can imagine that we impose a natural such translation of this axiom to such aggregators. Then the possibilities of (2i) are almost identical to the (1i) case: for each preference-approval profile, all alternatives must be approved or disapproved.

The case of (2ii) provides richer possibilities than (1ii). In particular, for every preference-approval profile, there is some threshold in the ranking dictators

ranking: alternatives above this threshold are approved, alternatives below this are disapproved. Unanimity, if applied here, might impose some restrictions on where this threshold can be placed for particular profiles. The (2iii) case is symmetric.

8 References

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